

*Dedicated to Lucian Bădescu with friendship and admiration on the occasion
of his 70th birthday*

ON THE UNIRATIONALITY OF HIGHER DIMENSIONAL UENO-TYPE MANIFOLDS

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We prove the unirationality of the Ueno-type manifold $X_{4,6}$. $X_{4,6}$ is the minimal resolution of the quotient of the Cartesian product $E(6)^4$, where $E(6)$ is the equianharmonic elliptic curve, by the diagonal action of a cyclic group of order 6 (having a fixed point on each copy of $E(6)$). We collect also other results, and discuss several related open questions.

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1. INTRODUCTION

Let k be any field of characteristic $\neq 3$ containing a primitive third root of unity ω , respectively a field of characteristic $\neq 2$ containing a primitive fourth root of unity i .

We shall work over k unless otherwise stated.

One of the standard normal forms for the function fields of elliptic curves is the following normal form, defining an affine plane curve with equation

$$E_\lambda = \{(x, y) | y^2 = x(x-1)(x-\lambda)\}.$$

The curve E_λ is said to be **harmonic** if the cross ratio $\lambda \in \{-1, 2, 1/2\}$, and **equianharmonic** if λ is a primitive 6-th root of 1, $\lambda \in \{\eta, \eta^{-1}\}$, $\eta := -\omega$.

These curves admit automorphisms of respective orders 4, 6, having a fixed point.

In the harmonic case if we take the above normal form with $\lambda = -1$, the automorphism of order 4 is given by

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$$g_4 : (x, y) \mapsto (-x, iy).$$

In the equianharmonic case it is easier to see the automorphism g_6 of order 6 by changing the normal form to

$$E_\eta \cong \{(x, y) | y^2 = x^3 - 1\},$$

so that

$$g_6 : (x, y) \mapsto (\omega x, -y),$$

while using the Fermat normal form things are more complicated,

$$E_\eta \cong \{(x, y) | y^3 = x^3 - 1\},$$

and

$$g_6 : (x, y) \mapsto (1/x, \eta y/x).$$

Using instead the normal form where E_η is birational to the singular plane curve

$$E'_\eta = \{(x, y) | y^6 = x^2(x - 1)\},$$

the automorphism takes the easier form

$$g_6 : (x, y) \mapsto (x, \eta y),$$

and one sees immediately that the field of g_6 -invariant rational functions is the field $k(x)$.

Let now $(x : y : z)$ be homogeneous coordinates on \mathbb{P}^2 , let

$$E(6) := \{(x : y : z) | y^2 z = x^3 - z^3\} \subseteq \mathbb{P}^2,$$

be the projective model of the elliptic curve E_η , on which the automorphism g_6 acts by

$$g_6(x : y : z) = (\omega x : -y : z)$$

and similarly let

$$E(4) := \{(x : y : z) | y^2 z = x(x^2 - z^2)\} \subseteq \mathbb{P}^2,$$

be the projective model of the elliptic curve E_{-1} , on which the automorphism g_4 acts by

$$g_4(x : y : z) = (-x : iy : z).$$

When k is the complex number field \mathbb{C} , we have

$$(E(6), g_6) \simeq (T_\omega, -\omega),$$

where $T_\omega = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ is the elliptic curve with period ω , and $-\omega$ is the automorphism induced by multiplication by $-\omega$ on \mathbb{C} . We have such an isomorphism because g_6 acts on the holomorphic 1-form dx/y via multiplication by $-\omega$.

Similarly

$$(E(4), g_4) \simeq (T_i, i).$$

Let now (C, g) be either $(E(6), g_6)$ or $(E(4), g_4)$, and let g act diagonally on the Cartesian product C^n . We set

$$Z_n(4) := E(4)^n, \quad Z_n(6) := E(6)^n.$$

Definition 1.1. We define the **Ueno-type manifold** $X_{n,6}$ of dimension n to be the minimal resolution of singularities of the normal variety $Y_{n,6} := Z_n(6)/g_6$, while we reserve the name of **Ueno-Campana manifold** for the Ueno-type manifold $X_{n,4}$ of dimension n , which is the minimal resolution of singularities of the normal variety $Y_{n,4} := Z_n(4)/g_4$.

Observe that the quotient n -fold

$$Y_{n,6} := Z_n(6)/g_6, \quad n \geq 2,$$

has finitely many singular points of type $(1, 1, 1, \dots, 1)/6$, of type $(1, 1, 1, \dots, 1)/3$ and of type $(1, 1, 1, \dots, 1)/2$, which are all k -rational. $X_{n,6}$ is the blow up of $Y_{n,6}$ at the maximal ideals of these singular points: it is a smooth projective n -fold defined over k .

It is classical that these manifolds are rational for $n \leq 2$, and the arguments of Ueno [12] show that:

- the Kodaira dimension of $X_{n,6}$ is 0 if $n \geq 6$ and $-\infty$ if $n \leq 5$,
- the Kodaira dimension of $X_{n,4}$ is 0 if $n \geq 4$ and $-\infty$ if $n \leq 3$.

Much later Kollár and Larsen [7] showed a more general result: if Z has trivial canonical bundle and a finite group G acts on Z , either the quotient Z/G has Kodaira dimension 0, or it is uniruled.

Ueno asked about separable unirationality of the manifold $X_{3,4}$, and Oguiso asked the similar question for $X_{n,6}$, $3 \leq n \leq 5$.

