THE VOLUME OF A SET OF ARCS ON A VARIETY

TOMMASO DE FERNEX and MIRCEA MUSTAŢĂ

We give a definition of volume for subsets in the space of arcs of an algebraic variety, and study its properties. Our main result relates the volume of a set of arcs on a Cohen-Macaulay variety to its jet-codimension, a notion which generalizes the codimension of a cylinder in the arc space of a smooth variety.

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1. **INTRODUCTION**

In this paper, we give a definition of volume for subsets in the space of arcs of an algebraic variety and study its properties. As our definition implies that the volume of a set of arcs is finite if and only if its projection to the variety is a finite set of closed points, we can restrict without loss of generality to the case of an affine variety. Suppose therefore that $X = \mathrm{Spec}(R)$ is an n-dimensional affine algebraic variety defined over an algebraically closed field of characteristic zero. For every ideal $\mathfrak{a}$ in R we denote by $\ell(R/\mathfrak{a})$ the length of the quotient ring $R/\mathfrak{a}$ and, if the cosupport consists of one point x defined by the ideal $\mathfrak{m}_x$, we denote by $e(\mathfrak{a})$ the Hilbert-Samuel multiplicity of $R_{\mathfrak{m}_x}$ with respect to $\mathfrak{a} R_{\mathfrak{m}_x}$.

Let X_∞ be the arc scheme of X. Recall that for every field extension K/k, the K-valued points of X_∞ are in natural bijection with the arcs $\mathrm{Spec}\, K[\![t]\!] \to X$ (see [10], Section 3). For every subset $C \subseteq X_\infty$ and any integer $m \geq 0$, we consider the ideal

$$\mathfrak{b}_m(C) := \{f \in R \mid \mathrm{ord}_\gamma(f) \geq m \text{ for all } \gamma \in C\}.$$

This defines a graded sequence of ideals $\mathfrak{b}_\bullet(C) = (\mathfrak{b}_m(C))_{m \geq 0}$. We then define the volume of C by the formula

$$\mathrm{vol}(C) := \mathrm{vol}(\mathfrak{b}_\bullet(C)) = \limsup_{m \to \infty} \frac{\ell(R/\mathfrak{b}_m(C))}{m^n/n!}.$$

It follows from [3] that the limsup is, in fact, a limit. It is easy to see that $\mathrm{vol}(C) < \infty$ if and only if the image π of C in X is a finite set of closed points. Here $\pi \colon X_\infty \to X$ is the canonical projection mapping an arc γ to its base point $\gamma(0)$. The volume satisfies the following inclusion/exclusion property.

PROPOSITION 1.1. *If $C_1, C_2 \subseteq X_\infty$, then*

$$\mathrm{vol}(C_1 \cup C_2) + \mathrm{vol}(C_1 \cap C_2) \leq \mathrm{vol}(C_1) + \mathrm{vol}(C_2).$$

The *contact locus* of order at least q of an ideal $\mathfrak{a} \subseteq R$ is defined to be

$$\mathrm{Cont}^{\geq q}(\mathfrak{a}) = \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(\mathfrak{a}) \geq q\}.$$

Contact loci form a special class of subsets in X_∞. For ideals cosupported at one point, the volumes of these sets relate to the Samuel multiplicities of the ideal in the following way.

PROPOSITION 1.2. *For every ideal $\mathfrak{a} \subseteq R$ whose cosupport consists of one point and for every $m, p \geq 1$, we have*

$$m^n \cdot \mathrm{vol}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) \leq (mp)^n \cdot \mathrm{vol}(\mathrm{Cont}^{\geq mp}(\mathfrak{a})) \leq e(\mathfrak{a})$$

for every $m, p \geq 1$. Furthermore, both inequalities are equalities for m sufficiently divisible.

Generalizing the definition of codimension of a cylinder in the space of arcs of a smooth variety, we define the *jet-codimension* of an irreducible closed subset C of X_∞ to be

$$\mathrm{jet\text{-}codim}(C) := \lim_{p \to \infty} \left((p+1)n - \dim \overline{\pi_p(C)}\right)$$

where $\pi_p \colon X_\infty \to X_p$ is the truncation map to the p-jet space. This definition extends to an arbitrary set $C \subseteq X_\infty$ by taking the smallest jet-codimension of the irreducible components of the closure $\overline{C}$ of C in X_∞. We will see, for instance, that if X is smooth, then the jet-codimension of a set C coincides with its Krull codimension $\mathrm{codim}(C)$ (which is similarly defined as the smallest Krull codimension of an irreducible component of $\overline{C}$).

Our main result relates the volume of a set of arcs on a Cohen-Macaulay variety to its jet-codimension.

THEOREM 1.3. *If X is Cohen-Macaulay, of dimension n, then for every subset $C \subseteq X_\infty$ whose image in X is a closed point we have*

$$\mathrm{vol}(C)^{1/n} \cdot \mathrm{jet\text{-}codim}(C) \geq n.$$

In particular, if X is smooth, then

$$\mathrm{vol}(C)^{1/n} \cdot \mathrm{codim}(C) \geq n.$$

The proof of this theorem requires a suitable extension of the main result of [6] to singular varieties, which we discuss next. Let $\mathfrak{a} \subseteq R$ be an $\mathfrak{m}$-primary ideal, where $\mathfrak{m} \subset R$ is a maximal ideal. If X is smooth, then the colength and the Hilbert-Samuel multiplicity of $\mathfrak{a}$ are related to the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ by the formulas

$$(1) \qquad\qquad (n! \cdot \ell(\mathcal{O}_X/\mathfrak{a}))^{1/n} \cdot \mathrm{lct}(\mathfrak{a}) \geq n,$$

$$(2) \qquad\qquad e(\mathfrak{a})^{1/n} \cdot \mathrm{lct}(\mathfrak{a}) \geq n.$$

We want to extend this result to all Cohen-Macaulay varieties. If X is singular, then the log canonical threshold (even when it is defined) is not the right invariant to consider. Instead, we look at the *Mather log canonical threshold* of the ideal [16], which is defined by

$$\widehat{\mathrm{lct}}(\mathfrak{a}) := \inf_{f,E} \frac{\mathrm{ord}_E(\mathrm{Jac}_f) + 1}{\mathrm{ord}_E(\mathfrak{a})}$$

where the infimum ranges over all birational morphisms $f \colon Y \to X$, with Y smooth, and all prime divisors $E \subset Y$, with Jac_f being the Jacobian ideal of f.

THEOREM 1.4. *With the above notation, if X is Cohen-Macaulay, of dimension n, then we have*

$$(3) \qquad\qquad (n! \cdot \ell(\mathcal{O}_X/\mathfrak{a}))^{1/n} \cdot \widehat{\mathrm{lct}}(\mathfrak{a}) \geq n,$$

$$(4) \qquad\qquad e(\mathfrak{a})^{1/n} \cdot \widehat{\mathrm{lct}}(\mathfrak{a}) \geq n.$$

The proofs of (1) and (2) rely on the reduction to monomial ideals via flat degeneration, where the inequality follows from Howard's description of log canonical thresholds of monomial ideals and the well-known inequality between arithmetic and geometric means. A slightly more general formulation of (2) is the key ingredient in the proof of a theorem of [5] on log canonical thresholds of generic projections. The proof of Theorem 1.4 follows the opposite direction: we first prove a theorem on Mather log discrepancies of generic projections (see Theorem 2.5 below), and then deduce (3) and (4) from it.

The paper is organized as follows. In the next section we prove Theorem 1.4. Section 3 is devoted to a discussion of volumes of graded sequences of ideals, with emphasis on sequences associated to pseudo-valuations. Then, in the last section we define the volume of a set of arcs and prove several properties including those stated above.

2. MATHER LOG DISCREPANCIES

Let X be a variety of dimension n defined over an algebraically closed field of characteristic zero. Recall that a *divisor over* X is a prime divisor E on a normal variety Y, with a birational morphism $f \colon Y \to X$. Such a divisor determines a valuation ord_E of $k(Y) = k(X)$ and as usual, we identify two divisors over X if they give the same valuation. The valuations that arise in this way are the *divisorial valuations* of $k(X)$ that have center on X (recall that the center of ord_E is the closure of $f(E)$).

Given a birational morphism $f \colon Y \to X$, with Y smooth, we consider $\mathrm{Jac}_f := \mathrm{Fitt}^0(\Omega_{Y/X}) \subseteq \mathcal{O}_Y$, the Jacobian ideal of the map.

Definition 2.1. Given a divisor E over X, the *Mather log discrepancy* $\widehat{a}_E(X)$ of E over X is defined as follows. Suppose that $f \colon Y \to X$ is a birational morphism, with Y normal, such that E is a prime divisor on Y. After possibly replacing Y by its smooth locus, we may assume that Y is smooth. If $\mathrm{Jac}_f \subseteq \mathcal{O}_Y$ is the Jacobian ideal of the map, then

$$\widehat{a}_E(X) := \mathrm{ord}_E(\mathrm{Jac}_f) + 1.$$

Given a nonzero ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ and a number $c \geq 0$, we define the *Mather log discrepancy* of E with respect to the pair $(X, \mathfrak{a}^c)$ to be

$$\widehat{a}_E(X, \mathfrak{a}^c) := \mathrm{ord}_E(\mathrm{Jac}_f) + 1 - c \cdot \mathrm{ord}_E(\mathfrak{a}).$$

When X is smooth, we write $a_E(X)$ and $a_E(X, \mathfrak{a}^c)$ instead of $\widehat{a}_E(X)$ and $\widehat{a}_E(X, \mathfrak{a}^c)$, respectively. It is clear that the definition of Mather log discrepancy only depends on the valuation ord_E that E defines on the function field of X, and not on the model Y. We say that the pair $(X, \mathfrak{a}^c)$ is Mather log canonical if for every E as above, we have $\widehat{a}_E(X, \mathfrak{a}^c) \geq 0$. The *Mather log canonical threshold* of the pair $(X, \mathfrak{a})$, with $\mathfrak{a}$ a proper nonzero ideal of R, is defined by

$$\widehat{\mathrm{lct}}(\mathfrak{a}) := \sup\{\, c \in \mathbb{R}_{\geq 0} \mid (X, \mathfrak{a}^c) \text{ is Mather log canonical}\,\}.$$

It is straightforward to check that this is equivalent to the definition of $\widehat{\mathrm{lct}}(\mathfrak{a})$ given in Introduction. We put, by convention, $\widehat{\mathrm{lct}}(0) = 0$ and $\widehat{\mathrm{lct}}(\mathcal{O}_X) = \infty$.

Remark 2.2. We refer to [16] for basic facts about Mather log discrepancies and Mather log canonical threshold. A useful fact is that if $f \colon Y \to X$ is a log resolution of $(X, \mathfrak{a})$ which factors through the Nash blow-up of X, then there is a divisor E on Y such that $\widehat{\mathrm{lct}}(\mathfrak{a}) = \frac{\widehat{a}_E(X)}{\mathrm{ord}_E(\mathfrak{a})}$.

