For my Professor Lucian Bădescu and for our teacher's Professor

RELATIVE GARSIDE ELEMENTS OF ARTIN MONOIDS

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Communicated by Vasile Brînzănescu

We introduce a relative Garside element, the quotient of the corresponding Garside elements $\Delta(\Gamma_{n-1})$ and $\Delta(\Gamma_n)$, for a pair of Artin monoids associated to Coxeter graphs $\Gamma_{n-1} \subset \Gamma_n$, the second graph containing a new vertex. These relative elements give a recurrence relation between Garside elements. As an application, we compute explicitly the Garside elements of Artin monoids corresponding to spherical Coxeter graphs or the longest elements of the associated finite Coxeter groups.

AMS 2010 Subject Classification: Primary 20F36; Secondary 20M05, 20F10.

Key words: Coxeter groups, Artin monoids, Garside elements.

1. INTRODUCTION

The Garside element for the braid group (and for the braid monoid) corresponding to Artin's presentation (see [2, 14])

$$\mathcal{B}_n = \left\langle x_1, x_2, \ldots, x_{n-1} \;\middle|\; \begin{array}{l} x_i x_j = x_j\, x_i \text{ if } \mid i-j \mid \geq 2 \\ x_{i+1}\, x_i\, x_{i+1} = x_i\, x_{i+1}\, x_i \text{ if } 1 \leq i \leq n-2 \end{array} \right\rangle$$

is given by (see [11])

$$\Delta_n = x_1(x_2x_1)(x_3x_2x_1)\ldots(x_{n-1}x_{n-2}\ldots x_1)$$

(to represent Δ_n, or, more general, to represent an element of a monoid as a product of generators, we chose the smallest word in the length-lexicographic order induced by the order between generators $x_1 < x_2 < \ldots < x_{n-1}$). For other representations of Δ_n, including the most used formula, see [4]. The right quotient $\Delta_n^{-1}\Delta_{n+1} = x_n x_{n-1}\ldots x_1$ will be called *the relative Garside element* corresponding to the embedding of the Coxeter graphs $A_{n-1} \subset A_n$:

This element and its left divisors play a central role in the construction of the Gröbner basis for the classical braid monoid (see [1, 3, 6]). In this paper we give few characterizations of the relative Garside element corresponding to an extension of Coxeter graphs by one new vertex, see Propositions 1–4, and we will use these to compute inductively the Garside elements of the Artin monoids of spherical type, see Corollaries 1–7.

We start to recollect some facts about Coxeter graphs, Coxeter and Artin groups, Artin monoids, and Garside elements. Let S be a set. A *Coxeter matrix* over S is a square matrix $M = (m_{st})_{s,t\in S}$ indexed by the elements of S such that

$$m_{ss} = 1 \text{ for all } s \in S \text{ and } m_{st} = m_{ts} \in \{2, 3, 4, \ldots, \infty\} \text{ for all } s, t \in S,\ s \neq t.$$

The associated *Coxeter graph* $\Gamma = \Gamma(M)$ is a labeled graph defined by the following data:

S is a set of vertices of Γ;

two vertices $s, t \in S$ are joined by an edge if $m_{st} \geq 3$, with label m_{st} if $m_{st} \geq 4$.

A Coxeter matrix $M = (m_{st})_{s,t\in S}$ is usually represented by its Coxeter graph $\Gamma(M)$.

Definition 1. Let $M = (m_{st})_{s,t\in S}$ be the Coxeter matrix of the Coxeter graph Γ. Then the group defined by

$$\mathcal{W}(\Gamma) = \big\langle s \in S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ satisfying } m_{st} \neq \infty \big\rangle$$

is called the *Coxeter group* of type Γ.

In an equivalent way we can write $\mathcal{W}(\Gamma) = \big\langle s \in S \mid s^2 = 1,\ \underbrace{sts\ldots}_{m_{st}\text{ factors}} = \underbrace{tst\ldots}_{m_{st}\text{ factors}} \big\rangle$.

We call Γ to be of *spherical type* if $\mathcal{W}(\Gamma)$ is a finite group. A graph is of spherical type if and only if it has finitely many connected components, any of them from the list in Fig. 1.1. (see [7]).

Definition 2. If Γ is a Coxeter graph, its associated *Artin group* is defined by

$$\mathcal{A}(\Gamma) = \big\langle s \in S \,\big|\, \underbrace{sts\ldots}_{m_{st}\text{ factors}} = \underbrace{tst\ldots}_{m_{st}\text{ factors}} \text{ for all } s, t \in S \text{ satisfying } m_{st} \neq \infty \big\rangle.$$

The set of positive elements (i.e. the elements which are product of generators with positive exponents) in an Artin group $\mathcal{A}(\Gamma)$ is called the associated *Artin monoid* and it can be also defined by the monoid presentation (see [15])

$$\mathcal{M}(\Gamma) = \big\langle s \in S \,\big|\, \underbrace{sts\ldots}_{m_{st}\text{ factors}} = \underbrace{tst\ldots}_{m_{st}\text{ factors}} \text{ for all } s, t \in S \text{ satisfying } m_{st} \neq \infty \big\rangle.$$

