To Lucian Bădescu on the occasion of his 70^{th} birthday, with admiration, gratitude and friendship

HILBERT SURFACES OF BIPOLARIZED VARIETIES

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Communicated by Marian Aprodu

Let X be a normal Gorenstein complex projective variety. We introduce the Hilbert variety V_X associated to the Hilbert polynomial $\chi(x_1\mathcal{L}_1, \ldots, x_\rho\mathcal{L}_\rho)$, where $\mathcal{L}_1, \ldots, \mathcal{L}_\rho$ is a basis of $\mathrm{Pic}(X)$, ρ being the Picard number of X, and $x_1, \ldots, x_\rho$ are complex variables. After reviewing general properties of V_X, we focus on the following specific topics. First, we consider the Hilbert surface of a bipolarized variety (X, L_1, L_2), namely, the surface of degree $\dim(X)$ in a 3-dimensional affine space, associated to $\chi(xK_X + yL_1 + zL_2)$. Special emphasis is given to the case of 3-folds. Next, we treat the case of the Hilbert curve of a polarized 4-fold (X, L), that is, the plane quartic curve associated to $\chi(xK_X + yL)$. We also study quotients of Hilbert surfaces under the Serre involution s induced by Serre duality, and we characterize surfaces in a 3-dimensional affine space which are invariant under s.

AMS 2010 Subject Classification: Primary 14C20, 14N30; Secondary 14M99, 14H50.

Key words: multipolarized variety, Hilbert variety, bidegree, Serre involution.

INTRODUCTION

Let X be an irreducible projective variety. Looking at the real vector space $\mathrm{N}(X)$ of numerical equivalence classes of divisors on X with real coefficients, following Kleiman's approach [7] and Mori's work [11], led to remarkable results in algebraic geometry. In particular, from the adjunction theoretic point of view, in the study of a polarized variety there are natural half-spaces arising in the dual vector space $\mathrm{N}(X)^*$ to looking at: those where a suitable adjoint bundle is negative. On the other hand, considering numerical equivalence classes with complex coefficients could suggest a new interesting point of view. This is exactly the idea pursued in [3] and [8], focusing on a complex algebraic plane curve which turns out to be naturally associated to any polarized variety. In this paper, inspired by [3], we take a natural step forward on this topic. In particular, we deal with the Hilbert surface of a bipolarized variety (X, L_1, L_2), with special attention to the case of 3-folds, and with the Hilbert quartic curve of a polarized 4-fold (X, L).

Let us make everything more precise, outlining the plan of the paper.

Let $\text{Pic}^0(X) \subset \text{Pic}(X)$ denote the subgroup of topologically trivial line bundles on X, so that $\text{Pic}(X)/\text{Pic}^0(X)$ is the Néron-Severi group $NS(X) \subseteq H^2(X, \mathbb{Z})$. The function sending every $\mathcal{L} \in \text{Pic}(X)$ to its Euler characteristic $\chi(\mathcal{L})$ gives rise to a polynomial function p from $\mathbf{N}(X) := \text{Pic}(X)/\text{Pic}^0(X) \otimes_{\mathbb{Z}} \mathbb{C}$ to $\mathbb{C}$. This is a polynomial of degree $\dim(X)$ with rational coefficients. We call the hypersurface $V_X \subset \mathbf{N}(X)$, defined by the vanishing of p, the *Hilbert variety of* X. Of course we can also regard V_X as a real hypersurface in $\mathrm{N}(X)$. Besides being invariant under conjugation, V_X is invariant under the linear map induced by Serre duality since $\chi(\mathcal{L}) = (-1)^{\dim(X)}\chi(K_X \otimes \mathcal{L}^*)$. We call this latter map, $s : \mathbf{N}(X) \to \mathbf{N}(X)$, the *Serre involution*.

Note that $\mathbf{N}(X) \cong \mathbb{A}^{\rho}_{\mathbb{C}}$, where $\rho := \rho(X)$ is the Picard number of X. Given a multipolarized variety $(X, L_1, \ldots, L_t)$, we have the vector subspace $\langle K_X, L_1, \ldots, L_t\rangle \subset \mathbf{N}(X)$ generated by $L_1, \ldots, L_t$ and K_X. This is a proper subspace if $t < \rho - 1$. Moreover, it is at least one dimensional since the L_i's are ample. We assume here that $\langle K_X, L_1, \ldots, L_t\rangle$ is isomorphic to $\mathbb{C}^{t+1}$, since if this is not true, then we fall in the degenerate case when there are integers x, $y_1, \ldots, y_t$ (not all zero) with $xK_X + y_1L_1 + \cdots + y_tL_t$ topologically trivial. We denote by $p(x, y_1, \ldots, y_t)$ the polynomial on $\mathbb{C}^{t+1}$ that $\chi(xK_X + y_1L_1 + \cdots + y_tL_t)$ extends to. We denote the *Hilbert variety of the multipolarized variety* $(X, L_1, \ldots, L_t)$ by $V_{(X,L_1,\ldots,L_t)}$. For $t = \rho - 1$ and $\langle K_X, L_1, \ldots, L_{\rho-1}\rangle = \mathbf{N}(X)$, note that $V_{(X,L_1,\ldots,L_{\rho-1})}$ is just the Hilbert variety V_X of X.

On $\langle K_X, L_1, \ldots, L_{\rho-1}\rangle$, the fixed point set of the involution s consists of $\frac{1}{2}K_X$; we call it the *central point* of s. The Taylor expansion of $p(x, y_1, \ldots, y_{\rho-1})$ at this point has all coefficients of powers whose parity is different from that of $\dim(X)$ equal to zero. In particular, $\left(\frac{1}{2}, 0, \ldots, 0\right) \in V_X$ if $\dim(X)$ is odd, and if the point belongs to V_X when $\dim(X)$ is even, it is a singular point. These and related general facts are discussed in Section 1.

In Section 2 we introduce some numerical invariants we need, the bidegrees of a bipolarized n-fold, following the same idea as in [4, §13.1]. We then prove some basic relations between them, which follow from the Hodge index theorem.

Let D be any divisor on X. In Section 3 we point out as the Riemann-Roch theorem provides for $\chi(D)$ a very useful expression to treat multipolarized manifolds. The idea is to write $D = E + \frac{1}{2}K_X$, and to express $\chi(D)$ in terms of E and the Chern classes of X in a quite effective way for our purposes. As a sample of the effectiveness, we shortly discuss the quadric Hilbert surface of a bipolarized surface (X, L_1, L_2).

Section 4 is devoted to the case of bipolarized 3-folds (X, L_1, L_2). We study some geometrical properties of the Hilbert cubic surface $\mathcal{S} = V_{(X,L_1,L_2)}$. In particular, we interpret the central point of the Serre involution as an Eckardt point; moreover, inspired by the study of singular points at infinity of the Hilbert curve of a polarized manifold, done in [3], we look at the singularities of the curve at infinity of $\mathcal{S}$.

In Section 5 we come back to Hilbert curves, studied in [3]. We provide explicit examples of quartic Hilbert curves of polarized 4-folds of any possible genus. A key result here is Lemma 2.5 which allows us to interpret a plane section of the Hilbert surface of a bipolarized n-fold as the Hilbert curve of a corresponding polarized n-fold. Precisely, let (X, L_1, L_2) be a bipolarized n-fold and let $\mathcal{S}$ be its Hilbert surface. For positive integers a, b, let $L := aL_1 + bL_2$ and let $\Gamma_{a,b}$ be the Hilbert curve of the polarized variety (X, L). Then $\Gamma_{a,b}$ is a suitable plane section of $\mathcal{S}$. We refer to [2] for the study of the j-invariant of the Hilbert cubic $\Gamma_{a,b}$ in the case of a bipolarized 3-fold (X, L_1, L_2).

In Section 6, quotients of projective Hilbert surfaces $\mathcal{S}$ with respect to the natural extension, $\overline{s}$, of the Serre involution s are analyzed. It is shown that the quotient is equipped with a natural map into $\mathbb{P}^6$. We show that, for suitable hyperplane sections h of $\mathcal{S}$, the curve $h/\langle\overline{s}\rangle$ is a Castelnuovo curve in $\mathbb{P}^3$, assuming $\mathcal{S}$ to be smooth.

Inspired by [3, Section 7], Section 7 is devoted to characterize surfaces in a 3-dimensional space which are invariant under the Serre involution. They provide a natural context which Hilbert surfaces fit into.

A lot of computations have been carried out with Maple 14 `algcurves` package; we simply refer for them to [10] throughout the paper, but we can make such computations available if necessary.

This paper grew up from some ideas developed in connection with the Master thesis of the third author [9], inspired by [3].

Notation and terminology. We work on the complex field $\mathbb{C}$ and use the standard terminology in algebraic geometry.

In particular, we denote by $\mathcal{O}_X$ the structure sheaf of a projective variety X. For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ stands for the complex dimension of $H^i(X, \mathcal{F})$. Moreover, $\chi(\mathcal{F}) := \sum_i (-1)^i h^i(\mathcal{F})$ is the Euler characteristic of $\mathcal{F}$.

Let L be a line bundle on X, and let $|L|$ be the complete linear system associated to it. The *Kodaira dimension*, $\kappa(L)$, of L is defined as $\kappa(L) = -\infty$ whenever $|mL| = \emptyset$ for every $m \in \mathbb{N}$, and $\kappa(L) = \max_{m>0}\{\dim(\phi_m(X))\}$, where ϕ_m is the rational map defined by $|mL|$, otherwise. Note that $\kappa(L) = \kappa(mL)$ for any positive integer m.

We say that L is *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on X. Moreover, L is said to be *big* if $\kappa(L) = \dim(X)$. If L is nef then this is equivalent to $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim(X)$. We say that L is *spanned* if it is spanned by global sections, i.e. globally generated, at all points of X by $H^0(X, L)$.

If L is spanned we say that L is *very ample* if the morphism $X \to \mathbb{P}^N$ defined by $|L|$ is an embedding, where $N = h^0(X, L) - 1$. We say that L is *ample* if there exsists $m > 0$ such that $L^{\otimes m}$ is very ample.

The pull-back $\iota^* L$ of L by an embedding $\iota : W \hookrightarrow X$ is denoted by L_W. We denote by K_X the canonical bundle of a Gorenstein variety X.

If L_1, L_2 are nef and big line bundles, the 3-tuple (X, L_1, L_2) is called *quasi-bipolarized variety.* If L_1, L_2 are ample line bundles, the 3-tuple (X, L_1, L_2) is called *bipolarized variety.*

When no confusion arises, we use the additive notation for the tensor product of line bundles. We freely use the notation $Y = 0$ to denote a cycle $Y \subset X$ numerically equivalent to zero.

1. THE HILBERT VARIETY: GENERALITIES

In this section, we will show how to obtain the Hilbert variety associated to a smooth n-fold and some properties closely related to it. The Hilbert variety does not depend on any polarization.

Let X be a complex projective irreducible variety. Let $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ denote the subgroup of topologically trivial line bundles. Set $\mathbf{N}(X) := \big(\mathrm{Pic}(X)/\mathrm{Pic}^0(X)\big) \otimes_{\mathbb{Z}} \mathbb{C}$. The Euler characteristic map

$$\chi : \mathrm{Pic}(X) \to \mathbb{Z},$$

defined by $L \mapsto \chi(L)$, gives rise to a polynomial function

$$p : \mathbf{N}(X) \to \mathbb{C}.$$

Note that $\mathbf{N}(X) \cong \mathbb{A}^{\rho}_{\mathbb{C}}$, where $\rho := \rho(X)$ is the Picard number of X. If $\mathbf{N}(X) = \langle L_1, \dots, L_\rho \rangle$ with $L_1, \dots, L_\rho \in \mathrm{Pic}(X)$, we can write $\mathcal{L} = \sum_{i=1}^{\rho} x_i L_i \in \mathbf{N}(X)$, $x_i \in \mathbb{C}$, for all $\mathcal{L} \in \mathbf{N}(X)$. Then the image

$$p(\mathcal{L}) = p(x_1, \dots, x_\rho)$$

is the evaluation in $\mathcal{L}$ of the polynomial $p \in \mathbb{C}[x_1, \dots, x_\rho]$, when we consider $x_1, \dots, x_\rho$ as complex variables. In other words, for $x_1, \dots, x_\rho$ integers, we consider the Hilbert polynomial

$$\chi(x_1, \dots, x_\rho) := \chi(x_1 L_1 + \dots + x_\rho L_\rho), \tag{1}$$

and we denote by $p(x_1, \dots, x_\rho)$ the polynomial $\chi(x_1, \dots, x_\rho)$ when we consider $x_1, \dots, x_\rho$ as complex variables.

Let us consider the affine algebraic set $V_X := V(p)$, which is a hypersurface of degree $\dim(X)$ in $\mathbf{N}(X) \cong \mathbb{A}^{\rho}_{\mathbb{C}}$. We also refer to V_X as the *(affine) Hilbert variety associated to* X. Note that the coefficients of the polynomial p are rational numbers; therefore V_X is defined over $\mathbb{Q}$, hence over $\mathbb{R}$. In particular, one can also consider V_X as a real affine hypersurface in $\mathbb{A}^{\rho}_{\mathbb{R}}$.

From now on, apart from the Hilbert variety and unless otherwise specified, we will use the word *variety* to mean a normal, Gorenstein, complex projective variety, X.

Up to a suitable choice of generators, we may assume $\mathbf{N}(X) = \langle K_X, \mathcal{L}_1, \ldots, \mathcal{L}_{\rho-1}\rangle$, provided that K_X is not numerically trivial. Thus, we can write an element $\mathcal{L} \in \mathbf{N}(X)$ as $\mathcal{L} = xK_X + \sum_i y_i\mathcal{L}_i$, with $\mathcal{L}_i \in \mathrm{Pic}(X)$ and $x, y_i \in \mathbb{C}$, $i = 1, \ldots, \rho - 1$. Then sending

$$\mathcal{L} = xK_X + \sum_{i=1}^{\rho-1} y_i\mathcal{L}_i \quad\mapsto\quad (1-x)K_X - \sum_{i=1}^{\rho-1} y_i\mathcal{L}_i$$

defines a map

$$s : \mathbf{N}(X) \to \mathbf{N}(X), \quad (x, y_1, \ldots, y_{\rho-1}) \;\mapsto\; (1-x, -y_1, \ldots, -y_{\rho-1}),$$

that we call *Serre involution.* More precisely, for integers x, y_i, look at the Hilbert polynomial $\chi(x, \ldots, y_i, \ldots) := \chi(xK_X + \sum_i y_i\mathcal{L}_i)$. By Serre duality,

$$\begin{aligned}
\chi(x, \ldots, y_i, \ldots) &= \chi\big(xK_X + \sum_{i=1}^{\rho-1} y_i\mathcal{L}_i\big) \\
&= (-1)^{\dim(X)}\chi\big((1-x)K_X - \sum_{i=1}^{\rho-1} y_i\mathcal{L}_i\big) \\
&= (-1)^{\dim(X)}\chi(1-x, \ldots, -y_i, \ldots).
\end{aligned}$$

According to the above notation, denote by $p(x, \ldots, y_i, \ldots)$ the polynomial $\chi(x, \ldots, y_i, \ldots)$ when we consider x, y_i as complex variables. Thus,

$$p(x, y_1, \ldots, y_{\rho-1}) = (-1)^{\dim(X)}p(1-x, -y_1, \ldots, -y_{\rho-1}).$$

Clearly, the Hilbert variety V_X is fixed under the Serre involution s, that is $s(V_X) = V_X$. Moreover the (unique) fixed point of the involution s is $C = \big(\frac{1}{2}, 0, \ldots, 0\big) \in \mathbb{A}^{\rho}_{\mathbb{C}}$ corresponding to $\frac{1}{2}K_X$. We express these facts saying that V_X is symmetric with respect to C. We also say that C is the *central point* of the Serre involution. Notice that

$$C \in V_X \;\text{ for }\; \dim(X) \;\text{ odd.} \tag{2}$$

Since, for any j-th partial derivative ∂^j, $j \geq 0$,

$$(\partial^j p)(1-x, -y_1, \ldots, -y_{\rho-1}) = (-1)^{\dim(X)+j}(\partial^j p)(x, y_1, \ldots, y_{\rho-1}),$$

we conclude that

$$(\partial^j p)\Big(\frac{1}{2}, 0, \ldots, 0\Big) = 0 \ \text{ if } \ n + j \ \text{ is odd.} \tag{3}$$

Summarizing, we have the following result.

