Dedicated to Professor Lucian Bădescu on the occasion of his 70^{th} birthday

ON THE TOPOLOGY OF SOME QUASI-PROJECTIVE SURFACES

ALEXANDRU DIMCA

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Let X be a surface with isolated singularities in the complex projective space $\mathbb{P}^3$ and let Y denote the smooth part of X. In this note we discuss, mostly on specific examples, some aspects of the topology of such quasi-projective surfaces Y: the fundamental groups and the associated Galois coverings, the second homotopy groups and the mixed Hodge structure on the first cohomology group.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

Surfaces theory is a classical subject, with a very rich history and a number of excellent textbooks, as for instance [3, 7]. It is quite surpring that new open questions arise even in such a classical subject, and one of the purposes of this note is to state such an open question in Example 2.2 related to the cubic surfaces in $\mathbb{P}^3$. These surfaces have been classified already by Cayley, see for a modern presentation [8]. Another open question in relation to the Zariski sextic with 6 cusps appears in Example 2.3.

Let X be a surface with isolated singularities in the complex projective space $\mathbb{P}^3$. Then it is well known that X is simply-connected, see for instance [11]. Let Y denote the smooth part of X. So Y is obtained from X be removing a finite number of points. However, unlike the case when X is smooth, this operation alters sometimes the fundamental groups and we may get quasi-projective surfaces Y with $\pi_1(Y) \neq 0$. We omit the base points in this note, since our spaces Y are path-connected, hence the isomorphism class of $\pi_1(Y, y)$ is independent of $y \in Y$. For related results on fundamental groups of surfaces we refer to [2, 16] and [17].

The first result describes the first integral homology group $H_1(Y)$ of the surface Y, which is exactly the abelianization of the fundamental group $\pi_1(Y)$. In the sequel Tors denotes the torsion part of a finitely generated abelian group.

THEOREM 1.1. *Let X be a surface with isolated singularities in $\mathbb{P}^3$, let Z be the singular set of X and set $Y = X \setminus Z$. Then one has the following.*

(i) $H_1(Y) = H^3(X)$. In particular, if X is a $\mathbb{Q}$-manifold, e.g. when X has only simple singularities of type A_n, D_n, E_6, E_7 and E_8, then $H_1(Y) = \operatorname{Tors} H_2(X)$ is a finite group.

(ii) $H_3(Y)$ is a free abelian group of rank given by $|Z| - 1$.

Note that the surface Y is never affine, since a regular function ϕ defined on Y extends to X, as X is normal, and hence ϕ has to be constant.

The next result shows that there is a geometrically induced epimorphism

$$\Gamma_g \to \pi_1(Y),$$

where Γ_g is the fundamental group of a smooth plane curve of genus

$$g = \frac{(d-1)(d-2)}{2}.$$

PROPOSITION 1.2. *(i) For a generic plane H in $\mathbb{P}^3$, the intersection $C = X \cap H$ is a smooth curve contained in Y and the inclusion $i : C \to Y$ induces an epimorphism*

$$i_\sharp : \pi_1(C) \to \pi_1(Y).$$

(ii) For any plane H in $\mathbb{P}^3$ such that the intersection $C = X \cap H$ is a (possibly singular) curve contained in Y, the inclusion $i : C \to Y$ induces an epimorphism

$$i_\sharp : \pi_1(C) \to \pi_1(Y).$$

In particular, if Y contains a rational cuspidal plane curve C, then $\pi_1(Y) = 0$.

COROLLARY 1.3. *Let X be a surface with isolated singularities in $\mathbb{P}^3$. If X is a surface of degree 3, then the fundamental group $\pi_1(Y)$ of its smooth part Y is abelian. More precisely, denote by $X(4A_1)$ the cubic surface in $\mathbb{P}^3$ having as singularities 4 nodes A_1, and similarly for $X(3A_2)$, $X(A_1A_5)$ and $X(2A_1A_3)$. Then the corresponding smooth quasi-projective surfaces $Y(4A_1)$, $Y(3A_2)$, $Y(A_1A_5)$ and $Y(2A_1A_3)$ have the following fundamental groups.*

$$\pi_1(Y(4A_1)) = \pi_1(Y(A_1A_5)) = \pi_1(Y(2A_1A_3)) = \mathbb{Z}/2\mathbb{Z} \text{ and } \pi_1(Y(3A_2)) = \mathbb{Z}/3\mathbb{Z}.$$

For a description of the associated Galois coverings and the second homotopy groups of these surfaces see Example 2.2.