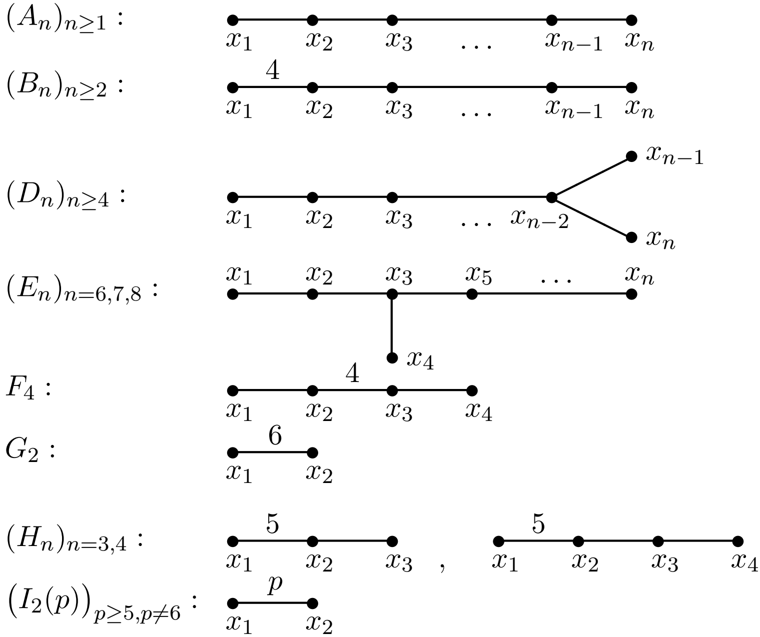


Fig. 1.1. The connected spherical type Coxeter graphs.

There are two obvious surjective morphisms:

$$\begin{array}{ccc} \mathcal{M}(\Gamma) & \hookrightarrow & \mathcal{A}(\Gamma) \\ & \searrow \quad W(\Gamma) \quad \swarrow & \end{array}$$

.

Now we recall some basic properties of the (absolute) Garside element of an Artin spherical monoid $\mathcal{M}(\Gamma)$ (see [8, 10, 11, 13, 15], and also Section 2 for notation). This element is the least common left-multiple of the set of generators $x_1, \ldots, x_n$: we have, for any $i = 1, \ldots, n$, $x_i \mid_L \Delta$ and if $x_i \mid_L \omega$ for all i, then $\Delta \mid_L \omega$. The element $\Delta(\Gamma)$ is square free (there is no generator x_i such that $x_i^2 \mid \Delta$). In some cases $\Delta(\Gamma)$ itself is a square: for example

$$\Delta(A_3) = x_1(x_2x_1)(x_3x_2x_1) = (x_1x_3x_2)^2;$$

$$\Delta\big(I_2(4k)\big) = (\underbrace{x_1x_2\ldots x_2}_{2k\,\text{times}})^2.$$

Also, there is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $x_i\Delta = \Delta x_{\sigma(i)}$. The image of $\Delta(\Gamma)$ in the corresponding Coxeter group $W(\Gamma)$ is the (unique) longest element of this group and it has order two (see [7] and [9]). The length $l(\Gamma)$ of the Garside element $\Delta(\Gamma)$ is equal to the number of the reflections (*i.e.* the conjugates of the generators in the Coxeter group) and it is given by the following table (see [12] and [5]):

Γ	A_n	B_n	D_n	E_6	E_7	E_8	F_4	G_2	H_3	H_4	$I_2(p)$
$l(\Gamma)$	$\binom{n+1}{2}$	n^2	n^2-n	36	63	120	24	6	15	60	p

2. RELATIVE GARSIDE ELEMENTS

Let us fix the notation. We already used the divisibility relation between two elements of a monoid $\mathcal{M}$, $\alpha \mid \beta$: this is equivalent to $\beta = \lambda\alpha\rho$ for some elements $\lambda, \rho \in \mathcal{M}$. If $\lambda = 1$, we write $\alpha \mid_L \beta$ and similarly, if $\rho = 1$, we write $\alpha \mid_R \beta$ and we say that α is a left and right divisor of β respectively. The element α in a monoid $\mathcal{M}$ generated by $x_1, x_2, \ldots$ is said to be *rigid* if α can be represented in a unique way as a word in $x_1, x_2, \ldots$.

Suppose we have an inclusion of Coxeter graphs $\Gamma_{n-1} \subset \Gamma_n$ with vertices $\{x_1, \ldots, x_{n-1}\}$ and $\{x_1, \ldots, x_n\}$, respectively. Using the definition of Garside elements we have

$$\Delta(\Gamma_{n-1}) \mid_L \Delta(\Gamma_n).$$

Definition 3. The relative Garside element $\Delta(\Gamma_n, \Gamma_{n-1})$ is defined as a right quotient:

$$\Delta(\Gamma_n) = \Delta(\Gamma_{n-1})\Delta(\Gamma_n, \Gamma_{n-1}).$$

If there is no ambiguity concerning the inclusion $\Gamma_{n-1} \subset \Gamma_n$, we will use the simple notation $\Delta_n = \Delta_{n-1}R_n$. Now we present some properties of the relative Garside element R_n which characterize this element.

PROPOSITION 1. *The relative Garside element $R_n = \Delta(\Gamma_n, \Gamma_{n-1})$ satisfies the properties:*

a) R_n is square free;

b) $x_i \mid_L R_n$ if and only if $i = n$;

c) there is a bijection $\sigma : \{1, \ldots, n-1\} \to \{1, \ldots, n\} \setminus \{\sigma_n(n)\}$ such that $x_i R_n = R_n x_{\sigma(i)}$;

d) $x_j \mid_R R_n$ if and only if $j = \sigma_n(n)$.