PROPOSITION 1.1. *Let V_X be the Hilbert variety of an n-dimensional variety X, and let C be the central point of the Serre involution.*

1. *V_X is symmetric with respect to C, and $C \in V_X$ for n odd;*
2. *For n even, if $C \in V_X$, then V_X is singular at C;*
3. *For any n, if $C \in V_X$ is a point of multiplicity $n-1$, then C is a point of multiplicity n of V_X.*

Proof. We have only to note that statements 2) and 3) are an immediate consequence of condition (3): take $j = 1$ to get 2), and $j - 1$ to get 3). □

Let us denote by $\overline{V_X} \subset \overline{\mathbf{N}(X)} (\cong \mathbb{P}^\rho_{\mathbb{C}})$ the projective closure of $V_X \subset \mathbf{N}(X)$. We also say that $\overline{V_X}$ is the (*projective*) *Hilbert variety* of X. Denoting by $u_0, u_1, \ldots, u_\rho$ the homogeneous coordinates in $\mathbb{P}^\rho_{\mathbb{C}}$, with $xu_\rho = u_0$, $y_i u_\rho = u_i$, $1 = 1, \ldots, \rho - 1$, the Serre involution extends to an involution

$$\overline{s} : \overline{\mathbf{N}(X)} \to \overline{\mathbf{N}(X)}, \ \ [u_0, u_1, \ldots, u_\rho] \mapsto [u_\rho - u_0, -u_1, \ldots, -u_{\rho-1}, u_\rho],$$

with the hyperplane at infinity $u_\rho = 0$ consisting of fixed points.

We note the following. For any affine linear subspace Λ of $\mathbf{N}(X)$ containing the point C, the variety $\Lambda \cap V_X$ cut out on V_X by Λ is invariant under the Serre involution s. The projective closure of $\Lambda \cap V_X$ in $\overline{\mathbf{N}(X)}$ is in turn invariant with respect to $\overline{s}$. This provides a motivation for the discussion in Section 7.

We refer to [3] for several illustrative basic examples.

2. BIPOLARIZED MANIFOLDS

To begin with, let us introduce some numerical invariants we need in the sequel (compare with [4, §13.1]).

2.1. **Bidegrees.** Following the notation as in [3, p. 462], we define the bidegrees of a 3-tuple (X, L_1, L_2). Let L_1, L_2 be two line bundles on an irreducible, normal, Gorenstein n-dimensional projective variety X. For $j, k \geq 0$ and $j + k \leq n$, define the (j,k)-th *bidegree of the* 3-*tuple* (X, L_1, L_2) as

$$d_{j,k}(L_1, L_2) := K_X^{n-j-k} \cdot L_1^j \cdot L_2^k.$$

Note that $d_{j,k}$ is an integer, since it is the intersection of cycles of complementary dimension. If no confusion will arise, we simply write $d_{j,k} := d_{j,k}(L_1, L_2)$.

Moreover, in each case when j, k are both assigned numbers, we avoid the comma in the symbol $d_{j,k}$, *e.g.*, simply writing d_{00}, d_{10}, d_{12}, $\dots$. From now on, we will consider the above condition $j+k \leq n$ as a blanket assumption. Just as a reminder,

$$d_{00} = K_X^n, \quad d_{n0} = L_1^n, \quad d_{0n} = L_2^n.$$

Here are some basic relations between the bidegrees, which follow from the Hodge index theorem (see [4, §2.5]).

PROPOSITION 2.2. *Let L_1, L_2 be two nef line bundles on an irreducible, normal, Gorenstein n-dimensional projective variety X. Then the following inequalities hold:*

1. $d_{j,n-1-j}^2 \geq d_{j-1,n-1-j}\ d_{j+1,n-1-j}$, *for* $j = 1, \dots, n-1$.
2. $d_{n-1-j,j}^2 \geq d_{n-1-j,j-1}\ d_{n-1-j,j+1}$, *for* $j = 1, \dots, n-1$.

Furthermore, assuming that K_X is nef, one has, for $j+k \leq n-1$,

3. $d_{j,k}^2 \geq d_{j-1,k}\ d_{j+1,k}$, *with* $j \geq 1$ *and* $k \geq 0$.
4. $d_{j,k}^2 \geq d_{j,k-1}\ d_{j,k+1}$, *with* $j \geq 0$ *and* $k \geq 1$.

Proof. It follows from [4, Proposition 2.5.1]. In particular, for $j = 1, \dots, n-1$, we get

$$\begin{aligned} d_{j,n-1-j}^2 &= (K_X \cdot L_1^j \cdot L_2^{n-1-j})^2 \geq (K_X^2 \cdot L_1^{j-1} \cdot L_2^{n-1-j})(L_1^{j+1} \cdot L_2^{n-1-j}) \\ &= d_{j+1,n-1-j}\ d_{j-1,n-1-j}, \end{aligned}$$

as well as the symmetric inequality obtained by exchanging the indices in each bidegree. This leads to 1) and 2) respectively.

If K_X is nef, the condition $k+j-1$ relaxes to $j+k \leq n-1$, that is, for $j \geq 1$ and $k \geq 0$,

$$\begin{aligned} d_{j,k}^2 &= (K_X^{n-j-k} \cdot L_1^j \cdot L_2^k)^2 = (L_1 \cdot K_X^{n-j-k} \cdot L_1^{j-1} \cdot L_2^k)^2 \\ &\geq (L_1^2 \cdot K_X^{n-j-k-1} \cdot L_1^{j-1} \cdot L_2^k)(K_X^{n-j-k+1} \cdot L_1^{j-1} \cdot L_2^k) \\ &= (K_X^{n-j-k-1} \cdot L_1^{j+1} \cdot L_2^k)(K_X^{n-j-k+1} \cdot L_1^{j-1} \cdot L_2^k) = d_{j+1,k}\ d_{j-1,k}. \end{aligned}$$

Symmetrically, for $k \geq 1$ and $j \geq 0$, we obtain $d_{j,k}^2 \geq d_{j,k+1}\ d_{j,k-1}$. □

Example 2.3. For $n = 3$, statements 1) and 2) of Proposition 2.2 yield, for $j = 1, 2$,

$$d_{11}^2 \geq d_{21}d_{01}, \quad d_{20}^2 \geq d_{30}d_{10} \quad \text{and} \quad d_{02}^2 \geq d_{03}d_{01}, \quad d_{11}^2 \geq d_{12}d_{10},$$

respectively. Under the further assumption that the canonical bundle is nef, statements 3) and 4) give, for $j+k = 1$, the two more conditions $d_{10}^2 \geq d_{20}d_{00}$ and $d_{01}^2 \geq d_{02}d_{00}$.

2.4. **The Hilbert surface.** Coming back to the case of interest, let X be a smooth projective variety of dimension n, let L_1 and L_2 be ample line bundles on X. The Hilbert polynomial $\chi(x,y,z) := \chi(xK_X + yL_1 + zL_2)$, with $x,y,z \in \mathbb{Z}$, arises naturally in the study of the bipolarized variety (X, L_1, L_2). As usual, denote by $p(x,y,z)$, sometimes by $p_{(X,L_1,L_2)}(x,y,z)$, the polynomial $\chi(x,y,z)$ when we consider x, y and z as complex variables. Then looking at the zeroes of $p(x,y,z)$ corresponds to taking a slice of the Hilbert variety V_X by the 3-dimensional vector subspace $\mathbb{C}^3_{(x,y,z)} \subseteq \mathbf{N}(X)$ ($\mathbb{C}^3_{(x,y,z)} = \langle K_X, L_1, L_2 \rangle$ whenever K_X, L_1 and L_2 are $\mathbb{C}$-linearly independent). We will also write

$$V_{(X,L_1,L_2)} := \mathbb{C}^3_{(x,y,z)} \cap V_X,$$

and we will say that the degree $n := \dim(X)$ affine surface $V_{(X,L_1,L_2)}$ is the *Hilbert surface of the bipolarized variety* (X, L_1, L_2).

Letting $\mathcal{S} := V_{(X,L_1,L_2)}$ we will denote by $\overline{\mathcal{S}}$ its projective closure in $\mathbb{P}^3$, where x, y, z, ζ are the homogeneous coordinates. According to Proposition 1.1, the degree n surface $\mathcal{S}$ is symmetric with respect to the central point $C = (\frac{1}{2}, 0, 0)$ of the Serre involution.

Unless otherwise specified, we make the blanket assumption that the numerical classes of L_1, L_2 and K_X are linearly independent in the vector space $\mathbf{N}(X)$.

We prove the following general fact.

Lemma 2.5 (Key lemma). *Let (X, L_1, L_2) be a bipolarized n-fold and let $\mathcal{S}$ be its Hilbert surface. For positive integers a, b, let $L := aL_1 + bL_2$ and let $\Gamma_{a,b}$ be the Hilbert curve of the polarized variety (X, L). Then $\Gamma_{a,b}$ is the section of $\mathcal{S}$ with the plane $az - by = 0$ in $\mathbb{C}^3_{(x,y,z)}$.*

Proof. Clearly, $L := aL_1 + bL_2$ is an ample line bundle for positive integers a, b. According to [3], the curve $\Gamma_{a,b}$ is defined in the $\mathbb{C}^2_{(x,t)}$ plane by the equation $p(x,t) = \chi(xK_X + tL) = 0$. Since $xK_X + tL = xK_X + atL_1 + btL_2$ and the Hilbert surface $\mathcal{S}$ of the bipolarized n-fold (X, L_1, L_2) is defined in the $\mathbb{C}^3_{(x,y,z)}$ space by the equation

$$p_{(X,L_1,L_2)}(x,y,z) = \chi(xK_X + yL_1 + zL_2) = 0,$$

we thus see that $\Gamma_{a,b}$ is the section of $\mathcal{S}$ with the plane defined by $az - by = 0$. Its equation in the plane $\mathbb{C}^2_{(x,t)}$ is obtained by specializing that of $\mathcal{S}$ letting $t = y/a = z/b$. Moreover, as b/a (or a/b) varies in $\mathbb{Q}$ we have that the corresponding Hilbert curve $\Gamma_{a,b}$ varies in the pencil of planes of $\mathbb{C}^3_{(x,y,z)}$ through the axis generated by K_X. □

Example 2.6 (The Hilbert surface of products). Let us consider a remarkable class of examples. Assume that the variety X is a product, $X = X' \times X''$, and consider the projections π' and π''

$$\begin{array}{ccc} & X & \\ {}^{\pi'}\swarrow & & \searrow^{\pi''} \\ X' & & X'' \end{array}$$

onto the factors. Set $L_i' \boxtimes L_i'' := (\pi')^* L_i' \otimes (\pi'')^* L_i''$, where $L_i' \in \operatorname{Pic}(X')$ and $L_i'' \in \operatorname{Pic}(X'')$ for $i = 1, 2$. By Künneth formulas one has $\chi(L_i' \boxtimes L_i'') = \chi(L_i')\ \chi(L_i'')$, $i = 1, 2$.

Assume L_i' and L_i'' nef and big, so that $L_i := L_i' \boxtimes L_i''$ is nef and big, $i = 1, 2$, and consider the quasi-bipolarized variety (X, L_1, L_2). We have

$$xK_X + yL_1 + zL_2 = (xK_{X'} + yL_1' + zL_2') \boxtimes (xK_{X''} + yL_1'' + zL_2''),$$

so we obtain

$$\chi(xK_X + yL_1 + zL_2) = \chi(xK_{X'} + yL_1' + zL_2')\ \chi(xK_{X''} + yL_1'' + zL_2'').$$

Thus, we find for the polynomial $p(x, y, z) := p_{(X,L_1,L_2)}(x, y, z)$ the expression

$$p(x, y, z) := \chi(xK_X + yL_1 + zL_2) = p_{(X',L_1',L_2')}(x, y, z)\ p_{(X'',L_1'',L_2'')}(x, y, z);$$

hence the Hilbert surface associated to X is reducible. In particular, if $X = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ is the product of $n = \dim(X)$ smooth curves, then the Hilbert surface $\mathcal{S}$ is the union of n planes, all containing the point C. Note however that this is not a sufficient condition for X being the product of n curves, *e.g.*, see [3, §3.5].

2.7. **The degenerate case.** Let (X, L_1, L_2) be a bipolarized n-fold such that $\dim_{\mathbb{C}}\langle K_X, L_1, L_2\rangle < 3$. Even in this case we can consider the polynomial

$$p(x, y, z) = \chi(xK_X + yL_1 + zL_2),$$

defining an affine surface, which we call the *degenerate Hilbert surface* of (X, L_1, L_2), and we denote again by $\mathcal{S}$. We observe that, for a polarized n-fold (X, L), degenerate case means $\dim_{\mathbb{C}}\langle K_X, L\rangle = 1$, while in the bipolarized case, degeneracy is equivalent to $1 \leq \dim_{\mathbb{C}}\langle K_X, L_1, L_2\rangle \leq 2$.

Clearly, if $\dim_{\mathbb{C}}\langle K_X, L_1, L_2\rangle < 3$, then the surface $\mathcal{S}$ is not a slice of type $\mathbb{C}^3 \cap V_X$ with $\mathbb{C}^3$ a vector subspace of $\mathbf{N}(X)$. In particular, if $\dim_{\mathbb{C}}\langle K_X, L_1, L_2\rangle = 1$, then $K_X = \lambda_i L_i$, for $\lambda_i \in \mathbb{Q}$, $i = 1, 2$, and letting $\lambda_2 t := \lambda_1\lambda_2 x + \lambda_2 y + \lambda_1 z$, one has

$$p(x, y, z) = \wp(t) \in \mathbb{C}[t].$$

This is a polynomial of degree $n = \dim(X)$ in t and its zeros correspond to the slice $\mathbb{C}_{(t)} \cap V_X$. Moreover, in this case, $\mathcal{S}$ is the union of n parallel planes, π_j, of equation $\lambda_1\lambda_2 x + \lambda_2 y + \lambda_1 z - \lambda_2 t_j = 0$, where t_j are the roots of $\wp(t)$, $j = 1, \ldots, n$.

The configuration of such planes π_j is symmetric with respect to the central point $C = (\frac{1}{2}, 0, 0)$ of the Serre involution. Moreover, according to (2), if n is odd, one of these planes passes through C.

A simple example is given by $(X, L_1, L_2) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a), \mathcal{O}_{\mathbb{P}^3}(b))$, for some positive integers a, b. In this case, for x, y, z integers, one has

$$p_{(\mathbb{P}^3, L_1, L_2)}(x, y, z) = \chi(xK_X + yL_1 + zL_2) = \chi(\mathcal{O}_{\mathbb{P}^3}(-4x + ay + bz)).$$

Recalling that $\chi(\mathcal{O}_{\mathbb{P}^3}(t)) = h^0(\mathcal{O}_{\mathbb{P}^3}(t))$ for $t \gg 0$ and that $h^0(\mathcal{O}_{\mathbb{P}^3}(t)) = \binom{t+3}{3}$ for $t \geq 0$, we have

$$\chi(xK_X + yL_1 + zL_2) = \chi(\mathcal{O}_{\mathbb{P}^3}(t)) = \binom{t+3}{t} = \frac{1}{3!}(t+3)(t+2)(t+1).$$

Thus, the polynomial $p_{(\mathbb{P}^3, L_1, L_2)}(x, y, z)$, with x, y, z complex variables, can be written in the form

$$p_{(\mathbb{P}^3, L_1, L_2)}(x, y, z) = \wp(t) = \frac{1}{3!} \prod_{i=1}^{3} (t + i), \quad i = 1, 2, 3,$$

where $t = -4x + ay + bz$. Therefore, $\mathcal{S}$ is the union of three parallel planes $\mathcal{S} = \pi_1 \cup \pi_2 \cup \pi_3$, where $\pi_i : -4x + ay + bz + i = 0$, for $i = 1, 2, 3$, and $C = (\frac{1}{2}, 0, 0) \in \pi_2$.

We want to stress the following numerical interpretation of degenerate cases when $K_X \in \langle L_1, L_2 \rangle \subset \mathbf{N}(X)$. The equivalences below are a limit case of statements 1) and 2) of Proposition 2.2 and can be regarded as an analog of [3, Lemma 2.4]. Here, $D = D'$ stands for numerical equivalence of divisors $D, D' \in \operatorname{Pic}(X) \otimes \mathbb{Q}$.

