Dedicated to Professor Lucian Bădescu on the occasion of his 70^{th} birthday

ARITHMETIC ANALOGUES OF SOME BASIC CONCEPTS FROM RIEMANNIAN GEOMETRY

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Following recent work of the author, partly in collaboration with T. Dupuy and M. Barrett, we describe arithmetic analogues of some key concepts from Riemannian geometry such as: metrics, Chern connections, curvature, etc. Theorems are stated to the effect that the spectrum of the integers has a non-vanishing curvature.

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1. INTRODUCTION

In previous work (initiated in [4] and partly summarized in [5, 6]) the author has developed an arithmetic analogue of differential calculus and, in particular, of differential equations. As explained in [3, 13], this theory can be viewed as an alternative approach to "absolute geometry" (or the "geometry over the field with one element, $\mathbb{F}_1$") and led to a series of diophantine applications [6]. Once an arithmetic analogue of differential calculus is available one can ask for arithmetic analogues of the basic concepts of differential geometry and, in particular, of Riemannian geometry. Such analogues were recently proposed in [1, 2, 8–10] and led to the somewhat surprising conclusion that the spectrum of the integers, *Spec* $\mathbb{Z}$, can be viewed as an (infinite dimensional) manifold which is naturally "curved" (although, as we shall see, only "mildly" curved). The aim of this note is to present, in a self contained manner, some of the ideas and results of this "arithmetic Riemannian" theory. For the details of the theory we refer to the papers cited above.

2. MAIN CONCEPTS AND RESULTS

The best way to present our material is by analogy with classical differential geometry. In classical differential geometry one starts with an m-dimensional smooth manifold M and its ring of smooth functions $C^{\infty}(M)$. For our purposes it is enough to think of M as being the Euclidean space $M = \mathbb{R}^m$. Also we would like to think of the dimension m as going to infinity, $m \to \infty$. In this paper the arithmetic analogue of $\mathbb{R}^m$, with $m \to \infty$, will be the scheme *Spec* $\mathbb{Z}$. Let

$$\mathcal{U} = \{u_1, u_2, ..., u_m\} \tag{2.1}$$

be the set of coordinate functions on $\mathbb{R}^m$. Then the analogue of the ring of polynomial functions

$$\mathbb{R}[u_1, ..., u_m] \tag{2.2}$$

on $\mathbb{R}^m$ will be the ring of integers $\mathbb{Z}$ or, more generally, the ring

$$\mathbb{Z}[1/N_0, \zeta_N] \tag{2.3}$$

where N_0 is an even integer, N is an integer, and ζ_N is a primitive N-th root of unity. The analogue of the coordinate functions 2.1 will be a set of primes,

$$\mathcal{V} = \{p_1, p_2, p_3, ...\} \subset \mathbb{Z}. \tag{2.4}$$

One can take all primes or, better, all primes not dividing N_0N. One can further ask for an analogue of the ring $C^{\infty}(\mathbb{R}^m)$; as a general rule, in this paper, C^{∞} objects will correspond, in arithmetic, to adelic objects.

Next one considers the partial derivative operators

$$\delta_i : C^{\infty}(\mathbb{R}^m) \to C^{\infty}(\mathbb{R}^m), \quad \delta_i f := \frac{\partial f}{\partial u_i}, \quad , i \in \{1, ..., m\}. \tag{2.5}$$

Following [4] we propose to take, as an analogue of 2.5, the operators

$$\delta_p : \mathbb{Z}[1/N_0, \zeta_N] \to \mathbb{Z}[1/N_0, \zeta_N], \quad \delta_p(a) = \frac{\phi_p(a) - a^p}{p}, \quad p \in \mathcal{V}, \tag{2.6}$$

where $\phi_p : \mathbb{Z}[1/N_0, \zeta_N] \to \mathbb{Z}[1/N_0, \zeta_N]$ is the unique ring automorphism sending ζ_N into ζ_N^p. More generally the concept of *derivation* on a ring B (by which we mean an additive map $B \to B$ that satisfies the Leibniz rule) has, as an arithmetic analogue, the concept of *p-derivation* defined as follows. Assume B is a ring and assume, for simplicity, that p is a non-zero divisor in B; then a *p-derivation* on B is a set theoretic map $\delta_p : B \to B$ with the property that the map $\phi_p : B \to B$ defined by $\phi_p(b) = b^p + p\delta_p b$ is a ring homomorphism; we will always denote by ϕ_p the ring homomorphism attached to a p-derivation δ_p and we shall refer to ϕ_p as the *Frobenius lift* attached to δ_p.