Proof. Let $\Delta_n = \Delta_{n-1}R_n$. Then we have:

a) R_n is square free because Δ_n is square free;

b) $x_i\,(1 \leq i \leq n-1)$ cannot be a left divisor of R_n: otherwise, $R_n = x_iU$. But $x_i \mid_R \Delta_{n-1}$ implies $\Delta_{n-1} = Vx_i$, therefore we have $\Delta_n = \Delta_{n-1}R_n = Vx_i^2U$, a contradiction.

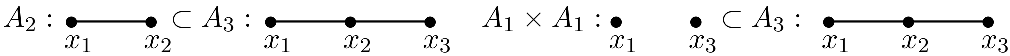
c) Let $\sigma_{n-1} : \{1,\dots,n-1\} \to \{1,\dots,n-1\}$ and $\sigma_n : \{1,\dots,n\} \to \{1,\dots,n\}$ be the bijections defined by the conjugation with Δ_{n-1} and Δ_n, respectively: $x_i\Delta_{n-1} = \Delta_{n-1}x_{\sigma_{n-1}(i)}$, $x_i\Delta_n = \Delta_n x_{\sigma_n(i)}$. Now, for $i \in \{1,\dots,n-1\}$, we have:

$$\Delta_{n-1}x_iR_n = x_{\sigma_{n-1}^{-1}(i)}\Delta_{n-1}R_n = x_{\sigma_{n-1}^{-1}(i)}\Delta_n = \Delta_n x_{\sigma_n\sigma_{n-1}^{-1}(i)} = \Delta_{n-1}R_n x_{\sigma_n\sigma_{n-1}^{-1}(i)},$$

therefore $x_iR_n = R_nx_{\sigma(i)}$, where $\sigma = \sigma_n \circ \sigma_{n-1}^{-1}$. The image of σ contains all the elements $1,\dots,n$, but not $\sigma_n(n) = m$.

d) If $x_j \neq x_m$, then there exists $i \in \{1,\dots,n-1\}$ such that $j = \sigma(i)$ and $x_iR_n = R_nx_j$. Suppose that $x_j \mid_R R_n$, *i.e.* $R_n = Sx_j$. We obtain a contradiction: $x_iR_n = R_nx_j = Sx_j^2$, because x_iR_n is a right divisor of Δ_n. $\square$

Remark 1. From this proposition, the first and the last factors of R_n are uniquely defined. In some cases R_n is completely rigid, for instance $R_3(A_3, A_2) = x_3x_2x_1$, in other cases only the interior factors can be changed, for instance $R_3(A_3, A_1 \times A_1) = x_2x_1x_3x_2 = x_2x_3x_1x_2$.



We will use the notation $m = \sigma_n(n)$.

Proposition 2. *If $U_n \in \mathcal{M}(\Gamma_n)$ satisfies the following three conditions:*

a) $x_m^2 \nmid U_n$,

b) $x_i \mid_L U_n$ if and only if $i = n$, and

c) there is a bijection $\tau : \{1,\dots,n-1\} \to \{1,\dots,\widehat{k},\dots,n\}$ such that $x_iU_n = U_nx_{\tau(i)}$,

then $U_n = R_n$ (and also $k = m$, $\tau = \sigma$).

Proof. Let us define $D = \Delta_{n-1}U_n$. Because of the relations $x_i \mid_L \Delta_{n-1}$, $i = 1,\dots,n-1$ and also from $x_n \mid_L D$, a consequence of b) and c), we have $x_i \mid_L \Delta_{n-1}U_n$ for any $i = 1,\dots,n$, therefore $\Delta_n \mid_L D$. This implies that $\Delta_{n-1}R_n \mid_L \Delta_{n-1}U_n$, hence $R_n \mid_L U_n$. If $R_n \neq U_n$, then $U_n = R_nx_jW$ for some j. If $j \neq m$, then $j = \sigma_n(i)$ for some $i \in \{1,\dots,n-1\}$, hence $U_n = R_nx_{\sigma(i)}W = x_iR_nW$ and this contradicts b). If $j = m$, then $U_n = (Sx_m)x_mW$ which contradicts a). $\square$

Proposition 3. *If $U_n \in \mathcal{M}(\Gamma_n)$ satisfies the following three conditions:*

a) $x_n \mid_L U_n$,

b) there is a bijection $\tau : \{1,\dots,n-1\} \to \{1,\dots,\widehat{k},\dots,n\}$ such that $x_iU_n = U_nx_{\tau(i)}$, and

c) U_n has minimal length among the words satisfying a) and b),

then $U_n = R_n$ (and also $k = m$, $\tau = \sigma$).

Proof. Let us define $D = \Delta_{n-1}U_n$. As in the previous proof, a) and b) implies $x_i \mid_L D$ for all $i = 1, \ldots, n$. Therefore $\Delta_n \mid_L D$, hence $R_n \mid_L U_n$. We have $|U_n| \geq |R_n|$ and R_n satisfies a) and b), therefore condition c) implies $R_n = U_n$. $\square$

With the same proof, we have another version of the previous proposition:

PROPOSITION 4. *If $U_n \in \mathcal{M}(\Gamma_n)$ satisfies the following three conditions:*
a) $x_n \mid_L U_n$,
b) there is a bijection $\tau : \{1, \ldots, n-1\} \to \{1, \ldots, \widehat{k}, \ldots, n\}$ such that $x_i U_n = U_n x_{\tau(i)}$, and
c) the length of U_n is the expected length $l(\Gamma_n) - l(\Gamma_{n-1})$ (from the table in Section 1),
then $U_n = R_n$ (and also $k = m$, $\tau = \sigma$).

