Dedicated to Professor Lucian Bădescu on the occasion of his 70^{th} birthday

ON REPRESENTATIONS BY EGYPTIAN FRACTIONS

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Communicated by Marian Aprodu

We bound the entries of the representations of a rational number as a sum of Egyptian fractions.

AMS 2010 Subject Classification: Primary 11D68, Secondary 14E30.

Key words: Egyptian fractions, log canonical models.

INTRODUCTION

Let (X, B) be a log canonical model with standard coefficients. That is X is a normal projective variety, $B = \sum_i b_i E_i$ is a $\mathbb{Q}$-Weil divisor with coefficients b_i belonging to the standard set $\{1 - \frac{1}{m}; m \in \mathbb{Z}_{\geq 1}\} \cup \{1\}$, $K_X + B$ is $\mathbb{Q}$-ample and (X, B) has at most log canonical singularities. The normalized volume of (X, B) is defined as $v = \sqrt[d]{(K_X + B)^d}$, where $d = \dim X$. By [1, 6, 3, 4], the volume v belongs to a DCC set, and there exists a positive integer r, bounded above only in terms of d and v, such that the linear system $|r(K_X + B)|$ is base point free (in particular, $r(K_X + B)$ is a Cartier divisor). The DCC property means that if t is a real number and $v > t$, then $v \geq t + \epsilon$, where ϵ depends only on d and t.

In this note, we estimate the gap and index bounds ϵ and r in the simplest possible case, when X is a projective space and the components of B are hyperplanes in general position, and t is rational. According to [6], the sharp bounds of the simplest case are possibly optimal in the general case.

To formulate our main result, we define a sequence of integers $(u_{p,q})_{p,q \geq 1}$ by the recursion $u_{1,q} = q$, $u_{p+1,q} = u_{p,q}(u_{p,q} + 1)$. Then $u_{p,q}$ is a polynomial in q with leading term $q^{2^{p-1}}$, and the following formulas hold:

$$\sum_{i=1}^{p} \frac{1}{1 + u_{i,q}} = \frac{1}{q} - \frac{1}{u_{p+1,q}}, \quad \prod_{i=1}^{p} (1 + u_{i,q}) = \frac{u_{p+1,q}}{q}.$$

The sequence $(1+u_{p,1})_{p\geq 1} = (2,3,7,43,...)$ is called the Sylvester sequence in the literature (see [5, 6]), and also the sequence $t_{p,q} = 1+u_{p,q}$ was considered in [7].

THEOREM 0.1. *Let $(\mathbb{P}^d, \sum_i b_i E_i)$ be a log structure such that the $(E_i)_i$ are general hyperplanes and the coefficients b_i belong to the standard set. Let $v = \deg(K+B)$. Let $t \geq 0$ be a rational number, with $qt \in \mathbb{Z}$ for some integer $q \geq 1$.*

a) If $v > t$, then $v \geq t + \frac{q(1-\{t\})}{u_{\lfloor t \rfloor + d+3,q}}$.

b) If $v = t$, then there exists an integer $1 \leq r \leq \frac{u_{\lfloor t \rfloor + d+2,q}}{q(1-\{t\})}$ such that the linear system $|r(K+B)|$ is base point free.

Theorem 0.1 is in fact combinatorial, about bounding the representations of a given rational number as a sum of Egyptian fractions. Any positive rational number x admits a representation as a sum of Egyptian fractions

$$x = \frac{1}{m_1} + \cdots + \frac{1}{m_k},$$

where m_i are positive integers and k is sufficiently large. If $x = \frac{p}{q}$ is the reduced form, we can write $x = \sum_{i=1}^{p} \frac{1}{q}$. From a representation with k terms we can construct another one with $k+1$ terms, using the formula

$$\frac{1}{m} = \frac{1}{m+1} + \frac{1}{m(m+1)}.$$

A canonical representation is provided by the *greedy algorithm*: if $x > 0$, let $m \geq 1$ be the smallest integer such that $mx \geq 1$, and replace x by $x - \frac{1}{m}$; if $x = 0$, stop. After each step, the numerator of the reduced fraction decreases strictly, and therefore the algorithm stops in finite time, and produces a representation of x as a sum of k Egyptian fractions ($k \leq \lfloor x \rfloor + q\{x\}$ if $qx \in \mathbb{Z}$).