We will use several times the following basic fact about divisorial valuations.

LEMMA 2.3. *Let $f\colon X \to Y$ be a dominant morphism of varieties. If E is a divisor over X, then the restriction of ord_E to $k(Y)$ is a multiple of a divisorial valuation, that is, we can write*

$$\operatorname{ord}_E\big|_{k(Y)} = q \cdot \operatorname{ord}_F$$

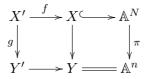*for some divisor F over Y and some positive integer q.*

Proof. Let $v = \operatorname{ord}_E$ and $w = v|_{k(Y)}$. Note that w is a valuation with center on Y, the center being the closure of the image of the center of v on X. We denote by R_v and R_w the valuation rings corresponding to v and w, respectively, and by k_v and k_w the corresponding residue fields. Note that $\operatorname{trdeg}(k_w/k) \leq \dim(Y)$, with equality if and only if w is the trivial valuation. Furthermore, w is a multiple of a divisorial valuation if and only if $\operatorname{trdeg}(k_w/k) = \dim(Y) - 1$ (see [18], Lemma 2.45). On the other hand, since v is a divisorial valuation, we know that $\operatorname{trdeg}(k_v/k) = \dim(X) - 1$. It follows from ([24], Chapter VI.6, Corollary 1) that $\operatorname{trdeg}(k_v/k_w) \leq \operatorname{trdeg}(k(X)/k(Y)) = \dim(X) - \dim(Y)$. We conclude that $\operatorname{trdeg}(k_w/k) \geq \dim(Y) - 1$. Since it is clear that w is not the trivial valuation, we conclude that in fact $\operatorname{trdeg}(k_w/k) = \dim(Y) - 1$, hence w is a multiple of a divisorial valuation. Since w only takes integer values, it is immediate to see that the multiple is a positive integer. □

The next result gives an alternative way of computing Mather log discrepancies. Suppose that E is a prime divisor over a normal n-dimensional affine variety X. Given a closed immersion $X \hookrightarrow \mathbb{A}^N$ and a general linear projection $\pi\colon \mathbb{A}^N \to Y := \mathbb{A}^n$, we may write $\operatorname{ord}_E|_{k(Y)} = q \cdot \operatorname{ord}_F$, for a prime divisor F over Y and a positive integer q, by Lemma 2.3.

PROPOSITION 2.4. *With the above notation, we have*

$$\widehat{a}_E(X) = q \cdot a_F(Y).$$

Proof. Consider a commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f\ } X \hookrightarrow & \mathbb{A}^N \\
{\scriptstyle g}\downarrow & \downarrow & \downarrow {\scriptstyle \pi} \\
Y' & \longrightarrow Y \ =\!=\!= & \mathbb{A}^n
\end{array}
$$

where $X' \to X$ and $Y' \to Y$ are resolutions such that E is a divisor on X' and F is a divisor on Y'. Note that $\operatorname{ord}_E(g^*F) = q$ and $\operatorname{ord}_E(K_{X'/Y'}) = q - 1$. Denoting by $h\colon X' \to Y$ the composition of f with the projection to Y, we have $\operatorname{ord}_E(K_{X'/Y}) = \operatorname{ord}_E(\operatorname{Jac}_h)$. If $x_1, \ldots, x_n$ is a regular system of parameters in X' centered at a general point of E and $y_1, \ldots, y_N$ are affine coordinates in $\mathbb{A}^N$, then f is locally given by equations $y_i = f_i(x_1, \ldots, x_n)$, and Jac_f is

locally defined by the $n \times n$ minors of the matrix $(\partial f_i/\partial x_j)$. For a linear projection $\pi\colon \mathbb{A}^N \to Y = \mathbb{A}^n$, Jac_h is locally defined by a linear combination of the $n \times n$ minors of $(\partial f_i/\partial x_j)$. If the projection is general, then so is the linear combination, and we have $\widehat{a}_E(X) = \mathrm{ord}_E(K_{X'/Y}) + 1$. Writing $K_{X'/Y} = K_{X'/Y'} + K_{Y'/Y}$, we get

$$\widehat{a}_E(X) = \mathrm{ord}_E(K_{X'/Y'}) + \mathrm{ord}_E(g^*K_{Y'/Y}) + 1 = q \cdot a_F(Y). \quad \square$$

The following theorem is a generalization of ([5], Theorem 1.1) to Cohen-Macaulay varieties.

THEOREM 2.5. *Let $X \subseteq \mathbb{A}^N$ be a Cohen-Macaulay variety of dimension n, and E a divisor over X. For some $1 \le r \le n$, consider the morphism*

$$\phi\colon X \to \mathbb{A}^{n-r+1}$$

induced by restriction of a general linear projection $\sigma\colon \mathbb{A}^N \to \mathbb{A}^{n-r+1}$. Write $\mathrm{ord}_E|_{k(\mathbb{A}^{n-r+1})} = q \cdot \mathrm{ord}_G$, where G is a prime divisor over $\mathbb{A}^{n-r+1}$ and q is a positive integer (cf. Lemma 2.3). Let $Z \hookrightarrow X$ a closed Cohen-Macaulay subscheme of pure codimension r such that $\phi|_Z$ is a finite morphism. Note that $\phi_*[Z]$ is a cycle of codimension one in $\mathbb{A}^{n-r+1}$; we regard $\phi_*[Z]$ as a Cartier divisor on $\mathbb{A}^{n-r+1}$. Then, for every $c \in \mathbb{R}_{\ge 0}$ such that $\widehat{a}_E(X, cZ) \ge 0$, we have

$$(5) \qquad q \cdot a_G\left(\mathbb{A}^{n-r+1}, \frac{r!\, c^r}{r^r} \cdot \phi_*[Z]\right) \le \widehat{a}_E(X, cZ).$$

Moreover, if the ideal defining Z in X is locally generated by a regular sequence, then

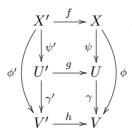$$(6) \qquad q \cdot a_G\left(\mathbb{A}^{n-r+1}, \frac{c^r}{r^r} \cdot \phi_*[Z]\right) \le \widehat{a}_E(X, cZ).$$

Proof. Our argument is similar to the one used in the proof of ([5], Theorem 1.1). We assume that $\mathrm{ord}_E(Z) > 0$ (the case $\mathrm{ord}_E(Z) = 0$ is easier and left to the reader). We factor σ as a composition of two general linear projections

$$\mathbb{A}^N \to U = \mathbb{A}^n \to V = \mathbb{A}^{n-r+1}.$$

By Lemma 2.3, we can write $\mathrm{ord}_E|_{k(U)} = p \cdot \mathrm{ord}_F$ for some prime divisor F over U and some positive integer p. Note that p divides q.

Let $h\colon V' \to V$ be a proper, birational morphism, with V' smooth, such that G is a prime divisor on V'. Let $X' := V' \times_V X$ and $U' := V' \times_V U$, and consider the induced commutative diagram with Cartezian squares

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f\ } & X \\
\end{array}
\qquad .
$$

Let $Z' := f^{-1}(Z) \hookrightarrow X'$ and $Z'' := \psi'(Z') \hookrightarrow U'$, both defined scheme-theoretically. In general, we have $Z'' \hookrightarrow g^{-1}(\psi(Z))$, but this may be a proper subscheme. First, note that ψ is a finite, flat morphism. Finiteness follows from the fact that it is induced by a generic projection, while flatness follows from the fact that it is finite, U is smooth, and X is Cohen-Macaulay. Since γ is clearly flat (in fact, smooth), we conclude that ϕ is flat. Therefore both X' and U' are varieties and f and g are proper, birational morphisms. Furthermore, the restriction $\phi'|_{Z'}$ is finite by base-change, and thus both $\psi'|_{Z'}$ and $\gamma'|_{Z''}$ are finite.

Note that Z' is a closed subscheme of $\psi'^{-1}(Z'')$, hence

(7) $\qquad p \cdot \mathrm{ord}_F(Z'') = \mathrm{ord}_E((\psi')^{-1}(Z'')) \geq \mathrm{ord}_E(Z') = \mathrm{ord}_E(Z).$

Since h, being a morphism between two smooth varieties, is a locally complete intersection morphism, it follows by flat base change that f is a locally complete intersection morphism as well. More explicitly, h factors as $h = h_1 \circ h_2$ where $h_1 \colon V' \times V \to V$ is the projection and $h_2 \colon V' \hookrightarrow V' \times V$ is the regular embedding given by the graph of h. By pulling back, we get a decomposition $f = f_1 \circ f_2$ where $f_1 \colon V' \times X \to X$ is smooth and $f_2 \colon X' \hookrightarrow V' \times X$ is a regular embedding of codimension equal to $\dim V = \dim V'$. Recall that the pull-back $f^*[Z] \in A_{n-r}(X')$ is defined as $f_2^![V' \times Z]$ (see [12], Section 6.6).

We now show that Z' is pure-dimensional, of the same dimension as Z, and $f^*[Z]$ is equal to the class of $[Z']$ in $A_{n-r}(X')$. Since $\phi'|_{Z'}$ is finite and $\phi'(Z')$ is a proper subset of V', we see that $\dim Z' \leq \dim V' - 1 = n - r$. On the other hand, Z' is locally cut out in $V' \times Z$ by $\dim V'$ equations, hence every irreducible component of Z' has dimension at least $\dim Z = n - r$. Therefore Z' is pure dimensional, of dimension $\dim Z$. Since $V' \times X$ is Cohen-Macaulay, it follows from ([12], Proposition 7.1) that $f^*[Z] = [Z']$ in $A_{n-r}(X')$.