The next step in the classical theory is to consider various fibrations $E \to M$ and various tensors on E and M. Depending on the nature of the tensors one is led to various flavors of differential geometry. Among these flavors we would like to mention contact geometry and metric geometry. Contact geometry has, as one of its motivating examples, the study of the case when E is a bundle of jets of M, equipped with contact forms; this example leads to the geometric theory of differential equations and of classical mechanics (Lagrange and Hamiltonian formalisms). Metric geometry leads, on the other hand, to Riemannian geometry (the geometry of gravity, where one has a metric on M) and gauge theory (the semi-classical theory of elementary particles, where one has a family of metrics on the fibers of E, a principal bundle over M). An arithmetic analogue of contact geometry was developed in a series of papers (reviewed, for instance, in [6]) where the fibration $E \to M$ had, as analogue, what were called *arithmetic jet spaces* of various arithmetically interesting varieties (especially elliptic curves and modular curves) over rings of integers; this led to a series of purely arithmetic applications (to topics related to the Manin-Mumford conjecture, congruences between classical modular forms, Heegner points, etc.). An analogue of the metric geometry (gauge-theoretic style) was proposed more recently in [1, 2, 8–10] and will be explained in what follows. Classically one starts with $E \to M$ the frame bundle of a rank n vector bundle over the manifold M. Then E is a principal homogeneous space for the group GL_n; and if the vector bundle is trivial (which we shall assume from now on) then E is identified with $M \times GL_n$. (Note that the rank n of the vector bundle and the dimension m of M in this picture are unrelated.) We want to review the classical concept of connection in E; we shall do it in a somewhat non-standard way so that the transition to arithmetic becomes more transparent. Indeed consider an $n \times n$ matrix $x = (x_{ij})$ of indeterminates and consider the ring of polynomials over $A = C^\infty(\mathbb{R}^m)$, with the determinant inverted,

$$B = A[x, \det(x)^{-1}]. \tag{2.7}$$

Note that B is naturally a subring of the ring $C^\infty(M \times GL_n)$. Then by a *connection* on $E = M \times GL_n$ we will understand a tuple $\delta = (\delta_i)$ of derivations

$$\delta_i : B \to B, \quad i \in \{1, ..., m\}, \tag{2.8}$$

lifting the derivations 2.5. We say the connection is *invariant* (or *linear*) if $\delta_i x = A_i x$ for some $n \times n$ matrix A_i with coefficients in A; invariance here corresponds to the concept of invariance of classical connections under the right action of GL_n on the frame bundle E. For an invariant connection δ as above one can define the *curvature* as the matrix (φ_{ij}) of commutators

$$\varphi_{ij} := [\delta_i, \delta_j] : B \to B, \quad i, j \in \{1, ..., m\}. \tag{2.9}$$

One has then $\varphi_{ij}(x) = F_{ij}x$ where F_{ij} is the matrix given by the classical formula

$$F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j]. \tag{2.10}$$

We would like to introduce now an arithmetic analogue of connection and curvature. The first step is clear: we consider a ring B defined as in 2.7 but where A is given now by $A = \mathbb{Z}[1/N_0, \zeta_N]$ as in 2.3. A first attempt to define arithmetic analogues of connections would be to consider families of p-derivations $\delta_p : B \to B$, $p \in \mathcal{V}$, lifting the p-derivations 2.6; one would then proceed by considering their commutators on B (or, if necessary, expressions derived from these commutators). But the point is that the examples of "arithmetic analogues of connections" we will encounter in practice (when we develop arithmetic analogues of the Chern connections of classical differential geometry) will never lead to p-derivations $B \to B$! What we shall be led to is, rather, an adelic concept we next introduce. (This is also in line with our "principle" that C^∞ geometric objects should correspond to adelic objects in arithmetic.) For each $p \in \mathcal{V}$ we consider the p-adic completion of B:

