SOME GEOMETRIC CONSTANTS AND THE EXTREME POINTS OF THE UNIT BALL OF BANACH SPACES

HIROYASU MIZUGUCHI

Communicated by Vasile Brînzănescu

In 2009, Mitani and Saito introduced and studied a geometric constant $\gamma_{X,\psi}$ of a Banach space X, by using the notion of ψ-ditect sum. For $t \in [0,1]$, the constant $\gamma_{X,\psi}(t)$ is defined as a supremum taken over all elements in the unit sphere of X. In this paper, we obtain that, for a Banach space which has a predual Banach space, the supremum can be taken over all extreme points of the unit ball. Then we calculate $\gamma_{X,\psi}(t)$ for some Banach spaces.

AMS 2010 Subject Classification: 46B20.

Key words: absolute normalized norm, ψ-direct sum, extreme point, von Neumann-Jordan constant, James type constant, von Neumann-Jordan type constant.

1. INTRODUCTION

There are several constants defined on Banach spaces such as the James constant [4] and von Neumann-Jordan constant [3]. It has been shown that these constants are very useful in the study of geometric structure of Banach spaces.

Throughout this paper, let X be a Banach space with $\dim X \geq 2$. By S_X and B_X, we denote the unit sphere and the unit ball of X, respectively. The von Neumann-Jordan constant is defined by

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x,y) \neq (0,0) \right\}$$

(Clarkson [3]), where the supremum can be taken over all $x \in S_X$ and $y \in B_X$. This constant has been considered in many papers ([3, 5, 8, 15, 17, 18] and so on). It is known that

(i) For any Banach space X, $1 \leq C_{NJ}(X) \leq 2$.

(ii) X is a Hilbert space if and only if $C_{NJ}(X) = 1$ ([5]).

(iii) X is uniformly non-square if and only if $C_{NJ}(X) < 2$ ([17]).

We note that the von Neumann-Jordan constant $C_{NJ}(X)$ is reformulated as

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in S_X, 0 \le t \le 1 \right\}.$$

In 2006, the function γ_X from $[0,1]$ into $[0,4]$ was introduced by Yang and Wang [21]:

$$\gamma_X(t) = \sup\left\{ \frac{\|x+ty\|^2 + \|x-ty\|^2}{2} : x, y \in S_X \right\}.$$

This function is useful to calculate the von Neumann-Jordan constant $C_{NJ}(X)$ for some Banach spaces. In fact, they computed $C_{NJ}(X)$ for X being Day-James spaces ℓ_∞-ℓ_1 and ℓ_2-ℓ_1 by using the function γ_X. In [11], Mitani and Saito introduced a geometrical constant $\gamma_{X,\psi}$ of a Banach space, by using the notion of ψ-direct sum.

Recall that a norm $\|\cdot\|$ on $\mathbb{C}^2$ is said to be *absolute* if

$$\|(z,w)\| = \|(|z|,|w|)\|$$

for all $(z,w) \in \mathbb{C}^2$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The family of all absolute normalized norms on $\mathbb{C}^2$ is denoted by AN_2. As in Bonsall and Duncan [2], AN_2 is in a one-to-one correspondence with the family Ψ_2 of all convex functions ψ on $[0,1]$ with $\max\{1-t,t\} \le \psi(t) \le 1$ for all $0 \le t \le 1$. Indeed, for any $\|\cdot\| \in AN_2$ we put $\psi(t) = \|(1-t,t)\|$. Then $\psi \in \Psi_2$. Conversely, for all $\psi \in \Psi_2$ let

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if} \quad (z,w) \ne (0,0), \\ 0 & \text{if} \quad (z,w) \ne (0,0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\psi(t) = \|(1-t,t)\|_\psi$ (cf. [15]). The functions corresponding to the ℓ_p-norms $\|\cdot\|_p$ on $\mathbb{C}^2$ are given by $\psi_p(t) = \{(1-t)^p + t^p\}^{1/p}$ if $1 \le p < \infty$, and $\psi_\infty(t) = \max\{1-t,t\}$ if $p = \infty$.

Takahashi, Kato and Saito [19] used the previous fact to introduce the notion of ψ-direct sum of Banach spaces X and Y as their direct sum $X \oplus Y$ equipped with the norm

$$\|(x,y)\|_\psi = \|(\|x\|,\|y\|)\|_\psi \quad ((x,y) \in X \oplus Y).$$

We denote by $X \oplus_\psi Y$ the direct sum $X \oplus Y$ with this norm. This notion has been studied by several authors (cf. [6, 7, 10, 14]).