3. THE RELATIVE GARSIDE AND GARSIDE ELEMENTS FOR THE INFINITE SERIES

After the computation of various relative Garside elements, we will describe the Garside elements associated to the classical list of the connected Coxeter diagrams: the infinite series in this section, the exceptional cases in the next section. Using the following obvious result, these will give formulae for the Garside elements of all Artin monoids of spherical type.

LEMMA 1. *If the graph Γ is the disjoint union of Γ_1 and Γ_2, then $\mathcal{M}(\Gamma) \cong \mathcal{M}(\Gamma_1) \times \mathcal{M}(\Gamma_2)$ and*

$$\Delta(\Gamma) = \Delta(\Gamma_1)\Delta(\Gamma_2) = \Delta(\Gamma_2)\Delta(\Gamma_1).$$

$\mathbf{A_n}$ series. We start with A_1: 

and its Garside element $\Delta(A_1) = \Delta_2 = x_1$.

Construction of $\Delta(A_n, A_{n-1})$: The element $M_n = x_n x_{n-1} \ldots x_1$ satisfies the conditions of Proposition 2:

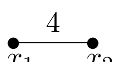
a) M_n is square free: obvious because M_n is rigid;

b) $x_i \mid_L M_n$ if and only if $i = n$ for the same reason;

c) $x_i M_n = M_n x_{i+1}$, therefore $\sigma(i) = i + 1$, where $i = 1, \ldots, n-1$, and $m = 1$.

COROLLARY 1 (Classical Garside element).

$$\Delta(A_n) = \Delta_{n+1} = x_1(x_2x_1)\ldots(x_{n-1}\ldots x_1)(x_n \ldots x_1).$$

$\mathbf{B_n}$ series. We start with B_2: 

and its Garside element $\Delta(B_2) = x_1(x_2x_1x_2)$, see the construction $\Delta(I_2(p), A_1)$.

Construction of $\Delta(B_n, B_{n-1})$: The element $N_n = x_n x_{n-1} \dots x_2 x_1 x_2 \dots x_n$ satisfies the conditions of Proposition 2:

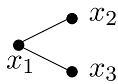
a) N_n is square free: obvious, because N_n is rigid in $\mathcal{M}(B_n)$;

b) $x_i \mid_L N_n$ if and only if $i = n$ for the same reason;

c) $x_i N_n = N_n x_i$, therefore $\sigma(i) = i$, where $i = 1, \dots, n-1$, and $m = n$.

Corollary 2.

$$\Delta(B_n) = x_1(x_2 x_1 x_2)(x_3 x_2 x_1 x_2 x_3) \dots (x_n x_{n-1} \dots x_2 x_1 x_2 \dots x_{n-1} x_n).$$

$\mathbf{D_n}$ series. We start with D_3: 

and its Garside element $\Delta(D_3) = x_1(x_2 x_1)(x_3 x_1 x_2)$ (this is $\Delta(A_3, A_2) = x_3 x_2 x_1$ with a change of notation).

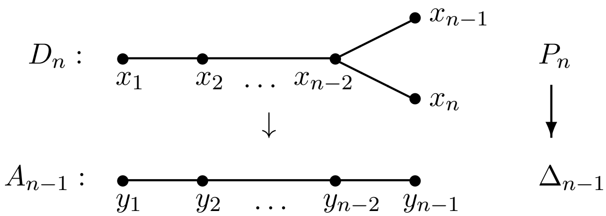
Construction of $\Delta(D_n, A_{n-1})$: Consider the product

$$P_n = (x_n x_{n-2} x_{n-3} .. x_1)(x_{n-1} x_{n-2} .. x_2)(x_n x_{n-2} x_{n-3} .. x_3)(x_{n-1} x_{n-2} .. x_4) ...$$

We prove by induction that $P_n = R_n$. The initial step is given by the previous formula: $\Delta(D_3, A_2) = (x_3 x_1) x_2$. We check the conditions in Proposition 2 in the case of n even, when the product ends with the generator x_n:

$$P_n = (x_n x_{n-2} x_{n-3} .. x_1)(x_{n-1} x_{n-2} x_{n-3} .. x_2)(x_n x_{n-2} x_{n-3} .. x_3) .. (x_{n-1} x_{n-2}) x_n.$$

a) First we prove that P_n is square free and also that $x_n x_{n-1}$ is not a divisor of P_n.



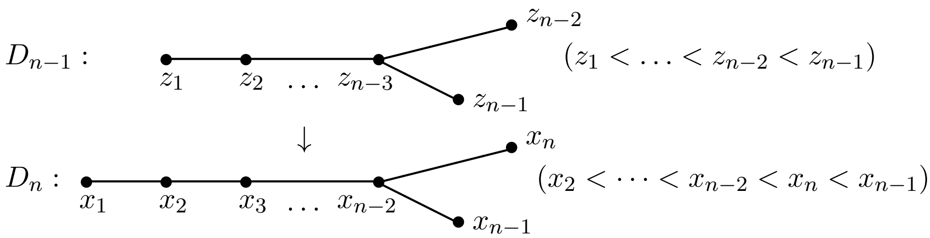
Define a projection $pr : \mathcal{M}(D_n) \to \mathcal{M}(A_{n-1})$ given by the diagram: $pr(x_i) = y_{\min(i, n-1)}$. The image of P_n under the map pr is Δ_{n-1} which is square free: if $x_j^2 \mid P_n$, then we have $pr(x_j^2) = y_h^2 \mid \Delta_{n-1}$ ($h = j$ if $j \le n-1$ and $h = n-1$ if $j = n$), a contradiction. Therefore P_n is square free. In the same way, if $x_n x_{n-1} \mid P_n$, then $pr(x_n x_{n-1}) = y_{n-1}^2 \mid \Delta_{n-1}$, a contradiction.