Since $\psi'|_{Z'} \colon Z' \to Z''$ is a finite, dominant morphism of schemes, we see that Z'' is also pure dimensional of the same dimension as Z', and $\psi'_*[Z'] \geq [Z'']$. Note that $h^* \phi_*[Z]$ and $\phi'_*[Z']$ are divisors on V'. Since f and h are locally complete intersection morphisms of the same codimension, and since we have seen that $f^*[Z] = [Z']$ in $A_{n-r}(X')$, it follows from ([12], Example 17.4.1) that $h^* \phi_*[Z] \sim \phi'_*[Z']$ (note that while ϕ and ϕ' are not proper morphisms, they are

proper when restricted to the supports of Z and Z', respectively). Since the two divisors are equal away from the exceptional locus of h, we deduce that $h^*\phi_*[Z] = \phi'_*[Z']$ by the Negativity Lemma (see [18], Lemma 3.39). We thus conclude that
$$h^*\phi_*[Z] = \phi'_*[Z'] \geq \gamma'_*[Z''].$$

On the other hand, the center C of ord_F in U' is contained in Z'' and dominates G. Since $\phi'|_{Z'}$ is finite, it follows that the map $\gamma'|_C \colon C \to G$ is finite. In particular, we have $\dim(C) = \dim(G) = n - r = \dim(Z'')$, hence C is an irreducible component of Z''. Therefore we have

$$(8) \quad \mathrm{ord}_G(\phi_*[Z]) = \mathrm{ord}_G(h^*\phi_*[Z]) \geq \mathrm{ord}_G(\gamma'_*[Z'']) \geq e_C([Z'']) = \ell(\mathcal{O}_{Z'',C}).$$

Let $b := k_G(V)$ denote the discrepancy of G over V, and let $H := (\gamma')^*G$. Note that $p \cdot \mathrm{val}_F(H) = q$ and since γ' is smooth, H is a smooth divisor at the generic point of C. Moreover, since $K_{U'/U} = (\gamma')^* K_{V'/V}$, we have $K_{U'/U} = bH + R$, where R is a divisor that does not contain C in its support. Then, by Proposition 2.4 and equation (7), we see that

$$\widehat{a}_E(X, cZ) \geq p \cdot a_F(U', cZ'' - K_{U'/U}) = p \cdot a_F(U', cZ'' - bH).$$

Setting $\alpha := \widehat{a}_E(X, cZ)/q$, we have

$$a_F(U', cZ'' - (b-\alpha)H) = a_F(U', cZ'' - bH) - \alpha \cdot \mathrm{ord}_F(H) \leq \alpha(1 - \mathrm{ord}_F(H)) \leq 0,$$

where the last inequality follows from the fact that $\mathrm{ord}_F(H) \geq 1$ and, by assumption, $\alpha \geq 0$. This in turn implies

$$(9) \qquad\qquad \ell(\mathcal{O}_{Z'',C}) \geq \frac{(b - \alpha + 1)r^r}{r!\, c^r}.$$

Indeed, if $b - \alpha \geq 0$, then (9) follows by ([5], Theorem 2.1). The case $b - \alpha < 0$ is easier, and follows from ([5], Lemma 2.4) using the same degeneration to monomial ideals (see [6], Section 2).

Combining (8) and (9), we get

$$q \cdot a_G\left(V, \frac{r!\, c^r}{r^r} \cdot \phi_*[Z]\right) = q \cdot a_G(V) - \frac{r!\, c^r}{r^r} \cdot \mathrm{ord}_G(\phi_*[Z])$$

$$\leq q(b + 1 - (b - \alpha + 1)) = \widehat{a}_E(X, cZ),$$

as stated in (5)

Suppose now that the ideal of Z in X is locally generated by a regular sequence. If $I_Z \subseteq \mathcal{O}_X$ is the ideal sheaf of Z and Z_i is an irreducible component of Z, then

$$(10) \qquad \ell(\mathcal{O}_{Z,Z_i}) = e(I_Z \mathcal{O}_{X,Z_i}) = \lim_{m \to \infty} \frac{r!}{m^r} \cdot \ell(\mathcal{O}_{X,Z_i}/I_Z^m \mathcal{O}_{X,Z_i}).$$

For every m, let $Z_m \hookrightarrow X$ be the subscheme defined by I_Z^m. Since I_Z is locally generated by a regular sequence, I_Z^m/I_Z^{m+1} is a locally free $\mathcal{O}_Z$-module,

and thus is Cohen-Macaulay (as an $\mathcal{O}_Z$-module, hence as an $\mathcal{O}_X$-module). Note that $\mathcal{O}_Z$ is also Cohen-Macaulay (as an $\mathcal{O}_Z$-module, hence as an $\mathcal{O}_X$-module). By applying ([2], Proposition 1.2.9) to the exact sequences of $\mathcal{O}_X$-modules

$$0 \to I_Z^m / I_Z^{m+1} \to \mathcal{O}_{Z_{m+1}} \to \mathcal{O}_{Z_m} \to 0,$$

we see by induction that $\mathcal{O}_{Z_m}$ is a Cohen-Macaulay $\mathcal{O}_X$-module, and therefore Z_m is a Cohen-Macaulay scheme. Note that

$$\widehat{a}_E \left(X, \frac{c}{m} \cdot Z_m \right) = \widehat{a}_E(X, cZ) \quad \text{for all } m,$$

and

$$\lim_{m \to \infty} \frac{r!}{m^r} \cdot [Z_m] = [Z]$$

by (10). Since $\phi|_{Z_m}$ if finite for every m, we may apply (5) with (Z, c) replaced by $(Z_m, c/m)$ to deduce, after letting m go to infinity, the inequality in (6). $\quad\square$

COROLLARY 2.6. *With the same assumptions as in the first part of Theorem 2.5, we have*

$$(11) \qquad \qquad \operatorname{lct}(\mathbb{A}^{n-r+1}, \phi_*[Z]) \le \frac{\widehat{\operatorname{lct}}(X, Z)^r}{r^r/r!}.$$

Moreover, if the ideal of Z in X is locally generated by a regular sequence, then

$$(12) \qquad \qquad \operatorname{lct}(\mathbb{A}^{n-r+1}, \phi_*[Z]) \le \frac{\widehat{\operatorname{lct}}(X, Z)^r}{r^r}.$$

Proof. We apply Theorem 2.5 for a divisor E computing $\widehat{\operatorname{lct}}(X, Z)$. $\quad\square$

We apply the first part of the corollary to prove Theorem 1.4.

Proof of Theorem 1.4. Let $x \in X$ be the cosupport of $\mathfrak{a}$. After replacing X by an affine neighborhood of x, we may assume that we have a closed immersion $X \hookrightarrow \mathbb{A}^N$. Let $m \ge 1$ be fixed and $Z_m \hookrightarrow X$ be the zero-dimensional scheme defined by $\mathfrak{a}^m$. Note that Z_m is Cohen-Macaulay, since it is zero dimensional.

Consider a general linear projection $\mathbb{A}^N \to \mathbb{A}^1$ and let $\phi \colon X \to \mathbb{A}^1$ be the induced map. Note that

$$\widehat{\operatorname{lct}}(X, Z_m) = \frac{1}{m} \cdot \widehat{\operatorname{lct}}(X, Z),$$

and since

$$\phi_*[Z_m] = \ell(\mathcal{O}_X / \mathfrak{a}^m) \cdot [f(x)],$$

we have

$$\operatorname{lct}(\mathbb{A}^1, \phi_*[Z_m]) = \frac{1}{\ell(\mathcal{O}_X / \mathfrak{a}^m)}.$$

Then (11) gives

$$\frac{\ell(\mathcal{O}_X/\mathfrak{a}^m)}{m^n/n!} \cdot \widehat{\mathrm{lct}}(X, Z)^n \geq n^n.$$

Setting $m = 1$ and taking n-th roots, we get (3). The formula (4) follows by taking the limit as m goes to infinity and then taking n-th roots. $\qquad\square$

3. THE VOLUME OF A GRADED SEQUENCE OF IDEALS

We recall, following [9] and [21], some basic facts about the volume of a graded sequence of ideals. Let k be an algebraically closed field of arbitrary characteristic and let $X = \mathrm{Spec}(R)$ be an n-dimensional affine variety over k (in particular, we assume that R is a domain). Recall that a sequence $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \geq 0}$ of ideals $\mathfrak{a}_m \subseteq R$ is a graded sequence of ideals if $\mathfrak{a}_0 = R$ and $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for every $p, q \geq 1$.

Definition 3.1. The *volume* of a graded sequence $\mathfrak{a}_\bullet$ is defined by

$$\mathrm{vol}(\mathfrak{a}_\bullet) := \limsup_{m \to \infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!}.$$

Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals in R. The main case for understanding the notion of volume is that when there is a closed point x in X such that for every $m \geq 1$, the cosupport of $\mathfrak{a}_m$ is equal to $\{x\}$ (we say that $\mathfrak{a}_\bullet$ is *cosupported at* x). Note that in this case we have $\mathrm{vol}(\mathfrak{a}_\bullet) < \infty$. Indeed, if N is a positive integer such that $\mathfrak{m}_x^N \subseteq \mathfrak{a}_1$, where $\mathfrak{m}_x$ is the ideal defining x, then $\mathfrak{m}_x^{pN} \subseteq \mathfrak{a}_p$ for every $p \geq 1$, hence $\mathrm{vol}(\mathfrak{a}_\bullet) \leq N^n \cdot e(\mathfrak{m}_x)$. In fact, under the same assumption, it follows from ([20], Theorem 3.8) that the volume of $\mathfrak{a}_\bullet$ can be computed as a limit of normalized Hilbert-Samuel multiplicities. More precisely, we have

$$(13) \qquad\qquad \mathrm{vol}(\mathfrak{a}_\bullet) = \lim_{m \to \infty} \frac{e(\mathfrak{a}_m)}{m^n}.$$

Moreover, the limit superior in the definition of volume is a limit

$$(14) \qquad\qquad \mathrm{vol}(\mathfrak{a}) = \lim_{m \to \infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!}$$

by ([3], Theorem 1).

Remark 3.2. Suppose that $\mathfrak{a}_\bullet$ is a graded sequence of ideals such that $\mathfrak{a}_p \subseteq \mathfrak{a}_q$ whenever $p \geq q$. If $\mathfrak{a}_\bullet$ is cosupported at a point $x \in X$, then

$$\mathrm{vol}(\mathfrak{a}_\bullet) = \inf_{m \geq 1} \frac{e(\mathfrak{a}_m)}{m^n}.$$

Indeed, this is a consequence of (13) and of the fact that

$$\lim_{m\to\infty} \frac{e(\mathfrak{a}_m)}{m^n} = \inf_{m\geq 1} \frac{e(\mathfrak{a}_m)}{m^n}.$$

This equality is a consequence of Lemma 3.7 below.

Remark 3.3. Suppose that $\mathfrak{a}_\bullet$ is a graded sequence of ideals and $\Gamma = \{x_1, \ldots, x_r\}$ is a finite set of closed points in X such that for every $m \geq 1$, the ideal $\mathfrak{a}_m$ has cosupport Γ. For every $m \geq 1$, let us consider the primary decomposition

$$\mathfrak{a}_m = \bigcap_{i=1}^{r} \mathfrak{a}_m^{(i)},$$

where each $\mathfrak{a}_m^{(i)}$ is an ideal with cosupport $\{x_i\}$. It is clear that each $\mathfrak{a}_\bullet^{(i)}$ is a graded sequence of ideals. Since

$$(15) \qquad \ell(R/\mathfrak{a}_m) = \sum_{i=1}^{r} \ell(R/\mathfrak{a}_m^{(i)}),$$

we deduce

$$(16) \qquad \mathrm{vol}(\mathfrak{a}_\bullet) = \sum_{i=1}^{r} \mathrm{vol}(\mathfrak{a}_\bullet^{(i)}).$$

In particular, we see that $\mathrm{vol}(\mathfrak{a}_\bullet) < \infty$ and the assertion in (14) also holds for $\mathfrak{a}_\bullet$.