$$B^{\widehat{p}} := \lim_{\leftarrow} B/p^n B. \tag{2.11}$$

Then we define an *adelic connection* on GL_n to be a family (δ_p) of p-derivations

$$\delta_p : B^{\widehat{p}} \to B^{\widehat{p}}, \quad p \in \mathcal{V}, \tag{2.12}$$

lifting the p-derivations in 2.6. We do not impose any condition analogous to invariance; instead, what happens is that our adelic connections of interest turn out to enjoy a certain invariance property with respect to right translations by the elements of the normalizer of the maximal (diagonal) torus of GL_n. Leaving the invariance issue aside we are facing, at this point, a more severe dilemma: our p-derivations δ_p in 2.12 do not act on the same ring, so there is no a priori way of considering their commutators and, hence, it does not seem possible to define, in this way, the notion of curvature. It will turn out, however, that our adelic connections of interest will satisfy an interesting property which we call "being global along the identity", and which will allow us to define curvature via commutators. Here is the definition of this property. Consider the matrix $T = x - 1$, where 1 is the identity matrix. We say that an adelic connection (δ_p) on GL_n, with attached family of Frobenius lifts (ϕ_p), is *global along* 1 if, for all p, ϕ_p sends the ideal of 1 into itself and, moreover, the induced homomorphism $\phi_p : A^{\widehat{p}}[[T]] \to A^{\widehat{p}}[[T]]$ sends the ring $A[[T]]$ into itself. If the above holds then one can consider the commutator $[\phi_p, \phi_{p'}] : A[[T]] \to A[[T]]$ for all $p, p' \in \mathcal{V}$; this commutator is divisible by pp' and one can define the *curvature* of (δ_p) as the matrix $(\varphi_{pp'})$ with entries

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[[T]] \to A[[T]], \quad p, p' \in \mathcal{V}. \tag{2.13}$$

The idea of comparing p-adic phenomena for different p's by "moving along the identity section" is borrowed from [7] where it was referred to as "analytic continuation along primes". Of course, in order for the above definitions to be interesting, we will need to:

1) find natural "metric" adelic connections on GL_n,

2) show that these adelic connections are global along 1, and

3) compute and interpret the curvatures of these adelic connections.

We explain now how this program can be achieved. First we go back to classical differential geometry and we "recall" the definition of the Chern connection [12]. We shall present this definition in the "real setting" only, where the Chern connections should be more appropriately referred to as Duistermaat connections [11]; for the "complex setting" we refer to [1, 9]. So let us consider the ring $A = C^\infty(\mathbb{R}^m)$ and let q be an $n \times n$ invertible matrix with coefficients in A which is either symmetric ($q^t = q$) or antisymmetric ($q^t = -q$), where the t superscript means *transposition*. Of course, a symmetric q as above is viewed as a "metric" while an antisymmetric q is viewed as a "2-form." Set $G = GL_n$ and consider the maps of schemes over A, $\mathcal{H}_q : G \to G$, $\mathcal{B}_q : G \times G \to G$ defined by $\mathcal{H}_q(x) = x^t q x$ and $\mathcal{B}_q(x, y) = x^t q y$. We continue to denote by the same letters the corresponding maps of rings $B \to B$ and $B \to B \otimes_A B$. Consider the *trivial* (invariant) connection $\delta_0 = (\delta_{0i})$ on G defined by $\delta_{0i} x = 0$. Then one can easily check (see below) that there is a unique invariant connection (δ_i) on G such that the following diagrams are commutative:

$$\begin{array}{ccccccc} B & \stackrel{\delta_i}{\longleftarrow} & B & \qquad & B & \stackrel{\delta_i \otimes 1 + 1 \otimes \delta_{0i}}{\longleftarrow} & B \otimes_A B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{H}_q & \qquad & \delta_{0i} \otimes 1 + 1 \otimes \delta_i \uparrow & & \uparrow \mathcal{B}_q \\ B & \stackrel{\delta_{0i}}{\longleftarrow} & B & \qquad & B \otimes_A B & \stackrel{\mathcal{B}_q}{\longleftarrow} & B \end{array} \tag{2.14}$$