Proposition 2.8. *Let (X, L_1, L_2) be a bipolarized n-fold. Then:*

1. $d_{j,n-1-j}^2 = d_{j-1,n-1-j}\, d_{j+1,n-1-j}$ *for some j, $1 \leq j \leq n-1$, if and only if $K_X = \lambda_1 L_1$ with $\lambda_1 \in \mathbb{Q}$.*
2. $d_{n-1-j,j}^2 = d_{n-1-j,j-1}\, d_{n-1-j,j+1}$ *for some j, $1 \leq j \leq n-1$, if and only if $K_X = \lambda_2 L_2$ with $\lambda_2 \in \mathbb{Q}$.*

Proof. Indeed, the equality $d_{j,n-1-j}^2 = d_{j-1,n-1-j}\, d_{j+1,n-1-j}$ can be rewritten as

$$\begin{aligned}
(K_X \cdot L_1^j \cdot L_2^{n-j-1})^2 &= (K_X \cdot L_1 \cdot L_1^{j-1} \cdot L_2^{n-j-1})^2 \\
&= (K_X^2 \cdot L_1^{j-1} \cdot L_2^{n-j-1})(L_1^{j+1} \cdot L_2^{n-j-1}) \\
&= (K_X^2 \cdot L_1^{j-1} \cdot L_2^{n-j-1})(L_1^2 \cdot L_1^{j-1} \cdot L_2^{n-j-1}).
\end{aligned}$$

If the above equality occurs for some j, then a consequence of the Hodge index theorem (see [4, Corollary 2.5.4]) applies to say that there exists a rational number λ_1 such that K_X is numerically equivalent to $\lambda_1 L_1$. Similarly,

$d_{n-1-j,j}^2 = d_{n-1-j,j-1}\, d_{n-1-j,j+1}$ can be rewritten as

$$\begin{aligned}(K_X \cdot L_1^{n-1-j} \cdot L_2^j)^2 &= (K_X \cdot L_2 \cdot L_1^{n-1-j} \cdot L_2^{j-1})^2 \\ &= (K_X^2 \cdot L_1^{n-1-j} \cdot L_2^{j-1})(L_1^{n-1-j} \cdot L_2^{j+1}) \\ &= (K_X^2 \cdot L_1^{n-1-j} \cdot L_2^{j-1})(L_2^2 \cdot L_1^{n-1-j} \cdot L_2^{j-1}).\end{aligned}$$

So, if this equality holds for some j, then there exists a rational number λ_2 such that K_X is numerically equivalent to $\lambda_2 L_2$.

In both cases, a straightforward check proves the converse. □

3. RIEMANN-ROCH FORMULA REVISITED

Let X be a smooth complex projective variety of dimension n and let D be any divisor on X. Recalling that $\frac{1}{2}K_X$ is the fixed point of the Serre involution $s : \mathbf{N}(X) \to \mathbf{N}(X)$, it is convenient to write $D = E + \frac{1}{2}K_X$. Then the Riemann-Roch theorem provides a very useful expression for $\chi(D)$ in terms of E and the Chern classes, $c_i(X)$, of X. Actually, let X be a smooth curve, *i.e.*, $n = 1$. Then

$$\chi(D) = \deg E.$$

If X is a smooth surface, we get

$$\chi(D) = \frac{1}{2}E^2 + \big(\chi(\mathcal{O}_X) - \frac{1}{8}K_X^2\big). \tag{4}$$

Now suppose that $n = 3$. The usual expression of the Riemann-Roch theorem for threefolds is

$$\chi(D) = \frac{1}{12}D \cdot (D - K_X) \cdot (2D - K_X) + \frac{1}{12}D \cdot c_2(X) + \chi(\mathcal{O}_X). \tag{5}$$

Hence, letting $D = E + \frac{1}{2}K_X$ we get

$$\chi(D) = \frac{1}{12}\big(E + \frac{1}{2}K_X\big) \cdot \big(E - \frac{1}{2}K_X\big) \cdot (2E) + \frac{1}{12}(E + \frac{1}{2}K_X) \cdot c_2(X) + \chi(\mathcal{O}_X).$$

Recalling that (*e.g.* see [5, Ex. 6.7, p. 437])

$$-\frac{1}{24}K_X \cdot c_2(X) = \chi(\mathcal{O}_X), \tag{6}$$

the sum of the last three terms in the above expression is simply $\frac{1}{12}E \cdot c_2(X)$. Then

$$\chi(D) = \frac{1}{12}\big(E^2 - \frac{1}{4}K_X^2\big) \cdot (2E) + \frac{1}{12}E \cdot c_2(X) = \frac{1}{12}\big(2E^3 - \frac{1}{2}E \cdot K_X^2 + E \cdot c_2(X)\big).$$

In conclusion,

$$\chi(D) = \frac{1}{6}E^3 + \frac{1}{24}E \cdot \big(2c_2(X) - K_X^2\big). \tag{7}$$

Now suppose that $n = 4$. The usual expression of the Riemann-Roch formula for 4-folds is the following

$$\chi(D) = \frac{1}{24}D^4 - \frac{1}{12}D^3 \cdot K_X + \frac{1}{24}D^2 \cdot K_X^2 + \frac{1}{24}c_2(X) \cdot D^2 - \frac{1}{24}c_2(X) \cdot K_X \cdot D + \chi(\mathcal{O}_X).$$

Grouping the first three summands and doing the same for the next two, we can rewrite it as

$$\begin{aligned}\chi(D) &= \frac{1}{24}D^2 \cdot (D^2 - 2D \cdot K_X + K_X^2) + \frac{1}{24}c_2(X) \cdot D \cdot (D - K_X) + \chi(\mathcal{O}_X) \\ &= \frac{1}{24}D^2 \cdot (D - K_X)^2 + \frac{1}{24}c_2(X) \cdot D \cdot (D - K_X) + \chi(\mathcal{O}_X),\end{aligned}$$

and by replacing D with $\frac{1}{2}K_X + E$, we get

$$\chi(D) = \frac{1}{24}\big(E^2 - \frac{1}{4}K^2\big)^2 + \frac{1}{24}c_2(X) \cdot \big(E^2 - \frac{1}{4}K_X^2\big) + \chi(\mathcal{O}_X).$$

In conclusion,

$$(8)\quad \chi(D) = \frac{1}{24}E^4 + \frac{1}{48}\big(2c_2(X) - K_X^2\big) \cdot E^2 + \frac{1}{384}\big(K_X^2 - 4c_2(X)\big) \cdot K_X^2 + \chi(\mathcal{O}_X).$$

A nice property of all these expressions is that $\chi(D)$ contains only powers of E of the same parity as n. They are very convenient to revisit the theory of the Hilbert curve of a polarized manifold developed in [3], as well as to extend it to the case of multipolarized manifolds. In particular, for a bipolarized manifold, we can construct the Hilbert surface in a parallel way, as follows.

Let X be a smooth projective variety of dimension n, let L_1 and L_2 be two ample line bundles on X, and suppose that the numerical classes of K_X, L_1, and L_2 are linearly independent. Consider the 3-dimensional vector subspace of $\mathbf{N}(X)$ generated by K_X, L_1, L_2. In line with paragraph 2.4 (just with a little change of perspective), we can consider the Hilbert surface $\mathcal{S} = \mathcal{S}_{(X,L_1,L_2)}$ of (X, L_1, L_2), namely, the affine surface $\mathcal{S} \subset \mathbb{A}^3$ defined by the complexified $p(x, y, z)$ of the polynomial expression of $\chi(D)$ given by the Riemann-Roch theorem letting $D = xK_X + yL_1 + zL_2$ and looking at x, y, z as complex variables (see (1)). Clearly, $\mathcal{S}$ has degree n. Let $C = (\frac{1}{2}, 0, 0) \in \mathbb{A}^3$ be the central point, corresponding to $\frac{1}{2}K_X$, of the Serre involution restricted to the plane $\langle K_X, L_1, L_2\rangle$. Thus, using new variables ($u = x - \frac{1}{2}, v = y, w = z$) centered at C and writing $D = E + \frac{1}{2}K_X$ where $E = uK_X + vL_1 + wL_2$, we can express the equation of $\mathcal{S}$ in terms of the coordinates u, v, w. In these coordinates C becomes the origin and, by Proposition 1.1(1), it is a center of symmetry for $\mathcal{S}$. We refer to

$$(9)\qquad f(u, v, w) = p\Big(\frac{1}{2} + u, v, w\Big) = 0$$

as the *canonical equation* of the Hilbert surface $\mathcal{S}$. Moreover, since E^k is a homogeneous polynomial of degree k in u, v, w for any positive integer k, the polynomial $f(u,v,w)$ is the sum of homogenous polynomials whose degrees have the same parity as n. Thus, we can write $f(u,v,w) = f_n + \cdots + f_0$, where $f_i = f_i(u,v,w)$ is homogeneous of degree i and it is identically zero if i and n have different parity. Clearly, $\overline{\mathcal{S}} \subset \mathbb{P}^3$, the projective closure of $\mathcal{S}$, is defined by the homogeneous polynomial $f(u,v,w)^{\text{hom}} = f_n + \zeta f_{n-1} + \cdots + \zeta^n f_0$, where ζ is the homogenizing coordinate.

3.1. **Bipolarized surfaces.** Let X be a smooth projective surface, bypolarized by two ample line bundles L_1 and L_2 such that $\operatorname{rk}\langle K_X, L_1, L_2\rangle = 3$. In this case, the Hilbert surface $\mathcal{S}$ is a quadric surface, since $\chi(D)$ has degree 2. Moreover, according to (4), the polynomial defining $\mathcal{S}$ is the sum of f_2, a homogeneous polynomial of degree two in u, v, w, and a constant term f_0. Actually, up to the constant factor $\frac{1}{2}$, $\mathcal{S}$ is the quadric surface associated to the matrix

$$A = \begin{pmatrix} A_\infty & 0 \\ 0 & a \end{pmatrix}, \tag{10}$$

where A_∞ is the submatrix

$$A_\infty := \begin{pmatrix} K_X^2 & K_X \cdot L & K_X \cdot L_2 \\ K_X \cdot L_1 & L_1^2 & L_1 \cdot L_2 \\ K_X \cdot L_1 & L_1 \cdot L_1 & L_2^2 \end{pmatrix} \tag{11}$$

and $a = 2\chi(\mathcal{O}_X) - \frac{1}{4}K_X^2$. Then, if $C \in \mathcal{S}$, it is a double point for $\mathcal{S}$, namely, $\mathcal{S}$ is a quadric cone with vertex C. Note that $C \in \mathcal{S}$ if and only if $a = 0$, *i.e.*,

$$K_X^2 = 8\chi(\mathcal{O}_X). \tag{12}$$

In [3, §3.5] we listed surfaces satisfying condition (12). However, here we are requiring that $\operatorname{rk}\langle K_X, L_1, L_2\rangle = 3$, which implies that X has Picard number ≥ 3. In particular this rules out the possibility that X is a $\mathbb{P}^1$-bundle over a curve. Here are two examples.

Example 3.2. Let X be the surface $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at a point, let $E \subset X$ be the exceptional curve of the blowing-up, let e' and f' be the fibers of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, and denote by e and f their total transforms on X respectively. Clearly, $K_X = -2(e+f)+E$; set $L_1 = e+2f-E$, $L_2 = 2e+f-E$ and note that both are ample line bundles. As e, f and E generate $\operatorname{Pic}(X)$ it is immediate to check that the condition $\operatorname{rk}\langle K_X, L_1, L_2\rangle = 3$ is satisfied. We have $K_X^2 = 8-1 = 7$ and $\chi(\mathcal{O}_X) = 1$. Moreover, recalling that $e^2 = f^2 = 0$, $e \cdot f = 1$, we get $K_X \cdot L_1 = K_X \cdot L_2 = -5$, $L_1^2 = L_2^2 = 3$ and $L_1 \cdot L_2 = 4$. Therefore the Hilbert surface $\mathcal{S}$ of (X, L_1, L_2) is the quadric affine surface defined by the matrix

$$A = \begin{pmatrix} 7 & -5 & -5 & 0 \\ -5 & 3 & 4 & 0 \\ -5 & 4 & 3 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Let A_∞ be the submatrix consisting of the first three rows and columns of A: then $\det A_\infty = 1$, hence $\det A = \frac{1}{4}$, so that $\mathcal{S}$ is smooth. From the complex point of view, $\mathcal{S}$ is a general affine quadric. On the other hand, $\det(A_\infty - tI) = -(t+1)(t^2 - 14t - 1)$, so that the signature of A_∞ is $(1, 2)$, hence from the real point of view $\mathcal{S}$ is a hyperbolic hyperboloid.

Example 3.3. Let X be the surface obtained by blowing-up $\mathbb{P}^2$ at three non-collinear points p_1, p_2, p_3, let e_i be the exceptional curve corresponding to p_i and let ℓ_i be the proper transform of the line in $\mathbb{P}^2$ joining p_j and p_k, with $j, k \neq i$. Note that $(X, -K_X)$ is a del Pezzo surface of degree 6, hence $\ell_1 + \ell_2 + \ell_3 + e_1 + e_2 + e_3 = -K_X$ is ample and $K_X^2 = 6$. Clearly, $\chi(\mathcal{O}_X) = 1$. Note that ℓ_i too is a (-1)-curve for $i = 1, 2, 3$ and $\ell_i \cdot e_j = 1 - \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Then the line bundles $L_1 = 2(\ell_1 + \ell_2 + \ell_3) + 2e_1 + 2e_2 + e_3$ and $L_2 = 2(\ell_1 + \ell_2 + \ell_3) + e_1 + 2e_2 + 2e_3$ are also ample and $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$. Moreover, $L_1^2 = L_2^2 = 19$, $L_1 \cdot L_2 = 20$, and $K_X \cdot L_1 = K_X \cdot L_2 = -11$. Therefore the Hilbert surface $\mathcal{S}$ of (X, L_1, L_2) is the affine quadric surface defined by the matrix

$$A = \begin{pmatrix} 6 & -11 & -11 & 0 \\ -11 & 19 & 20 & 0 \\ -11 & 20 & 19 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Here $\det A_\infty = 8$, hence $\det A = 4$, so that $\mathcal{S}$ is smooth. Moreover, $\det(A_\infty - tI) = -(t+1)(t^2 - 45t - 8)$, so that the signature of A_∞ is $(1, 2)$ again, and we get the same conclusion as before.

Coming back to the general case one can ask whether being a quadric cone with vertex C is the only possibility for $\mathcal{S}$ being singular. Note that the matrix A_∞ in (11) represents the quadratic form φ, obtained by restricting the intersection form on X to the real 3-dimensional vector subspace $U \subseteq \mathrm{N}(X)$ generated by the classes of K_X, L_1, L_2. Note that φ is positive definite on the 1-dimensional vector subspace $\langle L_1 \rangle$ of U. Then the Hodge index theorem implies that φ has signature $(1, 2)$ on U. Therefore $\det A_\infty > 0$; in particular, A_∞ is non-singular. Thus, for the matrix A in (10) we have

$$\mathrm{rk}(A) = \mathrm{rk}(A_\infty) + \varepsilon = 3 + \varepsilon, \quad \text{where} \quad \varepsilon = \begin{cases} 0, & \text{if } a = 0 \\ 1, & \text{if } a \neq 0. \end{cases}$$

In conclusion, we have proved the following

PROPOSITION 3.4. *Let* (X, L_1, L_2) *be a bipolarized surface such that* $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$. *Then the associated Hilbert surface* $\mathcal{S} \subset \mathbb{A}^3$ *is an irreducible quadric. Moreover, it is singular if and only if it contains* C, *in which case* $\mathcal{S}$ *is a quadric cone of vertex* C. *This happens if and only if* X *satisfies condition* (12).

4. BIPOLARIZED THREEFOLDS

Let (X, L_1, L_2) be a bipolarized 3-fold and let $\mathcal{S} = V_{(X,L_1,L_2)}$ be the corresponding Hilbert cubic surface. Assume that $\mathrm{rk}\langle K_X, , L_1, L_2\rangle = 3$. In this section, we provide two expressions for the equation of $\mathcal{S}$, and we study some geometrical properties of the surface. In particular, we describe $\mathrm{Sing}(\mathcal{S})$ when $\mathcal{S}$ is irreducible, we interpret the central point of the Serre involution as an Eckardt point if non-singular, and we look at the singularities of the curve at infinity of $\mathcal{S}$. At the end, we consider the special interesting case of bipolarized threefolds X in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, showing the effectiveness of the 1-cycle $2c_2(X) - K_X^2$, and discussing an explicit example.