Example 3.4. Suppose that $\mathfrak{a}_\bullet$ is a graded sequence of ideals such that each $\mathfrak{a}_m$, with $m \geq 1$, has cosupport equal to a finite set Γ. If $\overline{\mathfrak{a}}_\bullet$ is such that $\overline{\mathfrak{a}}_m$ is the integral closure of the ideal $\mathfrak{a}_m$, then $\overline{\mathfrak{a}}_\bullet$ is a graded sequence and $\mathrm{vol}(\overline{\mathfrak{a}}_\bullet) = \mathrm{vol}(\mathfrak{a}_\bullet)$. The first assertion follows from the fact that $\overline{\mathfrak{a}}_p \cdot \overline{\mathfrak{a}}_q$ is contained in the integral closure of $\mathfrak{a}_p \cdot \mathfrak{a}_q$, hence in $\overline{\mathfrak{a}}_{p+q}$. In order to see that $\mathrm{vol}(\mathfrak{a}_\bullet) = \mathrm{vol}(\overline{\mathfrak{a}}_\bullet)$, we may assume that all $\mathfrak{a}_m$ have cosupport at the same point $x \in X$ (see Remark 3.3). In this case, since $e(\mathfrak{a}_m) = e(\overline{\mathfrak{a}}_m)$ for every m, the assertion follows from (13).

Under a mild condition on $\mathfrak{a}_\bullet$ which is often satisfied, we give in the next proposition a new easy proof of the assertions (13) and (14) in the smooth case.

PROPOSITION 3.5. *Suppose that $X = \mathrm{Spec}(R)$ is smooth. If $\mathfrak{a}_\bullet$ is a graded sequence of ideals in R which is cosupported at a point in X, and $\mathfrak{a}_p \subseteq \mathfrak{a}_q$ whenever $p \geq q$, then*

$$(17) \qquad \mathrm{vol}(\mathfrak{a}_\bullet) = \lim_{m\to\infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!} = \inf_{m\geq 1} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!}$$

$$(18) \qquad = \lim_{m\to\infty} \frac{e(\mathfrak{a}_m)}{m^n} = \inf_{m\geq 1} \frac{e(\mathfrak{a}_m)}{m^n}.$$

Note that while the proposition recovers (13) and (14) in the smooth setting, it also implies the equality $\mathrm{vol}(\mathfrak{a}_\bullet) = \inf_{m \geq 1} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!}$, which needs the smoothness assumption. For the proof of the proposition we need two lemmas. The first one is a special case of ([19], Lemma 25); for completeness, we include the proof of this special case.

LEMMA 3.6. *If $X = \mathrm{Spec}(R)$ is smooth, $x \in X$ is a closed point defined by $\mathfrak{m}_x$, and $\mathfrak{a}$ is an $\mathfrak{m}_x$-primary ideal in R, then for every $p \geq 1$, we have*

$$\ell(R/\mathfrak{a}) \geq \frac{1}{p^n} \cdot \ell(R/\mathfrak{a}^p).$$

Proof. Since X is smooth, it is straightforward to reduce to the case when $X = \mathbb{A}^n$ and $\mathfrak{a}$ is an ideal supported at the origin. We choose a monomial order on $R = k[x_1, \ldots, x_n]$ and for every ideal $\mathfrak{b}$ in R, we consider the initial ideal

$$\mathrm{in}(\mathfrak{b}) = (\mathrm{in}(f) \mid f \in \mathfrak{b}).$$

We refer to ([11], Chapter 15) for the basic facts about initial ideals. Note that we have $\ell(R/\mathfrak{b}) = \ell(R/\mathrm{in}(\mathfrak{b}))$. It is clear that $\mathrm{in}(\mathfrak{a}^p) \supseteq \mathrm{in}(\mathfrak{a})^p$. It follows that if we know the assertion in the lemma for $\mathrm{in}(\mathfrak{a})$, then

$$\ell(R/\mathfrak{a}) = \ell(R/\mathrm{in}(\mathfrak{a})) \geq \frac{1}{p^n} \cdot \ell(R/\mathrm{in}(\mathfrak{a})^p) \geq \frac{1}{p^n} \cdot \ell(R/\mathrm{in}(\mathfrak{a}^p)) = \frac{1}{p^n} \cdot \ell(R/\mathfrak{a}^p),$$

hence we obtain the assertion for $\mathfrak{a}$.

The above argument shows that we may assume that $\mathfrak{a}$ is a monomial ideal. For every such ideal $\mathfrak{a}$, we consider the sets

$$Q(\mathfrak{a}) := \bigcup_{x^u \in \mathfrak{a}} (u + \mathbb{R}^n) \quad \text{and} \quad Q^c(\mathfrak{a}) := \mathbb{R}_{\geq 0}^n \smallsetminus Q(\mathfrak{a}).$$

Note that $Q^c(\mathfrak{a})$ is equal, up to a set of measure zero, to the union of $\ell(R/\mathfrak{a})$ disjoint open unit cubes. Therefore $\ell(R/\mathfrak{a})$ is equal to $\mathrm{vol}(Q^c(\mathfrak{a}))$, the Euclidean volume of $Q^c(\mathfrak{a})$. On the other hand, it is clear from definition that $Q(\mathfrak{a}^p) \supseteq p \cdot Q(\mathfrak{a})$, hence $Q^c(\mathfrak{a}^p) \subseteq p \cdot Q^c(\mathfrak{a})$. We thus conclude

$$\ell(R/\mathfrak{a}) = \mathrm{vol}(Q^c(\mathfrak{a})) \geq \mathrm{vol}\left(\frac{1}{p} \cdot Q^c(\mathfrak{a}^p)\right) = \frac{1}{p^n} \cdot \mathrm{vol}(Q^c(\mathfrak{a}^p)) = \frac{1}{p^n} \cdot \ell(R/\mathfrak{a}^p).$$

This completes the proof of the lemma. $\square$

The following is a variant of ([21], Lemma 2.2).

LEMMA 3.7. *If $(\alpha_m)_{m \geq 1}$ is a sequence of non-negative real numbers that satisfies the following two conditions:*
 i) $\alpha_{pq} \leq p \cdot \alpha_q$ for every $p, q \geq 1$, and
 ii) $\alpha_p \geq \alpha_q$ whenever $p \geq q$,

then

$$\lim_{m\to\infty}\frac{\alpha_m}{m}=\inf_{m\geq 1}\frac{\alpha_m}{m}.$$

Proof. Let $\lambda:=\inf_m\frac{\alpha_m}{m}$. We need to show that for every $\epsilon>0$, we have $\frac{\alpha_m}{m}\leq\lambda+\epsilon$ for all $m\gg 1$. By definition, there is $d>0$ such that $\frac{\alpha_d}{d}<\lambda+\frac{\epsilon}{2}$. Given m, we write $m=jd-i$, where $0\leq i<d$ (hence $j=\lceil m/d\rceil$). The hypotheses imply

$$\frac{\alpha_m}{m}\leq\frac{\alpha_{jd}}{jd-i}\leq\frac{\alpha_d}{d}\cdot\frac{jd}{jd-i}\leq\left(\lambda+\frac{\epsilon}{2}\right)\cdot\frac{jd}{jd-i}.$$

For $m\gg 1$, we have $j\gg 1$, hence $\frac{jd}{jd-i}<\frac{\lambda+\epsilon}{\lambda+\frac{\epsilon}{2}}$.

This completes the proof of the lemma. $\quad\square$

Proof of Proposition 3.5. Let $\alpha_m=\ell(R/\mathfrak{a}_m)$. If $p\geq q$, then by assumption $\mathfrak{a}_p\subseteq\mathfrak{a}_q$, hence $\alpha_p\geq\alpha_q$. Moreover, it follows from Lemma 3.6 that $\alpha_{pq}\leq p\cdot\alpha_q$ for all $p,q\geq 1$. The two equalities in (17) now follow from the definition of volume and Lemma 3.7.

Note now that Lemma 3.7 also gives the second equality in (18). Indeed, for $p\geq q$, we have $\mathfrak{a}_p\subseteq\mathfrak{a}_q$, hence $e(\mathfrak{a}_p)\geq e(\mathfrak{a}_q)$; moreover, the inclusion $\mathfrak{a}_q^p\subseteq\mathfrak{a}_{pq}$ implies $e(\mathfrak{a}_{pq})\leq e(\mathfrak{a}_q^p)=p^n\cdot e(\mathfrak{a}_q)$. In order to prove the first equality in (18), note first that by definition of Hilbert-Samuel multiplicity, for every m we have

$$e(\mathfrak{a}_m)=\lim_{q\to\infty}\frac{\ell(R/\mathfrak{a}_m^q)}{q^n/n!},$$

hence using Lemma 3.6 we conclude that $e(\mathfrak{a}_m)\leq\frac{\ell(R/\mathfrak{a}_m)}{n!}$. Dividing by m^n and passing to limit, we obtain

$$L:=\lim_{m\to\infty}\frac{e(\mathfrak{a}_m)}{m^n}\leq\mathrm{vol}(\mathfrak{a}_\bullet).$$

In order to prove the reverse inequality, note that given any $\epsilon>0$, by definition of L and of the Hilbert-Samuel multiplicity, we can find first $m\geq 1$ and then $q\geq 1$ such that $L>\frac{\ell(R/\mathfrak{a}_m^q)}{m^nq^n/n!}-\epsilon$. Since $\mathfrak{a}_m^q\subseteq\mathfrak{a}_{mq}$, it follows that

$$L>\frac{\ell(R/\mathfrak{a}_{mq})}{(mq)^n/n!}-\epsilon\geq\inf_p\frac{\ell(R/\mathfrak{a}_p)}{p^n/n!}-\epsilon.$$

Since this holds for every $\epsilon>0$, using (17) we conclude that $L\geq\mathrm{vol}(\mathfrak{a}_\bullet)$, completing the proof of the proposition. $\quad\square$

Remark 3.8. Suppose that $X=\mathrm{Spec}(R)$ is smooth and $\mathfrak{a}$ is an ideal in R which is cosupported at a point. Applying Proposition 3.5 in the case of the sequence given by the powers of $\mathfrak{a}$, we see that

$$e(\mathfrak{a})=\inf_{m\geq 1}\frac{\ell(R/\mathfrak{a}^m)}{m^n/n!}.$$

In this note, we will be interested in graded sequences that arise from pseudo-valuations.