Interest for these open questions was revived by Campana, who showed [1] that $X_{3,4}$ is rationally connected and asked about rationality of $X_{3,4}$; unirationality was proven by Catanese, Oguiso and Truong in [2], and later Colliot-Thélène proved rationality in [3] using the conic bundle description of [2].

In the case of the Ueno manifolds, Oguiso and Truong proved in [5] that $X_{3,6}$ is rational.

The main result of the present paper is the following.

THEOREM 1.2. $X_{4,6}$ is unirational.

Our description can be useful to attack the further questions:

Question 1.3. Is $X_{4,6}$ rational?

Question 1.4. Is $X_{5,6}$ unirational? Is $X_{5,6}$ rational?

The rebirth of interest in the rationality of these manifolds stems also from complex dynamics and entropy, since these manifolds admit an action by $GL(n, \mathbb{Z})$ (and indeed by $GL(n, R_m)$, where R_m is the cyclotomic ring $\mathbb{Z}[i]$, resp. $\mathbb{Z}[\omega]$).

In fact, $GL(n, \mathbb{Z})$ and $GL(n, R_m)$ act on the product $E(m)^n$; and, since we divide by a central automorphism, the action first of all descends to the

quotient, and then it extends biregularly to $X_{n,m}$ since the resolution is just obtained by blowing up the singular points of the quotient.

In the case of the Ueno manifolds, Uguiso and Truong proved in [5] that $X_{3,6}$ is rational. They not only proved the rationality of $X_{3,6}$, but also showed that in this way one gets a rational variety with a primitive automorphism of positive entropy. Here, according to a concept introduced by De-Qi-Zhang (see [13]), an automorphism $f : X \rightarrow X$ is said to be birationally imprimitive if there is a nontrivial rational fibration $\pi : X \rightarrow Y$, and a birational automorphism ϕ of Y such that $\pi \circ f = \phi \circ \pi$. De-Qi-Zhang showed that if a threefold X admits a primitive birational automorphism of positive entropy, then either X is a torus, or it is a \mathbb{Q} -Calabi-Yau manifold, or X is rationally connected.

Question 1.5. Does a similar result hold for $X_{4,6}$?

In another vein, our specific unirationality result lends itself to more general questions. To formulate these, we need to briefly describe the steps of the proof, and the analogy with the case of $X_{3,4}$.

The first step of the proof is computational, and consists in finding a minimal system of generators for the field of invariant rational functions on E_m^n : here the cases of $X_{3,4}$ and $X_{4,6}$ are treated quite similarly. For instance, in the case of $X_{3,6}$ one finds three generators, hence these three elements are algebraically independent and the variety is k -rational.

In the case of $X_{3,4}$ we found 4 generators t_1, t_2, u_1, u_2 and one equation, which can be written as a diagonal quadratic form of the form

$$u_1^2 - A(t_1, t_2)u_2^2 - B(t_1, t_2) = 0.$$

We thus got, birationally, a conic bundle over the projective plane, and the method of [2] consisted in showing that the conic bundle has a bisection Z which is rational: then the pull back of the conic bundle to Z is a conic bundle with a section hence it is rational.

Colliot-Thélène proved that the conic bundle does not have a section: in fact, if K is the function field of the plane, $A, B \in K$ and to such a diagonal conic over K one associates a central algebra over K , $M_{A,B}$, generated by $1, i, j, ij = -ji$ and defined by $i^2 = A, j^2 = B$.

By a general theorem the algebra is a division algebra if and only if the conic does not have any K -rational point (in the contrary case $M_{A,B} \cong M(2, 2, K)$). Moreover, two such conics are K -isomorphic if and only if the corresponding algebras are isomorphic (they yield the same element of the Brauer group).

Colliot-Thélène proved also that in this case the conic is isomorphic to one of the form

$$u_1^2 + t_1 u_2^2 + t_2 = 1,$$

hence the function field is generated by t_1, u_1, u_2 and $X_{3,4}$ is rational.

To prove here the unirationality of $X_{4,6}$, we show that it is birational to a diagonal cubic surface S over the function field $K := k(t_1, t_2)$

$$A(t_1, t_2)(u_1^3 - 1) + B(t_1, t_2)(u_2^3 - 1) + C(t_1, t_2)(u_3^3 - 1) = 0.$$

The surface S admits therefore 27 rational points (just let u_j be a cubic root of 1).

Then, by a theorem of B. Segre, it follows that S is unirational; we further observe here that the degree of unirationality is at most 6, and we conjecture it to be at most 2.

Using other classical results of B. Segre, Swinnerton-Dyer and Colliot-Thélène on cubic surfaces and on diagonal cubic surfaces we show finally that the surface S is K - unirational, but it is not K - rational.

We ask whether it is possible, like it was done for the conic bundle case, to change the cubic surface birationally and obtain equations which imply the rationality of $X_{4,6}$.

Observe that the coefficients $A(t_1, t_2), B(t_1, t_2), C(t_1, t_2)$ correspond to a very special system of plane cubics, yielding the Del Pezzo surface of degree 2 which is the double cover of \mathbb{P}^2 branched over a complete quadrilateral.

In particular, in the course of our proof, we show that our variety $X_{4,6}$ is birational to a hypersurface X of bidegree $(3, 3)$ inside $\mathbb{P}^2 \times \mathbb{P}^3$.