Let $D := xK_X + yL_1 + zL_2$, with x, y, z complex variables. In view of the Riemann-Roch formula for 3-fold (5), the polynomial $p(x, y, z) := p_{(X,L_1,L_2)}(x, y, z)$ is given by

$$p(x,y,z) = \frac{D\cdot\big((x-1)K_X + yL_1 + zL_2\big)\cdot\big((2x-1)K_X + 2(yL_1+zL_2)\big)}{12} + \frac{1}{12}(xK_X + yL_1 + zL_2)\cdot c_2(X) + \chi(\mathcal{O}_X).$$

Assume that there exist two smooth surfaces S_1 and S_2, in the linear systems $|L_1|$ and $|L_2|$, respectively. Then the relations

$$\begin{aligned} c_2(X)\cdot L_1 &= e(S_1) - (K_X + L_1)\cdot L_1^2 = e(S_1) - d_{20} - d_{30},\\ c_2(X)\cdot L_2 &= e(S_2) - (K_X + L_2)\cdot L_2^2 = e(S_2) - d_{02} - d_{03}, \end{aligned} \tag{13}$$

hold true, where $e(S_i)$ stands for the topological Euler characteristic of S_i, for $i = 1, 2$. Indeed, to compute $L_1 \cdot c_2(X)$, consider the exact tangent normal bundle sequence for $S_1 \subset X$,

$$0 \to T_{S_1} \to (T_X)_{S_1} \to L_{1S_1} \to 0.$$

From the properties of the Chern classes and adjunction formula we get

$$L_1 \cdot c_2(X) = c_2(S_1) = -K_{S_1} \cdot L_{1S_1} = -(K_X + L_1) \cdot L_1^2 + e(S_1).$$

This gives the first relation in (13) since $d_{30} = L_1^3$, $d_{20} = K_X \cdot L_1^2$, and $d_{10} = K_X^2 \cdot L_1$ The second equality follows similarly, considering the smooth surface $S_2 \in |L_2|$ (see also [4, §13.1]).

By using relations (13) and (6) we can express the Hilbert polynomial of the bipolarized 3-fold (X, L_1, L_2) in terms of the bidegrees $d_{j,k}$, as

$$
\begin{aligned}
p(x,y,z) = {} & \frac{1}{6}d_{00}x^3 + \frac{1}{2}d_{10}x^2y + \frac{1}{2}d_{20}xy^2 + \frac{1}{6}d_{30}y^3 + \frac{1}{2}d_{01}x^2z + d_{11}xyz \\
& + \frac{1}{2}d_{21}y^2z + \frac{1}{2}d_{02}xz^2 + \frac{1}{2}d_{12}yz^2 + \frac{1}{6}d_{03}z^3 - \frac{1}{4}d_{00}x^2 - \frac{1}{2}d_{10}xy \\
& - \frac{1}{4}d_{20}y^2 - \frac{1}{2}d_{01}xz - \frac{1}{2}d_{11}yz - \frac{1}{4}d_{02}z^2 \\
& + \Big(\frac{1}{12}d_{00} - 2\chi(\mathcal{O}_X)\Big)x + \frac{1}{12}\big(d_{10} - d_{20} - d_{30} + e(S_1)\big)y \\
& + \frac{1}{12}\big(d_{01} - d_{02} - d_{03} + e(S_2)\big)z + \chi(\mathcal{O}_X).
\end{aligned} \tag{14}
$$

Now, by using the coordinates u, v, w, we see from (7) that the polynomial defining $\mathcal{S}$ is the sum of two homogeneous parts, one of degree 3 and one of degree 1. In particular, this shows the known fact that $\mathcal{S}$ contains the centre C of the Serre involution; moreover, if C is a singular point for $\mathcal{S}$, then it is a triple point (see Proposition 1.1). This happens if and only if the term $E \cdot \big(2c_2(X) - K_X^2\big)$ is identically zero. Since $E = uK_X + vL_1 + wL_2$, this is in turn equivalent to the three "cone conditions"

$$K_X \cdot \big(2c_2(X) - K_X^2\big) = L_1 \cdot \big(2c_2(X) - K_X^2\big) = L_2 \cdot \big(2c_2(X) - K_X^2\big) = 0. \tag{15}$$

Moreover, under the assumption that both the linear systems $|L_1|$, $|L_2|$ on X contain smooth surfaces S_1, S_2 respectively, then the three above conditions rewrite as

$$
\begin{aligned}
d_{00} + 48\chi(\mathcal{O}_X) &= d_{10} + 2d_{20} + 2d_{30} - 2e(S_1) \\
&= d_{01} + 2d_{02} + 2d_{03} - 2e(S_2) = 0.
\end{aligned} \tag{16}
$$

Indeed, $K_X \cdot \big(2c_2(X) - K_X^2\big) = 0$ is equivalent to $48\chi(\mathcal{O}_X) + K_X^3 = 0$ by (6). Then relations (16) follow from conditions (15) by simply using relations (13).

Let us come back to the polynomial (14) that defines the Hilbert surface $\mathcal{S}$. Direct numerical computations allow to rewrite it in the useful and more expressive form:

$$
\begin{aligned}
f(u,v,w) = {} & \frac{1}{6}d_{00}u^3 + \frac{1}{2}d_{10}u^2v + \frac{1}{2}d_{20}uv^2 + \frac{1}{6}d_{30}v^3 + \frac{1}{2}d_{01}u^2w + d_{11}uvw \\
& + \frac{1}{2}d_{21}v^2w + \frac{1}{2}d_{02}uw^2 + \frac{1}{2}d_{12}vw^2 + \frac{1}{6}d_{03}w^3 \\
& - \frac{1}{24}\Big(d_{00} + 48\chi(\mathcal{O}_X)\Big)u - \frac{1}{24}\Big(d_{10} + 2d_{20} + 2d_{30} - 2e(S_1)\Big)v \\
& - \frac{1}{24}\Big(d_{01} + 2d_{02} + 2d_{03} - 2e(S_2)\Big)w.
\end{aligned} \tag{17}
$$

Recall that $f(u,v,w) = p\big(u+\frac{1}{2},v,w\big) = 0$ is the canonical equation of $\mathcal{S}$ (see Section 4). Moreover, whenever the surface $\mathcal{S}$ is smooth at C, the linear summand in (17) defines the tangent plane, $T_C(\mathcal{S})$, to $\mathcal{S}$ at C, that is,

$$T_C(\mathcal{S}) : \big(d_{00} + 48\chi(\mathcal{O}_X)\big)u + \big(d_{10} + 2d_{20} + 2d_{30} - 2e(S_1)\big)v \\ + \big(d_{01} + 2d_{02} + 2d_{03} - 2e(S_2)\big)w = 0.$$

PROPOSITION 4.1. *Let (X, L_1, L_2) be a bipolarized 3-fold and suppose that the associated Hilbert cubic surface $\mathcal{S}$ is irreducible. The following facts hold:*

1. *If $\dim(\mathrm{Sing}(\mathcal{S})) = 1$, then $\mathrm{Sing}(\mathcal{S})$ is a line (of double points for $\mathcal{S}$), which contains the central point C of the Serre involution.*
   Next suppose that $\mathcal{S}$ has isolated singularities at most.
2. *If C is a singular point of $\mathcal{S}$, then it is a triple point and $\mathcal{S}$ cannot have further singular points.*
3. *Suppose that C is a smooth point of $\mathcal{S}$: if $\mathcal{S}$ is singular, then it has exactly two double points which are symmetric with respect to C.*

Proof. 1) is obvious: the general hyperplane section, which is irreducible, is a singular plane cubic, hence with a single singular point. Therefore the 1-dimensional singular locus has degree one, *i.e.*, it is a line. Moreover it has to contain C, due to the symmetry.

2) The point C is of multiplicity three by Proposition 1.1. Suppose that $\mathcal{S}$ contains another singular point, say P. Then every plane containing the line $\langle C, P\rangle$ would cut $\mathcal{S}$ along a plane cubic with a triple point at C and a further singular point at P, which is impossible.

3) Suppose that P is a singular point of $\mathcal{S}$. Then P', the symmetric of P with respect to C, is also a singular point. Let $\ell = \langle P, P'\rangle$. Clearly, $\ell \subset \mathcal{S}$. The tangent plane $T_C(\mathcal{S})$ to $\mathcal{S}$ at C cuts out on $\mathcal{S}$ a plane cubic which is singular at P, P' and at the tangency point C. But $T_C(\mathcal{S})$ contains ℓ, since $\ell \subset \mathcal{S}$. Then, such a cubic must necessarily be of the form $2\ell + \ell'$, where ℓ' is another line through C, due to the symmetry. Now, suppose that $\mathcal{S}$ contains another singular point, say Q. Clearly Q cannot lie on ℓ (by the same argument as in 2)). Moreover, also its symmetric point Q' with respect to C is singular for $\mathcal{S}$. Letting $\lambda := \langle Q, Q'\rangle$ and arguing as before, we thus see that $T_C(\mathcal{S})$ contains the quartic $2\ell + 2\lambda$, which is impossible. □

The following can be viewed as an analogue of the fact that for the Hilbert curve of a general polarized threefold, the central point C of the Serre involution is a flex (see [3, Remark 4.6]). See *e.g.*, [1, p. 345]) for more on Eckardt points, and Example 4.8 for a further instance.

PROPOSITION 4.2. *Let (X, L_1, L_2) be a bipolarized threefold as above, and let $\mathcal{S}$ be its Hilbert surface. Suppose that the central point C of the Serre involution is a non-singular point of $\mathcal{S}$. Then C is an Eckardt point of $\mathcal{S}$.*

Proof. With the notation as in Section 3, let $f = f(u,v,w) = f_1(u,v,w) + f_3(u,v,w) = 0$ be the canonical equation of $\mathcal{S}$ where $f_j = f_j(u,v,w)$ is the homogeneous polynomial of degree j, $j = 1,3$, appearing in (17). Since $f_1 = 0$ is the equation of the tangent plane $T_C(\mathcal{S})$ to $\mathcal{S}$ at C. It follows that the plane cubic curve $\gamma =: \mathcal{S} \cap T_C(\mathcal{S})$ is described by $f_3 = f_1 = 0$. This implies that γ consists of three coplanar lines meeting at C. For instance, if $f_1 = u - av - bw = 0$, then projecting γ onto the plane $u = 0$, we get the plane cubic curve $f_3(0,v,w) = 0$, which has a triple point at $(v,w) = (0,0)$, since f_3 is homogeneous of degree 3. Thus, the same holds for γ as well. We then conclude that C is an Eckardt point of $\mathcal{S}$. $\square$

4.3. **Singular points at infinity.** Let $\mathcal{S} \subset \mathbb{A}^3_{(u,v,w)}$ be the Hilbert cubic surface of a bipolarized threefold (X, L_1, L_2) satisfying the condition $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$ and let $\overline{\mathcal{S}} \subset \mathbb{P}^3{}_{[u,v,w,\zeta]}$ be its projective closure, where ζ denotes the homogenizing coordinate. Then the curve at infinity of $\mathcal{S}$ is the cubic $\gamma_\infty := \overline{\mathcal{S}} \cap \pi_\infty$, defined by $f_3 = \frac{1}{6}E^3 = 0$ in the plane at infinity π_∞ of equation $\zeta = 0$ (see equation (7)). A natural question suggested by [3, Lemma 3.2] is: what about singularities of γ_∞? Using Greek letters for the coefficients, γ_∞ is described by the homogeneous equation

$$\begin{aligned} 6f_3 = E^3 &= \alpha u^3 + \beta v^3 + \gamma w^3 + \delta u^2 v + \varepsilon u^2 w \\ &\quad + \varphi u v^2 + \psi u w^2 + \lambda v^2 w + \mu v w^2 + \nu u v w = 0 \end{aligned}$$

together with $\zeta = 0$. Since $E = uK_X + vL_1 + wL_2$, then $\alpha, \beta, \ldots, \nu$ are nothing but the coefficients appearing in the expression

$$\begin{aligned} E^3 &= K_X^3 u^3 + L_1^3 v^3 + L_2^3 w^3 + 3K_X^2 \cdot L_1 u^2 v + 3K_X^2 \cdot L_2 u^2 w \\ &\quad + 3K_X \cdot L_1^2 u v^2 + 3K_X \cdot L_2^2 u w^2 + 3L_1^2 \cdot L_2 v^2 w \\ &\quad + 3L_1 \cdot L_2^2 v w^2 + 6K_X \cdot L_1 \cdot L_2 u v w. \end{aligned} \tag{18}$$

In the affine chart outside the line $u = 0$, by using v and w again as affine coordinates, we have to deal with the plane curve of equation

$$g := g(v,w) = \beta v^3 + \gamma w^3 + \lambda v^2 w + \mu v w^2 + \varphi v^2 + \psi w^2 + \nu v w + \delta v + \varepsilon w + \alpha = 0.$$

Suppose that γ_∞ has a triple point (v,w). A direct numerical check, computing the derivatives, shows that the system $g = g_v = g_w = g_{vv} = g_{vw} = g_{ww} = 0$ translates into the following set of relations between coefficients and solutions:

$$\begin{aligned} \alpha &= -\beta v^3 - \gamma w^3 - \mu v w^2 - \lambda v^2 w, \\ \delta &= 3\beta v^2 + \mu v w^2 + 2\lambda v^2 w, \end{aligned}$$

$$\begin{aligned}\varepsilon &= 3\gamma w^2 + \lambda v^2 + 2\mu v w,\\ \varphi &= -3\beta v - \lambda w,\\ \psi &= -3\gamma w - \mu v,\\ \nu &= -2\lambda v - 2\mu w.\end{aligned}$$

In particular, looking at the last three relations we see that our curve admits a triple point (outside the line $u = 0$) if and only if the linear system

$$\begin{cases} 3\beta v + \lambda w &= -\varphi\\ \mu v + 3\gamma w &= -\psi\\ 2\lambda v + 2\mu w &= -\nu \end{cases}$$

admits some solution. Taking into account relation (18), this implies that in the matrix

$$(\mathcal{A} \mid b) = \begin{pmatrix} L_1^3 & L_1^2 \cdot L_2 & -K_X \cdot L_1^2\\ L_1 \cdot L_2^2 & L_2^3 & -K_X \cdot L_2^2\\ L_1^2 \cdot L_2 & L_1 \cdot L_2^2 & -K_X \cdot L_1 \cdot L_2 \end{pmatrix},$$

where b denotes the last column, the submatrix $\mathcal{A}$ has rank ≤ 2. Note that $\mathrm{rk}(\mathcal{A}) \geq 1$, since L is ample. Moreover, equality occurs if and only if $(L_1^3)(L_2^3) = (L_1 \cdot L_2^2)(L_1^2 \cdot L_2)$ and $(L_1^3)(L_1 \cdot L_2^2) = (L_1^2 \cdot L_2)^2$. Suppose that there exists a smooth surface $S \in |L_1|$. Then the latter equality can be rewritten as $({L_1}_S{}^2)({L_2}_S{}^2) = ({L_1}_S \cdot {L_2}_S)^2$, which implies that $\mathrm{rk}\langle L_1, L_2\rangle = 1$ by the Hodge index theorem, combined with the injectivity of the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$ due to the Lefschetz theorem. But this contradicts our assumption that $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$. Therefore

$$\mathrm{rk}(\mathcal{A}) = 2. \tag{19}$$

It thus follows that the linear system above admits a solution (v, w) if and only if

$$\begin{vmatrix} L_1^3 & L_1^2 \cdot L_2 & -K_X \cdot L_1^2\\ L_1 \cdot L_2^2 & L_2^3 & -K_X \cdot L_2^2\\ L_1^2 \cdot L_2 & L_1 \cdot L_2^2 & -K_X \cdot L_1 \cdot L_2 \end{vmatrix} = 0.$$

Moreover, the solution is unique by (19), *i.e.*, there is a single triple point. Note that a similar argument works also on the affine charts outside the lines $v = 0$ and $w = 0$, due to the condition $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$, even though the coefficients are exchanged. We have only to require that there is a smooth surface in $|L_2|$. In particular, this proves the following fact, that can be regarded as an analogue of [3, Lemma 3.2] for the Hilbert curve of a polarized threefold.