If k is fixed, it is easy to see that x admits only finitely many representations with k Egyptian fractions. The following is an effective version of this fact, which is a restatement of Theorem 0.1.

THEOREM 0.2. *Let $1 \leq m_1 \leq \cdots \leq m_k$ be integers. Let $\delta \geq -1$ with $q\delta \in \mathbb{Z}$ for some integer $q \geq 1$.*

a) If $\sum_{i=1}^{k} \frac{1}{m_i} < k - \delta$, then $\sum_{i=1}^{k} \frac{1}{m_i} \leq k - \delta - \frac{q(1-\{\delta\})}{u_{\lfloor \delta \rfloor + 2,q}}$.

b) If $\sum_{i=1}^{k} \frac{1}{m_i} = k - \delta$, then $\operatorname{lcm}(m_1, \ldots, m_k) \leq \frac{u_{\lfloor \delta \rfloor + 1,q}}{q(1-\{\delta\})}$.

Moreover, equality holds in a) if and only if $\delta < 0$, or $\delta = \frac{r}{q} \in [0,1)$, $(m_i)_i = (1, \ldots, 1, \frac{1+q}{r})$, or $1 \leq \delta = s - \frac{1}{q}$, $(m_i)_i = (1, \ldots, 1, 1+u_{1,q}, \ldots, 1+u_{s,q})$. Equality holds in b) if and only if $\delta = s - \frac{1}{q}$, $(m_i)_i = (1, \ldots, 1, 1+u_{1,q}, \ldots, 1+u_{s,q}, u_{s+1,q})$, or $\delta = 2 - \frac{r}{q}$ and $(m_i)_i = (1, \ldots, 1, \frac{1+q}{r}, \frac{q(1+q)}{r})$.

The case $k-\delta=1$ is known (Kellogg [5], Curtiss [2], Soundararajan [8]), with b) replaced by the same bound for m_k instead of the least common multiple. We use the method of Soundararajan [8].

Note that in b), the positive integers $m_1,\dots,m_k$ are bounded above by $\frac{u_{\lfloor\delta\rfloor+1,q}}{q(1-\{\delta\})}$, a constant depending only on δ. Therefore $k-\delta$ admits at most finitely many representations as a sum of k Egyptian fractions.

1. PROOF OF ESTIMATES

LEMMA 1.1 ([8]). *Consider real numbers $x_1\ge x_2\ge\cdots\ge x_n>0$ and $y_1\ge y_2\ge\cdots\ge y_n>0$ such that $\prod_{i\le k}x_i\ge\prod_{i\le k}y_i$ for all k. Then $\sum_i x_i\ge\sum_i y_i$, with equality if and only if $x_i=y_i$ for all i.*

Proof. Soundararajan [8] deduces this lemma from Muirhead's inequality. We give here a direct proof, by induction on n. If $x_i=y_i$ for some i, we may remove the i-th terms from both n-tuples, and conclude by induction; therefore we may suppose $x_i\ne y_i$ for every i. If $x_i>y_i$ for all i, the conclusion is clear. Suppose $x_i<y_i$ for some i. Let $l=\min\{i;x_i<y_i\}$. Then $l>1$ and $x_i>y_i$ for every $i<l$. Let $t=\min\{\frac{x_{l-1}}{y_{l-1}},\frac{y_l}{x_l}\}>1$. Define $(x_i')_i$ by $x_i'=x_i$, for $i\notin\{l-1,l\}$, and $x_{l-1}'=\frac{x_{l-1}}{t}$, $x_l'=tx_l$. One checks that $x_1'\ge x_2'\ge\ldots\ge x_n'>0$, $\prod_{i=1}^k x_i'\ge\prod_{i=1}^k x_i$ for all k, and $x_{l-1}+x_l>x_{l-1}'+x_l'$, hence $\sum_{i=1}^n x_i>\sum_{i=1}^n x_i'$. Since either $x_{l-1}'=y_{l-1}$ or $x_l'=y_l$, $\sum_{i=1}^n x_i'\ge\sum_{i=1}^n y_i$ by induction. Therefore $\sum_{i=1}^n x_i>\sum_{i=1}^n y_i$. The claim on equality is clear. □