Viewing X as a cubic surface S over the function field $K := k(\mathbb{P}^2) = k(t_1, t_2)$ it follows, by the cited theorem of Beniamino Segre, that X is K - unirational if the surface S admits a K -rational point, and that there are many such surfaces which are K -rational (note that then X is k -rational).

Question 1.6. Let X be a very general hypersurface of bidegree $(3, 3)$ inside $\mathbb{P}^2 \times \mathbb{P}^3$.

Is X unirational? Is X rational?

2. Proof of Theorem (1.2)

Here $n = 4$ and we set $Z := Z_4(6)$ and $X := X_{4,6}$, $g := g_6$.

We write $Z := Z_4(6) = C_1 \times C_2 \times C_3 \times C_4$, and let $g := g_6$ be the diagonal action $g(1) \times g(2) \times g(3) \times g(4)$, where the curves $(C_i, g(i))$ ($i = 1, 2, 3, 4$) are birationally equivalent to (E'_η, g_6) .

Hence, we view C_i as birational to the singular curve C_i^0 in the affine space $\mathbb{A}^2 = \text{Spec } k[X_i, Y_i]$, and $g(i)$ as the automorphism of C_i^0 defined by

$$C_i^0 := \{(X_i, Y_i) | Y_i^6 = X_i^2(X_i - 1)\}, \quad g_i^* Y_i = -\omega Y_i, \quad g_i^* X_i = X_i.$$

The affine coordinate ring $k[C_i^0]$ of C_i^0 is

$$k[C_i^0] = k[X_i, Y_i]/(Y_i^6 - X_i^2(X_i - 1)) .$$

We set $x_i := X_i \bmod (Y_i^6 - X_i^2(X_i - 1))$, $y_i := Y_i \bmod (Y_i^6 - X_i^2(X_i - 1))$. Then y_i^j ($0 \leq j \leq 5$) form a free $k[x_i]$ -basis of $k[C_i^0]$ over a polynomial ring $k[x_i]$ and therefore $x_i^m y_i^j$ ($0 \leq j \leq 5$, $0 \leq m$) form a free k -basis of $k[C_i^0]$.

Then (Z, g) is birationally equivalent to the affine fourfold

$$V := C_1^0 \times C_2^0 \times C_3^0 \times C_4^0$$

with automorphism $g = (g(1), g(2), g(3), g(4))$, and with affine coordinate ring

$$k[V] = k[C_1^0] \otimes k[C_2^0] \otimes k[C_3^0] \otimes k[C_4^0] .$$

The subring $k[x_1, x_2, x_3, x_4]$ of $k[V]$ is a polynomial ring with four free variables x_1, x_2, x_3, x_4 and $k[V]$ is a free $k[x_1, x_2, x_3, x_4]$ -module with free basis

$$y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} , \text{ where } 0 \leq m_i \leq 5 .$$

The rational function field $k(Z)$ of Z is

$$k(Z) = k(V) = k(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) .$$

In both $k[V]$ and $k(Z) = k(V)$, we have

$$(2.1) \quad y_i^6 = x_i^2(x_i - 1) ,$$

$$(2.2) \quad g^* y_i = -\omega y_i , \quad g^* x_i = x_i .$$

Here and hereafter each equation shall be viewed as an equation in $k(V)$.

The affine coordinate ring $V/\langle g \rangle$ is $k[V]^{g^*}$, the invariant subring of $k[V]$. Thus, by (2.2),

$$(2.3) \quad k[V]^{g^*} = k[x_1, x_2, x_3, x_4][y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} \mid \sum_{i=1}^4 m_i \equiv 0 \pmod{6}] .$$

If $\sum_{i=1}^4 m_i = 6k$, then

$$y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} = (y_1^6)^k \left(\frac{y_2}{y_1}\right)^{m_2} \left(\frac{y_3}{y_1}\right)^{m_3} \left(\frac{y_4}{y_1}\right)^{m_4} = (x_1^2(x_1 - 1))^k \left(\frac{y_2}{y_1}\right)^{m_2} \left(\frac{y_3}{y_1}\right)^{m_3} \left(\frac{y_4}{y_1}\right)^{m_4} .$$

Note that $k(X) = Q(k[V]^{g^*})$, the field of fractions of $k[V]^{g^*}$.

Hence,

LEMMA 2.1. $k(X) = k(x_1, x_2, x_3, x_4, t_2 := \frac{y_2}{y_1}, t_3 := \frac{y_3}{y_1}, t_4 := \frac{y_4}{y_1})$ with relations precisely

$$t_i^6 = \frac{x_i^2(x_i - 1)}{x_1^2(x_1 - 1)} , \text{ where } i = 2, 3, 4 .$$

Proof. We only need to observe that the three relations above are all the relations. Since $[k(V) : k(x_1, x_2, x_3, x_4)] = 6^4$ and $[k(V) : k(X)] = 6$, it follows that

$$[k(X) : k(x_1, x_2, x_3, x_4)] = 6^3 .$$

Thus, the equation above for $i = 2$ is the minimal equation of t_2 over $k(x_1, x_2, x_3, x_4)$, the equation above for $i = 3$ is the minimal equation of t_3 over $k(x_1, x_2, x_3, x_4)(t_2)$ and the equation above for $i = 4$ is the minimal equation of t_4 over $k(x_1, x_2, x_3, x_4)(t_2, t_3)$, as desired. \square