Definition 3.9. A function $v\colon R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is said to be a *pseudo-valuation* of R if it satisfies the following conditions:

(i) $\qquad\qquad v(0) = \infty \quad \text{and} \quad v(\lambda) = 0 \quad \text{for every } \lambda \in k,$

(ii) $\qquad\qquad v(f + g) \geq \min\{v(f), v(g)\} \quad \text{for every } f, g \in R, \text{ and}$

(iii) $\qquad\qquad v(fg) \geq v(f) + v(g) \quad \text{for every } f, g \in R.$

We say that a pseudo-valuation v is *radical* if, in addition, it satisfies

(iv) $\qquad\qquad v(f^r) = r \cdot v(f) \quad \text{for every } f \in R, r \in \mathbb{Z}_{>0}.$

The *support* of a pseudo-valuation v is the closed subscheme $\mathrm{Supp}(v) \hookrightarrow X$ defined by the ideal

$$\mathfrak{b}_\infty(v) := \{f \in R \mid v(f) = \infty\}.$$

Given a pseudo-valuation v and an ideal $\mathfrak{a}$ in R, we put

$$v(\mathfrak{a}) := \inf\{v(f) \mid f \in \mathfrak{a}\}.$$

We say that v *has center at* the closed subscheme Y defined by the ideal $\mathfrak{b}$ in R if we have $\mathfrak{b} = \{f \in R \mid v(f) > 0\}$.

Remark 3.10. Note that if $\mathfrak{b}$ defines the center of v, then $v(\mathfrak{b}) > 0$. Indeed, if we put $I_m := \{f \in R \mid v(f) \geq 1/m\}$, then each I_m is an ideal in R and we have $I_m \subseteq I_{m+1}$. Since $\mathfrak{b} = \bigcup_m I_m$ and R is Noetherian, it follows that $\mathfrak{b} = I_m$ for $m \gg 0$.

Remark 3.11. There are two other related notions. A *semi-valuation* of R is a pseudo-valuation with the property that the inequality in (iii) is an equality for all f and g (in this case, condition (iv) is automatically satisfied). A semi-valuation v is a *valuation* if, in addition, we have $v(f) < \infty$ for all $f \in R \smallsetminus \{0\}$. It is clear that in this case we can extend v to a valuation of the function field of R by putting $v(f/g) = v(f) - v(g)$ for every nonzero $f, g \in R$. Note that if v is a semi-valuation, then the ideal $\mathfrak{b}_\infty(v)$ is a prime ideal and we have a valuation $\overline{v}$ on $R/\mathfrak{b}_\infty$ such that $v = \overline{v} \circ \pi$, where $\pi\colon R \to R/\mathfrak{b}_\infty$ is the canonical projection.

Remark 3.12. If $(v_\alpha)_{\alpha \in \Lambda}$ is a family of semi-valuations of R and we put $v(f) := \inf_{\alpha \in \Lambda} v_\alpha(f)$, then v is a radical pseudo-valuation. Note that the support of v is the union of the supports of the v_α and if Λ is finite, then the center of v is the union of the centers of the v_α. In particular, these sets

are not necessarily irreducible. It is a theorem of Bergman that every radical pseudo-valuation arises in this way. More precisely, for every radical pseudo-valuation w of R, there is a family $(w_i)_{i \in I}$ of semi-valuations of R such that $w(f) = \inf_i w_i(f)$ for every $f \in R$ (see ([1], Theorem 2)).

Remark 3.13. There is a canonical way to obtain a radical pseudo-valuation of R from an arbitrary pseudo-valuation. Indeed, if v is any pseudo-valuation, then we put

$$\widetilde{v}(f) := \inf_{m \geq 1} \frac{v(f^m)}{m} = \lim_{m \to \infty} \frac{v(f^m)}{m},$$

where the second equality follows from property (iii) and a version of Lemma 3.7 (see [21], Lemma 1.4). It is easy to see that $\widetilde{v}$ is a radical pseudo-valuation such that $\widetilde{v}(f) \leq v(f)$ for every $f \in R$. Moreover, if w is another radical pseudo-valuation such that $w(f) \leq v(f)$ for every $f \in R$, then $w(f) \leq \widetilde{w}(f)$ for every $f \in R$.

Suppose that v is a pseudo-valuation of R. We define for every $m \in \mathbb{Z}_{\geq 0}$

$$\mathfrak{b}_m(v) := \{f \in R \mid v(f) \geq m\}.$$

It follows from (ii) and (iii) that $\mathfrak{b}_\bullet(v) = (\mathfrak{b}_m(v))_{m \geq 0}$ is a graded sequence of ideals.

Remark 3.14. The sequence $\mathfrak{b}_\bullet(v)$ clearly satisfies the condition $\mathfrak{b}_p(v) \subseteq \mathfrak{b}_q(v)$ for $p \geq q$.

Example 3.15. Suppose that $I \neq R$ is an ideal of R. If for every $f \in R$, we put $v_I(f) := \min\{m \geq 0 \mid f \in I^m\}$, then v_I is a pseudo-valuation of R, with support X and whose center is defined by I. It follows from definition that in this case $\mathfrak{b}_m(v_I) = I^m$.

Remark 3.16. It is clear that for every pseudo-valuation v and every $m \geq 1$, if $\mathfrak{b}$ is the ideal defining the center of v, then $\mathfrak{b}_m(v) \subseteq \mathfrak{b}$ and the two ideals have the same radical. In fact, if d is an integer such that $d \cdot v(\mathfrak{b}) \geq 1$ (see Remark 3.10), then $\mathfrak{b}^{dm} \subseteq \mathfrak{b}_m(v)$ for every $m \geq 1$.

We will be mostly interested in pseudo-valuations with 0-dimensional center.

Definition 3.17. The *volume* of a pseudo-valuation v of R is defined to be the volume

$$\mathrm{vol}(v) := \mathrm{vol}(\mathfrak{b}_\bullet(v))$$

of the graded sequence $\mathfrak{b}_\bullet(v)$. Recall that by (13) and (14), we have

$$\mathrm{vol}(v) = \lim_{m \to \infty} \frac{\ell(R/\mathfrak{b}_m(v))}{m^n/n!} = \lim_{m \to \infty} \frac{e(\mathfrak{b}_m(v))}{m^n}.$$

Remark 3.18. We have $\mathrm{vol}(v) < \infty$ if and only if the center of v is a finite set. Indeed, if the latter condition holds, then the finiteness of the volume follows from Remark 3.3. On the other hand, if the center of v has positive dimension, then $\ell(R/\mathfrak{b}_m(v)) = \infty$ for all $m \geq 1$ by Remark 3.16.

Example 3.19. If $I \neq R$ is an ideal whose cosupport consists of one point and v_I is the pseudo-valuation associated to I in Remark 3.15, then $\mathrm{vol}(v_I) = e(I)$.

Example 3.20. Let $I \neq R$ be an ideal in R. Recall that there are finitely many divisorial valuations $w_1, \ldots, w_r$ of R (the *Rees valuations* of I) with the property that for every $m \geq 0$, the integral closure $\overline{I^m}$ of I^m is equal to

$$\{f \in R \mid w_i(f) \geq m \cdot w_i(I) \text{ for } 1 \leq i \leq r\}.$$

We refer to [23] for an introduction to Rees valuations. In particular, we see that if w is the pseudo-valuation given by $w = \min_i \frac{1}{w_i(I)} w_i$, then $b_m(w) = \overline{I^m}$ for every m. In particular, it follows from Example 3.4 that if the cosupport of I consists of one point, then $\mathrm{vol}(w) = e(I)$.

Example 3.21. Suppose that v and w are pseudo-valuations of R such that $v(f) \geq w(f)$ for all $f \in R$. In this case we have $\mathfrak{b}_m(w) \subseteq \mathfrak{b}_m(v)$ for all m. By taking the colength, dividing by $m^n/n!$, and passing to limit, we obtain $\mathrm{vol}(w) \geq \mathrm{vol}(v)$.

Example 3.22. If v is a pseudo-valuation of R and α is a positive real number, then αv is a pseudo-valuation such that $\mathrm{vol}(\alpha v) = \frac{1}{\alpha^n} \cdot \mathrm{vol}(v)$. Indeed, note that we have

$$\mathfrak{b}_m(\alpha v) \supseteq \mathfrak{b}_{\lceil m/\alpha \rceil}(v),$$

hence

$$\mathrm{vol}(\alpha v) \leq \lim_{m \to \infty} \frac{\ell(R/\mathfrak{b}_{\lceil m/\alpha \rceil}(v))}{\lceil m/\alpha \rceil^n/n!} \cdot \frac{\lceil m/\alpha \rceil^n}{m^n} = \mathrm{vol}(v) \cdot \frac{1}{\alpha^n}.$$

By writing $v = \frac{1}{\alpha}(\alpha v)$ and applying the inequality already proved, we obtain $\mathrm{vol}(v) \leq \alpha^n \cdot \mathrm{vol}(\alpha v)$. By combining the two inequalities, we obtain $\mathrm{vol}(\alpha v) = \frac{1}{\alpha^n} \cdot \mathrm{vol}(v)$.

Remark 3.23. Suppose that v is a pseudo-valuation of R and $h \in R$ is nonzero and such that $v(fh^m) = v(f)$ for every $f \in R$ (in particular, $v(h) = 0$, and this condition is sufficient if v is a valuation). In this case, v extends uniquely to a pseudo-valuation $\widetilde{v}$ of R_h, given by $\widetilde{v}(f/h^m) = v(f)$ for every positive integer m and every $f \in R$. It is clear that we have $\mathfrak{b}_m(\widetilde{v}) = \mathfrak{b}_m(v) \cdot R_h$ for every $m \geq 0$. We now show that $\mathrm{vol}(v) = \mathrm{vol}(\widetilde{v})$. Note first that if $\mathfrak{b}$ is the ideal defining the center of v, then $\mathfrak{b} \cdot R_h$ defines the center of $\widetilde{v}$. Let $\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be an irredundant primary decomposition of $\mathfrak{b}$. If $h \in \sqrt{\mathfrak{q}_i}$ for some

i, then we obtain a contradiction: indeed, by choosing $g \in \left(\bigcap_{j \neq i} \mathfrak{q}_j \right) \smallsetminus \mathfrak{q}_i$, we see that $gh^m \in \mathfrak{b}$ for some $m \geq 1$. However, $v(gh^m) = v(g)$ by the assumption on h, and $v(g) > 0$ since $g \notin \mathfrak{b}$. Therefore $h \notin \sqrt{\mathfrak{q}_i}$ for any i. First, this implies that if $\dim R/\mathfrak{b} > 0$, then also $\dim(R/\mathfrak{b})_h > 0$. Second, it implies that if $\dim R/\mathfrak{b} = 0$, then for every m, the class of h in $R/\mathfrak{b}_m(v)$ is invertible, hence $\ell(R/\mathfrak{b}_m(v)) = \ell(R_h/\mathfrak{b}_m(\widetilde{v}))$. We thus conclude that $\mathrm{vol}(v) = \mathrm{vol}(\widetilde{v})$.

The following proposition gives an important example of valuation with positive volume.