For a Banach space X and $\psi \in \Psi_2$, the function $\gamma_{X,\psi}$ on $[0,1]$ is defined by

$$\gamma_{X,\psi}(t) = \sup\{\|(x+ty, x-ty)\|_\psi : x, y \in S_X\}.$$

Mitani and Saito [11] showed that

Proposition 1.1 ([11]).

(1) *For any Banach space X, $\psi \in \Psi_2$ and $t \in [0,1]$,*

$$2\psi\left(\frac{1-t}{2}\right) \le \gamma_{X,\psi}(t) \le 2(1+t)\psi\left(\frac{1}{2}\right).$$

(2) *For a Banach space X, $\psi \in \Psi_2$ and $t \in [0,1]$,*

$$\gamma_{X,\psi}(t) = \sup\left\{\|(x+ty, x-ty)\|_\psi : x, y \in B_X\right\}.$$

(3) *Let $\psi \in \Psi_2$ with $\psi \neq \psi_\infty$. Then a Banach space X is uniformly non-square if and only if $\gamma_{X,\psi}(t) < 2(1+t)\psi(1/2)$ for any (or some) t with $0 < t \le 1$.*

They also gave the value of $\gamma_{X,\psi}$ when X is a Hilbert space and an ℓ_p-space, and obtained a sufficient condition for uniform normal structure of Banach spaces in terms of $\gamma_{X,\psi}$.

Our aim in this paper is to study some properties of $\gamma_{X,\psi}$. In particular, we prove that for a Banach space X with a predual Banach space X_*, the function $\gamma_{X,\psi}(t)$ can be calculated as the supremum taken over all extreme points of the unit ball. Then we calculate $\gamma_{X,\psi}(t)$ for X being Day-James spaces ℓ_∞-ℓ_1 and ℓ_2-ℓ_1.

2. SOME PROPERTIES OF $\gamma_{X,\psi}$

We easily obtain the following properties of $\gamma_{X,\psi}$.

Proposition 2.1. *Let X be a Banach space, and let $\psi \in \Psi_2$. Then*

(1) *$\gamma_{X,\psi}(t)$ is a non-decreasing function;*

(2) *$\gamma_{X,\psi}(t)$ is a convex function;*

(3) *$\gamma_{X,\psi}(t)$ is continuous on $[0,1]$.*

(4) *The function*

$$\frac{\gamma_{X,\psi}(t) - \gamma_{X,\psi}(0)}{t}$$

is non-decreasing on $(0,1]$.

Proof. (1) Let $0 \le t_1 \le t_2 \le 1$. Take any $x, y \in S_X$. Since $\frac{t_1}{t_2}y \in B_X$, by Proposition 1.1 (2), we have

$$\|(x+t_1y, x-t_1y)\|_\psi = \left\|\left(x + t_2\cdot\frac{t_1}{t_2}y, x - t_2\cdot\frac{t_1}{t_2}y\right)\right\|_\psi \le \gamma_{X,\psi}(t_2).$$

Thus, we obtain $\gamma_{X,\psi}(t_1) \le \gamma_{X,\psi}(t_2)$.

(2) Let $t_1, t_2 \in [0,1]$ and $\lambda \in (0,1)$. Then, from the convexity of $\|\cdot\|_\psi \in AN_2$, we obtain

$$\gamma_{X,\psi}((1-\lambda)t_1 + \lambda t_2)) \le (1-\lambda)\gamma_{X,\psi}(t_1) + \lambda\gamma_{X,\psi}(t_2).$$

(3) Since (2) implies that $\gamma_{X,\psi}(t)$ is continuous on $(0,1)$, it suffices to show that $\gamma_{X,\psi}(t)$ is continuous at $t=0$ and $t=1$. From Proposition 1.1 (1), it follows that $\gamma_{X,\psi}(t)-\gamma_{X,\psi}(0)\le 2t\psi(1/2)$ for any $t\in(0,1)$. Thus, $\gamma_{X,\psi}(t)$ is continuous at $t=0$.