Define:

$$(2.4) \quad u_2 := \frac{x_1 t_2^3}{x_2}, \quad u_3 := \frac{x_1 t_3^3}{x_3}, \quad u_4 := \frac{x_1 t_4^3}{x_4} .$$

Then

LEMMA 2.2. $k(X) = k(x_1, u_2, u_3, u_4, t_2, t_3, t_4)$ with relations precisely

$$\frac{u_i}{t_i^3} = \frac{x_1}{u_i^2(x_1 - 1) + 1}, \quad \text{where } i = 2, 3, 4 .$$

Proof. By the definition

$$u_i := \frac{x_1 t_i^3}{x_i},$$

the equation in Lemma (5.1) is

$$u_i^2 = \frac{x_i - 1}{x_1 - 1} .$$

Thus,

$$x_i = u_i^2(x_1 - 1) + 1 .$$

Therefore,

$$k(X) = k(x_1, u_2, u_3, u_4, t_2, t_3, t_4)$$

with relations precisely

$$u_i (= \frac{x_1 t_i^3}{x_i}) = \frac{x_1}{u_i^2(x_1 - 1) + 1} t_i^3 .$$

Dividing both sides by $t_i^3 \neq 0$ in $k(V)$, we complete the proof. \square

LEMMA 2.3. $k(X) = k(u_2, u_3, u_4, t_2, t_3, t_4)$ with relations precisely

$$\frac{u_2 - t_2^3}{u_2^3 - u_2} = \frac{u_3 - t_3^3}{u_3^3 - u_3} = \frac{u_4 - t_4^3}{u_4^3 - u_4} .$$

Proof. The equations in Lemma (2.2) are linear with respect to x_1 .

In fact, by clearing the denominator

$$u_i^3(x_1 - 1) + u_i = t_i^3 x_1 ,$$

that is (by adding $-t_i^3x_1 + u_i^3 - u_i$ to both sides),

$$(u_i^3 - t_i^3)x_1 = u_i^3 - u_i .$$

Observe that $u_i^3 - u_i \neq 0$ in $k(V)$. Thus, this is equivalent to

$$\frac{1}{x_1} = \frac{u_i^3 - t_i^3}{u_i^3 - u_i} ,$$

that is, (by adding -1 to both sides),

$$\frac{1}{x_1} - 1 = \frac{u_i - t_i^3}{u_i^3 - u_i} .$$

This implies the result. \square

Observe that $u_i \neq 0$ in $k(V)$ and define:

$$(2.5) \quad v_2 := \frac{1}{u_2} , w_2 := \frac{t_2}{u_2} , v_3 := \frac{1}{u_3} , w_3 := \frac{t_3}{u_3} , v_4 := \frac{1}{u_4} , w_4 := \frac{t_4}{u_4} .$$

LEMMA 2.4. $k(X) = k(v_2, v_3, v_4, w_2, w_3, w_4)$ with relations precisely

$$(v_3^2 - 1)(w_2^3 - 1) = (v_2^2 - 1)(w_3^3 - 1) , (v_4^2 - 1)(w_2^3 - 1) = (v_2^2 - 1)(w_4^3 - 1) .$$

Proof. By the definition, $v_i = 1/u_i$, $w_i = t_i/u_i$ and $u_i = 1/v_i$, $t_i = w_i u_i = w_i/v_i$, it follows that

$$k(u_2, u_3, u_4, t_2, t_3, t_4) = k(v_2, v_3, v_4, w_2, w_3, w_4) .$$

Observe that

$$\frac{u_i - t_i^3}{u_i^3 - u_i} = \frac{(1/u_i)^2 - (t_i/u_i)^3}{1 - (1/u_i)^2} = \frac{v_i^2 - w_i^3}{1 - v_i^2} = -1 + \frac{1 - w_i^3}{1 - v_i^2} .$$

Hence, the precise relations in Lemma (2.3) are rewritten in terms of the new variables as

$$\frac{1 - w_2^3}{1 - v_2^2} = \frac{1 - w_3^3}{1 - v_3^2} = \frac{1 - w_4^3}{1 - v_4^2} .$$

By clearing the denominators, we obtain the precise relations that we claimed. \square

Observe that $v_i - 1 \neq 0$ in $k(V)$ and define:

$$(2.6) \quad s_3 := \frac{v_3 - 1}{v_2 - 1} , s_4 := \frac{v_4 - 1}{v_2 - 1} .$$

LEMMA 2.5. $k(X) = k(s_3, s_4, w_2, w_3, w_4)$ with relations precisely

$$(s_3 - s_4)s_3s_4(w_2^3 - 1) - (s_3 - 1)s_3(w_4^3 - 1) + (s_4 - 1)s_4(w_3^3 - 1) = 0 .$$