LEMMA 1.2. *Consider real numbers $x_1\ge x_2\ge\cdots\ge x_n>0$ and $y_1\ge y_2\ge\cdots\ge y_n>0$ such that $\sum_{i\ge k}x_i\ge\sum_{i\ge k}y_i$ for all k. Then $\prod_i x_i\ge\prod_i y_i$, with equality if and only if $x_i=y_i$ for all i.*

Proof. As in the previous lemma we use induction on n, so that we may suppose $x_i\ne y_i$ for every i. In particular, $x_n>y_n$. If $x_i>y_i$ for every i, the claim is clear. So suppose that $x_i<y_i$ for some i. Let $k=\max\{i;x_i<y_i\}$. Then $k<n$ and $x_i>y_i$ for every $i\ge k+1$. In particular,

$$y_{k+1}<x_{k+1}\le x_k<y_k.$$

Define $(y_i')_i$ by $y_i'=y_i$ for $i\notin\{k,k+1\}$, $y_k'=y_k-\epsilon, y_{k+1}'=y_{k+1}+\epsilon$, where $\epsilon=\min\{x_{k+1}-y_{k+1},y_k-x_k\}>0$. The following hold:

- $y_1'\ge\cdots\ge y_n'>0$.
- $\sum_{i\ge j}y_i\le\sum_{i\ge j}y_i'$, with equality for $j\ne k+1$. And $\sum_{i\ge j}x_i\ge\sum_{i\ge j}y_i'$ for all j.
- $y_k'y_{k+1}'-y_ky_{k+1}=\epsilon(y_k-y_{k+1}-\epsilon)>0$. Therefore $\prod_i y_i'>\prod_i y_i$.

By induction, the claim holds for (x_i) and (y'_i), since either $x_k = y'_k$ or $x_{k+1} = y'_{k+1}$. Therefore $\prod_i x_i \geq \prod_i y'_i$, so that $\prod_i x_i > \prod_i y_i$. $\square$

For the next proposition we need the following lemma whose proof is obvious.

Lemma 1.3. *Let n, p, q be positive integers with $1 - \frac{1}{n} \leq \frac{p}{q} < 1$. Then $n \leq q$.*

Proposition 1.4. *Let $s \geq 0, 1 \leq r \leq q$ be integers. If $1 \leq n_1 \leq \cdots \leq n_k$ are integers such that $\sum_{i=1}^k \frac{1}{n_i} < k - s + \frac{r}{q}$, then $\sum_{i=1}^k \frac{1}{n_i} \leq k - s + \frac{r}{q} - \frac{r}{u_{s+1,q}}$. Equality holds if and only if $n_i = 1$ for $i \leq k - s$ and $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $i > k - s$.*

Proof. We use induction on s to prove that if $1 \leq n_1 \leq \cdots \leq n_k$ are integers such that $k - s + \frac{r}{q} - \frac{r}{u_{s+1,q}} \leq \sum_{i=1}^k \frac{1}{n_i} < k - s + \frac{r}{q}$, then $n_i = 1$ for $i \leq k - s$ and $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $i > k - s$.

If $s = 0$, then $k \leq \sum_{i=1}^k \frac{1}{n_i} < k + \frac{r}{q}$, so that $n_i = 1$ for all i.