Remark 1.4. If X is a cubic surface with simple singularities in $\mathbb{P}^3$, then X is an exemple of a log del Pezzo surface, see [1] for the corresponding definition. It is known that the fundamental group of the smooth part Y of such a log del Pezzo surface is always finite, see [20, 21, 32]. For an extension of this result to higher dimensional log Fano varieties see [19, 33].

We have also the following.

PROPOSITION 1.5. *Let X be a surface with isolated singularities in $\mathbb{P}^3$, let Z be the singular set of X and set $Y = X \setminus Z$. For each singular point $z \in Z$, let L_z denote the link of the singularity (X, z). Then there is a morphism*

$$\amalg_{z\in Z}\pi_1(L_z) \to \pi_1(Y),$$

where $\amalg_{z\in Z}\pi_1(L_z)$ denotes the free product of the family of groups $(\pi_1(L_z))_{z\in Z}$, whose image is not contained in any normal subgroup of $\pi_1(Y)$.

One can ask about the mixed Hodge structures on the surfaces Y. Here is the answer.

THEOREM 1.6. *Let X be surface with isolated singularities in $\mathbb{P}^3$, let Z be the singular set of X and set $Y = X \setminus Z$. Then $H^3_c(Y) = H^3(X)$ is a pure Hodge structure of weight 3 and by duality, $H^1(Y)$ is a pure Hodge structure of weight 1.*

The cohomology groups $H^2_c(Y) = H^2(X)$ have a more complicated mixed Hodge structure, in general with several possible weights, for more on this subject see [12] and [14].

The Hodge theoretic result in Theorem 1.6 has the following pure topological consequence. Let $C : g(x, y, z) = 0$ be a reduced curve in $\mathbb{P}^2$ of degree d. Consider the surfaces $X_C : g(x, y, z) + t^d = 0$, whose singularities are the degree d suspension of the singularities of the curve C. Let Y_C be the smooth part of X_C as above and denote by F_C the Milnor fiber of g, namely the affine smooth surface

$$F_C : \quad g(x, y, z) = 1$$

in $\mathbb{C}^3$. Then clearly $F_C \subset Y_C$ is a Zariski open subset and hence the inclusion $j : F_C \to Y_C$ induces an epimorphism $\pi_1(F_C) \to \pi_1(Y_C)$. By duality, we get a monomorphism $j^* : H^1(Y_C, \mathbb{Q}) \to H^1(F_C, \mathbb{Q})$.

PROPOSITION 1.7. *The image of the monomorphism $j^* : H^1(Y_C, \mathbb{Q}) \to H^1(F_C, \mathbb{Q})$ is exactly $H^1(F_C, \mathbb{Q})_{\neq 1}$, the non-fixed part of $H^1(F_C, \mathbb{Q})$ under the monodromy action.*

This result allows us to construct many examples of surfaces X_C such that $H^1(Y_C, \mathbb{Q})$ (and presumably $\pi_1(Y)$) is quite large, *e.g.* using as C various line arrangements in $\mathbb{P}^2$, see [25, 30, 31] for various monodromy computations in this case. To increase these groups, one may also use non-linear arrangements as well, as described for instance in Example 5.14 in [13]. One example using Zariski sextic curve with 6 cusps is given in Example 2.3 below.

Some open questions appear in Example 2.2 and Example 2.3. A major open question is to develop a general strategy for the computation of the fundamental groups for this class of surfaces.

This note gives a number of (very limited) answers to questions that Ciro Ciliberto asked me some time ago. I would like to thank him for asking the questions and for explaining to me the role of del Pezzo surfaces in Example 2.2 and Remark 2.5. Many thanks also to De-Qi Zhang for his comments on the fundamental groups of log del Pezzo surfaces and open $K3$ surfaces, incorporated essentially in Remarks 1.4 and 2.6 below.