Proof. The defining equation of s_i is linear in both s_i and v_i , hence it follows that

$$k(X) = k(v_2, v_3, v_4, w_2, w_3, w_4) = k(v_2, s_3, s_4, w_2, w_3, w_4) .$$

Since

$$v_i + 1 = (v_i - 1) + 2 , \quad v_i^2 - 1 = (v_i - 1)(v_i + 1) = s_i(v_2 - 1)(s_i(v_2 - 1) + 2)$$

and $v_2 - 1 \neq 0$ in $k(V)$, the relations in Lemma (2.4) are precisely

$$s_3(s_3(v_2 - 1) + 2)(w_2^3 - 1) = (v_2 + 1)(w_3^3 - 1) ,$$

$$s_4(s_4(v_2 - 1) + 2)(w_2^3 - 1) = (v_2 + 1)(w_4^3 - 1) .$$

Both are linear in terms of v_2 , more explicitly, these two equations are equivalent to

$$(s_3^2(w_2^3 - 1) - (w_3^3 - 1))(v_2 - 1) = -2s_3(w_2^3 - 1) + 2(w_3^3 - 1) .$$

$$(s_4^2(w_2^3 - 1) - (w_4^3 - 1))(v_2 - 1) = -2s_4(w_2^3 - 1) + 2(w_4^3 - 1) .$$

Observe that

$$-s_i^2(w_2^3 - 1) + (w_i^3 - 1) \neq 0$$

in $k(V)$ and recall that k is not of characteristic 2. Then these two equations are equivalent to

$$\frac{v_2 - 1}{2} = \frac{-s_3(w_2^3 - 1) + (w_3^3 - 1)}{s_3^2(w_2^3 - 1) - (w_3^3 - 1)} = \frac{-s_4(w_2^3 - 1) + (w_4^3 - 1)}{s_4^2(w_2^3 - 1) - (w_4^3 - 1)} .$$

Thus,

$$k(X) = k(s_3, s_4, w_2, w_3, w_4) ,$$

with the above precise relation, that is (by taking the inverse and multiply by -1)

$$\frac{s_3^2(w_2^3 - 1) - (w_3^3 - 1)}{s_3(w_2^3 - 1) - (w_3^3 - 1)} = \frac{s_4^2(w_2^3 - 1) - (w_4^3 - 1)}{s_4(w_2^3 - 1) - (w_4^3 - 1)} .$$

Observe that

$$s_i^2(w_2^3 - 1) - (w_i^3 - 1) = (s_i^2 - s_i)(w_2^3 - 1) + (s_i(w_2^3 - 1) - (w_i^3 - 1)) .$$

Thus, the equation above is equivalent to

$$\frac{(s_3^2 - s_3)(w_2^3 - 1)}{s_3(w_2^3 - 1) - (w_3^3 - 1)} + 1 = \frac{(s_4^2 - s_4)(w_2^3 - 1)}{s_4(w_2^3 - 1) - (w_4^3 - 1)} + 1 ,$$

whence, equivalent to

$$\frac{(s_3^2 - s_3)}{s_3(w_2^3 - 1) - (w_3^3 - 1)} = \frac{(s_4^2 - s_4)}{s_4(w_2^3 - 1) - (w_4^3 - 1)} ,$$

by $w_2^3 - 1 \neq 0$ in $k(V)$. By clearing denominators, the last equation is equivalent to

$$(s_3^2 - s_3)(s_4(w_2^3 - 1) - (w_4^3 - 1)) - (s_4^2 - s_4)(s_3(w_2^3 - 1) - (w_3^3 - 1)) = 0,$$

which is nothing but the equation claimed (just make the equation as an equation with respect to $w_i^3 - 1$). \square

To proceed with the proof of Theorem (1.2), consider the affine hypersurface H defined by

$$(s_3 - s_4)s_3s_4(w_2^3 - 1) - (s_3 - 1)s_3(w_4^3 - 1) + (s_4 - 1)s_4(w_3^3 - 1) = 0$$

in the affine space \mathbb{A}^5 with affine coordinates $(s_3, s_4, w_2, w_3, w_4)$.

Since X is of dimension 4, Lemma (2.5) means that X is birational to H .

Remark 2.1. The projection $\pi : H \rightarrow \mathbb{A}^2$ defined by

$$(s_3, s_4, w_2, w_3, w_4) \mapsto (s_3, s_4)$$

makes H a fibration of cubic surfaces over the affine space \mathbb{A}^2 with affine coordinates (s_3, s_4) .

Let η be the generic point of \mathbb{A}^2 and let H_η be the generic fiber. Then the projective completion \overline{H}_η of H_η in \mathbb{P}_η^3 is a smooth cubic surface S over the field $k(s_3, s_4)$ (by the Jacobian criterion and the fact that k is not of characteristic 3) with a rational point $(1, 1, 1) \in \overline{H}_\eta(k(s_3, s_4))$.

The above remark shows that it is sufficient to show that \overline{H}_η is unirational over $K := k(s_3, s_4)$ (i.e., there is a dominant rational map $\mathbb{P}_\eta^2 - - \rightarrow \overline{H}_\eta$ over $k(s_3, s_4)$): because then, $k(s_3, s_4)$ being purely transcendental, this means that there is a dominant rational map $\mathbb{P}_k^4 - - \rightarrow X$ over k , that is, X is unirational over k .

The K -unirationality of $S = \overline{H}_\eta$ follows by a theorem of Segre (see [8], see also the extension to higher dimensional cubic hypersurfaces done by Kollár in [6], Theorem 1), asserting that a smooth cubic surface with a K -rational point is K -unirational.

In the next section we shall give an upper bound of the K -unirationality degree, and show that S is not K -rational.

3. DIAGONAL CUBIC SURFACES AND THEIR RATIONALITY

We begin this section recalling several known results, which immediately imply the claimed assertion that our cubic surface S is not K -rational.