b) Obviously $x_n \mid_L P_n$. Let us suppose that $x_i \mid_L P_n$ for some $i \le n-1$. By induction we know that R_{n-1} is given by the next formula and we will analyze the following three cases:

1) $i = n-1$, 2) $i = n-2$, and 3) $i \le n-3$:

$$R_{n-1} = (x_{n-1} x_{n-3} x_{n-4} \dots x_1)(x_{n-2} x_{n-3} \dots x_2)(x_{n-1} x_{n-3} x_{n-4} \dots x_3) \dots x_{n-2}.$$

Using the inclusion morphism $in : \mathcal{M}(D_{n-1}) \to \mathcal{M}(D_n)$, $z_1 \mapsto x_2$, $z_2 \mapsto x_3, \ldots, z_{n-3} \mapsto x_{n-2}$, $z_{n-2} \mapsto x_n$, $z_{n-1} \mapsto x_{n-1}$, given by the inclusion of graphs:



the image of

$$R_{n-1} = (z_{n-1}z_{n-3}z_{n-4}..z_1)(z_{n-2}z_{n-3}..z_2)(z_{n-1}z_{n-3}..z_3)..(z_{n-1}z_{n-3})z_{n-2}$$

is $in(R_{n-1}) = R'_{n-1}$, given by

$$R'_{n-1} = (x_{n-1}x_{n-2}x_{n-3}..x_2)(x_nx_{n-2}x_{n-3}..x_3)(x_{n-1}x_{n-2}..x_4)..(x_{n-1}x_{n-2})x_n$$

and we have $P_n = (x_nx_{n-2}x_{n-3}\ldots x_1)R'_{n-1}$. If $x_i \mid_L R'_{n-1}$, then $i = n-1$ (by induction this is true for $i \in \{2, 3, \ldots, n, n-1\}$ and also $x_1 \mid_L R'_{n-1}$ is impossible because R'_{n-1} does not contain x_1). In the case 1), $x_{n-1} \mid_L P_n$ implies $x_{n-1}x_n\alpha_1 = P_n$ (by Garside Lemma, see Section 5) and $pr(P_n)$ contains y_{n-1}^2, a contradiction. In the case 2), $x_{n-2} \mid_L P_n$, and Garside Lemma implies that $x_nx_{n-2}x_n\alpha_2 = x_nx_{n-2}x_{n-3}\ldots x_1R_{n-1}$, hence $x_n\alpha_2 = x_{n-3}x_{n-4}\ldots x_1R_{n-1}$ and $x_nx_{n-3}\ldots x_1\alpha_3 = x_{n-3}\ldots x_1R_{n-1}$, and this gives a contradiction: $x_n\alpha_3 = R_{n-1}$. In the last case, 3), if $x_i \mid_L P_n$ $(i = 1, \ldots, n-3)$, then

$$x_nx_i\beta_1 = x_nx_{n-2}\ldots x_{i+1}x_i\ldots x_1R_{n-1},$$

and using Garside Lemma we have $x_i\beta_1 = x_{n-2}\ldots x_{i+1}x_i\ldots x_1R_{n-1}$, and also $x_ix_{n-2}\ldots x_{i+2}\beta_2 = x_{n-2}\ldots x_{i+2}x_{i+1}x_i\ldots x_1R_{n-1}$. We obtain

$$x_i\beta_2 = x_{i+1}x_i\ldots x_1R_{n-1},$$

next $x_{i+1}x_ix_{i+1}\beta_3 = x_{i+1}x_ix_{i-1}..x_1R_{n-1}$ and $x_{i+1}x_{i-1}..x_1\beta_4 = x_{i-1}..x_1R_{n-1}$; this gives another contradiction: $x_{i+1} \mid_L R_{n-1}$.

c) We have $x_iP_n = P_nx_{n-i}$, hence $\sigma(i) = n-i$. Therefore $P_n = R_n$.

Similarly one can check the conditions a), b), c) of Proposition 2 for n odd:

$$R_n = (x_nx_{n-2}x_{n-3}..x_1)(x_{n-1}x_{n-2}..x_2)(x_nx_{n-2}x_{n-3}..x_3)..(x_nx_{n-2})x_{n-1}.$$

Corollary 3.

$$\Delta(D_n) = x_1(x_2x_1)..(x_{n-1}..x_1)(x_nx_{n-2}x_{n-3}..x_1)(x_{n-1}x_{n-2}..x_2)(x_nx_{n-2}..x_3)..$$

$\mathbf{I_2(p)}$ series, $\mathbf{B_2}$ and $\mathbf{G_2}$. Construction of $\Delta(I_2(p), A_1)$: Let us define $Q_2(p) = x_2x_1x_2x_1x_2\ldots$, $(p-1$ factors). This element satisfies the conditions in Proposition 3:

a) clearly $x_2 \,|_L\, Q_2(p)$;

b) we have $x_1Q_2(2p+1) = Q_2(2p+1)x_2$ $(m = 1)$ and $x_1Q_2(2p) = Q_2(2p)x_1$ $(m = 2)$;

c) any relation $x_1V = Vx_{\sigma(1)}$ should involve the unique defining relation $x_2x_1x_2..$ (p factors) $= x_1x_2x_1..$(p factors), so the length of V is greater than or equal to $p-1$ and $Q_2(p)$ has minimal length among the words satisfying a) and b).