PROPOSITION 4.4. *Let (X, L_1, L_2) be a bipolarized threefold with $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$ and suppose that both the linear systems $|L_1|$ and $|L_2|$ contain a smooth surface. Then the curve at infinity of the Hilbert surface cannot be a line with multiplicity three.*

4.5. **Bipolarized 3-folds in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.** Let $P := \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, let $p_i : P \to \mathbb{P}^1$ be the i-th projection, and set $A_i = p_i^*\mathcal{O}_{\mathbb{P}^1}(1)$. Let $H := \mathcal{O}_P(1,1,1,1) = \sum_{i=1}^4 A_i$. Clearly, $K_P = -2H$.

For simplicity of notation, we will omit from now on the dot symbol "$\cdot$" when intersecting the pullback divisors A_i's. Note that $A_i^2 = 0$ for every i, while $A_1A_2A_3A_4 = 1$. In particular, this gives

$$\begin{aligned} H^2 &= 2\sum_{i<j} A_iA_j \qquad \text{(6 summands)},\\ H^3 &= 6\sum_{i<j<k} A_iA_jA_k \qquad \text{(4 summands)},\\ H^4 &= 24(A_1A_2A_3A_4) = 24. \end{aligned}$$

Now, let $X \subset P$ be a connected smooth threefold. Then $X \in |\mathcal{O}_P(a_1, a_2, a_3, a_4)|$ for some non-negative integers a_i, not all zeroes; moreover, the connectedness requirement implies that if $a_i = 0$ for three indices, then $a_j = 1$ for the remaining index j. We point out the following fact.

Proposition 4.6. *Let $X \subset P$ be any smooth connected threefold as above. Then $2c_2(X) - K_X^2$ is an effective 1-cycle. Moreover, it is nontrivial unless $(a_1, a_2, a_3, a_4) = (0, 0, 0, 1)$, up to reordering, i.e., unless $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let us compute the explicit expression of $2c_2(X) - K_X^2$. As $K_P = -2H$, we get by adjunction $K_X = \big(\sum_{i=1}^4 (a_i - 2)A_i\big)_X$, so that

$$K_X^2 = 2\Big(\sum_{i<j}(a_i - 2)(a_j - 2)A_iA_j\Big)_X.$$

To compute $c_2(X)$ we proceed as follows. The tangent-normal bundle sequence of $X \subset P$ is

$$0 \to T_X \to (T_P)_X \to \mathcal{O}_X(X) \to 0,$$

where $\mathcal{O}_X(X) =: [X]_X$ denotes the normal bundle, since X is a divisor inside P. From the relation between the Chern polynomials

$$c((T_P)_X; t) = \big(1 + c_1(X)t + c_2(X)t^2 + c_3(X)t^3\big)(1 + [X]_Xt)$$

we get

$$c_2(X) = c_2((T_P)_X) - c_1(X) \cdot [X]_X. \tag{20}$$

Clearly $c_1(X) = -K_X = -\big(\sum_{i=1}^4 (a_i - 2)A_i\big)_X$. On the other hand, since P is the product of four copies of $\mathbb{P}^1$, we have $T_P = \oplus_i p_i^*T_{\mathbb{P}^1}$, hence

$$c(T_P; t) = \prod_{i=1}^4 (1 + 2A_it) = 1 + 2\Big(\sum_i A_i\Big)t + 4\Big(\sum_{i<j} A_iA_j\Big)t^2 + \cdots .$$

Thus, $c_2(T_P) = 4\sum_{i<j} A_iA_j = 2H^2$, so that (20) gives

$$\begin{aligned} c_2(X) &= 4\Big(\sum_{i<j} A_iA_j\Big)_X + \Big(\sum_{i=1}^{4}(a_i-2)A_i\Big)_X\Big(\sum_{i=1}^{4} a_jA_j\Big)_X \\ &= \Big(4\sum_{i<j} A_iA_j + 2\sum_{i<j}(a_ia_j - a_i - a_j)A_iA_j\Big)_X \\ &= 2\Big(\sum_{i<j}(a_ia_j - a_i - a_j + 2)A_iA_j\Big)_X. \end{aligned}$$

Therefore,

$$2c_2(X) - K_X^2 = 2\Big(\sum_{i<j} a_ia_jA_iA_j\Big)_X. \tag{21}$$

This is always an effective 1-cycle, since $a_i \geq 0$ for every i. Moreover, it is trivial if and only if $a_ia_j = 0$ for every pair (i,j) with $i<j$. This happens if and only if three of the a_i's are zeroes, but in this case, as observed before, the connectedness of X implies that the remaining degree is 1. This is enough to conclude. □

Now, let L_1 and L_2 be any two ample line bundles on X such that $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$. If X is as in the exceptional case of Proposition 4.6, then $\big(2c_2(X) - K_X^2\big)\cdot E = 0$ for any $E = uK_X + vL_1 + wL_2$, *i.e.*, the linear term f_1 in the equation (17) of the Hilbert surface $\mathcal{S}$ is identically zero. This means that C is a singular point of $\mathcal{S}$ of multiplicity 3. However, we know that $\mathcal{S}$ is in fact reducible into three planes passing through C, since X is a product of three factors. Apart from this case, $2c_2(X) - K_X^2$ is an effective non trivial 1-cycle, hence it has positive intersection with any ample line bundle on X. This implies that f_1 is not identically zero, hence C is a smooth point of $\mathcal{S}$. This proves the following result.

Corollary 4.7. *Let (X, L_1, L_2) be any bipolarized threefold with $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$. If $X \subset P$, then the Hilbert surface $\mathcal{S}$ can never be an irreducible cone.*

Thus, in order to produce an example in which $\mathcal{S}$ is an irreducible cone we have to look for a threefold X not in P.

In fact one can expect that $\mathcal{S}$ is a smooth surface for a general multidegree (a_1, a_2, a_3, a_4) and general L_1 and L_2. Here is an example.

Example 4.8. Let P, H be as in paragraph 4.5 and let $X \in |H|$, *i.e.*, $a_1 = a_2 = a_3 = a_4 = 1$. From the exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_P \to \mathcal{O}_X \to 0,$$

we get $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_P) - \chi(-H) = \chi(\mathcal{O}_P) = 1$, since $h^j(-H) = 0$ for $j \leq 3$, and $h^4(-H) = h^0(K_P + H) = h^0(-H) = 0$ by Serre duality. We have $K_X = (K_P + H)_X = -H_X$, by adjunction. Therefore $K_X^3 = -H_X^3 = -H^4 = -24$. In particular,

$$48\chi(\mathcal{O}_X) + K_X^3 = 24,$$

showing that condition $48\chi(\mathcal{O}_X) + K_X^3 = 0$ is not satisfied. This is enough to grant that for any bipolarization we fix on X the corresponding Hilbert surface is not a cubic cone (see conditions (15) and (16)). Actually, the linear term f_1 in the equation of $\mathcal{S}$ is not identically zero. Note that H is ample, hence $\mathrm{Pic}(P) \cong \mathrm{Pic}(X)$ under the restriction homomorphism, by the Lefschtez theorem. Set, *e.g.*, $L_1 = \mathcal{O}_X(2,1,1,1)$ and $L_2 = \mathcal{O}_X(1,1,1,3)$. Then the basic condition $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$ is clearly satisfied. To compute intersection indices note that $L_1 = (A_1 + H)_X$ and $L_2 = (H + 2A_4)_X$. By using [10], we find:

$$\begin{array}{lcccccc}
d_{30} & = & L_1^3 & = & (A_1+H)^3 \cdot H & = & 42 \\
d_{03} & = & L_2^3 & = & (H+2A_4)^3 \cdot H & = & 60 \\
d_{10} & = & K_X^2 \cdot L_1 & = & H^2 \cdot (A_1+H) \cdot H & = & 30 \\
d_{01} & = & K_X^2 \cdot L_2 & = & H^2 \cdot (H+2A_4) \cdot H & = & 36 \\
d_{20} & = & K_X \cdot L_1^2 & = & -H \cdot (A_1+H)^2 \cdot H & = & -36 \\
d_{02} & = & K_X \cdot L_2^2 & = & -H \cdot (H+2A_4)^2 \cdot H & = & -48 \\
d_{21} & = & L_1^2 \cdot L_2 & = & (A_1+H)^2 \cdot (H+2A_4) \cdot H & = & 56 \\
d_{12} & = & L_1 \cdot L_2^2 & = & (A_1+H) \cdot (H+2A_4)^2 \cdot H & = & 62 \\
d_{11} & = & K_X \cdot L_1 \cdot L_2 & = & -H^2 \cdot (A_1+H) \cdot (H+2A_4) & = & -46.
\end{array}$$

On the other hand, $2c_2(X) - K_X^2 = H_X^2$ by (21), and this allows us to compute the terms (to get the equalities on the first line we use the relation (6))

$$\begin{array}{lcccc}
K_X \cdot (2c_2(X) - K_X^2) & = & -H_X^3 = -H^4 & = & -24; \\
L_1 \cdot \big(2c_2(X) - K_X^2\big) & = & (A_1+H) \cdot H^3 & = & 30; \\
L_2 \cdot \big(2c_2(X) - K_X^2\big) & = & (H+2A_4) \cdot H^3 & = & 36.
\end{array}$$

Finally, set $E = uK_X + vL_1 + wL_2$, consider $D := E + \frac{1}{2}K_X$, and recall the expression of $\chi(D)$ provided by (7),

$$\chi(\mathcal{O}_X(D)) = \frac{1}{6}E^3 + \frac{1}{24}E \cdot \big(2c_2(X) - K_X^2\big),$$

where E^3 is as in (18).

Now we have all we need to write the equation of the Hilbert cubic surface $\mathcal{S}$ of our bipolarized threefold (X, L_1, L_2) explicitly. Actually, $\mathcal{S}$ is defined by the equation

$$\begin{aligned}
&-4u^3 + 7v^3 + 10w^3 + 15u^2v + 18u^2w - 18uv^2 \\
&\qquad - 24uw^2 + 28v^2w + 26vw^2 - 46uvw - u + \frac{5}{4}v + \frac{3}{2}w = 0.
\end{aligned}$$

A check carried out by using [10] proves that $\mathcal{S}$ is smooth. In particular, it follows from what we said in the general case that C is an Eckardt point of $\mathcal{S}$ (see Proposition 4.2).

We note that the coefficients of v and w in the above expression are not integral. This fact, however, should not be surprising since $D = \frac{1}{2}K_X + E$ and K_X belong to $\mathrm{Pic}(X)$, while E doesn't. Recalling that $K_X = -H_X = (-\sum_{i=1}^4 A_i)_X$, $L_1 = (A_1 + H)_X$, $L_2 = (H + 2A_4)_X$, we have

$$D = \big(u + \frac{1}{2}\big)K_X + vL_1 + wL_2$$
$$= \big(-u+2v+w+\frac{1}{2}\big)A_{1X}+\big(-u+v+w-\frac{1}{2}\big)(A_2+A_3)_X+\big(-u+v+2w-\frac{1}{2}\big)A_{4X}.$$

So, the only condition is that the cubic polynomial above takes integral values when the four coefficients of the A_{iX} are integers.

5. HILBERT QUARTIC CURVES

In this section, we come back to Hilbert curves, studied in [3]. We discuss several examples and, in particular, we produce quartic Hilbert curves of polarized 4-folds having each possible genus.

Let's start considering a bipolarized 4-fold (X, L_1, L_2) and let $\mathcal{S} = V_{(X,L_1,L_2)}$ be the corresponding Hilbert quartic surface. Assume that $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$. In this section we deal with the general case when $\mathcal{S}$ is irreducible. Recall from Section 3 that the equation of the corresponding Hilbert quartic surface $\mathcal{S}$ (in coordinates (u, v, w) centered in C) is given by (9), namely,

$$f(u, v, w) = f_4 + f_2 + f_0 = 0,$$

the homogeneous parts of f of the various degree being

$$f_4 = \frac{1}{24}E^4, f_2 = \frac{1}{48}\big(2c_2(X)-K_X^2\big)\cdot E^2, \text{and} f_0 = \frac{1}{384}\big(K_X^4-4c_2(X)\cdot K_X^2\big)+\chi(\mathcal{O}_X),$$

where $E = uK_X + vL_1 + wL_2$. Note that f_0 is the constant term depending only on X.

The following class of examples provides quartic Hilbert surfaces with isolated singularities. Essentially, this is in line with the contents of Section 4, except for the fact that here $n = 4$, and it is the key to provide examples of quartic Hilbert curves we are looking for (see Example 5.6).

Example 5.1. Let $P = \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$; let $p_1 = P \to \mathbb{P}^2$ with $p_i : P \to \mathbb{P}^1$ the i-th projection; let $A_1 = p_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$; let $A_i = p_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $i = 2, 3, 4$; and set $H = \mathcal{O}_P(k, h, 1, 1)$, where h and k are positive integers. Let X be a smooth element of $|H|$. Note that $A_i^2 = 0$ for $i = 2, 3, 4$, and $A_1^3 = 0$, while $A_1^2 A_2 A_3 A_4 = 1$. Now consider the ample line bundles $L_1 := \mathcal{O}_X(1, 1, 1, 2)$ and $L_2 := \mathcal{O}_X(3, 1, 1, 1)$.

Recall that $\mathrm{Pic}(P) \cong \mathrm{Pic}(X)$ under the restriction homomorphism, by the Lefschetz theorem. By the choice of L_1, L_2, we see that the basic condition $\mathrm{rk}\langle K_X, L_1, L_2\rangle = 3$ is satisfied.

To compute the coefficients of $f_2(u, v, w)$, the part of degree two of the canonical equation of $\mathcal{S}$, it is necessary to compute the explicit expression of $2c_2(X) - K_X^2$. As $K_P = \mathcal{O}_P(-3, -2, -2, -2)$, we get by adjunction

$$K_X = (K_P + X)_X = \mathcal{O}_X(k - 3, h - 2, -1, -1),$$

hence

$$K_X^2 = (\mathcal{O}_X(k - 3, h - 2, -1, -1))^2 = \big(((k - 3)A_1 + (h - 2)A_2 - A_3 - A_4)^2\big)_X.$$

To compute $c_2(X)$ we proceed as follows. The tangent-normal bundle sequence of $X \subset P$ is

$$0 \to T_X \to (T_P)_X \to \mathcal{O}_X(X) \to 0,$$

where $\mathcal{O}_X(X) =: [X]_X$ is the normal bundle, since X is a divisor inside P. From the relation between the Chern polynomials

$$c((T_P)_X; t) = \big(1 + c_1(X) + c_2(X)t^2 + c_3(X)t^3 + c_4(X)t^4\big)(1 + [X]_X t)$$

we get

$$c_2(X) = c_2((T_P)_X) - c_1(X) \cdot [X]_X. \tag{22}$$

Clearly,

$$c_1(X) = -K_X = -\big((k - 3)A_1 + (h - 2)A_2 - A_3 - A_4\big)_X,$$

and $[X]_X = (hA_1 + kA_2 + A_3 + A_4)_X$. On the other hand, since P is the product of $\mathbb{P}^2$ and three copies of $\mathbb{P}^1$, we have $T_P = p_1^* T_{\mathbb{P}^2} \oplus_i p_i^* T_{\mathbb{P}^1}$. Then

$$c(T_P; t) = (1 + 3A_1 t + 3A_1^2 t^2)(1 + 2A_2 t)(1 + 2A_3 t)(1 + 2A_4 t),$$

so that

$$c_2(T_P) = 3A_1^2 + 6(A_1A_2 + A_1A_3 + A_1A_4) + 4(A_2A_3 + A_2A_4 + A_3A_4).$$

In conclusion, (22) gives the expression

(23)
$$\begin{aligned} c_2(X) &= \big[3A_1^2 + 6(A_1A_2 + A_1A_3 + A_1A_4) + 4(A_2A_3 + A_2A_4 + A_3A_4) \\ &\qquad - ((3 - k)A_1 + (2 - h)A_2 + A_3 + A_4)(kA_1 + hA_2 + A_3 + A_4)\big]_X \\ &= \big[A_1^2(k^2 - 3k + 3) + A_1A_2(2hk - 3h - 2k + 6) + 3(A_1A_3 + A_1A_4) \\ &\qquad + 2(A_2A_3 + A_2A_4 + A_3A_4)\big]_X. \end{aligned}$$

Therefore,

$$\begin{aligned} 2c_2(X)-K_X^2 &= \big[2\big(A_1^2(k^2-3k+3)+A_1A_2(2hk-3h-2k+6)\\ &\quad +3(A_1A_3+A_1A_4)+2(A_2A_3+A_2A_4+A_3A_4)\big)\\ &\quad -\big((k-3)A_1+(h-2)A_2-A_3-A_4\big)^2\big]_X\\ &= \big[(k^2-3)A_1^2+2khA_1A_2+2kA_1A_3+2kA_1A_4+2hA_2A_3\\ &\quad +2hA_2A_4+2hA_2A_4+2A_3A_4\big]_X. \end{aligned}$$

We may note that the cycle $2c_2(X)-K_X^2$ is always an effective 2-cycle for $k>1$, since in that case each coefficient in the last expression above is non-negative (compare with Proposition 4.6).