Take any $t\in(0,1)$ and any $x,y\in S_X$. Since $tx\in B_X$, by Proposition 1.1 (2), one can have that

$$t\|(x+y,x-y)\|_\psi=\|(tx+ty,tx-ty)\|_\psi\le\gamma_{X,\psi}(t)$$

and hence, $t\gamma_{X,\psi}(1)\le\gamma_{X,\psi}(t)$. Thus, we obtain $\gamma_{X,\psi}(1)-\gamma_{X,\psi}(t)\le(1-t)\gamma_{X,\psi}(1)$, which completes the proof.

(4) This is an easy consequence of (2). □

In [21], the sufficient condition of uniform smoothness was obtained in terms of γ_X. Related to this result, we obtain the following

PROPOSITION 2.2. *Let $\psi\in\Psi_2$. Assume that ψ takes the minimum at $t=1/2$. Then, a Banach space X is uniformly smooth if*

$$\lim_{t\to 0_+}\frac{\gamma_{X,\psi}(t)-\gamma_{X,\psi}(0)}{t}=0.$$

Proof. Put $M=1/\min_{0\le t\le 1}\psi(t)$. Then, for any $t\in(0,1]$ and any $x,y\in S_X$, we have

$$\begin{aligned}\frac{\|x+ty\|+\|x-ty\|}{2}-1&=\frac{\|(x+ty,x-ty)\|_1}{2}-1\\&\le\frac{M\gamma_{X,\psi}(t)}{2}-1\\&=\frac{M}{2}\left(\gamma_{X,\psi}(t)-\frac{\|(1,1)\|_1}{M}\right)\\&=\frac{M}{2}(\gamma_{X,\psi}(t)-\|(1,1)\|_\psi)\le\gamma_{X,\psi}(t)-\gamma_{X,\psi}(0),\end{aligned}$$

which implies $\rho_X(t)\le\gamma_{X,\psi}(t)-\gamma_{X,\psi}(0)$. Thus, we obtain

$$\lim_{t\to 0_+}\frac{\rho_X(t)}{t}\le\lim_{t\to 0_+}\frac{\gamma_{X,\psi}(t)-\gamma_{X,\psi}(0)}{t}=0$$

and then X is uniformly smooth. □

An element $x\in S_X$ is called an extreme point of B_X if $y,z\in S_X$ and $x=(y+z)/2$ implies $x=y=z$. The set of all extreme points of B_X is denoted by $\mathrm{ext}(B_X)$. There exists some infinite-dimensional Banach spaces whose unit ball has no extreme point. However, from the Banach-Alaoglu Theorem and

Krein-Milman Theorem, we have that for any Banach space, the unit ball of the dual space is the weakly* closed convex hull of its set of extreme points.

For $\psi \in \Psi_2$, the dual function ψ^* of ψ is defined by

$$\psi^*(s) = \sup_{t\in[0,1]} \frac{(1-s)(1-t)+st}{\psi(t)}$$

for $s \in [0,1]$. Then we have $\psi^* \in \Psi_2$ and that $\|\cdot\|_{\psi^*}$ is the dual norm of $\|\cdot\|_\psi$. It is easy to see that $\psi^{**} = \psi$. Let Y, Z be Banach spaces. Then according to [10], the dual of $Y \oplus_\psi Z$ is isomorphic to $Y^* \oplus_{\psi^*} Z^*$.

Suppose that X is a Banach space which has the predual Banach space X_*. Then the unit ball B_X is the weakly* closed convex hull of $\mathrm{ext}(B_X)$, and the direct sum $X \oplus_\psi X$ is isomorphic to the dual of $X_* \oplus_{\psi^*} X_*$.

THEOREM 2.3. *Let X be a Banach space with the predual Banach space. Then*

$$\gamma_{X,\psi}(t) = \sup\{\|(x+ty, x-ty)\|_\psi : x, y \in \mathrm{ext}(B_X)\}$$

for any $\psi \in \Psi_2$ and any $t \in [0,1]$.