2. THE PROOFS AND ADDITIONAL EXAMPLES

We consider first Theorem 1.1. We apply Lefschetz duality theorem, see for instance [28], p. 297 to the compact relative 4-manifold (X, Z), where Z is the finite set of singular points of the surface X. Since any pair of algebraic sets is triangulable, it follows that the pair (X, Z) is taut, see [28], p. 291, and hence we have an isomorphism $H_k(Y) = H^{4-k}(X, Z)$. To prove the first claim (i), we take $k = 1$ and the long exact sequence of the pair (X, Z) yields an isomorphism $H^3(X, Z) = H^3(X)$. If X is a $\mathbb{Q}$-manifold, it follows that $b_3(X) = b_1(X) = 0$ and hence $H^3(X)$ is a finite group. It remains to use the standard fact that $\operatorname{Tors} H_2(X) = \operatorname{Tors} H^3(X)$, see [28], p. 244. For the classification of simple singularities A, D, E and their properties we refer to [10, 11]. To prove the claim (ii), we take $k = 3$ in the above isomorphism and get $H_3(Y) = H^1(X, Z)$. Then the long exact sequence of the pair (X, Z) yields

$$b_3(Y) = |Z| - 1.$$

The proof of Proposition 1.2 follows from the Zariski theorem of Lefschetz type stated for instance in [11], p. 25 and the fact that X admits the obvious Whitney regular stratification given by Y and the finite set Z, see for instance in [11], p. 5. For the part (ii), one has to use the careful description of "good" hyperplanes in this case given in loc.cit. Moreover, an irreducible projective curve is simply-connected if and only if it is a rational cuspidal curve, *i.e.* C is rational and any singular point of C is unibranch.

To prove Corollary 1.3, for the first claim we just apply Proposition 1.2 and the fact that $\Gamma_1 = \mathbb{Z}^2$ to get that $\pi_1(Y)$ is abelian.

For the computation of the fundamental groups of the surfaces, $Y(4A_1)$, $Y(3A_2)$, $Y(A_1A_5)$ and $Y(2A_1A_3)$, we use the corresponding results for the second homology groups described in [11], p. 165.

The proof of Proposition 1.5 follows by applying the van Kampen theorem to the open covering Y, Z' of X, where Z' is obtained as follows. Take a mimal tree T formed of simple non-intersecting arcs connecting the points in Z. Add for each $z \in Z$ a small contractible neighborhood B_z of z in X such that $B_z^* = B_z \setminus \{z\}$ is homotopically equivalent to the link L_z. Then Z' is a tubular open neighborhood of $T \cup \cup_{z \in Z} B_z$. It follows that $Y \cap Z'$ has the homotopy type of the join of the links L_z, and hence

$$\pi_1(Y \cap Z') = \Pi_{z \in Z} \pi_1(L_z).$$

On the other hand, Z' is contractible and X is simply-connected, so van Kampen theorem implies that the inclusion $Y \cap Z' \to Y$ induces a morphism $\pi_1(Y \cap Z') \to \pi_1(Y)$ whose image is not contained in any normal subgroup of $\pi_1(Y)$.

We consider now Theorem 1.6. Note that the first part of this proof gives an alternative proof for the claim (i) in Theorem 1.1. The exact sequence with compact supports for the pair (X, Z) yields the isomorphism $H^3_c(Y) = H^3(X)$ of MHS (short for mixed Hodge structures). The result follows using the fact that $H^3(X)$ is a pure Hodge structure, see [29]. For duality between $H^3_c(Y)$ and $H^1(Y)$ we refer to [26].