PROPOSITION 3.24. *If v is a divisorial valuation of R having center at a closed point $x \in X$ and X is analytically unramified[*] at x, then $\mathrm{val}(v) > 0$.*

Proof. This is an immediate consequence of Izumi's theorem (see for example ([14], Theorem 1.2). This says that since the local ring $\mathcal{O}_{X,x}$ is analytically unramified, there is a constant $c = c(v)$ such that for every other divisorial valuation v' with center $\{x\}$, we have $v(f) \leq c \cdot v'(f)$ for every $f \in R$. Let $w_1, \ldots, w_r$ be the Rees valuations corresponding to the maximal ideal $\mathfrak{m}_x$ defining x. If $w = \min_i \frac{1}{w_i(\mathfrak{m}_x)} w_i$ and $\alpha = c \cdot \max_i w_i(\mathfrak{m}_x)$, then we see that $v(f) \leq \alpha \cdot w(f)$ for every $f \in R$. By combining Examples 3.20, 3.21, and 3.22, we conclude that

$$\mathrm{vol}(v) \geq \mathrm{vol}(\alpha \cdot w) = \frac{\mathrm{vol}(w)}{\alpha^n} = \frac{e(\mathfrak{m}_x)}{\alpha^n} > 0. \quad \square$$

4. THE VOLUME OF A SUBSET IN THE SPACE OF ARCS

Suppose, as in the previous section, that $X = \mathrm{Spec}(R)$ is an n-dimensional, affine algebraic variety over an algebraically closed field k. We now assume that $\mathrm{char}(k) = 0$.

Let X_∞ be the scheme of arcs of X (for an introduction to spaces of arcs, see for example [10]). Since X is affine, X_∞ is affine as well, but in general not of finite type over k. Note that if $\gamma \in X_\infty$ is a point with residue field $k(\gamma)$, then we can identify γ with a morphism $\mathrm{Spec}(k(\gamma)[\![t]\!]) \to X$. We denote by $\pi \colon X_\infty \to X$ the canonical projection taking γ to $\gamma(0)$, the image by γ of the closed point.

Remark 4.1. While X_∞ is not a Noetherian scheme, if C is a closed subset of X_∞, we may still consider the irreducible components of C: these correspond to the prime ideals in $\mathcal{O}(X_\infty)$ which are minimal over the ideal of C. Note that we can still write C as the union of its irreducible components: this is an immediate application of Zorn's Lemma.

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[*]This means that the completion $\widehat{\mathcal{O}_{X,x}}$ is a domain (note that it is always reduced, since $\mathcal{O}_{X,x}$ is a reduced excellent ring). The condition is satisfied, for example, if X is normal.

For every $\gamma \in X_\infty$, we define the function $\mathrm{ord}_\gamma \colon R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by $\mathrm{ord}_\gamma(f) = \mathrm{ord}_t(\gamma^*(f))$. It is clear that ord_γ is a semi-valuation of R.

Given a subset $C \subseteq X_\infty$, we consider the function $\mathrm{ord}_C \colon R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined by

$$\mathrm{ord}_C(f) = \min_{\gamma \in C} \mathrm{ord}_\gamma(f).$$

It follows from the definition that ord_C is a radical pseudo-valuation. For short, we denote $\mathfrak{b}_m(C) := \mathfrak{b}_m(\mathrm{ord}_C) \subseteq R$ and, similarly, let $\mathfrak{b}_\bullet(C) := \mathfrak{b}_\bullet(\mathrm{ord}_C)$.

LEMMA 4.2. *If $\overline{C}$ is the closure of a subset $C \subseteq X_\infty$, then $\mathrm{ord}_{\overline{C}} = \mathrm{ord}_C$.*

Proof. The assertion follows from the fact that for every $f \in R$ and every $m \in \mathbb{Z}$, the set $\{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(f) \geq m\}$ is closed. $\square$

The assertion in the next lemma follows directly from definition.

LEMMA 4.3. *If $C = \bigcup_{i \in I} C_i$, then $\mathrm{ord}_C(f) = \min_{i \in I} \mathrm{ord}_{C_i}(f)$ for every $f \in R$.*

Remark 4.4. If C is irreducible, then ord_C is a semi-valuation. Indeed, it follows from Lemma 4.2 that if δ is the generic point of $\overline{C}$, then $\mathrm{ord}_C = \mathrm{ord}_\delta$, hence ord_C is a semi-valuation.

Remark 4.5. The center of the pseudo-valuation ord_C is equal to $\overline{\pi(C)}$, with the reduced scheme structure. Indeed, this follows from the fact that for $f \in R$ and $\gamma \in X_\infty$, we have $\mathrm{ord}_\gamma(f) \geq 1$ if and only if f lies in the ideal defining $\overline{\pi(\gamma)}$.

Definition 4.6. We define the *volume* $\mathrm{vol}(C)$ of a set $C \subseteq X_\infty$ to be the volume

$$\mathrm{vol}(C) := \mathrm{vol}(\mathrm{ord}_C) = \mathrm{vol}(\mathfrak{b}_\bullet(C))$$

of the pseudo-valuation ord_C.

PROPOSITION 4.7. *For every $C \subseteq X_\infty$, we have $\mathrm{vol}(C) < \infty$ if and only if $\pi(C)$ is a finite set of closed points.*

Proof. The assertion follows by combining Remarks 3.18 and 4.5. $\square$

From now on, we restrict our attention to subsets $C \subseteq X_\infty$ whose image in X is a finite set of closed points. In the next propositions, we give some basic properties of volumes of subsets of X_∞.

PROPOSITION 4.8. *If $C_1 \subseteq C_2$, then $\mathrm{vol}(C_1) \leq \mathrm{vol}(C_2)$.*

Proof. If $C_1 \subseteq C_2$ then it is clear that $\mathrm{ord}_{C_1}(f) \geq \mathrm{ord}_{C_2}(f)$ for every $f \in R$. The assertion then follows from Example 3.21. $\square$

The next proposition allows us to reduce to considering subsets lying in a fiber of $\pi\colon X_\infty \to X$. For every closed point $x \in X$, we denote the fiber $\pi^{-1}(x)$ by $X_\infty(x)$.

PROPOSITION 4.9. *Let $C \subseteq X_\infty$ be such that $\pi(C)$ is a finite set of closed points. If we consider the unique decomposition $C = C_1 \cup \ldots \cup C_r$ such that the $\pi(C_i)$ are pairwise distinct points, then we have*

$$\operatorname{vol}(C) = \sum_{i=1}^{r} \operatorname{vol}(C_i).$$

Proof. If $\pi(C_i) = \{x_i\}$, then it is clear that

$$\mathfrak{b}_m(C) = \bigcap_{i=1}^{r} \mathfrak{b}_m(C_j)$$

and $\mathfrak{b}_m(C_j)$ is cosupported at x_j for every $m \geq 1$. Therefore the assertion follows from Remark 3.3. □

PROPOSITION 4.10. *If $C \subseteq X_\infty(x)$, for some closed point $x \in X$, then*

$$\operatorname{vol}(X) \leq e_x(X).$$

Proof. Note that if $\mathfrak{m}_x$ is the ideal defining x, then $\mathfrak{m}_x \subseteq \mathfrak{b}_1(C)$. Therefore $\mathfrak{m}_x^p \subseteq \mathfrak{b}_1(C)^p \subseteq \mathfrak{b}_p(C)$ for every p, and we obtain $\operatorname{vol}(C) \leq e(\mathfrak{m}_x) = e_x(X)$. □

The following definition extends the notions of thin and fat arcs introduced in [8, 15] to arbitrary sets of arcs.

Definition 4.11. A subset C of X_∞ is said to be *thin* if there exists a proper closed subscheme $Z \hookrightarrow X$ such that $C \subseteq Z_\infty$. If C is not thin, then we say that C is *fat*. A subset C of X_∞ is a *cylinder* if $C = \pi_m^{-1}(S)$ for some m and some constructible subset $S \subseteq X_m$, where $\pi_m\colon X_\infty \to X_m$ is the canonical projection. It is a basic fact that a cylinder C is thin if and only if $C \subseteq (X_{\text{sing}})_\infty$, where X_{sing} is the singular locus of X (see [10], Lemma 5.1).

PROPOSITION 4.12. *Let C be a subset of X_∞ whose image in X is a finite set of closed points. If C is thin, then $\operatorname{vol}(C) = 0$, and if the closure of C is a fat cylinder and X is analytically unramified at every point, then $\operatorname{vol}(C) > 0$.*

Proof. Suppose first that there exists a proper closed subscheme Z of X such that $C \subseteq Z_\infty$. Let $I_Z \subseteq \mathcal{O}_X$ be the ideal of Z. We have $I_Z \subseteq \mathfrak{b}_m(C)$ for every m, hence

$$\ell(\mathcal{O}_X/\mathfrak{b}_m(C)) = \ell(\mathcal{O}_Z/\mathfrak{b}_m(C)\mathcal{O}_Z) = o(m^n)$$

since $\dim Z < n$. This implies that $\operatorname{vol}(C) = 0$.

Let us assume now that $\overline{C}$ is a fat cylinder. Since $\mathrm{ord}_C = \mathrm{ord}_{\overline{C}}$ by Lemma 4.2, we may replace C by $\overline{C}$ and thus assume that C is closed. Since C is a cylinder, it has finitely many irreducible components (see [4], Proposition 3.5). One of these, say C', has to be fat, in which case $\mathrm{ord}_{C'}$ is a divisorial valuation by ([4], Propositions 2.12 and 3.9). Of course, the image of C' in X consists of one closed point. Using Propositions 4.8 and 3.24, we conclude that

$$\mathrm{vol}(C) \geq \mathrm{vol}(C') = \mathrm{vol}(\mathrm{ord}_{C'}) > 0. \quad \square$$

We now address the results stated in the introduction. We begin with the first two propositions.