Proof. Let $\psi \in \Psi_2$ and $t \in [0,1]$. Take arbitrary elements $x, y \in B_X$. It follows from $y \in B_X = \overline{\mathrm{co}}^{w*}(\mathrm{ext}(B_X))$ that there exists a net $\{y_\alpha\}$ in $\mathrm{co}(\mathrm{ext}(B_X))$ which weakly* converges to y. Since the net $\{(x+ty_\alpha, x-ty_\alpha)\}$ weakly* converges to $(x+ty, x-ty) \in X \oplus_\psi X$, we obtain

$$\begin{aligned}\|(x+ty, x-ty)\|_\psi &\le \varliminf_\alpha \|(x+ty_\alpha, x-ty_\alpha)\|_\psi \\ &\le \sup_\alpha \|(x+ty_\alpha, x-ty_\alpha)\|_\psi \\ &= \sup\{\|(x+tv, x-tv)\|_\psi : v \in \mathrm{co}(\mathrm{ext}(B_X))\}.\end{aligned}$$

For any $v \in \mathrm{co}(\mathrm{ext}(B_X))$, since $x \in B_X = \overline{\mathrm{co}}^{w*}(\mathrm{ext}(B_X))$, as in the preceding paragraph, we have

$$\|(x+tv, x-tv)\|_\psi \le \sup\{\|(u+tv, u-tv)\|_\psi : u \in \mathrm{co}(\mathrm{ext}(B_X))\}.$$

Hence, we obtain

$$\|(x+ty, x-ty)\|_\psi \le \sup\{\|(u+tv, u-tv)\|_\psi : u, v \in \mathrm{co}(\mathrm{ext}(B_X))\}.$$

On the other hand, from the convexity of $\|\cdot\|_\psi \in AN_2$, we directly have

$$\begin{aligned}&\sup\{\|(x+ty, x-ty)\|_\psi : x, y \in \mathrm{co}(\mathrm{ext}(B_X))\} \\ &= \sup\{\|(x+ty, x-ty)\|_\psi : x, y \in \mathrm{ext}(B_X)\}.\end{aligned}$$

Thus, we obtain this theorem. $\square$

In [16], Takahashi introduced the James and von Neumann-Jordan type constants of Banach spaces. For $t \in [-\infty, \infty)$ and $\tau \geq 0$, the James type constant is defined as

$$J_{X,t}(\tau) = \begin{cases} \sup\left\{ \left(\dfrac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2} \right)^{1/t} : x, y \in S_X \right\} & \text{if} \quad t \neq -\infty, \\ \sup\{\min(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\} & \text{if} \quad t = -\infty \end{cases}$$

(cf. [20, 22]). The von Neumann-Jordan type constant is defined as

$$C_t(X) = \sup\left\{ \frac{J_{X,t}(\tau)^2}{1+\tau^2} : 0 \leq \tau \leq 1 \right\}.$$

For $q \in [1, \infty)$ and $t \in [0, 1]$, it is easy to see that $J_{X,q}(t) = 2^{-1/q}\gamma_{X,\psi_q}(t)$. Thus, we have the following results on the James and von Neumann-Jordan type constants.

Corollary 2.4. *Let X be a Banach space with the predual Banach space.*
(1) *For any $q \in [1, \infty)$ and any $t \in [0, 1]$,*

$$J_{X,q}(t) = \sup\left\{ \left(\frac{\|x+ty\|^q + \|x-ty\|^q}{2} \right)^{1/q} : x, y \in \text{ext}(B_X) \right\}$$

(2) *For any $q \in [1, \infty)$,*

$$C_q(X) = \sup\left\{ \frac{(\|x+ty\|^q + \|x-ty\|^q)^{2/q}}{2^{2/q}(1+t^2)} : x, y \in \text{ext}(B_X), 0 \leq t \leq 1 \right\}.$$

In particular, one can easily has

$$\rho_X(t) = J_{X,1}(t) - 1 = \frac{\gamma_{X,\psi_1}(t)}{2} - 1$$

for any $t \in [0, 1]$, and

$$C_{NJ}(X) = C_2(X) = \sup\left\{ \frac{\gamma_{X,\psi_2}(t)^2}{2(1+t^2)} : 0 \leq t \leq 1 \right\}.$$

Hence, we obtain

Corollary 2.5. *Let X be a Banach space with the predual Banach space. Then,*

$$\rho_X(t) = \sup\left\{ \frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in \text{ext}(B_X) \right\}$$

for all $t \in [0, 1]$, and

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in \text{ext}(B_X),\ 0 \leq t \leq 1 \right\}.$$

3. EXAMPLES

In this section, we calculate $\gamma_{X,\psi}(t)$ for some two-dimensional Banach spaces. Then we mention a geometric constant which can not be expressed by $\gamma_{X,\psi}(t)$.