Finally, to prove Proposition 1.7, we recall that we have a splitting

$$H^1(F_C, \mathbb{Q}) = H^1(F_C, \mathbb{Q})_1 \oplus H^1(F_C, \mathbb{Q})_{\neq 1},$$

where $H^1(F_C, \mathbb{Q})_1$ has pure weight 2 and $H^1(F_C, \mathbb{Q})_{\neq 1}$ has pure weight 1, see [14]. Moreover, it is shown in [14] and in [15] that

$$\dim H^1(Y_C, \mathbb{Q}) = \dim H^3(X_C, \mathbb{Q}) = \dim H^1(F_C, \mathbb{Q})_{\neq 1}.$$

This clearly completes the proof. □

Remark 2.1. It follows from Theorem 1.1 (ii) that the three surfaces $Y(4A_1)$, $Y(A_1A_5)$ and $Y(2A_1A_3)$ have distinct third Betti numbers, namely 3, 1 and 2, hence they are not homotopically equivalent to each other.

Example 2.2. It follows from Corollary 1.3, that each of the surfaces, $Y(4A_1)$, $Y(3A_2)$, $Y(A_1A_5)$ and $Y(2A_1A_3)$ has a finite unramified cover which is a simply-connected surface, *i.e.* the corresponding universal covering. In the case of the surface $Y(3A_2)$, we can chose the equation for $X(3A_2)$ to be

$$f = xyz - t^3 = 0$$

and hence the map $p : \mathbb{P}^2 \to X(3A_2)$ given by

$$p([u, v, w]) = [u^3, v^3, w^3, uvw]$$

is ramified precisely over the singular set Z, see also [11], p. 166. Hence, the universal cover of $Y(3A_2)$ is obtained from $\mathbb{P}^2$ by deleting 3 points. This also implies $\pi_2(Y(3A_2)) = \pi_2(\mathbb{P}^2) = \mathbb{Z}$.

For the other three universal covering surfaces, the construction involves Cremona transformations and del Pezzo surfaces of degree 6, *i.e.* surfaces obtained from $\mathbb{P}^2$ by blowing up 3 points. The easiest case to describe is for the surface $X(4A_1)$. Consider the classical Cremena rational map

$$c_1 : \mathbb{P}^2 \to \mathbb{P}^2, \quad [x, y, z] \mapsto [yz, xz, xy].$$

This map has $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$ and $p_3 = [0, 0, 1]$ as indeterminacy points, see Example 1.5.1 in [18]. Let S denote the del Pezzo surface obtained by blowing-up these 3 points and note that c_1 lifts to a regular map $c_1' : S \to S$, which is an involution, *i.e.* $c_1'^2 = Id$, and has 4 fixed points, namely the points in S corresponding to the points $[1, \pm 1, \pm 1]$ in $\mathbb{P}^2$. It follows that the quotient surface $S/ < 1, c_1' >$, which has 4 nodes, can be identified to $X(4A_1)$. This implies $\pi_2(Y(4A_1)) = \pi_2(S) = \mathbb{Z}^4$, by an easy application of Hurewicz Theorem.

To get the universal cover of $Y(2A_1A_3)$, we start with the *first degenerate standard quadratic transformation*

$$c_2 : \mathbb{P}^2 \to \mathbb{P}^2, \quad [x, y, z] \mapsto [y^2, xy, xz],$$

see [18], p. 15, which is also an involution, *i.e.* $c_2^2 = Id$, and has p_1 and p_3 as indeterminacy points. This map has 2 lines of fixed points, namely the lines $x \pm y = 0$ meeting at the point p_3. After blowing up p_1, p_3 and an infinitely near point p_2', we get as above the degree 2 covering over $X(2A_1A_3)$. The A_3 singularity occurs as the $\mathbb{Z}/(2)$-quotient of an A_1 singularity, which in turn comes from the contraction of a (-2)-curve obtained in the blowing-up process, see [18], p. 15.

Finally, to get the universal cover of $Y(A_1A_5)$, we start with the *second degenerate standard quadratic transformation*

$$c_3 : \mathbb{P}^2 \to \mathbb{P}^2, \quad [x, y, z] \mapsto [x^2, xy, y^2 - xz],$$

see [18], p. 16, which is also an involution, *i.e.* $c_3^2 = Id$, and has p_3 as its unique indeterminacy point. This map has a smooth conic of fixed points, namely $y^2 - 2xz = 0$, passing through the point p_3. After blowing up p_3 and 2 infinitely near points p_1' and p_2', we get as above the degree 2 covering over $X(A_1A_5)$. The A_5 singularity occurs as the $\mathbb{Z}/(2)$-quotient of an A_2 singularity, which in turn comes from the contraction of a chain of two (-2)-curves obtained in the blowing-up process, see [18], p. 16.