Following the same approach as in Section 4, set $E=uK_X+vL_1+wL_2$, consider $D=E+\frac{1}{2}K_X$, and recall the expression of $\chi(D)$ provided by (8),

$$\chi(D)=\frac{1}{24}E^4+\frac{1}{48}\big(2c_2(X)-K_X^2\big)\cdot E^2+\frac{1}{384}\big(K_X^2-4c_2(X)\big)\cdot K_X^2+\chi(\mathcal{O}_X).$$

We have all we need to write the canonical equation $f(u,v,w)=0$ of the Hilbert cubic surface $\mathcal{S}$ of our bipolarized 4-fold (X,L_1,L_2) explicitly.

Note that $\chi(\mathcal{O}_X)=1$. Indeed, $\chi(\mathcal{O}_X)=\chi(\mathcal{O}_P)-\chi(-H)$, and $\chi(-H)=h^0(K_P+H)=h^0(\mathcal{O}_P(-3+k,-2+h,-1,-1))=0$.

We compute all the other coefficients of the Hilbert polynomial by using [10]. Setting $\alpha:=2c_2(X)-K_X^2$, we have (to calculate the third an the second summand of $\chi(D)$):

$$\begin{aligned} \alpha\cdot K_X^2 &= -24+72k+60h-12k^2h\\ \alpha\cdot K_X\cdot L_1 &= 24-30k-18h-18k^2+18k^2h-54kh\\ \alpha\cdot K_X\cdot L_2 &= 18-54k-54h-12k^2+12k^2h-36kh\\ \alpha\cdot L_1^2 &= -18+12k-6h+18k^2+12k^2h+36kh\\ \alpha\cdot L_2^2 &= -12+36k+48h+12k^2+6k^2h+72kh\\ \alpha\cdot L_1\cdot L_2 &= -15+24k+9h+15k^2+9k^2h+66kh. \end{aligned}$$

Moreover, for the bidegrees $d_{j,k}$ we need to calculate the E^4 term, we find

$$\begin{aligned} d_{00} &= K_X^4 &&= 432-144k-108h+12k^2h\\ d_{10} &= K_X^3\cdot L_1 &&= -288+30k+18h+18k^2-18k^2h+54kh\\ d_{01} &= K_X^3\cdot L_2 &&= -378+54k+54h+12k^2-12k^2h+36kh\\ d_{03} &= K_X\cdot L_2^3 &&= -270-54k-54h+6k^2+36kh\\ d_{12} &= K_X\cdot L_1\cdot L_2^2 &&= -210-64k-60h+8k^2+40kh\\ d_{21} &= K_X\cdot L_1^2\cdot L_2 &&= -138-66k-54h+10k^2+36kh\\ d_{30} &= K_X\cdot L_1^3 &&= -78-60k-36h+24kh+12k^2 \end{aligned}$$

$$
\begin{array}{lllll}
d_{04} & = & L_2^4 & = & 216+72k+108h \\
d_{13} & = & L_1 \cdot L_2^3 & = & 171+78k+99h \\
d_{22} & = & L_1^2 \cdot L_2^2 & = & 118+76k+74h \\
d_{31} & = & L_1^3 \cdot L_2 & = & 69+66k+45h \\
d_{40} & = & L_1^4 & = & 36+48k+24h \\
d_{02} & = & K_X^2 \cdot L_2^2 & = & 324+12k-12k^2+6k^2h-48kh \\
d_{11} & = & K_X^2 \cdot L_1 \cdot L_2 & = & 249+28k+21h-17k^2+9k^2h-58kh \\
d_{20} & = & K_X^2 \cdot L_1^2 & = & 158+44k+34h-22k^2+12k^2h-60kh.
\end{array}
$$

Thus, the projective closure $\overline{\mathcal{S}} \subset \mathbb{P}^3{}_{[u,v,w,\zeta]}$ of the Hilbert surface $\mathcal{S}$ of (X, L_1, L_2) has equation

$$
\begin{aligned}
f(u,v,w)^{\mathrm{hom}} &= \frac{1}{24}(432-144k-108h+12k^2h)u^4 + \frac{1}{24}(36+48k+24h)v^4 \\
&+\frac{1}{24}(216+72k+108h)w^4 \\
&+\frac{1}{6}(-288+30k+18h-18k^2h+18k^2+54kh)u^3v \\
&+\frac{1}{6}(-378+54k+54h-12k^2h+12k^2+36kh)u^3w \\
&+\frac{1}{6}(-78-60k-36h+12k^2+24kh)uv^3 \\
&+\frac{1}{6}(-270-54k-54h+6k^2+36kh)uw^3 \\
&+\frac{1}{6}(69+66k+45h)v^3w + \frac{1}{6}(171+78k+99h)vw^3 \\
&+\frac{1}{4}(158+44k+34h+12k^2h-22k^2-60kh)u^2v^2 \\
&+\frac{1}{4}(324+12k+6k^2h-12k^2-48kh)u^2w^2 + \frac{1}{4}(118+76k+74h)v^2w^2 \\
&+\frac{1}{2}(249+28k+21h+9k^2h-17k^2-58kh)u^2vw \\
&+\frac{1}{2}(-138-66k-54h+10k^2+36kh)uv^2w \\
&+\frac{1}{2}(-210-64k-60h+8k^2+40kh)uvw^2 \\
&+\frac{1}{48}(-24+72k+60h-12k^2h)u^2\zeta^2 \\
&+\frac{1}{48}(-18+12k-6h+12k^2h+18k^2+36kh)v^2\zeta^2 \\
&+\frac{1}{48}(-12+36k+48h+6k^2h+12k^2+72kh)w^2\zeta^2
\end{aligned}
$$

$$+\frac{1}{24}(24-30k-18h+18k^2h-18k^2-54kh)uv\zeta^2$$
$$+\frac{1}{24}(18-54k-54h+12k^2h-12k^2-36kh)uw\zeta^2$$
$$+\frac{1}{24}(-15+24k+9h+9k^2h+15k^2+66kh)vw\zeta^2-\frac{1}{32}h\zeta^4+\frac{1}{32}k^2h\zeta^4=0.$$

For example, for $(h,k)=(2,2)$, it turns out that the surface $\overline{\mathcal{S}}$ has seven double points:

$$\Big[\frac{3}{2},0,1,\pm 1\Big],\quad \Big[\frac{3}{2},1,0,\pm 1\Big],\quad [1,1,0,0],\quad [5,2,1,0],\quad [2,-1,1,0].$$

The first four of them belong to the affine Hilbert surface $\mathcal{S}$ and are symmetric with respect to C, the origin in coordinates u, v, w.

Now, let's consider a polarized 4-fold (X,L). Following the same argument as above, letting $D=E+\frac{1}{2}K_X$ and $E=uK_X+vL$, we have for the quartic Hilbert curve Γ of (X,L) the canonical equation $f(u,v)=0$, where, as usual, $f(u,v)$ is the polynomial $\chi(D)$ expressed by (8), when we consider u, v as complex variables.

From Proposition 1.1 we know that if the central point of the Serre involution C belongs to Γ, then C is a double point; moreover, if C is a triple point, then it is a point of multiplicity 4, so Γ splits into four lines through C. Furthermore, if Γ has a singular point Q, then, for symmetry, it must have another singular point Q', symmetric to Q with respect to C.

In conclusion, assuming Γ to be irreducible, either Γ has C as a double point and no more singular points, or Γ does not pass through C and in this case it can have two double points Q, Q', symmetric with respect to C.

The fact that

$$f(u,v)=p\Big(\frac{1}{2}+u,v\Big)=f_0+f_2+f_4,$$

with $f_i=f_i(u,v)$ homogeneous polynomial of degree $i=0,2,4$, suggests one more comment. Assume that Γ is irreducible, and let's consider the special case when the constant term f_0 is zero, which translates into the condition

$$\big(K_X^2-4c_2(X)\big)\cdot K_X^2+384\chi(\mathcal{O}_X)=0. \tag{24}$$

One has $f_2(u,v)=(au+bv)(cu+dv)$ for some complex numbers a, b, c, d. Then either Γ has a nodal point or a cuspidal point at the origin C, according to whether the tangent lines $\ell_1: au+bv=0$, $\ell_2: cu+dv=0$ are distinct or not. We observe that the intersection multiplicity of Γ with ℓ_i, $i=1,2$, at C is at least four. Therefore, the double point C is a *biflecnode* (which decreases the genus by 1) in the former case, and a *tacnode* (which decreases the genus by 2) in the latter case. Accordingly, if ($f_0 = 0$ and) $f_4(u,v)$ is general, the Hilbert curve Γ has either genus 2 or 1.

Remark 5.2 (The real case). Let's look at the quartic Hilbert curve Γ from the real point of view, assuming $f_0 = 0$. Write $f_2(u,v) = Au^2+2Buv+Cv^2$ and recall that A, B, C are rational numbers. As above, let $f_2(u,v) = (au+bv)(cu+dv)$ be the factorization over $\mathbb{C}$. In the tacnode case, one has $B^2 - AC = 0$, so that a, b, c, d are real numbers with $ad - bc = 0$. Up to this case, one has that either a, b, c, d are real (if $B^2 - AC > 0$), or they are complex conjugate, with $c = \overline{a}$ and $d = \overline{b}$ (if $B^2 - AC < 0$). In the former case, Γ presents a loop at the origin $(0,0)$, while, in latter case, $(0,0)$ is an isolated double point.

The above argument recovers as well the fact that the general Hilbert curve is a plane quartic of genus 3. We refer to [6, Chapter XVIII] for the geometry of quartic plane curves.

Coming to examples, let us first produce some reducible Hilbert quartic curves.

Example 5.3. Consider the 4-fold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let A_i denote the pullback to X of $\mathcal{O}(1)$ via the i-th projection. Note that $A_i^2 = 0$ for every $i = 1,2,3,4$ and $A_1A_2A_3A_4 = 1$. We have $\chi(\mathcal{O}_X) = 1$, $K_X^2 = 4\big(\sum_i A_i\big)^2 = 8(A_1A_2 + \cdots + A_3A_4)$, $K_X^4 = 16\big(\sum_i A_i\big)^4 = 16 \times 24$. Moreover, $c_2(X) = 4(A_1A_2 + \cdots + A_3A_4)$. Then $c_2(X) \cdot K_X^2 = 192$. Thus, the constant term of $f(u,v)$ is

$$\frac{1}{384}K_X^4 - \frac{1}{96}c_2(X) \cdot K_X^2 + \chi(\mathcal{O}_X) = 1 - 2 + 1 = 0.$$

This means that the Hilbert curve $\Gamma = \Gamma_{(X,L)}$ contains C regardless of any polarization L on X. The term of the second degree of $f(u,v)$ in (8) is

$$\frac{1}{48}\big(2c_2(X) - K_X^2\big) \cdot E^2,$$

and the above computations says that $2c_2(X) - K_X^2 = 0$ as a 2-cycle. Therefore, the only surviving term in the equation of Γ is E^4 (up to a multiplicative constant), regardless of the polarization. In other words, for any polarization L on X, the curve Γ consists of four lines through C.

Here is an example of a quartic Hilbert curve reducible into four lines having a different configuration.

Example 5.4. Let $P = \mathbb{P}^2 \times \mathbb{P}^3$, let X be a smooth element in $|\mathcal{O}_P(1,1)|$, and let $L = \mathcal{O}_X(1,k)$ be an ample divisor on X, with k a positive integer. Let A_i denote the pullback to P of $\mathcal{O}(1)$ via the i-th projection, $i = 1,2$. Note that $A_1^3 = 0$, $A_2^4 = 0$ and $A_1^2A_2^3 = 1$. By the adjunction formula, we obtain

$K_X = (K_P + X)_X = \mathcal{O}_X(-2,-3)$. Moreover, $c(T_P;t) = (1+A_1t)^3(1+A_2t)^4$. Hence, in view of (22), we have

$$c_2(X) = (A_1^2 + 7A_1A_2 + 3A_2^2)_X.$$

By using [10], we find for the Hilbert polynomial the expression

$$\begin{aligned} f(u,v) = 18u^4 &+ \Big(\frac{1}{4}k^2 + \frac{1}{6}k^3\Big)v^4 - \Big(\frac{5}{2}k^2 + \frac{1}{3}k^3 + \frac{3}{2}k\Big)uv^3 \\ &- \Big(15k + \frac{27}{2}\Big)u^3v + \Big(4k^2 + \frac{9}{4} + \frac{21}{2}k\Big)u^2v^2 \\ &- \frac{1}{2}u^2 - \Big(\frac{1}{16} + \frac{1}{24}k\Big)v^2 + \Big(\frac{3}{8} + \frac{1}{12}k\Big)uv. \end{aligned}$$

A close inspection shows that $f(u,v)$ factors as

$$f(u,v) = -\frac{1}{48}(2u-v)(-12u+2kv+3v)(-6u+1+2kv)(-6u-1+2kv),$$

so that the Hilbert curve splits into four lines symmetric with respect to the origin, two of them being parallel, and one of remaining two not depending on k.

Further examples of reducible Hilbert quartic curves come from general results in [3, Theorem 6.1] and [8]. The following example shows that condition (24) is not necessary to have a quartic Hilbert curve of genus 1, 2.

Example 5.5. Let $P = \mathbb{P}^2 \times \mathbb{P}^3$, let X a smooth element in $|\mathcal{O}_P(4,3)|$, and let $L = \mathcal{O}_X(3,k)$ be an ample divisor on X, with k a positive integer. Let A_i denote the pullback to P of $\mathcal{O}(1)$ via the i-th projection, $i = 1,2$. In this case one has

$$\begin{aligned} X &\in |4A_1 + 3A_2|, \quad L = (3A_1 + kA_2)_X, \\ K_X &= (K_P + X)_X = \mathcal{O}_X(1,-1) = (A_1 - A_2)_X, \\ E &= (uK_X + vL) = \big(u(A_1 - A_2) + v(3A_1 + kA_2)\big)_X. \end{aligned}$$

Referring to (8) to obtain the equation of the quartic Hilbert curve, we have to compute $c_2(X)$. To this end, consider the tangent-normal bundle sequence of $X \subset P$,

$$0 \to T_X \to (T_P)_X \to \mathcal{O}(X)_X \to 0,$$

where $\mathcal{O}_X(X) =: [X]_X$ is the normal bundle, since X is a divisor inside P. From the relation between the Chern polynomials

$$c((T_P)_X;t) = \big(1 + c_1(X)t + c_2(X)t^2 + c_3(X)t^3 + c_4(X)t^4\big)(1 + [X]_X t),$$

we get

$$c_2(X) = c_2((T_P)_X) - c_1(X)\cdot[X]_X. \tag{25}$$

Clearly, $c_1(X) = -K_X = (A_2 - A_1)_X$. On the other hand, we have

$$T_P = p_1^* T_{\mathbb{P}^2} \oplus p_2^* T_{\mathbb{P}^3},$$

so that

$$\begin{aligned} c(T_P; t) &= (1 + 3A_1 t + 3A_1^2 t^2)(1 + 4A_2 t + 6A_2^2 t^2 + 4A_2^3 t^3) \\ &= 1 + (3A_1 + 4A_2)t + (3A_1^2 + 12A_1A_2 + 6A_2^2)t^2 + \cdots \end{aligned}$$

Thus,

$$c_2(T_P) = 3A_1^2 + 12A_1A_2 + 6A_2^2.$$

In conclusion, (25) gives

$$\begin{aligned} c_2(X) &= \big((3A_1^2 + 12A_1A_2 + 6A_2^2) - (A_2 - A_1)(4A_1 + 3A_2)\big)_X \\ &= \big(7A_1^2 + 11A_1A_2 + 3A_2^2\big)_X. \end{aligned}$$

By combining (8) with this expression, and carrying out all the computations by using [10], we obtain the equation of the Hilbert curve Γ_k of (X, L), that is,

$$\begin{aligned} f(u,v) = &-\frac{43}{24}kuv - 3ku^2v^2 - \frac{3}{2}k^2uv^3 - \frac{5}{4}k^2u^2v^2 - \frac{27}{2}kuv^3 + \frac{2}{3}k^3uv^3 \\ &+ \frac{1}{2}ku^3v - \frac{17}{24}u^2 + \frac{27}{4}k^2v^4 + 2k^3v^4 + \frac{23}{2}kv^2 + \frac{45}{16}v^2 + \frac{1}{12}u^4 \\ &+ \frac{27}{4}u^2v^2 + \frac{5}{2}u^3v - \frac{77}{8}uv + \frac{45}{16}k^2v^2 + \frac{75}{64} = 0. \end{aligned}$$

Furthermore, one checks that Γ_k has only one double point $[k, 1, 0] \in \mathbb{P}^2$ at infinity, and therefore it is a curve of genus $g = 2$.