This δ can be referred to as the *Chern connection* attached to q. The definition just given may look non-standard. It turns out that the Chern connection we just defined is a real analogue [11] of the usual Chern connection in differential geometry [12] (in which δ_0 is an analogue of a complex structure). To see this set $\Gamma_i = -A_i^t$, let Γ_{ij}^k be the (j, k)-entry of Γ_i (the Cristoffel symbols), and set $\Gamma_{ijk} := \Gamma_{ij}^l q_{lk}$ (Einstein notation). Assume we are in the symmetric case, $q^t = q$. Then the commutativity of the left diagram in 2.14 is equivalent to the condition

$$\delta_i q_{jk} = \Gamma_{ijk} + \Gamma_{ikj}, \tag{2.15}$$

and the commutativity of the right diagram in 2.14 is equivalent to the condition

$$\Gamma_{ijk} = \Gamma_{ikj}; \tag{2.16}$$

so the Chern connection attached to q is given by

$$\Gamma_{ijk} = \frac{1}{2}\delta_i q_{jk}. \tag{2.17}$$

The Chern connection will have an arithmetic analogue to be explained presently. The condition 2.15 expresses the fact that q is *parallel* with respect to the connection δ. It is important to note, however, that, in our setting, the *torsion* is not defined and, in particular, the symmetry in 2.16 has nothing to do with the vanishing of the torsion. On the other hand, if one takes E to be the tangent bundle of M (so in particular $n = m$), then the condition that the *torsion of δ vanishes* is given by

$$\Gamma_{ijk} = \Gamma_{jik} \tag{2.18}$$

which is a symmetry condition rather different from 2.16. By the way there is a unique connection δ such that conditions 2.15 and 2.18 are satisfied; this connection is referred to as the *Levi-Civita connection* and is given by the formula

$$\Gamma_{kij} = \frac{1}{2}\left(\delta_k q_{ij} + \delta_i q_{jk} - \delta_j q_{ki}\right). \tag{2.19}$$

The Levi-Civita connection does not seem to have an arithmetic analogue in our theory, at this point.

Now we move to the arithmetic situation. So let $A = \mathbb{Z}[1/N_0, \zeta_N]$. Let $q \in GL_n(A)$ with $q^t = \pm q$. Set $G = GL_n = Spec\ B$, viewed as a group scheme over A. Attached to q we have, again, maps $\mathcal{H}_q : G \to G$ and $\mathcal{B}_q : G \times G \to G$ defined by $\mathcal{H}_q(x) = x^t q x$ and $\mathcal{B}_q(x,y) = x^t q y$. We continue to denote by $\mathcal{H}_q, \mathcal{B}_q$ the maps induced on the p-adic completions $G^{\widehat{p}}$ and $G^{\widehat{p}} \times G^{\widehat{p}}$. Consider the unique adelic connection $\delta_0 = (\delta_{0,p})$ on G with $\delta_{0,p}x = 0$ and denote by (ϕ_p) and $(\phi_{0,p})$ the families of lifts of Frobenius attached to δ and δ_0 respectively. Then one has the following:

THEOREM 2.1 ([9]). *For any $q \in GL_n(A)$ with $q^t = \pm q$ there exists a unique adelic connection δ such that the following diagrams are commutative:*

$$\begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_p} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \xrightarrow{\phi_{0,p}} & G^{\widehat{p}} \end{array} \qquad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_{0,p}\times\phi_p} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \phi_p \times \phi_{0,p} \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mathcal{B}_q} & G^{\widehat{p}} \end{array}$$