For p, q with $1 \leq p, q \leq \infty$, the Day-James space ℓ_p-ℓ_q is defined as the space $\mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\|_{p,q} = \begin{cases} \|(x_1, x_2)\|_p & \text{if} \quad x_1 x_2 \geq 0, \\ \|(x_1, x_2)\|_q & \text{if} \quad x_1 x_2 \leq 0. \end{cases}$$

Yang and Wang [21] calculated the von Neumann-Jordan constant of the Day-James spaces ℓ_∞-ℓ_1 and ℓ_2-ℓ_1 by using the notion of $\gamma_X(t)$. We compute $\gamma_{X,\psi}(t)$ of these spaces for all $\psi \in \Psi_2$ and all $t \in [0, 1]$. Remark that ℓ_∞-ℓ_1 and ℓ_2-ℓ_1 have the predual spaces ℓ_1-ℓ_∞ and ℓ_2-ℓ_∞, respectively (cf. [13]). Thus, from Theorem 2.3, we obtain

$$\gamma_{X,\psi}(t) = \sup\{\|(x + ty, x - ty)\|_\psi : x, y \in \mathrm{ext}(B_X)\}$$

for X being $\ell_\infty - \ell_1$ or $\ell_2 - \ell_1$. We note that, for $x, y \in \mathrm{ext}(B_X)$, $\|(x - ty, x + ty)\|_\psi$ does not necessarily coincide with $\|(x + ty, x - ty)\|_\psi$.

Example 3.1. Let X be the Day-James space $\ell_\infty - \ell_1$, $\psi \in \Psi_2$ and $t \in [0, 1]$. Then

$$\gamma_{X,\psi}(t) = (2 + t) \max\left\{\psi\left(\frac{1}{2+t}\right), \psi\left(\frac{1+t}{2+t}\right)\right\}.$$

In particular, for $q \in [1, \infty)$,

$$J_{X,q}(t) = \left(\frac{1 + (1+t)^q}{2}\right)^{1/q} \quad \text{and} \quad C_q(X) = \frac{\{1 + (1+t_0)^q\}^{2/q}}{2^{2/q}(1 + t_0^2)},$$

where $t_0 \in (0, 1)$ such that $(1 + t_0)^{q-1}(1 - t_0) - t_0 = 0$.

Proof. It is easy to check

$$\mathrm{ext}(B_X) = \{\pm(1, 1),\ (\pm 1, 0),\ (0, \pm 1)\}.$$

By the definition of $\|\cdot\|_{\infty,1}$, we may consider $\|(x + ty, x - ty)\|_\psi$ and $\|(x - ty, x + ty)\|_\psi$ only in the following three cases.

Case 1. $x = (1, 0)$, $y = (0, 1)$. We have

$$\|x + ty\|_{\infty,1} = \|(1, t)\|_{\infty,1} = 1 \quad \text{and} \quad \|x - ty\|_{\infty,1} = \|(1, -t)\|_{\infty,1} = 1 + t.$$

Case 2. $x = (1, 1)$, $y = (1, 0)$. We have

$$\|x + ty\|_{\infty,1} = \|(1 + t, 1)\|_{\infty,1} = 1 + t$$

and

$$\|x - ty\|_{\infty,1} = \|(1 - t, 1)\|_{\infty,1} = 1.$$

Case 3. $x = (1,0)$, $y = (1,1)$. We have

$$\|x + ty\|_{\infty,1} = \|(1+t, t)\|_{\infty,1} = 1 + t$$

and

$$\|x - ty\|_{\infty,1} = \|(1-t, -t)\|_{\infty,1} = 1.$$

Thus, we obtain

$$\begin{aligned}\gamma_{X,\psi}(t) &= \max\{\|(1+t,1)\|_\psi, \|(1,1+t)\|_\psi\}\\ &= (2+t)\max\left\{\psi\left(\frac{1}{2+t}\right), \psi\left(\frac{1+t}{2+t}\right)\right\},\end{aligned}$$

and hence, for $q \in [1,\infty)$,

$$J_{X,q}(t) = \left(\frac{1+(1+t)^q}{2}\right)^{1/q}$$

and

$$C_q(X) = \sup\left\{\frac{\{1+(1+t)^q\}^{2/q}}{2^{2/q}(1+t^2)} : 0 \le t \le 1\right\}.$$