Let S be a nonsingular cubic surface defined over a field K : then Swinnerton-Dyer completed earlier results by B. Segre in [10] showing

THEOREM 3.1 (Swinnerton-Dyer [11]). *A smooth cubic surface S defined over a field K is birational to \mathbb{P}_K^2 if and only if*

1) *it has a K -rational point*

and

2) *S contains a set Σ_n of pairwise disjoint lines which is defined over K (i.e., invariant under the Galois group $\text{Gal}(\bar{K}, K)$) and has cardinality $n \in \{2, 3, 6\}$.*

If one drops the second condition, then one has

THEOREM 3.2 (Segre [9]). *A smooth cubic surface S defined over a field K is K -unirational if and only if it has a K -rational point.*

Recall now that a **diagonal cubic surface** S is a cubic surface

$$S \subset \mathbb{P}_K^3, \quad S = \{(x_1, x_2, x_3, x_4) \mid \sum_{i=1}^4 a_i x_i^3 = 0\},$$

and one says that S is defined over K if $a_i \in K, \forall i = 1, \dots, 4$.

Observe that, in the case where the field K is moreover algebraically closed, then such a diagonal cubic surface S is projectively equivalent to the Fermat cubic surface. For diagonal cubic surfaces it is easy to find the lines lying on it, moreover they have a special geometry; recall for this the definition of **Eckardt points**: these are the points P where the tangent plane intersects the surface S in a set of three lines passing through P .

PROPOSITION 3.3. *Let K be a field of characteristic $\neq 2, 3$ and containing a primitive third root of unity ω . Let S be, as above, a diagonal cubic surface defined over K (i.e., $a_i \in K$).*

$$S \subset \mathbb{P}_K^3, \quad S = \{(x_1, x_2, x_3, x_4) \mid \sum_{i=1}^4 a_i x_i^3 = 0\}.$$

Then the 27 lines of S , defined over a field extension of K , are the three sets of 9 lines obtained, for each partition $\{1, 2, 3, 4\} = \{i, j\} \cup \{h, k\}$, by the equations

$$a_i x_i^3 + a_j x_j^3 = a_h x_h^3 + a_k x_k^3 = 0,$$

i.e.,

$$x_i = \lambda_{ij} x_j, \quad x_h = \lambda_{hk} x_k, \quad a_i \lambda_{ij}^3 + a_i = 0, \quad a_h \lambda_{hk}^3 + a_k = 0.$$

Moreover, the surface possesses exactly 18 Eckardt points, defined on the algebraic closure of K by the equations $x_i = x_j = 0, 1 \leq i, j \leq 4$.

Proof. The first assertion being clear, we prove the second assertion. Since the coefficients a_i are not zero, due to the smoothness of S , the complete

intersection Γ of S with the Hessian surface $H_S = \{x_1x_2x_3x_4 = 0\}$ of S consists of 4 smooth plane cubics, which intersect in the nine points $x_i = x_j = 0, a_hx_h^3 + a_kx_k^3 = 0$, which are therefore the singular points of Γ . Notice that an Eckardt point must be a singular point of Γ ; conversely, at one such point the tangent plane is the plane

$$\{y|y_ha_hx_h^2 + y_ka_kx_k^2 = 0\}.$$

Since $a_hx_h^3 + a_kx_k^3 = 0$, the intersection of S with the tangent plane is the union of the three lines

$$\{y|y_kx_h = y_hx_k, a_iy_i^3 + a_ky_k^3 = 0\}. \quad \square$$

Remark 3.1. The maximum number of Eckardt points that a smooth cubic surface can have is exactly 18.

This can be shown using the model of the cubic surface as the blow-up of the plane in six points not lying on a conic. The analysis using the parabolic curve Γ , which is a complete intersection of type $(3, 3)$, hence has arithmetic genus equal to 19, and degree equal to 12, seems more complicated.

THEOREM 3.4. *Let K be a field of characteristic $\neq 2, 3$ and containing a primitive third root of unity ω . And let S be a smooth diagonal cubic surface defined over K and with equation*

$$S = \{(x_1, x_2, x_3, x_4) | a_1(x_1^3 - x_4^3) + a_2(x_2^3 - x_4^3) + a_3(x_3^3 - x_4^3) = 0\}.$$

Then there is a dominant rational map $\mathbb{P}_K^2 \rightarrow S$ of degree at most 6.

Proof. Observe first of all that S is smooth if and only if $a_1a_2a_3(a_1 + a_2 + a_3) \neq 0$.

Secondly, S contains the 27 points whose coordinates are cubic roots of 1. As we have shown, these are not Eckardt points.

Assume now that $P \in S$ is a K -rational point, so that the curve

$$C_P = S \cap (T_P S)$$

intersection of the surface with the tangent plane in P is defined over K , and has a singular point at P .

Case 1): C_P is irreducible.

Then C_P is birational to \mathbb{P}_K^1 , and we have a birational parametrization $\lambda \in K \mapsto P_\lambda \in C_P$. Observe that the plane $(T_P S)$ meets the lines contained in S in a finite set, hence through the general point P_λ passes no line contained in S .

Then we let $D_\lambda = S \cap (T_{P_\lambda} S)$, which is in general an irreducible cubic, singular in P_λ : hence there is a dominant rational map $\psi : \mathbb{P}_K^1 \times \mathbb{P}_K^1 \rightarrow S$.

In this case the degree of ψ is equal to 6.