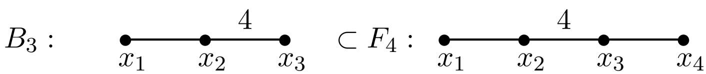
Corollary 4.

$$\begin{aligned} \Delta\big(G_2\big) &= x_1x_2x_1x_2x_1x_2, \\ \Delta\big(I_2(p)\big) &= x_1x_2x_1x_2\ldots\ (p \textit{ factors}). \end{aligned}$$

4. THE RELATIVE GARSIDE ELEMENTS FOR THE EXCEPTIONAL SERIES

As a consequence of the results in Section 3 we obtained a new proof for the lengths of the Garside elements corresponding to the infinite series A_*, B_*, D_* and $I(*)$ (including G_2). In this section we will use the lengths of the Garside elements to find the relative Garside elements and also the Garside elements of the monoids corresponding to the exceptional Coxeter graphs E_6, E_7, E_8, F_4, H_3 and H_4.

$\mathbf{F_4}$ case. We consider the inclusion:



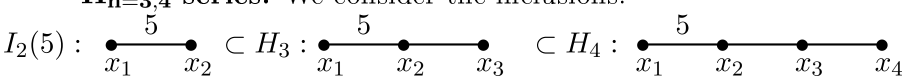
Construction of $\Delta(F_4, B_3)$: Let us define $T_3 = x_3x_2x_1x_3x_2x_3$; the Garside element of B_3 (with the change of marking $x_1 \leftrightarrow x_3$) is $\Delta(B_3) = x_1x_2x_1T_3$. Defining $R_4 = x_4T_3x_4T_3x_4$, we have $x_iR_4 = R_4x_i$, $i = 1,2,3$, and $l(R_4) = 15$; from the table in Section 1 we have $l(F_4) - l(B_3) = 24 - 9 = 15$, hence, using Proposition 4, we obtain

$$\Delta(F_4, B_3) = x_4T_3x_4T_3x_4.$$

Corollary 5.

$$\Delta(F_4) = x_1x_2x_1(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4.$$

$\mathbf{H_{n=3,4}}$ series. We consider the inclusions:



Construction of $\Delta(H_3, I_2(5))$ and $\Delta(H_4, H_3)$: The element

$$S_3 = (x_3x_2x_1x_2x_1)(x_3x_2x_1x_2)x_3$$

satisfies the commutation rules $x_1S_3 = S_3x_2$, $x_2S_3 = S_3x_1$, and its length is 10. From the length table we find that $l(\Delta(H_3, I_2(5))) = 15 - 5 = 10$, so

$$\Delta(H_3, I_2(5)) = S_3.$$

Similarly, the element

$$S_4 = x_4S_3x_4S_3x_4S_3x_4S_3x_4$$

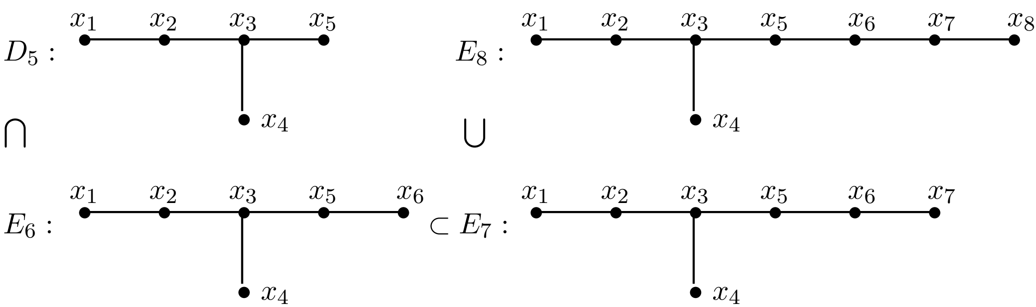
verifies $x_iS_4 = S_4x_i$, $i = 1, 2, 3$, and it has the expected length: $l(S_4) = 45 = 60 - 15 = l(H_4) - l(H_3)$, therefore

$$\Delta(H_4, H_3) = S_4.$$

Corollary 6.

$$\begin{aligned}\Delta(H_3) &= x_1x_2x_1x_2x_1S_3 = x_1x_2x_1x_2x_1 \cdot (x_3x_2x_1x_2x_1)(x_3x_2x_1x_2)x_3,\\ \Delta(H_4) &= x_1x_2x_1x_2x_1S_3S_4 = x_1x_2x_1x_2x_1S_3x_4S_3x_4S_3x_4S_3x_4S_3x_4.\end{aligned}$$

$\mathbf{E_{n=6,7,8}}$ series. We consider the inclusions:



Construction of $\Delta(E_6, D_5)$, $\Delta(E_7, E_6)$, and $\Delta(E_8, E_7)$: Let us define the element

$$V_6 = x_6\Delta(D_5, D_4)x_6x_5x_3x_2x_1 = (x_6x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(x_6x_5x_3x_2x_1).$$

This verifies the commutation relations:

$$x_1V_6 = V_6x_6, x_2V_6 = V_6x_5, x_3V_6 = V_6x_3, x_4V_6 = V_6x_2 \text{ and } x_5V_6 = V_6x_4.$$