The adelic connection δ is referred to as the *Chern connection* (on $G = GL_n$) attached to q. The various q's with $q^t = \pm q$ lead to the various forms of the classical groups Sp_n and SO_n. A similar theorem is proved in [9] for the classical groups SL_n. Note the following relation between the "Christoffel symbols" defining our Chern connection and the Legendre symbol. We explain this in a special case. Let $q \in GL_1(A) = A^\times$, $A = \mathbb{Z}[1/N_0]$, and let $\delta = (\delta_p)$ be the Chern connection associated to q. Then it turns out that $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ is defined by $\phi_p : \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}} \to \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}}$,

$$\phi_p(x) = q^{(p-1)/2} \left(\frac{q}{p}\right) x^p, \tag{2.20}$$

where $\left(\frac{q}{p}\right)$ is the Legendre symbol of $q \in A^\times \subset \mathbb{Z}_{(p)}$.

Next one can ask which of these adelic connections admit curvatures. One has:

THEOREM 2.2 ([1]). *If all the entries of q are roots of unity or 0 then the Chern connection δ attached to q is global along 1. In particular δ has a well defined curvature.*

Next we address the question of computing the curvature of Chern connections. Let us say that a matrix $q \in GL_n(A)$ is split if it is one of the following:

$$\begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix}, \tag{2.21}$$

where 1_r is the $r \times r$ identity matrix and $n = 2r, 2r, 2r+1$ respectively. These are matrices that define the classical split groups $Sp_{2r}, SO_{2r}, SO_{2r+1}$, respectively. One has the following:

THEOREM 2.3 ([1]). *Let q be split and let $(\varphi_{pp'})$ be the curvature of the Chern connection on G attached to q.*

1) Assume $n \geq 4$. Then for all $p \neq p'$ we have $\varphi_{pp'} \neq 0$.

2) Assume n even. Then for all p, p' we have $\varphi_{pp'}(T) \equiv 0$ mod $(T)^3$.

3) Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi_{pp'} = 0$.

4) Assume $n = 1$. Then for all p, p' we have $\varphi_{pp'} = 0$.

In assertion 2) we denoted by $(T)^3$ the cube of the ideal in $A[[T]]$ generated by the entries of the matrix T. Assertion 1) morally says that $Spec\, \mathbb{Z}$ is "curved", while assertion 2) morally says that $Spec\, \mathbb{Z}$ is only "mildly curved". Note that the theorem says nothing about the vanishing of the curvature $\varphi_{pp'}$ in case $n = 2, 3$ and $q^t = q$; our method of proof does not seem to apply to these cases.

The theory explained above has a "complex analogue" (or, rather, a $(1,1)$-analogue) for which we refer to [1, 9].

This theory (that largely follows [1]) was based on what we called “analytic continuation between primes”; this was the key to making Frobenius lifts corresponding to different primes act on a same ring. There is a different approach towards making Frobenius lifts comparable; this approach was developed in [2] and is based on the following “algebraization” result:

THEOREM 2.4 ([2]). *Let* $\delta = (\delta_p)$ *be the Chern connection on* $G = GL_n$ *attached to a matrix* $q \in GL_n(A)$ *with* $q^t = \pm q$. *Then one can find maps of* A*-schemes* $\pi_p : Y_p \to G$ *and* $\varphi_p : Y_p \to G$ *such that:*

i) π_p *are affine and étale,*

ii) $\pi_p^{\widehat{p}} : Y_p^{\widehat{p}} \to G^{\widehat{p}}$ *are isomorphisms, and*

iii) $\varphi_p^{\widehat{p}} = \phi_p \circ \pi_p^{\widehat{p}} : Y_p^{\widehat{p}} \to G^{\widehat{p}}$.