In fact, C_P maps to a curve C' of degree 6 on the dual surface S^\vee of S , and the condition for a point $z \in S$ that $z \in T_y S$ means that the dual plane z^* contains the point $(T_y S)^*$.

So, if we intersect the curve C' with the plane z^* we obtain 6 points: this shows that the degree of the map ψ is 6.

Case 2): C_P has two components, a line L and a conic Q .

In this case L is K -rational. In general, if a cubic S contains such a K -rational line, then the pencil of planes $\pi_t \supset L$ yields a dominant rational map $\phi : \mathbb{P}_K^1 \times \mathbb{P}_K^1 \rightarrow S$ of degree 2. Because for each point $P' \in L$ the tangent plane in P' intersects S in $L \cup Q_{P'}$, and for general $P' \in L$, K -rational, the conic $Q_{P'}$ is irreducible and contains P' , hence we obtain a dominant rational map $\phi : \mathbb{P}_K^1 \times \mathbb{P}_K^1 \rightarrow S$. For a general $x \in S$, the plane spanned by x and L intersects S as $L \cup Q$, and Q intersects L in two points: hence the degree of ϕ equals 2.

We shall however see in theorem 3.5 that for a diagonal cubic surface with a K -rational point the existence of a K -rational line implies the birationality of S with \mathbb{P}_K^2 .

Case 3): C_P consists of three lines.

Then, since P is singular for C_P , at least two of these lines contain P . If there is one of the three lines which does not contain P , then this line is K -rational, and we are done as in case 2). There remains only

Case 4): C_P consists of three lines passing through P , and none of them is K -rational.

Then P is classically called an **Eckardt point** and since P is K -rational the three lines passing through P form a Galois orbit.

We conclude that the number of such points is at most 9, hence we have shown that there is a dominant rational map $\psi : \mathbb{P}_K^2 \rightarrow S$ of degree at most 6. \square

The following theorem was first stated by Segre in [9] and [10], and was then also proven by Colliot-Thélène, Kanevsky, Sansuc in [4].

THEOREM 3.5 (Segre, Colliot-Thélène, Kanevsky and Sansuc). *Let K be a field of characteristic $\neq 2, 3$ and containing a primitive third root of unity ω . And let S be a smooth diagonal cubic surface defined over K and with equation*

$$S = \{(x_1, x_2, x_3, x_4) \mid a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = 0\}.$$

Then S is K -rational if and only if it has a K -rational point and there is a permutation a, b, c, d of the four coefficients a_1, \dots, a_4 such that ac/bd has a cubic root in K .

Proof. Without loss of generality we may assume that $a_1 = 1$.

Recall that for a diagonal cubic surface S the 27 lines are grouped into 3 subsets \mathcal{R}_i of 9 elements, corresponding to indices $i = 2, 3, 4$.

In fact, write $\{1, 2, 3, 4\} = \{1, i\} \cup \{j, k\}$: then any line has the form

$$x_1 + \beta_i x_i = 0, x_j + \beta_{j,k} x_k = 0,$$

where $\beta_i^3 = a_i$, $\beta_{j,k}^3 = a_k/a_j$.

This shows that S has a K -rational line if and only if there is a permutation a, b, c, d of the four coefficients a_1, \dots, a_4 such that a/b and c/d have a cubic root in K .

Consider now the field extension K' generated by the cubic roots of the coefficients a_2, a_3, a_4 (recall that we are assuming that $a_1 = 1$).

Then the Galois group $G := \text{Gal}(K', K)$ is $(\mathbb{Z}/3)^m$, $m = 1, 2, 3$. Each set \mathcal{R}_i is a union of Galois orbits, and it is a single Galois orbit if the field extension K_i generated by $a_i, a_k/a_j$ has degree 9.

We already observed that $K_i = K$ for some i if and only if there is a K -rational line.

In this case two lines

$$x_1 + \beta_i x_i = 0, x_j + \beta_{j,k} x_k = 0,$$

$$x_1 + \beta'_i x_i = 0, x_j + \beta'_{j,k} x_k = 0,$$

are K -rational and skew if $\beta_i \neq \beta'_i, \beta_{j,k} \neq \beta'_{j,k}$.

In this way we have found three pairwise disjoint and K -rational lines, and S is rational by the criterion of Segre and Swinnerton-Dyer.

Now, if there is no K -rational line, then all the Galois orbits inside \mathcal{R}_i have either cardinality 9, or are three orbits of cardinality 3. But in this case the field extension K_i generated by $a_i, a_k/a_j$ has degree 3. If $a_i \in K$ or $a_k/a_j \in K$ then the orbits consist of incident lines, and one cannot apply the criterion of Segre and Swinnerton-Dyer. The only possibility is that both $a_i \notin K, a_k/a_j \notin K$ but $a_k/a_j a_i \in K$.

Then we get a Galois orbit of 3 pairwise disjoint lines, and we need to have a K -rational point in order to apply the criterion of Segre and Swinnerton-Dyer. \square

We can now show that our surface S is not K -rational.

THEOREM 3.6. *Let $K := k(s_3, s_4)$ and let S be the diagonal cubic surface of equation*

$$(s_3 - s_4)s_3s_4(x_1^3 - x_4^3) - (s_3 - 1)s_3(x_2^3 - x_4^3) + (s_4 - 1)s_4(x_3^3 - x_4^3) = 0.$$

Then S is not K -rational.