Let $s \geq 1$. The right inequality yields $s \leq k$. Denote $m_i = 1$ for $1 \leq i \leq k - s$ and $m_i = \frac{1+u_{i-k+s,q}}{r}$ for $k - s < i \leq k$. We have

$$\sum_{i=1}^k \frac{1}{m_i} = k - s + \frac{r}{q} - \frac{r}{u_{s+1,q}}, \ \prod_{i=1}^k m_i = \frac{u_{s+1,q}}{r^s q}.$$

Our hypothesis can be rewritten as

$$1 - \frac{q}{u_{s+1,q}} \leq \frac{q}{r}(s - k + \sum_{i=1}^k \frac{1}{n_i}) < 1.$$

The middle term can be represented as a fraction with denominator $r \prod_i n_i$. By Lemma 1.3, $\frac{u_{s+1,q}}{q} \leq r \prod_{i=1}^k n_i$. Therefore $\prod_{i=1}^k m_i \leq \prod_{i=1}^k n_i$. Then we can define

$$j = \max\{1 \leq l \leq k; \prod_{i \geq l} m_i \leq \prod_{i \geq l} n_i\}.$$

Assume $j = k$, that is $m_k \leq n_k$. Then $\sum_{i=1}^{k-1} \frac{1}{m_i} \leq \sum_{i=1}^{k-1} \frac{1}{n_i} < (k-1) - (s-1) + \frac{r}{q}$. By induction, $n_i = m_i$ for every $i \leq k-1$. It follows that $n_k = m_k$.

Assuming $j < k$, we derive a contradiction. Then $\prod_{i \geq j} n_i \geq \prod_{i \geq j} m_i$ and $\prod_{i \geq p} n_i < \prod_{i \geq p} m_i$ for every $j < p \leq k$. It follows that $\prod_{i=j}^p n_i > \prod_{i=j}^p m_i$ for every $j \leq p < k$. We rewrite this as

$$\prod_{i=j}^p \frac{1}{m_i} \geq \prod_{i=j}^p \frac{1}{n_i} \ (j \leq p \leq k),$$

with strict inequality for $p \neq k$. By Lemma 1.1, $\sum_{i=j}^{k} \frac{1}{m_i} > \sum_{i=j}^{k} \frac{1}{n_i}$. On the other hand, $\sum_{i=1}^{j-1} \frac{1}{n_i} < k-s+\frac{r}{q}$. By induction, $\sum_{i=1}^{j-1} \frac{1}{n_i} \leq \sum_{i=1}^{j-1} \frac{1}{m_i}$. Therefore $\sum_{i=1}^{k} \frac{1}{n_i} < \sum_{i=1}^{k} \frac{1}{m_i}$, a contradiction. □

Remark 1.5. Notice that since $1+u_{1,q}$ and $1+u_{2,q}$ are relatively prime, if $s \geq 2$ equality is achieved only for $r=1$.

Proposition 1.6. *Let $s \geq 0$ and $1 \leq r \leq q$ be integers. If $1 \leq n_2 \leq \cdots \leq n_k$ are integers such that $\sum_{i=1}^{k} \frac{1}{n_i} = k-s+\frac{r}{q}$, then* $\operatorname{lcm}(n_1,\ldots,n_k) \leq \frac{u_{s,q}}{r}$. *Equality holds if and only if $n_i = 1$ for $1 \leq i \leq k-s$, $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $k-s < i < k$ and $n_k = \frac{u_{s,q}}{r}$.*

Proof. We prove by induction on s that if $1 \leq n_2 \leq \cdots \leq n_k$ are integers such that $\sum_{i=1}^{k} \frac{1}{n_i} = k-s+\frac{r}{q}$ and $\operatorname{lcm}(n_1,\ldots,n_k) \geq \frac{u_{s,q}}{r}$, then $n_i = 1$ for $1 \leq i \leq k-s$, $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $k-s < i < k$ and $n_k = \frac{u_{s,q}}{r}$.

It follows that $s \geq 1$. If $s=1$, we must have $(n_i) = (1,\ldots,1,\frac{q}{r})$, so the conclusion holds. Suppose $s \geq 2$. Let $m_i = 1$ for $1 \leq i \leq k-s$, $m_i = \frac{1+u_{i-k+s,q}}{r}$ for $k-s < i < k$ and $m_k = \frac{u_{s,q}}{r}$. We have $m_1 \leq \cdots \leq m_k$, $\sum_{i=1}^{k} \frac{1}{m_i} = k-s+\frac{r}{q}$ and $\prod_{i=1}^{k} m_i = \frac{u_{s,q}^2}{r^s q}$.