In other words the *correspondences* $\Gamma_p := (Y_p, \pi_p, \varphi_p)$ on G are “algebraizations” of our Frobenius lifts ϕ_p; the system (Γ_p) is referred to as a *correspondence structure* for (δ_p); it is not unique but does have some “uniqueness features” (cf. [2]). On the other hand any correspondence Γ_p acts on the field E of rational functions of G by the formula $\Gamma_p^* : E \to E$,

$$\Gamma_p^*(z) = \mathrm{Tr}_{\pi_p}(\varphi_p^*(z)), \quad z \in E, \tag{2.22}$$

where $\mathrm{Tr}_{\pi_p} : F_p \to E$ is the trace of the extension $\pi_p^* : E \to F_p := Y_p \otimes_G E$ and $\varphi_p^* : E \to F_p$ is induced by φ_p. By the way the degrees of the extensions $\pi_p^* : E \to F_p$ and $\varphi_p^* : E \to F_p$ will be referred to as the *left degree* and the *right degree* of Γ_p respectively. Also we say Γ_p is *irreducible* if F_p is a field. So one can define the $*$-*curvature* of the adelic connection (δ_p) as the matrix $(\varphi_{pp'}^*)$ where

$$\varphi_{pp'}^* := \frac{1}{pp'}[\Gamma_p^*, \Gamma_{p'}^*] : E \to E, \quad p, p' \in \mathcal{V}. \tag{2.23}$$

Note that, in this way, we have defined a concept of “curvature” for Chern connections attached to arbitrary q’s (that do not necessarily have entries zeroes or roots of unity). There is a $(1,1)$-version of the above as follows. Given one more adelic connection $\overline{\delta} = (\overline{\delta}_p) =: (\delta_{\overline{p}})$ with correspondence structure $(\overline{\Gamma}_p) =: (\Gamma_{\overline{p}})$ one can define the $(1,1)$-$*$-*curvature* of (Γ_p) with respect to $(\Gamma_{\overline{p}})$ as the family $(\varphi_{p\overline{p}'}^*)$ where $\varphi_{p\overline{p}'}^*$ is the additive endomorphism

$$\varphi_{p\overline{p}'}^* := \frac{1}{pp'}[\Gamma_{\overline{p}'}^*, \Gamma_p^*] : E \to E \ \text{ for } p \neq p', \text{ and } \ \varphi_{p\overline{p}}^* := \frac{1}{p}[\Gamma_{\overline{p}}^*, \Gamma_p^*] : E \to E. \tag{2.24}$$

In what follows we let $\overline{\delta}$ be equal to $\delta_0 = (\delta_{0,p})$, where $\delta_{0,p}x = 0$; we give $\overline{\delta}$ the correspondence structure $(\Gamma_{\overline{p}}) = (G, \pi_{\overline{p}}, \varphi_{\overline{p}})$, $\pi_{\overline{p}}$ the identity, and $\varphi_{\overline{p}}(x) = x^{(p)}$.

THEOREM 2.5 ([2]).

1) Assume $n = 2$ and q is split with $q^t = -q$. Then Γ_p is irreducible and has left degree 2 and right degree $2p^4$. Moreover the $*$-curvature satisfies $\varphi^*_{pp'} = 0$ for all p, p' while the $(1,1)$-$*$-curvature satisfies $\varphi^*_{p\overline{p}'} \neq 0$ for all p, p'.

2) Assume $n = 2$ and q is split with $q^t = q$. Then Γ_p is irreducible and has left degree 4. Moreover the $(1,1)$-$*$-curvature satisfies $\varphi^*_{p\overline{p}'} \neq 0$ for all p, p'.

Once again, the theorem says nothing about the $*$-curvature in case $n = 2$ and $q^t = q$; our method of proof does not seem to apply to this case.

3. FINAL REMARKS

The theory outlined above should be viewed as a first step in a program of developing a *differential geometry of Spec* $\mathbb{Z}$. Other types of curvature (*Ricci, mean, scalar*) are developed in [1] and lead to some interesting Dirichlet series. An *arithmetic Maurer-Cartan connection* and a Galois theory attached to it is given in [9, 10]; this Galois theory should be viewed as an *arithmetic gauge theory* and should be further developed. It might be possible to attach *deRham cohomology classes* to our curvatures and to link them to the étale cohomology of *Spec* $\mathbb{Z}$. Links between adelic connections and Galois representations might exist that mimic the link between flat connections on vector bundles over manifolds and representations of the fundamental group of those manifolds. We hope to come back to these issues in future work.

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