Example 2.3. Here is an example of a surface X such that the fundamental group $\pi_1(Y)$ is infinite. Let X be the surface given by

$$(x^2 + y^2)^3 + (y^3 + z^3)^2 + t^6 = 0,$$

the degree 6 cover of $\mathbb{P}^2$ ramified over the Zariski sextic with 6 cusps on a conic. It is well known that $b_3(X) = 2$, see for instance in [11], p. 210. It would be interesting to check whether $H^3(X) = \mathbb{Z}^2$ and also to determine the fundamental group $\pi_1(Y)$ in this classical example.

Remark 2.4. When $b_1(Y) = b_3(X) > 0$, then one can use Theorem 2 in [17] to compute the germs at the origin of the characteristic varieties $V_r^1(Y)_1$ in terms of the resonance varieties germs $R_r^1(\tilde{X})_0$, where $\tilde{X}$ is a resolution of singularities for X. Note that there is no dominant morphism from Y to non proper curve C, since such a curve is necessarily affine and we can repeat the map extension described after Theorem 1.1.

Remark 2.5. Some of the considerations in this paper can be applied to a complete intersection surface X with isolated singularities in $\mathbb{P}^n$ with $n \geq 4$. The case when X is the intersection of two quadrics in $\mathbb{P}^4$ is considered in Proposition 4.3 in [9], where the corresponding group $H_1(Y)$ is computed and found to be $\mathbb{Z}/2\mathbb{Z}$ in two cases, namely for $Y(4A_1)$ and for $Y(2A_1A_3)$, with an obvious notation. A generic hyperplane section of X in this case is again a curve of genus one, hence we get as in Corollary 1.3 that one has

$$\pi_1(Y) = H_1(Y) = \mathbb{Z}/2\mathbb{Z},$$

in these cases as well. It would be interesting to find an analog of Proposition 1.7 in the case of complete intersections of codimension > 1. The corresponding degree 2 universal covering spaces associated with $Y(4A_1)$ and $Y(2A_1A_3)$ have a simple geometrical description in terms of del Pezzo surfaces of degree 8. There are two types of such surfaces.

The first type is the product $S = \mathbb{P}^1 \times \mathbb{P}^1$, which has the involution $\iota : S \to S$ given by $\iota([x,y],[u,v]) = ([y,x],[v,u])$, which has 4 fixed points, namely $([1,\pm 1],[1,\pm 1])$. It can be shown that the quotient $S/ < 1, \iota >$ is nothing else but the surface $X(4A1)$. This also implies $\pi_2(Y(4A_1)) = \pi_2(S) = \mathbb{Z}^2$.

The universal covering space of $Y(2A_1A_3)$ can be described as follows. On the minimal resolution X' of $X = X(2A_1A_3)$ we have the three (-2)-curves $E_1+E_2+E_3$ (E_2 being the "central" one) in the resolution of the A_3 singularity and two (-2)-curves E_4, E_5 resolving the two A_1 points. The double cover of Y gives rise to a double cover of X' branched along the sum of some of the curves E_i, which must be divisible by 2 in $Pic(X')$. The branch divisor is $E_1 + E_3 + E_4 + E_5$, thus the double cover X'' possesses two (-1)-curves E_4', E_5' over E_4, E_5, and a cycle $E_1' + E_2' + E_3'$ over $E_1 + E_2 + E_3$, with E_1', E_3' being (-1)-curves and $E_2'^2 = -4$. By contracting all the (-1) curves we see that E_2' becomes a (-2)-curve and in fact we get in this way a Hirzebruch surface $\mathbb{F}_2$. If we contract this last (-2)-curve, we get the cone S in $\mathbb{P}^3$ over a smooth conic Q, say $Q : x^2 - yz = 0$. Then S is a singular (log) del Pezzo surface of degree 8 and it is the double cover of X ramified exactly over the singular points. The corresponding involution of S can be taken to be $[x,y,z,t] \mapsto [-x,y,z,-t]$, whose fixed points are precisely