Proof of Proposition 1.1. For every p, we have

$$(19) \qquad\qquad \mathfrak{b}_p(C_1 \cup C_2) = \mathfrak{b}_p(C_1) \cap \mathfrak{b}_p(C_2) \quad \text{and}$$

$$(20) \qquad\qquad \mathfrak{b}_p(C_1 \cap C_2) \supseteq \mathfrak{b}_p(C_1) + \mathfrak{b}_p(C_2).$$

The exact sequence

$$0 \to \mathcal{O}_X/(\mathfrak{b}_p(C_1) \cap \mathfrak{b}_p(C_2)) \to \mathcal{O}_X/\mathfrak{b}_p(C_1) \oplus \mathcal{O}_X/\mathfrak{b}_p(C_2) \to \mathcal{O}_X/(\mathfrak{b}_p(C_1) + \mathfrak{b}_p(C_2)) \to 0$$

implies

$$\ell(\mathcal{O}_X/\mathfrak{b}_p(C_1)) + \ell(\mathcal{O}_X/\mathfrak{b}_p(C_2)) = \ell(\mathcal{O}_X/\mathfrak{b}_p(C_1) \cap \mathfrak{b}_p(C_2)) + \ell(\mathcal{O}_X/\mathfrak{b}_p(C_1) + \mathfrak{b}_p(C_2)).$$

Using (19) and (20), we conclude

$$\ell(\mathcal{O}_X/\mathfrak{b}_p(C_1)) + \ell(\mathcal{O}_X/\mathfrak{b}_p(C_2)) \geq \ell(\mathcal{O}_X/\mathfrak{b}_p(C_1 \cup C_2)) + \ell(\mathcal{O}_X/\mathfrak{b}_p(C_1 \cap C_2)).$$

Then the assertion follows by dividing by $p^n/n!$ and letting p go to infinity. Note that this step uses the property that the limsup in the definition of the volume is, in fact, a limit. $\square$

Proof of Proposition 1.2. Let $C_m = \mathrm{Cont}^{\geq m}(\mathfrak{a})$. It follows from definition that $\mathfrak{a}^p \subseteq \mathfrak{b}_{mp}(C_m)$ for every $p \geq 1$. By (13), we have

$$m^n \cdot \mathrm{vol}(C_m) = \lim_{p\to\infty} \frac{e(\mathfrak{b}_{mp}(C_m))}{p^n} \leq \lim_{p\to\infty} \frac{e(\mathfrak{a}^p)}{p^n} = e(\mathfrak{a}).$$

Using the characterization of volume in Remark 3.2, we deduce from the inclusion $\mathfrak{a} \subseteq \mathfrak{b}_m(C_m)$ that

$$\mathrm{vol}(C_m) \leq \frac{e(\mathfrak{b}_m(C_m))}{m^n} \leq \frac{e(\mathfrak{a})}{m^n}.$$

Note that if $\gamma(t) \in C_m$, then $\gamma(t^p) \in C_{mp}$. This implies that we have an inclusion

$$\mathfrak{b}_{mpq}(C_{mp}) \subseteq \mathfrak{b}_{mq}(C_m) \quad \text{for every } q,$$

and therefore

$$m^n \cdot \frac{e(\mathfrak{b}_{mq}(C_m))}{(mq)^n} \leq (mp)^n \cdot \frac{e(\mathfrak{b}_{mpq}(C_{mp}))}{(mpq)^n}.$$

By letting q go to infinity, we obtain

$$m^n \cdot \mathrm{vol}(C_m) \leq (mp)^n \cdot \mathrm{vol}(C_{mp}).$$

In order to complete the proof, it is enough to show that when m is divisible enough, we have $\mathrm{vol}(C_m) \geq \frac{e(\mathfrak{a})}{m^n}$. Suppose that $E_1, \ldots, E_r$ are the divisors over X corresponding to the Rees valuations associated to the ideal $\mathfrak{a}$ (see Example 3.20). We put $q_i = \mathrm{ord}_{E_i}(\mathfrak{a})$ and assume that m is divisible by every q_i. Recall that if E is a divisor over X, then there is a sequence of irreducible closed subsets $C_X^q(E)$, for $q \geq 1$, called the *maximal divisorial sets*, which are defined as follows. If $\pi \colon Y \to X$ is a birational map such that Y is smooth and E is a smooth divisor on Y, then $C_X^q(E)$ is the closure of $\pi_\infty(\mathrm{Cont}^{\geq q}(E))$. It is easy to see that $\mathrm{ord}_{C_X^q(E)} = q \cdot \mathrm{ord}_E$. For a discussion of these subsets of X_∞, we refer to [8] and [4]. With this notation, we consider the closed subset

$$T_m := \bigcup_{i=1}^r C_X^{m/q_i}(E_i).$$

Note that we have $T_m \subseteq C_m$, hence

$$\mathfrak{b}_{jm}(C_m) \subseteq \mathfrak{b}_{jm}(T_m) = \bigcap_{i=1}^r \{f \in R \mid \mathrm{ord}_{E_i}(f) \geq jq_i\} = \overline{\mathfrak{a}^j},$$

where we denote by $\bar{\mathfrak{c}}$ the integral closure of an ideal $\mathfrak{c}$. We conclude that

$$e(\mathfrak{b}_{jm}(C_m)) \geq e(\overline{\mathfrak{a}^j}) = j^n \cdot e(\mathfrak{a}).$$

Dividing by $(jm)^n$ and letting j go to infinity, we get $\mathrm{vol}(C_m) \geq \frac{e(\mathfrak{a})}{m^n}$. This completes the proof of the proposition. $\square$

Next, we review the definition of jet-codimension and prove two more preliminary properties before addressing the proof of Theorem 1.3. Recall that the Krull codimension of a closed irreducible set $C \subseteq X_\infty$ is the dimension of the local ring $\mathcal{O}_{X_\infty, C}$, and is denoted by $\mathrm{codim}(C)$. The definition extends to an arbitrary set $C \subseteq X_\infty$ by taking the smallest codimenion of an irreducible component of the closure $\overline{C}$.

While the Krull codimension is computed from the local rings of X_∞, the jet-codimension is computed from the finite levels X_m. In order to define it, we need the following lemma.

LEMMA 4.13. *For every subset $C \subseteq X_\infty$, the limit*

$$\lim_{m \to \infty} \left((m+1)n - \dim \overline{\pi_m(C)} \right)$$

exists.

Proof. It follows from ([7], Lemma 4.3) that for every m, the fibers of the map $\pi_{m+1}(X_\infty) \to \pi_m(X_\infty)$ have dimension $\leq n$ (note that both sets are constructible by a result due to Greenberg [13]). It follows from Lemma 4.14 below that $\dim \overline{\pi_{m+1}(C)} \leq \dim \overline{\pi_m(C)} + n$, hence the sequence $(a_m)_{m\geq 1}$ with $a_m = (m+1)n - \dim \overline{\pi_m(C)}$ is a non-decreasing sequence of integers. Therefore it either stabilizes or it has limit infinity. $\square$

LEMMA 4.14. *Let $f: V \to W$ be a morphism of algebraic varieties over k and suppose that d is a non-negative integer and A is a constructible subset of V such that for every $y \in f(A)$, we have $\dim(f^{-1}(y) \cap A) \leq d$. For every subset $B \subseteq A$, we have*

$$\dim(\overline{B}) \leq d + \dim(\overline{f(B)}).$$

Proof. We can write $A = \bigcup_{i=1}^r A_i$, with each A_i a locally closed subset of V. If $B_i = B \cap A_i$, then $B = \bigcup_{i=1}^r B_i$, $\overline{B} = \bigcup_{i=1}^r \overline{B_i}$, and $\overline{f(B)} = \bigcup_{i=1}^r \overline{f(B_i)}$. Since it is enough to prove the assertion for each B_i, it follows that we may assume that A is a locally closed subset. In this case A is open in $\overline{A}$, hence $A \cap \overline{B}$ is a dense open subset of $\overline{B}$. Since $\dim(\overline{B}) = \dim(A \cap \overline{B})$ and the fibers of the morphism $A \cap \overline{B} \to \overline{f(B)}$ have dimension $\leq d$, we obtain the assertion in the lemma. $\square$

Definition 4.15. The *jet-codimension* of an irreducible closed subset C of X_∞ is defined to be

$$\text{jet-codim}(C) := \lim_{m\to\infty} \left((m+1)n - \dim \overline{\pi_m(C)} \right).$$

For an arbitrary subset $C \subseteq X_\infty$, we define jet-codim(C) to be the smallest jet-codimension of an irreducible component of $\overline{C}$.

Remark 4.16. It follows from the proof of Lemma 4.13 that if C is closed and irreducible, then jet-codim$(C) \geq n - \dim \overline{\pi(C)} \geq 0$. This implies that for every $C \subseteq X$, we have jet-codim$(C) \geq 0$.

Remark 4.17. If $C_1 \subseteq C_2 \subseteq X_\infty$, then jet-codim$(C_1) \geq$ jet-codim(C_2). Indeed, if C_1' is an irreducible component of $\overline{C_1}$, then there is an irreducible component C_2' of C_2 such that $C_1' \subseteq C_2'$. In this case, for every m we have

$$(m+1)n - \dim \overline{\pi_m(C_1')} \geq (m+1)n - \dim \overline{\pi_m(C_2')}.$$

By letting m go to infinity, we conclude that jet-codim$(C_1') \geq$jet-codim(C_2') $\geq$ codim(C_2). Since this holds for every irreducible component of $\overline{C_1}$, we conclude that jet-codim$(C_1) \geq$ jet-codim(C_2).

Remark 4.18. For any subset $C \subseteq X_\infty$, we have $\text{codim}(C) = \text{codim}(\overline{C})$ and $\text{jet-codim}(C) = \text{jet-codim}(\overline{C})$.

If X is smooth and $C \subseteq X_\infty$ is a cylinder, then we have $\text{jet-codim}(C) = \text{codim}(\pi_m(C), X_m)$ for all $m \gg 1$. As the next proposition shows, this is equal to the Krull codimension $\text{codim}(C)$. More generally, we have the following property.

PROPOSITION 4.19. *If X is smooth and $C \subseteq X_\infty$ is any set, then we have* $\text{jet-codim}(C) = \text{codim}(C)$.

Proof. The proof of the proposition follows immediately by applying the next lemma to the irreducible components of $\overline{C}$. $\square$

LEMMA 4.20. *If X is smooth and $C \subseteq X_\infty$ is a closed irreducible subset, then*
$$\text{jet-codim}(C) = \text{codim}(C),$$
and this number is finite if and only if C is a cylinder.

Proof. If C is a cylinder, then it follows from ([8], Corollary 1.9) that
$$\text{jet-codim}(C) = \text{codim}(\pi_m(C), X_m) = \text{codim}(C) \quad \text{for } m \gg 1.$$

Therefore it suffices to show that if C is not a cylinder then
$$\text{jet-codim}(C) = \dim(C) = \infty.$$

In order to check this, consider the sequence of closed irreducible cylinders
$$F_i := \pi_i^{-1}(\overline{\pi_i(C)}), \quad i \geq 0.$$

We have inclusions
$$C \subseteq \cdots \subseteq F_{i+1} \subseteq F_i \subseteq \cdots \subseteq F_1 \subseteq F_0 \subseteq X_\infty.$$

Moreover, since C is closed, we have $C = \bigcap_{i \geq 0} F_i$.

Since C is not a cylinder, the sequence $(F_i)_{i \geq 0}$ does not stabilize. Therefore we can pick a subsequence $(F_{i_m})_{m \geq 0}$ such that
$$C \subsetneq F_{i_m} \subsetneq F_{i_{m-1}} \subsetneq \cdots \subsetneq F_{i_1} \subsetneq F_{i_0} \subsetneq X_\infty,$$

which clearly implies that $\text{codim}(C) = \infty$. In fact, for every m, if $p \geq i_m$, then we also have the sequence
$$\overline{\pi_p(C)} \subseteq \pi_p(F_{i_m}) \subsetneq \pi_p(F_{i_{m-1}}) \subsetneq \cdots \subsetneq \pi_p(F_{i_1}) \subsetneq \pi_p(F_{i_0}) \subsetneq X_p.$$

Note that for every $k \leq m$, the subset $\pi_p(F_{i_k})$ of X_p is irreducible and closed since $p \geq i_k$. Therefore $\text{codim}(\overline{\pi_p(C)}, X_p) \geq m$ and we conclude that $\text{jet-codim}(C) = \infty$. $\square$

Remark 4.21. The definition of jet-codimension generalizes to all sets the definition of codimension of a quasi-cylinder given in [4]. In general, if X is singular and $C \subseteq X_\infty$ is a closed irreducible set, then there is only an inequality $\mathrm{codim}(C) \leq \mathrm{jet\text{-}codim}(C)$ which can be strict (*e.g.*, see [17], Example 2.8).