Proof. The four coefficients are $(s_3 - s_4)s_3s_4, -(s_3 - 1)s_3, (s_4 - 1)s_4$ and their sum $f = (s_3 - s_4)s_3s_4 - (s_3 - 1)s_3 + (s_4 - 1)s_4$.

Take a permutation a, b, c of the first three coefficients and consider fa/bc . This is a fraction with relatively prime numerator and denominator, and such that the denominator is not a cube; hence fa/bc is not a cube in K , and by the previous theorem 3.5 S is not K -rational. \square

4. SPECIAL GEOMETRY OF $X_{4,6}$

Let us analyse the equation of the hypersurface H .

It is a diagonal cubic surface in the variables w_i , with equation

$$a(s_3, s_4)(w_2^3 - 1) + b(s_3, s_4)(w_4^3 - 1) + c(s_3, s_4)(w_3^3 - 1) = 0$$

where the coefficients a, b, c are cubic polynomials

$$a := (s_3 - s_4)s_3s_4, \quad b := -(s_3 - 1)s_3, \quad c := (s_4 - 1)s_4$$

The coefficients a, b, c yield a rational map ϕ of the plane \mathbb{P}^2 into \mathbb{P}^2 , given by the system of cubics which, in homogeneous coordinates (u_2, u_3, u_4) (here $s_3 = u_3/u_2, s_4 = u_4/u_2$) reads out as

$$\phi(u_2, u_3, u_4) = [(u_3 - u_4)u_3u_4 : (u_2 - u_3)u_2u_3 : (u_4 - u_2)u_4u_2].$$

The cubic polynomials a, b, c are products of linear forms and the indeterminacy locus of ϕ is the set of 7 points

$$\{(1, 1, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Hence, ϕ is a double covering, and induces a birational involution on the source \mathbb{P}^2 , which is classically called the **Geiser involution**.

These seven base points are simple base points and blowing up the plane \mathbb{P}^2 at them one obtains a Del Pezzo surface of degree 2, F , which is a double cover of the plane branched on a curve \mathcal{B} of degree 4.

In order to compute effectively \mathcal{B} , we calculate the ramification divisor of ϕ , which is given by the determinant of the Jacobian matrix of ϕ .

Thus, the equation $R(u_2, u_3, u_4)$ of the ramification divisor is the determinant of the following matrix

$$\begin{pmatrix} 0 & 2s_3s_4 - s_4^2 & s_3^2 - 2s_3s_4 \\ s_4^2 - 2s_2s_4 & 0 & 2s_2s_4 - s_2^2 \\ 2s_2s_3 - s_3^2 & s_2^2 - 2s_2s_3 & 0 \end{pmatrix}$$

An elementary calculation yields

$$R(u_2, u_3, u_4) = 6u_2u_3u_4(u_2 - u_3)(u_3 - u_4)(u_4 - u_2).$$

Thus, the ramification divisor consists of six lines, which are contracted to the 6 points

$$\{(0, 1, -1), (1, 0, 1), (1, 0, -1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

The six lines of the ramification divisor intersect in the four points

$$\{e, e_1, e_2, e_3\} = \{(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\},$$

through each of which three of the lines meet, and in the further three points

$$\{P_1, P_2, P_3\} = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\},$$

where only two lines of the sextuple pass.

In the Del Pezzo surface F the strict transforms of the six lines are (-2) (smooth) rational curves, the points e, e_1, e_2, e_3 yield (-1) (smooth) rational curves, which are seen to be part of the ramification divisor of the degree 2 morphism

$$\varphi : F \rightarrow \mathbb{P}^2$$

since the strict transform of R contains these curves with multiplicity 3. Since the strict transform of R contains the (-1) (smooth) rational curves, blow up of the points P_i , with multiplicity 2, these curves are not in the ramification divisor of φ .

The conclusion is that the branch locus is the image of the 4 curves blow up of e, e_1, e_2, e_3 , *i.e.*,

$$\mathcal{B} = \{(a, b, c) \mid abc(a + b + c) = 0\}.$$

Hence, the Del Pezzo surface F , the double cover of \mathbb{P}^2 branched on \mathcal{B} , is contained in the line bundle \mathbb{L} over \mathbb{P}^2 whose sheaf of sections is $\mathcal{O}_{\mathbb{P}^2}(2)$, and is defined there by the equation

$$s^2 = abc(a + b + c).$$

5. SOME REMARKS IN THE CASE $N = 5$

Our computation for $n = 4$ is quite symmetric with respect to the variables with index ≥ 3 . So exactly the same computation yields:

LEMMA 5.1. $k(X_5) = k(s_3, s_4, s_5, w_2, w_3, w_4, w_5)$ with relations precisely

$$(s_3 - s_4)s_3s_4(w_2^3 - 1) = (s_3 - 1)s_3(w_4^3 - 1) - (s_4 - 1)s_4(w_3^3 - 1),$$

$$(s_3 - s_5)s_3s_5(w_2^3 - 1) = (s_3 - 1)s_3(w_5^3 - 1) - (s_5 - 1)s_5(w_3^3 - 1).$$

In other words, X_5 is birational to the above complete intersection in \mathbb{A}^7 with affine coordinate

$$(s_3, s_4, s_5, w_2, w_3, w_4, w_5)$$

defined by the above equations of bidegree $(3, 3)$.

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