Now, by using Lemma 2.5 and Example 5.1, we construct Hilbert quartic curves of each possible genus $g = 0, 1, 2, 3$.

Example 5.6. Let X, L_1, L_2, A_i, $i = 1, 2, 3, 4$, be as in Example 5.1, with $h = k = 2$. We then have $X \in |2A_1 + 2A_2 + A_3 + A_4|$, and

$$\begin{aligned} L_1 &= (A_1 + A_2 + A_3 + 2A_4)_X, \quad L_2 = (3A_1 + A_2 + A_3 + A_4)_X, \\ K_X &= (K_P + X)_X = \mathcal{O}_X(-1, 0, -1, -1) = (-A_1 - A_3 - A_4)_X, \\ E &= (uK_X + vL_1 + wL_2) \\ &= \big((-A_1 - A_3 - A_4)u + v(A_1 + A_2 + A_3 + 2A_4) \\ &\quad + w(3A_1 + A_2 + A_3 + A_4)\big)_X. \end{aligned}$$

To compute $c_2(X)$, we follow the same argument as in Example 5.5. Expression (23) reads, for $h = k = 2$,

$$c_2(X) = (4A_1A_2 + 3A_1A_3 + 3A_1A_4 + 2A_2A_3 + 2A_2A_4 + 2A_3A_4 + A_1^2)_X.$$

Now, let $\mathcal{S}$ be the Hilbert surface of the bipolarized 4-fold (X, L_1, L_2). Keeping the notation as in Lemma 2.5, set $L_{a,b} := aL_1 + bL_2$ and let $\Gamma_{a,b}$ be the Hilbert curve of the polarized 4-fold $(X, L_{a,b})$ obtained by cutting out $\mathcal{S}$ with the plane $\pi_{a,b} : aw - bv = 0$ in $\mathbb{C}^3_{(u,v,w)}$. Take on X the ample line bundles

$$L_{1,0} = L_1, \quad L_{0,1} = L_2, \quad L_{2,1} = 2L_1 + L_2, \quad L_{1,1} = L_1 + L_2,$$

and let $\Gamma_{1,0}, \Gamma_{0,1}, \Gamma_{2,1}, \Gamma_{1,1}$ be the Hilbert curves of the polarized 4-folds (X, L_1), (X, L_2), $(X, 2L_1 + L_2)$, $(X, L_1 + L_2)$ obtained as section of $\mathcal{S}$ with the planes

$$\pi_{1,0} : w = 0, \quad \pi_{0,1} : v = 0, \quad \pi_{2,1} : 2w - v = 0, \quad \pi_{1,1} : w - v = 0,$$

respectively. We denote with $\overline{\Gamma}_{a,b}$ the projective closure, in the projective plane $\overline{\pi}_{a,b} \subset \mathbb{P}^3{}_{[u,v,w,\zeta]}$, of the Hilbert curve $\Gamma_{a,b}$ occurring above. As pointed out in Example 5.1, the projective Hilbert surface $\overline{\mathcal{S}}_{2,2} \subset \mathbb{P}^3{}_{[u,v,w,\zeta]}$ of equation $f(u,v,w)^{\text{hom}} = 0$, with $h = k = 2$, has the seven singular points

$$\left[\frac{3}{2}, 0, 1, \pm 1\right], \quad \left[\frac{3}{2}, 1, 0, \pm 1\right], \quad [1,1,0,0], \quad [5,2,1,0], \quad [2,-1,1,0].$$

We observe that $\pi_{1,0} : w = 0$ contains the points $[1,1,0,0]$ and $\left[\frac{3}{2}, 1, 0, \pm 1\right]$. Therefore the projective Hilbert curve $\overline{\Gamma}_{1,0} \subset \mathbb{P}^2{}_{[u,v,\zeta]}$ has three double points. Similarly, $\pi_{0,1} : v = 0$ contains the points $\left[\frac{3}{2}, 0, 1, \pm 1\right]$, $\pi_{2,1} : 2w - v = 0$ contains the point $[5,2,1,0]$, while the plane $\pi_{1,1} : w - v = 0$ does not contain singular points of $\overline{\mathcal{S}}_{2,2}$. Thus, $\overline{\Gamma}_{0,1} \subset \mathbb{P}^2{}_{[u,w,\zeta]}$ has (at least) two double points, $\overline{\Gamma}_{2,1} \subset \mathbb{P}^2{}_{[u,w,\zeta]}$ has (at least) one double point, and $\overline{\Gamma}_{1,1} \subset \mathbb{P}^2{}_{[u,w,\zeta]}$ is a possibly non-singular plane quartic.

By putting $w = 0$ in the equation of $\overline{\mathcal{S}}_{2,2}$, we find for $\Gamma_{1,0} \subset \mathbb{C}^2_{(u,v)}$ the equation

$$p_1\Big(\frac{1}{2} + u, v\Big) = u^4 + 3u^2 + \frac{51}{8}v^2 + \frac{15}{2}v^4 - 9uv - 21uv^3 - 8u^3v + \frac{41}{2}u^2v^2 + \frac{3}{16} = 0.$$

Furthermore, we check that $\overline{\Gamma}_{1,0}$ has indeed the three double points $\left[\frac{3}{2}, 1, \pm 1\right]$ and $[1,1,0]$, whence $\overline{\Gamma}_{1,0}$ has genus 0. Similarly, for $v = 0$ we find for $\Gamma_{0,1} \subset \mathbb{C}^2_{(u,w)}$ the equation

$$p_2\Big(\frac{1}{2} + u, w\Big) = 3u^2 + \frac{45}{4}w^2 + 24w^4 + u^4 - \frac{49}{4}uw - 53uw^3 - 11u^3w + 39u^2w^2 + \frac{3}{16} = 0,$$

and $\overline{\Gamma}_{1,0}$ has in fact the two double points $\left[-\frac{3}{2}, -1, 1\right]$, $[\frac{3}{2}, 1, 1]$, so that it is a curve of genus 1. Let $\Gamma'_{2,1}$ be the projection of the curve $\Gamma_{2,1}$ onto the plane $\langle u, w \rangle$. By putting $v = 2w$ in the equation of $\overline{\mathcal{S}}_{2,2}$, we get for $\Gamma'_{2,1}$ the equation

$$p_3\Big(\frac{1}{2} + u, w\Big) = 3u^2 + 74w^2 + 1125w^4 + u^4 - \frac{121}{4}uw - 875uw^3 - 27u^3w + 240u^2w^2 + \frac{3}{16} = 0,$$

and $\Gamma'_{2,1}$ has the only double point $(5,0)$, whence $\overline{\Gamma}_{2,1}$ has genus 2. This provides a further example of quartic Hilbert curve of genus 2 (compare with

Example 5.5). The same procedure finally yields for the projection of the curve $\Gamma_{1,1}$ onto the plane $\langle u, v\rangle$ the equation

$$p_4\Big(\frac{1}{2}+u,v\Big) = 3u^2+\frac{145}{4}v^2+272v^4+u^4-\frac{85}{4}uv-304uv^3-19u^3v+119u^2v^2+\frac{3}{16}.$$

and the usual numerical check shows that $\Gamma_{1,1}$ is a non-singular quartic curve of genus 3.

6. IMAGE OF THE HILBERT SURFACE IN $\mathbb{P}^6$

Let (X, L_1, L_2) be an n-dimensional bipolarized variety. In this section we work in the projective space. For simplicity of notation, we then use the symbol $\mathcal{S}$ to denote the Hilbert surface of (X, L_1, L_2) in $\mathbb{P}^3_{[x,y,z,\zeta]}$. Keeping for the rest the notation as in the previous sections, we make the change of homogeneous coordinates $[x, y, z, \zeta] \mapsto \big[x - \frac{\zeta}{2}, y, z, \zeta\big] =: [u, v, w, \zeta]$, so that the central point becomes $C = [0, 0, 0, 1]$, and we consider the map $\Phi : \mathbb{P}^3{}_{[u,v,w,\zeta]} \to \mathbb{P}^6_{[T_0,T_1,\dots,T_6]}$ defined by

$$[u, v, w, \zeta] \mapsto [u^2, uv, v^2, uw, vw, w^2, \zeta^2]. \tag{26}$$

Proposition 6.1. *Let (X, L_1, L_2) be an n-dimensional bipolarized variety, $n \geq 3$. Consider the map $\Phi : \mathbb{P}^3 \to \mathcal{Q} = \mathbb{P}^3/\langle\overline{s}\rangle \subset \mathbb{P}^6$ defined as in* (26). *Then Φ is a two-to-one immersion outside the central point C of the Serre involution s and the plane $\pi_\infty : \zeta = 0$.*

Proof. Express the morphism Φ locally around C in affine coordinates as $(u, v, w) \mapsto (u^2, uv, v^2, uw, vw, w^2)$. Then the Jacobian matrix

$$\begin{pmatrix} 2u & v & 0 & w & 0 & 0 \\ 0 & u & 2v & 0 & w & 0 \\ 0 & 0 & 0 & u & v & 2w \end{pmatrix}$$

has rank 3 except at C. Similarly, fix a point on the plane at infinity, $\pi_\infty : \zeta = 0$, *e.g.* $[0, 0, 1, 0]$, and take (u, v, ζ) as local coordinates around it. Then Φ is locally given by $(u, v, \zeta) \mapsto (u^2, uv, v^2, u, v, \zeta^2)$. Therefore the Jacobian matrix

$$\begin{pmatrix} 2u & v & 0 & 1 & 0 & 0 \\ 0 & u & 2v & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\zeta \end{pmatrix}$$

has rank 3 except where $\zeta = 0$. These local computations prove the result. □

Let $\Sigma := \Phi(\mathcal{S})$. Note that $\langle\overline{s}\rangle$ acts on $\mathcal{S}$, and $\Sigma = \mathcal{S}/\langle\overline{s}\rangle$. The morphism $\varphi := \Phi_{|\mathcal{S}} : \mathcal{S} \to \Sigma$ is two-to-one by Proposition 6.1. Moreover, Σ has a double point at (the image of) the central point C for n odd, and it is smooth for n

even. To have a global picture summarizing the situation as in [3, Formula (17)], consider the Veronese embedding $\mathbb{P}^3{}_{[u,v,w,\zeta]} \hookrightarrow \mathbb{P}^9{}_{[x_0,x_1,\dots,x_9]}$ defined by

$$[u,v,w,\zeta] \mapsto [u^2, uv, uw, u\zeta, v^2, vw, v\zeta, w^2, w\zeta, \zeta^2].$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{S} \subset \mathbb{P}^3 & \hookrightarrow & \mathcal{V} \subset \mathbb{P}^9 \\ \varphi \downarrow & \overset{\Phi}{\searrow} & \downarrow \rho \quad \vdots \\ \Sigma & \hookrightarrow & \mathcal{Q} \subset \mathbb{P}^6, \end{array} \tag{27}$$

where $\rho : \mathcal{V} \to \mathcal{Q}$ is the two-to-one morphism obtained by projection of the Veronese 3-fold $\mathcal{V}$ from the plane $x_0 = x_1 = x_2 = x_4 = x_5 = x_7 = x_9 = 0$ onto the quartic cone $\mathcal{Q} := \mathbb{P}^3/\langle \overline{s} \rangle \subset \mathbb{P}^6$ defined in $\mathbb{P}^6$ by the equations $T_0T_2 - T_1^2 = T_0T_5 - T_3^2 = T_2T_5 - T_4^2 = 0$. Precisely, $\mathcal{Q}$ is the cone over the Veronese surface in $\mathbb{P}^5$ of equation $T_6 = 0$ defined by the condition

$$\operatorname{rk} \begin{pmatrix} T_0 & T_1 & T_3 \\ T_1 & T_2 & T_4 \\ T_3 & T_4 & T_5 \end{pmatrix} = 1.$$

The branch locus of ρ consists of the images of the central point C and the plane at infinity $\pi_\infty : \zeta = 0$, via Φ.

Proposition 6.2. *Let (X, L_1, L_2) be an n-dimensional bipolarized variety, $n \geq 3$. Assume that the projective Hilbert surface $\mathcal{S}$ is smooth. Then, for suitable hyperplane sections h of $\mathcal{S}$, the curve $h/\langle \overline{s} \rangle$ is a Castelnuovo curve in $\mathbb{P}^3$.*

Proof. Let $\widetilde{\Sigma}$ be the desingularization of Σ. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\psi} & \widetilde{\Sigma} \\ & \searrow & \downarrow \nu \\ & & \Sigma, \end{array}$$

where $\psi : \mathcal{S} \to \widetilde{\Sigma}$ is a two-to-one map ramified in C (if $C \in \mathcal{S}$) and along $\mathcal{S} \cap \pi_\infty$, with $\pi_\infty : \zeta = 0$ the plane at infinity, and $\nu : \widetilde{\Sigma} \to \Sigma$ is a generically one-to-one map. Let $\psi' := \nu \circ \psi : \mathcal{S} \to \Sigma$.

Set $L := aL_1 + bL_2$ for positive integers a, b. Then (X, L) is a polarized 3-fold whose Hilbert curve, say h, lies in a plane of $|\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}_C|$, where $\mathcal{I}_C$ is the ideal sheaf of the point C in $\mathbb{P}^3$ (see Lemma 2.5).

Let $h' := \psi'(h)$, so that $\widetilde{h} := \nu^{-1}(h') = \psi(h)$. Note that $h' = h/\langle \overline{s} \rangle$. The restriction $\psi_{|h} : h \to \widetilde{h}$ is then a two-to-one map ramified at either $n+1$ points (the n points $h \cap \pi_\infty$ and the point C) or n points according to whether n is odd or even. Hence, the assertion follows by [3, Proposition 5.2]. $\square$

Remark 6.3. Look again at diagram (27). We claim that $\Sigma \hookrightarrow \mathcal{Q} \subset \mathbb{P}^6$ embeds with degree $2n$. To see this, note that a hyperplane section of Σ in $\mathbb{P}^6$ corresponds to a quadric surface in $\mathbb{P}^3$, whose equation only involves the terms appearing in formula (26) (*i.e.*, a quadric cone with vertex $[0,0,1,0]$). Since $\varphi : \mathcal{S} \to \Sigma$ is of degree two, one sees that $\deg(\Sigma)$, *i.e.*, the degree of the 0-cycle cut out on Σ by two general hyperplanes of $\mathbb{P}^6$, is then given by

$$\frac{1}{2}\,\mathcal{S}\cdot Q_1 \cdot Q_2 = \frac{1}{2}\,4\deg(\mathcal{S}) = 2n,$$

for two general elements Q_1, Q_2 belonging to $|\mathcal{O}_{\mathbb{P}^3}(2)|$. Since a quadric section of $\mathcal{S}$ is linearly equivalent to $2h$, we conclude that the general hyperplane section of Σ is numerically equivalent to twice the Castelnuovo curve h' as in the proof of Proposition 6.2.

7. SERRE INVARIANT SURFACES

This section is inspired by [3, Section 7]. Let $\mathbb{A}^3 = \mathbb{A}^3{}_{(x,y,z)}$, $\mathbb{P}^3 = \mathbb{P}^3{}_{[x,y,z,\zeta]}$, and let $s : \mathbb{A}^3 \to \mathbb{A}^3$, $\overline{s} : \mathbb{P}^3 \to \mathbb{P}^3$ be the Serre involutions defined in Section 1.