Let $t_0 \in (0,1)$ with $(1+t_0)^{q-1}(1-t_0) - t_0 = 0$. Then the function

$$\frac{\{1+(1+t)^q\}^{2/q}}{2^{2/q}(1+t^2)}$$

takes the maximum at $t = t_0$. This completes the proof. □

To calculate $\gamma_{X,\psi}(t)$ for X being the Day-James space ℓ_2-ℓ_1, we note that for any $\psi \in \Psi_2$, if $|z| \le |u|$ and $|w| \le |v|$, then $\|(z,w)\|_\psi \le \|(u,v)\|_\psi$, and if $|z| < |u|$ and $|w| < |v|$, then $\|(z,w)\|_\psi < \|(u,v)\|_\psi$ (cf. [2]). More results on the monotonicity of absolute normalized norms can be found in [12, 10, 19] and so on.

Example 3.2. Let X be the Day-James space ℓ_2-ℓ_1, $\psi \in \Psi_2$ and $t \in [0,1]$. Then

$$\begin{aligned}&\gamma_{X,\psi}(t)\\ &= (1+t+\sqrt{1+t^2})\max\left\{\psi\left(\frac{1+t}{1+t+\sqrt{1+t^2}}\right), \psi\left(\frac{\sqrt{1+t^2}}{1+t+\sqrt{1+t^2}}\right)\right\}.\end{aligned}$$

In particular, for $q \in [1,\infty)$,

$$J_{X,q}(t) = \left(\frac{(1+t)^q + (1+t^2)^{q/2}}{2}\right)^{1/q} \quad \text{and} \quad C_q(X) = \left(\frac{1+2^{q/2}}{2}\right)^{2/q}.$$

Proof. One can easily have that

$$\mathrm{ext}(B_X) = \{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_1 x_2 \geq 0\}.$$

Let $t \in [0,1]$. For θ_1, θ_2 with $0 \leq \theta_1 \leq \theta_2 \leq \pi/2$, put $x = (\cos\theta_1, \sin\theta_1)$ and $y = (\cos\theta_2, \sin\theta_2)$. Then $x + ty = (\cos\theta_1 + t\cos\theta_2, \sin\theta_1 + t\sin\theta_2)$ and $x - ty = (\cos\theta_1 - t\cos\theta_2, \sin\theta_1 - t\sin\theta_2)$. Thus, we have

$$\|x + ty\|_{2,1} = \|x + ty\|_2 = \sqrt{1 + t^2 + 2t\cos(\theta_2 - \theta_1)}$$

and hence, $\sqrt{1+t^2} \leq \|x + ty\|_{2,1} \leq 1 + t$.

If $\sin\theta_1 \geq t\sin\theta_2$, then one has

$$\|x - ty\|_{2,1} = \|x - ty\|_2 = \sqrt{1 + t^2 - 2t\cos(\theta_2 - \theta_1)} \leq \sqrt{1+t^2}.$$

Hence, from the monotonicity of $\|\cdot\|_\psi \in AN_2$,

$$\|(x + ty, x - ty)\|_\psi \leq \|(1 + t, \sqrt{1+t^2})\|_\psi$$

and

$$\|(x - ty, x + ty)\|_\psi \leq \|(\sqrt{1+t^2}, 1 + t)\|_\psi.$$

Suppose that $\sin\theta_1 < t\sin\theta_2$. Then we have

$$\|x - ty\|_{2,1} = \|x - ty\|_1 = \cos\theta_1 - t\cos\theta_2 - \sin\theta_1 + t\sin\theta_2.$$

One can show that $\|x + ty\|_{2,1} + \|x - ty\|_{2,1} \leq 1 + t + \sqrt{1+t^2}$. Indeed, putting $e_1 = (1,0)$ and $e_2 = (0,1)$, by the triangle inequality, we have

$$\begin{aligned}
\|x + ty\|_{2,1} - \sqrt{1+t^2} &= \|x + ty\|_{2,1} - \|e_1 + te_2\|_{2,1} \\
&\leq \|(x + ty) - (e_1 + te_2)\|_{2,1} \\
&\leq \|x - e_1\|_{2,1} + t\|y - e_2\|_{2,1} \\
&= 1 - \cos\theta_1 + \sin\theta_1 + t(\cos\theta_2 + 1 - \sin\theta_2) \\
&= 1 + t - \|x - ty\|_{2,1}.
\end{aligned}$$