If E is a prime exceptional divisor over X and $C_X^q(E) \subseteq X_\infty$ is the maximal divisorial set associated to the divisorial valuation $q \cdot \mathrm{ord}_E$, then we have

$$(21) \qquad \mathrm{jet\text{-}codim}(C_X^q(E)) = q \cdot \widehat{a}_E(X)$$

by ([4], Theorem 3.8). Using this fact, it is easy to extend ([22], Corollary 0.2) to the singular setting, as follows. This proposition is also proved in ([16], Proposition 3.5), but since the proof is short, we include it for the convenience of the reader.

PROPOSITION 4.22. *For every proper, nonzero ideal* $\mathfrak{a} \subseteq R$ *and every positive integer* m, *we have*

$$\mathrm{jet\text{-}codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) \geq m \cdot \widehat{\mathrm{lct}}(\mathfrak{a}),$$

with equality if m *is sufficiently divisible.*

Proof. By ([4], Propositions 3.5 and 2.12), $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ has finitely many fat irreducible components, and any such component C is a maximal divisorial set. In particular, there is a fat irreducible component of the form $C = C_X^q(E)$ for some divisorial valuation $q \cdot \mathrm{ord}_E$, such that

$$\mathrm{jet\text{-}codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) = \mathrm{jet\text{-}codim}(C_X^q(E)) = q \cdot \widehat{a}_E(X),$$

by (21). Note that $q \cdot \mathrm{ord}_E(\mathfrak{a}) \geq m$, since $C_X^q(E) \subseteq \mathrm{Cont}^{\geq m}(\mathfrak{a})$. On the other hand, we have

$$\widehat{\mathrm{lct}}(\mathfrak{a}) \leq \frac{\widehat{a}_E(X)}{\mathrm{ord}_E(\mathfrak{a})}$$

by the definition of Mather log canonical threshold. We conclude that

$$\mathrm{jet\text{-}codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) \geq m \cdot \widehat{\mathrm{lct}}(\mathfrak{a}).$$

On the other hand, suppose that F is a divisor over X such that $\widehat{\mathrm{lct}}(\mathfrak{a}) = \frac{\widehat{a}_F(X)}{\mathrm{ord}_F(\mathfrak{a})}$ and suppose that $m = q \cdot \mathrm{ord}_F(\mathfrak{a})$ for some positive integer q. In this case $C_X^q(F) \subseteq \mathrm{Cont}^{\geq m}(\mathfrak{a})$, hence

$$\mathrm{jet\text{-}codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) \leq \mathrm{jet\text{-}codim}(C_X^q(F)) = q \cdot \widehat{a}_F(X) = m \cdot \widehat{\mathrm{lct}}(\mathfrak{a}).$$

By combining this with what we have already proved, we conclude that in this case we have $\mathrm{jet\text{-}codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a})) = m \cdot \widehat{\mathrm{lct}}(\mathfrak{a})$. $\square$

Proof of Theorem 1.3 For every $p \geq 1$, we have $C \subseteq \mathrm{Cont}^{\geq p}(\mathfrak{b}_p(C))$. Note that if C lies over the closed point $x \in X$, defined by the maximal ideal $\mathfrak{m}_x$, the ideal $\mathfrak{b}_p(C)$ is $\mathfrak{m}_x$-primary. It follows from Proposition 4.22 that

$$(22) \qquad \mathrm{jet\text{-}codim}(C) \geq \mathrm{jet\text{-}codim}\,\mathrm{Cont}^{\geq p}(\mathfrak{b}_p(C))) \geq p \cdot \widehat{\mathrm{lct}}(\mathfrak{b}_p(C)).$$

On the other hand, Theorem 1.4 implies that

$$(23) \qquad (n! \cdot \ell(\mathcal{O}_X/\mathfrak{b}_p(C)))^{1/n} \cdot \widehat{\mathrm{lct}}(\mathfrak{b}_p(C)) \geq n.$$

By combining (22) and (23), we get

$$\left(\frac{\ell(\mathcal{O}_X/\mathfrak{b}_p(C))}{p^n/n!} \right)^{1/n} \cdot \mathrm{jet\text{-}codim}(C) \geq n.$$

We conclude that

$$\mathrm{vol}(C)^{1/n} \cdot \mathrm{jet\text{-}codim}(C) = \lim_{p \to \infty} \left(\frac{\ell(\mathcal{O}_X/\mathfrak{b}_p(C))}{p^n/n!} \right)^{1/n} \cdot \mathrm{jet\text{-}codim}(C) \geq n.$$

This gives the first part of the statement of the theorem. The second part follows from Proposition 4.19. $\square$

For simplicity, we have always assumed that the ambient variety is affine. However, as Remark 4.24 below shows, one can easily extend the notion of volume for subsets of spaces of arcs to the case of possibly non-affine varieties.

Remark 4.23. Suppose that $C \subseteq X_\infty$ is contained in the fiber over a closed point $x \in X$. If $h \in R$ is such that $h(x) \neq 0$, then $\mathrm{ord}_C(fh^m) = \mathrm{ord}_C(f)$ for every $f \in R$ and every $m \geq 1$. By Remark 3.23, we may extend ord_C as a pseudo-valuation of R_f; in fact, this is the pseudo-valuation associated to C when considered as a subset of U_∞, where U is the open subset $\mathrm{Spec}\,R_h \subseteq X$. It follows from Remark 3.23 that C has the same volume, whether regarded as a subset of X_∞ or U_∞.

Remark 4.24. Suppose now that X is an arbitrary variety over k, possibly not affine, and $C \subseteq X_\infty$ is a subset. We can define $\mathrm{vol}(C)$ as follows. If the image of C in X does not consist of finitely many closed points, then we put $\mathrm{vol}(C) = \infty$. If the image of C in X consists of the closed points $x_1, \ldots, x_r$, then we choose affine open neighborhoods U_i of x_i in X. If $C_i = C \cap (U_i)_\infty$, then $C = \bigsqcup_{i=1}^r C_i$ and we may consider each C_i as a subset of $(U_i)_\infty$. Therefore $\mathrm{vol}(C_i)$ is defined and independent of the choice of U_i by Remark 4.23 and we put

$$\mathrm{vol}(C) := \sum_{i=1}^r \mathrm{vol}(C_i).$$

It is easy to deduce from Proposition 4.9 and Remark 4.23 that if there is an affine open subset $U \subseteq X$ such that $C \subseteq U_\infty$ (this is the case, for

example, whenever X is quasiprojective), then $\mathrm{vol}(C)$ can be also computed by considering C as a subset of U_∞ and using our original definition.

REFERENCES

[1] George M. Bergman, *A weak Nullstellensatz for valuations*. Proc. Amer. Math. Soc. **28** (1971), 32–38.

[2] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*. Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.

[3] Steven Dale Cutkosky, *Asymptotic multiplicities of graded families of ideals and linear series*. Adv. Math. **264** (2014), 55–113.

[4] Tommaso de Fernex, Lawrence Ein and Shihoko Ishii, *Divisorial valuations via arcs*. Publ. Res. Inst. Math. Sci. **44** (2008), *2*, 425–448.

[5] Tommaso de Fernex, Lawrence Ein and Mircea Mustaţă, *Bounds for log canonical thresholds with applications to birational rigidity*. Math. Res. Lett. **10** (2003), *2–3*, 219–236.

[6] Tommaso de Fernex, Lawrence Ein and Mircea Mustaţă, *Multiplicities and log canonical threshold*. J. Algebraic Geom. **13** (2004), *3*, 603–615.

[7] Jan Denef and François Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*. Invent. Math. **135** (1999), *1*, 201–232.

[8] Lawrence Ein, Robert Lazarsfeld and Mircea Mustaţă, *Contact loci in arc spaces*. Compos. Math. **140** (2004), *5*, 1229–1244.

[9] Lawrence Ein, Robert Lazarsfeld and Karen E. Smith, *Uniform approximation of Abhyankar valuation ideals in smooth function fields*. Amer. J. Math., **125** (2003), *2*, 409–440.

[10] Lawrence Ein and Mircea Mustaţă, *Jet schemes and singularities*. Algebraic geometry, Seattle 2005. Part 2, 2009, pp. 505–546.

[11] David Eisenbud, *Commutative algebra. With a view toward algebraic geometry*. Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.

[12] William Fulton, *Intersection theory*. 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. **3**. Folge, Springer-Verlag, Berlin,1998.

[13] Marvin J. Greenberg, *Rational points in Henselian discrete valuation rings*. Inst. Hautes Études Sci. Publ. Math. **31** (1966), 59–64.

[14] Reinhold Hübl and Irena Swanson, *Discrete valuations centered on local domains*. J. Pure Appl. Algebra **161** (2001), *1–2*, 145–166.

[15] Shihoko Ishii, *Arcs, valuations and the Nash map*. J. Reine Angew. Math. **588** (2005), 71–92.

[16] Shihoko Ishii, *Mather discrepancy and the arc spaces*. Ann. Inst. Fourier (Grenoble) **63** (2013), *1*, 89–111.

[17] Shihoko Ishii and Ana J. Reguera, *Singularities with the highest Mather minimal log discrepancy*. Math. Z. **275** (2013), *3–4*, 1255–1274.

[18] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*. Cambridge Tracts in Mathematics. **134**, Cambridge University Press, Cambridge, 1998. With the collaboration of C.H. Clemens and A. Corti; Translated from the 1998 Japanese original.

[19] János Kollár and András Némethi, *Durfee's conjecture on the signature of smoothings of surface singularities, with an appendix by Tommaso de Fernex*. (2014). `arXiv:1411.1039`.

[20] Robert Lazarsfeld and Mircea Mustaţă, *Convex bodies associated to linear series*. Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), *5*, 783–835.

[21] Mircea Mustaţă, *On multiplicities of graded sequences of ideals*. J. Algebra **256** (2002), *1*, 229–249.

[22] Mircea Mustaţă, *Singularities of pairs via jet schemes*. J. Amer. Math. Soc. **15** (2002), *3*, 599–615 (electronic).

[23] Irena Swanson, *Rees valuations*. Commutative algebra – Noetherian and non-Noetherian perspectives, New York Springer, 2011, pp. 421–440.

[24] Oscar Zariski and Pierre Samuel, *Commutative algebra*. Vol. II. The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N.J.–Toronto–London–New York, 1960.

University of Utah,
Department of Mathematics,
Salt Lake City,
UT 48112, USA
defernex@math.utah.edu

University of Michigan,
Department of Mathematics,
Ann Arbor,
MI 48109, USA
mmustata@umich.edu