It is natural to consider a family of surfaces in a 3-dimensional space larger than that of Hilbert surfaces; namely, the family of surfaces that are invariant under the Serre involution. Let $\mathcal{S}$ be a possibly reducible and non-reduced surface in $\mathbb{P}^3$ (respectively $\mathbb{A}^3$) of given degree d. We say that $\mathcal{S}$ is a *Serre-invariant surface* if $\overline{s}(\mathcal{S}) = \mathcal{S}$ (respectively $s(\mathcal{S}) = \mathcal{S}$). The Serre involution acts on $\mathcal{S}$, so that we can consider the quotient $\mathcal{S}/\langle\, \overline{s}\,\rangle$ and identify it with its image on the cone over the Veronese surface, $\mathcal{Q} = \mathbb{P}^3/\langle\, \overline{s}\,\rangle \subset \mathbb{P}^6$.

Clearly, a Hilbert surface of a d-dimensional bipolarized variety is a Serre-invariant surface of degree d.

A noteworthy property is that Serre-invariant surfaces are in fact zero sets of polynomials with the same Serre-invariance as the Hilbert polynomial.

Claim 7.1. *Let $\mathcal{S}$ be a Serre-invariant surface on $\mathbb{A}^3$, defined by a polynomial $F(x,y,z)$ of degree d. Then*

$$F(x,y,z) = (-1)^d F(1-x,-y,-z).$$

Proof. Since $s(\mathcal{S}) = \mathcal{S}$, and $\mathcal{S}$ is defined by a single polynomial up to multiplication by a constant, we know that $F(s(x,y,z)) = \lambda F(x,y,z)$ for some constant $\lambda \neq 0$. Thus,

$$F(x,y,z) = F(s^2(x,y,z)) = F(s(s(x,y,z))) = \lambda F(s(x,y,z)) = \lambda^2 F(x,y,z).$$

But $s^2(x,y,z) = (x,y,z)$, so that $\lambda^2 = 1$, *i.e.*, $\lambda = \pm 1$. To determine λ it is enough to compare a non-zero monomial of maximal degree d, say $cx^a y^b z^{d-a-b}$, of $F(x,y,z)$ with its corresponding monomial in $F(s(x,y,z))$. That is,

$$c(-x)^a(-y)^b(-z)^{d-a-b} = (-1)^d c x^a y^b z^{d-a-b},$$

so that $\lambda = (-1)^d$. $\square$

Remark 7.2. With the notation as above, break up $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_m$, where $\mathcal{S}_\mu$ is the union of all components of multiplicity $\mu = 1, 2, \ldots, m$. Then $s(\mathcal{S}_\mu) = \mathcal{S}_\mu$, and so $\mathcal{S}_\mu$ and $\left(\mathcal{S}_\mu\right)_{\mathrm{red}}$ are also Serre-invariant surfaces. We thus conclude that if Z is an irreducible and reduced component of $\mathcal{S}$ that contains the central point $C = \left(\frac{1}{2}, 0, 0\right)$, and if $\deg(Z)$ is even, then Z is singular at $\left(\frac{1}{2}, 0, 0\right)$ (compare with Proposition 1.1).

Let us point out some consequences of Claim (7.1) (compare with (3) and Proposition 1.1(2)).

1. If d is odd, then

$$\left(\left(\frac{\partial}{\partial x}\right)^r \left(\frac{\partial}{\partial y}\right)^s \left(\frac{\partial}{\partial z}\right)^t F\right)\left(\frac{1}{2}, 0, 0\right) = 0$$

   for all non-negative integers r, s, t with $r+s+t$ even.
2. If d is even, then the above equality holds for all non-negative integers r, s, t with $r+s+t$ odd.
3. The central point C of the Serre involution belongs to a smooth Serre-invariant surface of degree d if and only if d is odd.

Let $\mathcal{V}_d$ be the closure in $|\mathcal{O}_{\mathbb{P}^3}(d)|$ of the family of Serre invariant surfaces of degree d, and identify the group $\mathcal{A}$ of affine transformations of $\mathbb{A}^3_{(x,y,z)}$ with the subgroup of $\mathrm{PGL}(4;\mathbb{C})$ fixing the plane at infinity $\pi_\infty : \zeta = 0$. Let G be the subgroup of $\mathcal{A}$ defined by

$$G := \{g \in \mathcal{A} \mid g \circ \overline{s} = \overline{s} \circ g\}.$$

We then have the following result.

Theorem 7.3. *Let G and $\mathcal{V}_d$ be as above. Then*

1. $\dim(G) = 9$*;*
2. *For d even,*

$$\dim(\mathcal{V}_d) = \frac{1}{3}\, a(a+1)(2a+1) + \frac{3}{2}\, a(a+1) + a, \quad \textit{where} \quad a = \frac{d}{2};$$

   while, for d odd,

$$\dim(\mathcal{V}_d) = \frac{1}{3}a(a+1)(2a+1) + \frac{5}{2}a(a+1) + 3a + 2, \quad \textit{where} \quad a = \frac{d-1}{2}.$$

Proof. We make the usual change of homogeneous coordinates $[x,y,z,\zeta] \mapsto \left[x - \frac{\zeta}{2}, y, z, \zeta\right] =: [u,v,w,\zeta]$, so that the central point becomes $C = [0,0,0,1]$.

With respect to the new coordinates, the Serre involution is represented by the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}.$$

On the other hand, any affine transformation $g \in G$ is represented by a matrix of $\mathrm{PGL}(4;\mathbb{C})$, of the form

$$M = \begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} N & \mathbf{d} \\ \mathbf{0} & 1 \end{pmatrix},$$

where, with clear meaning of the symbols, N is the non-singular matrix

$$N = \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} d \\ d' \\ d'' \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} -N & -\mathbf{d} \\ \mathbf{0} & 1 \end{pmatrix} = AM = MA = \begin{pmatrix} -N & \mathbf{d} \\ \mathbf{0} & 1 \end{pmatrix}.$$

This gives the three linearly independent conditions $d = d' = d'' = 0$. We then conclude that $\dim(G) = 9$.

Let $\Phi : \mathbb{P}^3 \to \mathcal{Q}$ be the double cover defined by the Serre involution, where $\mathcal{Q} \subset \mathbb{P}^6$ is a cone over the Veronese surface $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)\big)$. Let v be its vertex; so $v = \Phi(C)$. Recall that Φ is ramified at the central point C and along the plane at infinity π_∞.

Let $\beta : P \to \mathbb{P}^3$ be the blowing-up at C, and let E_1 be the exceptional divisor. Note that P is a $\mathbb{P}^1$ bundle over $\mathbb{P}^2$; in fact, $P = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$. Denoting by $\pi : P \to \mathbb{P}^2$ the bundle projection and letting $M_1 = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, we have that the classes of E_1 and M_1 generate the Picard group of P, and $\beta^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_P(E_1+M_1)$. Now let $\nu : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ be the minimal desingularization of the Veronese cone, and let $\alpha : P \to \widetilde{\mathcal{Q}}$ be the double cover induced by Φ, which gives rise to the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha\ } & \widetilde{\mathcal{Q}} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\nu} \\ \mathbb{P}^3 & \xrightarrow{\ \Phi\ } & \mathcal{Q}. \end{array}$$

Note that $\widetilde{\mathcal{Q}} = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)\big)$. Call $\pi' : \widetilde{\mathcal{Q}} \to \mathbb{P}^2$ the bundle projection, let $E_2 = \nu^{-1}(v)$ and set $M_2 = \pi'^*\mathcal{O}_{\mathbb{P}^2}(1)$. Note that E_2 is the section of π'

corresponding to the surjection onto the trivial summand. In particular, $M_2^3 = 0$ and $E_2 \cdot M_2^2 = 1$. Clearly, the classes of E_2 and M_2 generate the Picard group of $\widetilde{\mathcal{Q}}$; moreover, $\alpha^* E_2 = 2E_1$, $\alpha^* M_2 = M_1$, and $\nu^*\mathcal{O}_{\mathcal{Q}}(1) = \mathcal{O}_{\widetilde{\mathcal{Q}}}(E_2 + 2M_2)$, where $\mathcal{O}_{\mathcal{Q}}(1) = \mathcal{O}_{\mathbb{P}^6}(1)_{\mathcal{Q}}$. Now let $\mathcal{S} \subset \mathbb{P}^3$ be a Serre invariant smooth surface of degree d.

First suppose that d is even. Then $\mathcal{S}$ does not contain C; moreover, $\widetilde{\mathcal{S}} := \beta^{-1}(\mathcal{S}) \in |d(E_1 + M_1)|$. On the other hand, $\widetilde{\mathcal{S}} = \alpha^*\mathcal{S}'$, where $\mathcal{S}' \subset \widetilde{\mathcal{Q}}$ is a surface not intersecting E_2 (because $\Phi(\mathcal{S}) = \nu(\mathcal{S}')$ does not contain the vertex v of $\mathcal{Q}$). In other words,

$$\mathcal{O}_{E_2}(\mathcal{S}') = \mathcal{O}_{E_2}. \tag{28}$$

By what we said before we can write (up to linear equivalence) $\mathcal{S}' = aE_2 + bM_2$ for some integers a and b. Recalling that $\mathcal{O}_{E_2}(E_2) = \mathcal{O}_{E_2}(-2)$, while $M_{2E_2} = \mathcal{O}_{E_2}(1)$, condition (28) gives $b = 2a$. In conclusion, $\mathcal{S}' \in |a(E_2 + 2M_2)|$. Next, let us relate a and d. Since $\mathcal{S}'_{E_2}$ is trivial, we have

$$d = \mathcal{S} \cdot \big(\mathcal{O}_{\mathbb{P}^3}(1)\big)^2 = \widetilde{\mathcal{S}} \cdot (E_1 + M_1)^2 = \widetilde{\mathcal{S}} \cdot M_1^2 = \alpha^*\mathcal{S}' \cdot (\alpha^* M_2)^2 = (\deg \alpha)\, \mathcal{S}' \cdot M_2.$$

Hence,

$$d = 2a(E_2 + 2M_2) \cdot M_2^2 = 2a,$$

i.e., $d = 2a$ also in this case. In other words,

$$\mathcal{S}' \in \Big|\frac{d}{2}(E_2 + 2M_2)\Big|.$$

Now we are ready to proceed with the computation as in [3, Section 7].

The surfaces on $\widetilde{\mathcal{Q}}$ which pull back to an $\widetilde{\mathcal{S}}$ on P constitute a family of dimension $h^0\big(\widetilde{\mathcal{Q}}, a(E_2 + 2M_2)\big) - 1$, where $a = \frac{d}{2}$. Thus, this is the dimension $\dim(\mathcal{V}_d)$ we are looking for, when d is even. Recall that $E_2 + 2M_2$ is the tautological line bundle of $\mathcal{E} := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\pi'_*\big(a(E_2 + 2M_2)\big) = S^a(\mathcal{E})$, the a-th symmetric power of $\mathcal{E}$. We have

$$S^a(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(4) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^2}(2a). \tag{29}$$

Therefore,

$$h^0\big(\mathbb{P}^2, S^a(\mathcal{E})\big) = 1 + 6 + 15 + \cdots + \binom{2a+2}{2} = 1 + \sum_{k=1}^{a} \binom{2k+2}{2}.$$

In conclusion, recalling that

$$\sum_{k=1}^{a} k^2 = \frac{1}{6} a(a+1)(2a+1), \tag{30}$$

we get

$$\begin{aligned}\dim(\mathcal{V}_d) &= h^0(\widetilde{\mathcal{Q}}, a(E_2+2M_2))-1 = h^0(\mathbb{P}^2, S^a(\mathcal{E}))-1 = \sum_{k=1}^{a}\binom{2k+2}{2} \\ &= \sum_{k=1}^{a}(2k^2+3k+1) = \frac{1}{3}\,a(a+1)(2a+1)+\frac{3}{2}\,a(a+1)+a,\end{aligned}$$

where $a=\frac{d}{2}$.

Now suppose that d is odd. In this case the surface $\mathcal{S}$ contains C as a simple point (being smooth). Hence, its proper transform via β, $\widetilde{\mathcal{S}}=\beta^*\mathcal{S}-E_1$, belongs to $|\beta^*\mathcal{O}_{\mathbb{P}^3}(d)-E_1| = |(d-1)(E_1+M_1)+M_1|$. Note that

$$\begin{aligned} d &= \mathcal{S}\cdot\big(\mathcal{O}_{\mathbb{P}^3}(1)\big)^2 = (\widetilde{\mathcal{S}}+E_1)\cdot\big(\beta^*\big(\mathcal{O}_{\mathbb{P}^3}(1)\big)\big)^2 \\ &= \widetilde{\mathcal{S}}\cdot(E_1+M_1)^2 = \widetilde{\mathcal{S}}\cdot\big((E_1+M_1)\cdot E_1+M_1\cdot E_1+M_1^2\big) \ = 1+\widetilde{\mathcal{S}}\cdot M_1^2\end{aligned}$$

(here we use the relation $\mathcal{O}_{E_1}=(\beta^*\mathcal{S})_{E_1}=\mathcal{O}_{E_1}(\widetilde{\mathcal{S}}+E_1)$, so that $\widetilde{\mathcal{S}}_{E_1}=\mathcal{O}_{E_1}(1)$). As before, $\widetilde{\mathcal{S}}=\alpha^*\mathcal{S}'$, where $\mathcal{S}'\subset\widetilde{\mathcal{Q}}$ is a surface. Note however that $\widetilde{\mathcal{S}}$ intersects E_2, since now $\Phi(\mathcal{S})=\nu(\mathcal{S}')$ contains v. Up to linear equivalence we can write again $\mathcal{S}'=aE_2+bM_2$. Thus, the above relation gives

$$\begin{aligned} d-1 &= \mathcal{S}\cdot M_1^2 = \alpha^*(\mathcal{S}')\cdot\big(\alpha^*(M_2)\big)^2 \\ &= (\deg\alpha)\mathcal{S}'\cdot M_2^2 = 2(aE_2+bM_2)\cdot M_2^2 \ = 2a.\end{aligned}$$

On the other hand, we have

$$\widetilde{\mathcal{S}}\cdot E_1\cdot M_1 = \widetilde{\mathcal{S}}_{E_1}\cdot M_{1E_1} = \big(\mathcal{O}_{E_1}(1)\big)^2 = 1.$$

Therefore,

$$\begin{aligned} 1 &= \widetilde{\mathcal{S}}\cdot E_1\cdot M_1 = \alpha^*\mathcal{S}'\cdot\frac{1}{2}\alpha^*E_2\cdot\alpha^*M_2 \\ &= \frac{1}{2}(\deg\alpha)\mathcal{S}'\cdot E_2\cdot M_2 = (aE_2+bM_2)\cdot E_2\cdot M_2 \\ &= \mathcal{O}_{E_2}(-2a+b)\cdot\mathcal{O}_{E_2}(1) = -2a+b.\end{aligned}$$

In conclusion, $a=\frac{d-1}{2}$ and $b=2a+1=d$. In other words,

$$\mathcal{S}'\in\Big|\frac{d-1}{2}(E_2+2M_2)+M_2\Big|.$$

Now, arguing in the same way as in the d even case, we find for d odd

$$\dim(\mathcal{V}_d) = h^0\big(\widetilde{\mathcal{Q}}, a(E_2+2M_2)+M_2\big)-1,$$

where $a=\frac{d-1}{2}$. Noting that $\pi'_*\big(a(E_2+2M_2)+M_2\big) = S^a(\mathcal{E})\otimes\mathcal{O}_{\mathbb{P}^2}(1)$ and taking into account (29), we get

$$S^a(\mathcal{E})\otimes\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(5)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^2}(2a+1).$$

Therefore,

$$h^0\big(\mathbb{P}^2, S^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^2}(1)\big) = 3 + 10 + \cdots + \binom{2a+3}{2} = 3 + \sum_{k=1}^{a} \binom{2k+3}{2}.$$

Eventually, recalling (30) again, we obtain

$$\dim(\mathcal{V}_d) = \frac{1}{3}a(a+1)(2a+1) + \frac{5}{2}a(a+1) + 3a + 2,$$

where $a = \frac{d-1}{2}$. $\square$

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Received 19 April 2015

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