On the other hand, we have already obtained that $\sqrt{1+t^2} \leq \|x+ty\|_{2,1} \leq 1 + t$. Thus, by the monotonicity and convexity of $\|\cdot\|_\psi \in AN_2$, we have

$$\|(x + ty, x - ty)\|_\psi \leq \max\{\|(\sqrt{1+t^2}, 1 + t)\|_\psi, \|(1 + t, \sqrt{1+t^2})\|_\psi\}$$

and

$$\|(x - ty, x + ty)\|_\psi \leq \max\{\|(\sqrt{1+t^2}, 1 + t)\|_\psi, \|(1 + t, \sqrt{1+t^2})\|_\psi\}.$$

Therefore we obtain

$$\gamma_{X,\psi}(t) \leq \max\{\|(\sqrt{1+t^2}, 1 + t)\|_\psi, \|(1 + t, \sqrt{1+t^2})\|_\psi\}.$$

Finally, for $e_1=(1,0)$ and $e_2=(0,1)$, one has

$$\|(e_1+te_2,e_1-te_2)\|_\psi=\|(\sqrt{1+t^2},1+t)\|_\psi$$

and

$$\|(e_1-te_2,e_1+te_2)\|_\psi=\|(1+t,\sqrt{1+t^2})\|_\psi.$$

Thus, we obtain

$$\begin{aligned}&\gamma_{X,\psi}(t)\\&=\max\{\|(\sqrt{1+t^2},1+t)\|_\psi,\|(1+t,\sqrt{1+t^2})\|_\psi\}\\&=(1+t+\sqrt{1+t^2})\max\left\{\psi\left(\frac{1+t}{1+t+\sqrt{1+t^2}}\right),\psi\left(\frac{\sqrt{1+t^2}}{1+t+\sqrt{1+t^2}}\right)\right\}\end{aligned}$$

and hence, for $q\in[1,\infty)$,

$$J_{X,q}(t)=\left(\frac{(1+t)^q+(1+t^2)^{q/2}}{2}\right)^{1/q}$$

and

$$C_q(X)=\sup\left\{\frac{\{(1+t)^q+(1+t^2)^{q/2}\}^{2/q}}{2^{2/q}(1+t^2)}:0\le t\le 1\right\}.$$

Since the function

$$\frac{\{(1+t)^q+(1+t^2)^{q/2}\}^{2/q}}{2^{2/q}(1+t^2)}$$

is increasing on the interval $[0,1]$, one has

$$C_q(X)=\frac{(2^q+2^{q/2})^{2/q}}{2\cdot 2^{2/q}}=\frac{(1+2^{q/2})^{2/q}}{2^{2/q}},$$

as desired. $\square$

Although Theorem 2.3 holds, some geometric constants does not necessarily coincide with the supremum taken over all extreme points of the unit ball. We show a such example.

The constant

$$C_Z(X)=\sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2+\|y\|^2}:x,y\in X,\ (x,y)\ne(0,0)\right\}.$$

was introduced by Zbăganu [23]. As in the von Neumann-Jordan constant, this constant is reformulated as

$$C_Z(X)=\sup\left\{\frac{\|x+ty\|\|x-ty\|}{1+t^2}:x,y\in S_X,\ 0\le t\le 1\right\}.$$

Example 3.3. Let X be the Day-James space ℓ_∞-ℓ_1. Then

$$\sup\left\{\frac{\|x+ty\|\|x-ty\|}{1+t^2} : x,y\in \mathrm{ext}(B_X),\ 0\le t\le 1\right\} < C_Z(X).$$

Proof. According to [1], we have $C_Z(X)=5/4$.
On the other hand, as in Example 3.1, we obtain

$$\sup\left\{\frac{\|x+ty\|\|x-ty\|}{1+t^2} : x,y\in \mathrm{ext}(B_X),\ 0\le t\le 1\right\} = \max_{0\le t\le 1}\frac{1+t}{1+t^2} = \frac{1}{2(\sqrt{2}-1)},$$

and hence, this supremum is less than the Zbăganu constant $C_Z(X)$. □

Remark 3.4. From [16], Zbăganu constant $C_Z(X)$ coincide with the von Neumann-Jordan type constant $C_0(X)$.

We do not know whether, for any q less than 1, there exist a Banach space X in which the von Neumann-Jordan type constant $C_q(X)$ does not coincide with the supremum taken over all extreme points of the unit ball B_X.