Define also the elements

$$V_7 = x_7V_6(x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7 \text{ and } V_8 = x_8V_7x_8V_7x_8.$$

These elements verify the commutation relations:

$$x_iV_7 = \begin{cases} V_7x_{7-i}, & i = 1, 2, 5, 6,\\ V_7x_i, & i = 3, 4,\end{cases} \qquad \text{and} \qquad x_iV_8 = V_8x_i, \ \ i = 1, \dots, 7.$$

Counting the lengths we obtain

$$\begin{aligned} l(V_6) &= 16 = 36 - 20 = l(E_6) - l(D_5), \\ l(V_7) &= 27 = 63 - 36 = l(E_7) - l(E_6), \\ l(V_8) &= 57 = 120 - 63 = l(E_8) - l(E_7). \end{aligned}$$

From Proposition 4 we obtain the relative Garside elements:

$$\Delta(E_6, D_5) = V_6, \Delta(E_7, E_6) = V_7, \Delta(E_8, E_7) = V_8.$$

COROLLARY 7.

$$\begin{aligned} \Delta(E_6) &= \Delta(D_5)V_6 = \Delta(A_4)\Delta(D_5, A_4)V_6, \\ \Delta(E_7) &= \Delta(E_6)V_7 = \Delta(A_4)\Delta(D_5, A_4)V_6V_7, \\ \Delta(E_8) &= \Delta(E_7)V_8 = \Delta(A_4)\Delta(D_5, A_4)V_6V_7V_8. \end{aligned}$$

5. GARSIDE LEMMA AND FEW COMPUTATIONS

The next lemma was proved by Garside for the braid monoid (or A_n series), see [11], and generalized for an arbitrary Artin monoid by Brieskorn and Saito, see [8]:

LEMMA 2 (Garside Lemma). *Let W be an element in the Artin monoid $\mathcal{M}$ such that $x_i \mid_L W$ and $x_j \mid_L W$ $(i \neq j)$. Then there is an element $Z \in \mathcal{M}$ such that*

$$W = (\underbrace{x_i x_j x_i x_j \dots}_{m_{ij}\ times})Z = (\underbrace{x_j x_i x_j x_i \dots}_{m_{ij}\ times})Z.$$

Now we give the details for the proof of two commutation relations described in Section 4. First a short computation:

LEMMA 3. *In F_4 we have $x_1 R_4 = R_4 x_1$.*

Proof. The factors which are transformed under Coxeter relations are written in bold characters:

$$\begin{aligned} &\mathbf{x_1} \cdot x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 = \\ &= x_4(x_3\mathbf{x_1}x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 = \\ &= x_4(x_3x_2x_1\mathbf{x_2}x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 = \\ &= x_4(x_3x_2x_1x_3x_2x_3)\mathbf{x_2}x_4(x_3x_2x_1x_3x_2x_3)x_4 = \\ &= x_4(x_3x_2x_1x_3x_2x_3)x_4(\mathbf{x_2}x_3x_2x_3x_1x_2x_3)x_4 = \\ &= x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_3\mathbf{x_2}x_1x_2x_3)x_4 = \\ &= x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_3x_1x_2\mathbf{x_1}x_3)x_4 = \\ &= x_4(x_3x_2x_1x_3x_2x_3)x_4(x_3x_2x_1x_3x_2x_3)x_4 \cdot \mathbf{x_1}. \quad \square \end{aligned}$$

And now a long computation:

Lemma 4. *In E_8 we have $x_7V_8 = V_8x_7$.*

Proof. In E_8 we have the following sequence of equalities:

$$\begin{aligned}\alpha &\equiv x_3x_2x_4x_3x_2\mathbf{x_4} = x_3x_2\mathbf{x_3}x_4x_3x_2 = \mathbf{x_2}x_3x_2x_4x_3x_2 \equiv \\ &\equiv x_2x_3\mathbf{x_2}x_4x_3x_2 = x_2x_3x_4x_3x_2\mathbf{x_3} = x_2\mathbf{x_3}x_4x_3x_2x_3 = \\ &= x_2x_4x_3x_2\mathbf{x_4}x_3 \equiv \beta,\end{aligned}$$

and from the equality $\alpha = \beta$ we get

$$\begin{aligned}\gamma &\equiv x_5(x_3x_2x_4x_3x_5)(\mathbf{x_3}x_2x_1x_4x_3x_2) = x_5(x_3\mathbf{x_5}x_2x_4x_3x_5)(x_2x_1x_4x_3x_2) = \\ &= \mathbf{x_3}x_5(x_3x_2x_4x_3\mathbf{x_5})(x_2x_1x_4x_3x_2) = x_3x_5(x_3x_2x_4x_3)(x_2x_1x_4\mathbf{x_5}x_3x_2) \equiv \\ &\equiv x_3x_5(x_3x_2x_4x_3)(x_2\mathbf{x_1}x_4x_5x_3x_2) = x_3x_5(x_3x_2x_4x_3)(x_2x_4\mathbf{x_1}x_5x_3x_2) \equiv \\ &\equiv x_3x_5\alpha\mathbf{x_1}x_5x_3x_2 = x_3x_5\beta\mathbf{x_1}x_5x_3x_2 \equiv x_3x_5(x_2x_4x_3x_2x_4x_3\mathbf{x_1}x_5x_3x_2) = \\ &= x_3x_5(x_2x_4x_3x_2\mathbf{x_1}x_4x_3x_5\mathbf{x_3}x_2) = x_3\mathbf{x_5}(x_2x_4x_3\mathbf{x_5})(x_2x_1x_4x_3x_5x_2) = \\ &= x_3(x_2x_4\mathbf{x_5}x_3x_5)(x_2x_1x_4x_3x_5\mathbf{x_2}) = (x_3x_2x_4x_3x_5)(\mathbf{x_3}x_2x_1x_4x_3\mathbf{x_2})x_5 \equiv \delta,\end{aligned}$$