Step 1: We claim that $\sum_{i \geq l} \frac{1}{n_i} \geq \sum_{i \geq l} \frac{1}{m_i}$ for every $1 \leq l \leq k$.

Indeed, equality holds for $l=1$. Let $1 < l \leq k-s+1$. Then $\sum_{i<l} \frac{1}{n_i} \leq l-1 = \sum_{i<l} \frac{1}{m_i}$. Therefore $\sum_{i \geq l} \frac{1}{n_i} \geq \sum_{i \geq l} \frac{1}{m_i}$. Let $k-s+1 < l \leq k$. Then $\sum_{i=1}^{l-1} \frac{1}{n_i} < k-s+\frac{r}{q} = (l-1)-(l+s-k-1)+\frac{r}{q}$. By Proposition 1.4, $\sum_{i=1}^{l-1} \frac{1}{n_i} \leq \sum_{i=1}^{l-1} \frac{1}{m_i}$. Therefore $\sum_{i \geq l} \frac{1}{n_i} \geq \sum_{i \geq l} \frac{1}{m_i}$.

Step 2: By Step 1 and Lemma 1.2, we obtain $\prod_i \frac{1}{n_i} \geq \prod_i \frac{1}{m_i}$, that is $\prod_i n_i \leq \prod_i m_i$. And equality holds if and only if $n_i = m_i$ for all i.

Step 3: Denote $L = \operatorname{lcm}(n_1,\ldots,n_k)$. We claim that $L^2 \leq q \prod_{i=1}^{k} n_i$.

Indeed, $q \mid L$ and $n_i \mid \operatorname{lcm}(q,n_1,\ldots,\widehat{n_i},\ldots,n_k)$ for all i. Fix a prime p. The power of p in L is the highest power of p occuring in the prime decomposition of $q,n_1,\ldots,n_k$. From above, the maximum is attained at least twice. Therefore $L^2 \leq q \prod_{i=1}^{k} n_i$.

Step 4: We obtain $L^2 \leq q \prod_{i=1}^{k} n_i \leq q \prod_{i=1}^{k} m_i = \frac{u_{s,q}^2}{r^s}$. Since $s \geq 2$, we obtain $L \leq \frac{u_{s,q}}{r}$. We assumed the opposite inequality, so $L = \frac{u_{s,q}}{r}$. It follows that $\prod_{i=1}^{k} n_i = \prod_{i=1}^{k} m_i$, so $n_i = m_i$ for all i. □

Remark 1.7. Note that equality is achieved if $\frac{1+u_{i-k+s,q}}{r}$ for $k-s < i < k$ and $\frac{u_{s,q}}{r}$ are integers, that is if and only if $s=1$ and $r \mid q$, or $s=2$ and $r \mid 1+q$, or $s \geq 3$ and $r=1$.

Proof of Theorem 0.2. Write $\delta = s - \frac{r}{q}$, where $s = \lfloor \delta \rfloor + 1$ and $r = q(1 - \{\delta\})$. Then $k - \delta = k - s + \frac{r}{q}$, and we may apply Propositions 1.4 and 1.6. □

Proof of Theorem 0.1. Order the coefficients of B as $0 \le b_1 \le \cdots \le b_k < 1 = b_{k+1} = \cdots = b_{k+c}$. Let $b_i = 1 - \frac{1}{m_i}$, for $1 \le i \le k$. Then $\sum_{i=1}^{k} \frac{1}{m_i} = k - (d - c + 1 + v)$. Denote $r = \mathrm{lcm}(m_i)$. By Theorem 0.2, $r \le \frac{u_{\lfloor t \rfloor + d - c + 2, q}}{q(1 - \{t\})}$. Then rB is a divisor with integer coefficients, and since the ambient space is $\mathbb{P}^d$, the semipositive Cartier divisor $r(K + B)$ is base point free. □

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Received 29 April 2015

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