REFERENCES

[1] J. Alonso and P. Martin, *A counterexample for a conjecture of G. Zbăganu about the Neumann-Jordan constant.* Rev. Roumaine Math. Pures Appl. **51** (2006), 135–141.

[2] F.F. Bonsall and J. Duncan, *Numerical Ranges II.* London Math. Soc. Lecture Note Series **10**, Cambridge University Press, Cambridge, 1973.

[3] J.A. Clarkson, *The von Neumann-Jordan constant for the Lebesgue spaces.* Ann. of Math. **38** (1937), 114–115.

[4] J. Gao and K.S. Lau, *On the geometry of spheres in normed linear spaces.* J. Austral. Math. Soc. Ser. A **48** (1990), 101–112.

[5] P. Jordan and J. von Neumann, *On inner products in linear metric spaces.* Ann. of Math. **36** (1935), 719–723.

[6] M. Kato, K.-S. Saito and T. Tamura, *On ψ-direct sums of Banach spaces and convexity.* J. Aust. Math. Soc. **75** (2003), 413–422.

[7] M. Kato, K.-S. Saito and T. Tamura, *Uniform non-squareness of ψ-direct sums of Banach space $X\oplus_\psi Y$.* Math. Inequal. Appl. **7** (2004), 429–437.

[8] M. Kato and Y. Takahashi, *On sharp estimates concerning the von Neumann-Jordan and James constants for a Banach space.* Rend. Circ. Mat. Palermo, Serie II, Suppl. **82** (2010), 75–91.

[9] R.E. Megginson, *An Introduction to Banach Space Theory.* Grad. Texts in Math. **183**, Springer, New York, 1998.

[10] K.-I. Mitani, S. Oshiro and K.-S. Saito, *Smoothness of ψ-ditect sums of Banach spaces.* Math. Inequal. Appl. **8** (2005), 147–157.

[11] K.-I. Mitani and K.-S. Saito, *A new geometrical constant of Banach spaces and the uniform normal structure.* Comment. Math. **49** (2009), 3–13.

[12] K.-I. Mitani, K.-S. Saito and N. Komuro, *The monotonicity of absolute normalized norms on $\mathbb{C}^n$.* Nihonkai Math. J. **22** (2011), 91–102.

[13] W. Nilsrakoo and S. Saejung, *The James constant of normalized norms on* $\mathbb{R}^2$. J. Inequal. Appl. 2006, Art. ID 26265, 12pp.

[14] K.-S. Saito and M. Kato, *Uniform convexity of* ψ*-direct sums of Banach spaces*. J. Math. Anal. Appl. **277** (2003), 1–11.

[15] K.-S. Saito, M. Kato and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$. J. Math. Anal. Appl. **244** (2000), 515–532.

[16] Y. Takahashi, *Some geometric constants of Banach spaces – a unified approach*. Proc. of 2$^{\text{nd}}$ International Symposium on Banach and Function Spaces **II** (2008), 191–220.

[17] Y. Takahashi and M. Kato, *von Neumann-Jordan constant and uniformly non-square Banach spaces*. Nihonkai. Math. J. **9** (1998), 155–169.

[18] Y. Takahashi and M. Kato, *A simple inequality for the von Neumann-Jordan and James constants of a Banach space*. J. Math. Anal. Appl. **359** (2009), 602–609.

[19] Y. Takahashi, M. Kato and K.-S. Saito, *Strict convexity of absolute norms on* $\mathbb{C}^2$ *and direct sums of Banach spaces*. J. Inequal. Appl. **7** (2002), 179–186.

[20] C. Yang, *An inequality between the James type constant and the modulus of smoothness*. J. Math. Anal. Appl. **398** (2013), 622–629.

[21] C. Yang and F. Wang, *On a new geometric constant related to the von Neumann-Jordan constant*. J. Math. Anal. Appl. **324** (2006), 555–565.

[22] C. Yang and F. Wang, *Some properties of James type constant*. Appl. Math. Lett. **25** (2012), 538–544.

[23] G. Zbăganu, *An inequality of M. Rădulescu and S. Rădulescu which characterizes the inner product spaces*. Rev. Roumaine Math. Pures Appl. **47** (2002), 253–257.

Received 29 October 2013

Niigata University,
Department of Mathematical Sciences,
Graduate School of Science
and Technology,
Niigata 950-2181, Japan
mizuguchi@m.sc.niigata-u.ac.jp