and also, from $\gamma = \delta$, we obtain

$$\begin{aligned}\eta &\equiv x_6(x_5x_3x_2x_4x_3x_5x_6)(\mathbf{x_5}x_3x_2x_1x_4x_3x_2x_5x_3x_4) = \\ &\equiv x_6(x_5\mathbf{x_6}x_3x_2x_4x_3x_5\mathbf{x_6})(x_3x_2x_1x_4x_3x_2x_5x_3x_4) = \\ &= \mathbf{x_5}x_6x_5(x_3x_2x_4x_3x_5)(x_3x_2x_1x_4x_3x_2)\mathbf{x_6}x_5x_3x_4 \equiv \\ &\equiv x_5x_6\gamma x_6x_5x_3x_4 = x_5x_6\delta x_6x_5x_3x_4 \equiv \\ &\equiv x_5x_6(x_3x_2x_4x_3x_5)(x_3x_2x_1x_4x_3x_2)x_5x_6\mathbf{x_5}x_3x_4 = \\ &= x_5x_6(x_3x_2x_4x_3x_5)(x_3x_2x_1x_4x_3x_2\mathbf{x_6}x_5x_6x_3x_4) = \\ &= x_5\mathbf{x_6}(x_3x_2x_4x_3x_5\mathbf{x_6})(x_3x_2x_1x_4x_3x_2x_5\mathbf{x_6}x_3x_4) = \\ &= (x_5x_3x_2x_4x_3x_5x_6)(\mathbf{x_5}x_3x_2x_1x_4x_3x_2x_5x_3x_4)\mathbf{x_6} \equiv \theta.\end{aligned}$$

For the final step we use the next equality

$$\begin{aligned}\lambda &\equiv x_7(x_6x_5x_3x_2x_4x_3x_5x_6x_7)V_6 \equiv \\ &\equiv x_7(x_6x_5x_3x_2x_4x_3x_5x_6x_7)(\mathbf{x_6}x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(x_6x_5x_3x_2x_1) = \\ &= x_7(x_6\mathbf{x_7}x_5x_3x_2x_4x_3x_5x_6\mathbf{x_7})(x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(x_6x_5x_3x_2x_1) = \\ &= \mathbf{x_6}x_7x_6(x_5x_3x_2x_4x_3x_5x_6)(x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(\mathbf{x_7}x_6x_5x_3x_2x_1) \equiv \\ &\equiv x_6x_7\eta x_7x_6x_5x_3x_2x_1 = x_6x_7\theta x_7x_6x_5x_3x_2x_1 \equiv \\ &\equiv x_6\mathbf{x_7}(x_5x_3x_2x_4x_3x_5x_6)(x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)x_6x_7\mathbf{x_6}x_5x_3x_2x_1 = \\ &= (x_6x_5\mathbf{x_7}x_3x_2x_4x_3x_5x_6\mathbf{x_7})(x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(x_6\mathbf{x_7}x_5x_3x_2x_1) = \\ &= (x_6x_5x_3x_2x_4x_3x_5x_6x_7)(\mathbf{x_6}x_5x_3x_2x_1x_4x_3x_2x_5x_3x_4)(x_6x_5x_3x_2x_1)\mathbf{x_7} \equiv \\ &\equiv (x_6x_5x_3x_2x_4x_3x_5x_6x_7)V_6x_7 \equiv \mu,\end{aligned}$$

and we find

$$\begin{aligned}&x_7V_8 \equiv x_7(x_8V_7x_8V_7x_8) \equiv \\ &\equiv \mathbf{x_7}[x_8x_7V_6(x_7x_6x_5x_3x_2x_4x_3x_5x_6)\mathbf{x_7}x_8x_7V_6(x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8] = \\ &= x_8x_7V_6(\mathbf{x_8}x_7x_6x_5x_3x_2x_4x_3x_5x_6)\mathbf{x_8}x_7V_6(\mathbf{x_8}x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8 = \\ &= x_8x_7V_6\mathbf{x_7}x_8(x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7V_6)(x_8x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8 \equiv \\ &\equiv x_8x_7V_6x_7x_8\lambda(x_8x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8 = \\ &= x_8x_7V_6x_7x_8\mu(x_8x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8 \equiv\end{aligned}$$

$$\equiv x_8x_7V_6x_7x_8(x_6x_5x_3x_2x_4x_3x_5x_6x_7V_6)(x_7x_8\mathbf{x_7}x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8 =$$
$$= x_8x_7V_6(x_7\mathbf{x_8}x_6x_5x_3x_2x_4x_3x_5x_6)x_7\mathbf{x_8}V_6(x_7\mathbf{x_8}x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8 =$$
$$= [x_8x_7V_6(x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8\mathbf{x_7}V_6(x_7x_6x_5x_3x_2x_4x_3x_5x_6)x_7x_8]\mathbf{x_7} \equiv$$
$$\equiv (x_8V_7x_8V_7x_8)x_7 \equiv V_8x_7.$$

Acknowledgments. This research is partially supported by Higher Education Commission, Pakistan.

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Received 3 March 2015

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