(i) $[0,0,0,1]$, the vertex of the cone, which gives the A_3- singularity in the quotient X, and

(ii) the two points $[0,1,0,0]$ and $[0,0,1,0]$, which give the two A_1 singularities of X. This proof also implies $\pi_2(Y(2A_1A_3)) = \pi_2(Q) = \mathbb{Z}$.

Remark 2.6. Corollary 1.3 and Remark 2.5 describe the fundamental groups of the smooth part Y of some singular del Pezzo surfaces X, *i.e.* surfaces X with an ample anticanonical divisor $-K_X$. For the reader convenience, and for their beauty, we recall below some similar results on the fundamental groups in the case of singular $K3$-surfaces, *i.e.* surfaces with $K_X = 0$.

(i) Let first X be a $K3$-surface with 16 A_1-singularities, for instance the surface in $\mathbb{P}^3$ given by the equation

$$x^4 + y^4 + z^4 - (x^2 + y^2 + z^2)t^2 - x^2y^2 - x^2z^2 - y^2z^2 + 1 = 0.$$

Such a surface X can be obtained as the quotient of an Abelian surface A under the involution $a \mapsto -a$, see [23]. Since A is a quotient $\mathbb{C}^2/\Lambda$, with Λ a lattice of rank 4, we get a map $p : \mathbb{C}^2 \to X$, presenting X as the quotient of $\mathbb{C}^2$ under the non-commutative subgroup $\tilde{\Lambda}$ spanned by Λ (regarded as translations) and the involution $v \mapsto -v$ inside the group of affine transformations of the plane $\mathbb{C}^2$. It follows that $\mathbb{C}^2 \setminus p^{-1}(Z) \to Y$ is a universal covering for Y, where Z is the singular part of X, with deck transformation group $\pi_1(Y) = \tilde{\Lambda}$. We also get $\pi_2(Y) = \pi_2(\mathbb{C}^2 \setminus B) = 0$, since $B = p^{-1}(Z)$ is discrete, hence of real codimension 4.

(ii) Let now X be a $K3$-surface with 8 cusps A_2 and no other singularities. Some of these surface can embedded in $\mathbb{P}^3$ and the corresponding equations can be found in [5]. For such surfaces, one has either $\pi_1(Y) = (\mathbb{Z}/3\mathbb{Z})$, or $\pi_1(Y) = (\mathbb{Z}/3\mathbb{Z})^2$, see [22], Table 1, case $p = 3$, $c = 8$.

(iii) Let now X be a $K3$-surface with 9 cusps A_2 and no other singularities. None of these surface can embedded in $\mathbb{P}^3$, see [5]. For such surfaces, one has an extension

$$0 \to \Lambda \to \pi_1(Y) \to \mathbb{Z}/3\mathbb{Z} \to 0$$

where Λ a lattice of rank 4 as above, see [22], Table 1, case $p = 3$, $c = 8$. Moreover, one has as above $\pi_1(Y) = \tilde{\Lambda}$, where $\tilde{\Lambda}$ is the subgroup of the group of affine transformations of the plane $\mathbb{C}^2$ generated by Λ and an order 3 linear automorphism of $\mathbb{C}^2$ preserving Λ, see [24], Example 1 and [27]. An alternative approach is described in [4], and one may wonder if the fundamental groups $\pi_1(Y)$ are the same for these two distinct examples.

As above, we also get $\pi_2(Y) = 0$.

For other low degree cuspidal surfaces X in $\mathbb{P}^3$ admiting degree 3 unramified covering over their smooth part Y, in particular a discussion of irreducible families of such surfaces, see [6].

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Univ. Nice Sophia Antipolis,
CNRS, LJAD, UMR 7351,
06100 Nice, France
dimca@unice.fr