Dedicated to Professor Lucian Bădescu on the occasion of his 70^{th} birthday

ON A CLASS OF ALGEBRAIC SURFACES CONSTRUCTED FROM ARRANGEMENTS OF LINES

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We analyse the numerical invariants of a class of complex algebraic surfaces constructed as finite abelian coverings of the projective plane ramified over certain arrangements of lines. This class includes two classical families of surfaces of general type discovered by P. Burniat and respectively by F. Hirzebruch.

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1. INTRODUCTION

Covering spaces are a fundamental construction in complex geometry. For a given connected complex manifold Z, a finite branched covering of Z is by definition a finite proper holomorphic mapping $h : Y \to Z$ of an irreducible normal complex space Y onto Z. The ramification locus R is the hypersurface in Y consisting of the points around which h is not biholomorphic. The image of R under h is the branching locus B, which is a hypersurface in Z. The restriction $h_0 : Y \setminus h^{-1}(B) \to Z \setminus B$ of h is a finite unbranched covering. The Grauert-Remmert correspondence $h \leftrightarrow h_0$ gives an equivalence between finite unbranched coverings of $Z \setminus B$ and finite coverings of Z branched at most at B, in the holomorphic category, cf. [24], and further on in the algebraic category, as shown by O. Zariski [42], cf. [38]. The topological version is due to R. Fox [23].

In this paper we aim to revisit and enlarge two classes of algebraic surfaces that appear as desingularizations of branched coverings of the projective plane $\mathbb{P}^2$. The particularity of the construction is that the coverings are Galois and finite abelian and the ramification locus consists of a union of lines. From this deceivingly simple situation one obtains special, sophisticated surfaces of general type S, such as the examples with vanishing geometric genus and irregularity $p_g(S) = q(S) = 0$, due to P. Burniat [12], or those attaining the Miyaoka-Yau bound $c_1^2(S) = 3c_2(S)$, due to F. Hirzebruch [27].

An algebraic surface S, or simply a *surface*, will be here a complex projective variety of dimension 2. We refer to the monograph [7] for the fundamentals of the theory of algebraic surfaces over the complex field, and to L. Bădescu [5, 6] for algebraically closed fields of arbitrary characteristic.

A smooth surface S is of *general type* if the canonical divisor K_S is big. Every such surface has a unique birational minimal model, *i.e.*, it does not contain any rational curve of self intersection -1, and for which the canonical divisor is big and nef.

To a minimal surface S of general type with structure sheaf $\mathcal{O}_S$, one associates several integer invariants (we use the notation $h^i(\mathcal{F}) := \dim_{\mathbb{C}} H^i(S, \mathcal{F})$, for a sheaf $\mathcal{F}$ on S):

- the *irregularity* $q(S) := h^1(\mathcal{O}_S)$,
- the *geometric genus* $p_g(S) := h^0(K_S)$,
- the *holomorphic Euler characteristic* $\chi(S) := h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S)$,
- the *Chern numbers* $c_1^2(S)$ and $c_2(S)$.

All these numbers are invariants under deformations and they are determined by the oriented topological type of S, more precisely by the Euler number $e(S)$ and the signature $\sigma(S)$ of the intersection form on $H^2(S)$. They are not independent as they satisfy a number of relations among themselves. The Chern number $c_1^2(S)$ coincides with the self-intersection K_S^2 of the canonical divisor, whereas $c_2(S)$ is the topological Euler characteristic $e(S)$. The holomorphic Euler characteristic may be expressed as $\chi(S) = 1 - q(S) + p_g(S)$. The Noether's formula gives $c_1^2(S) + c_2(S) = 12\chi(S)$, and by the index theorem we have that $3\sigma(S) = c_1^2(S) - 2c_2(S)$. The irregularity $q(S)$ is equal to half the first Betti number $b_1(S)$, by Hodge theory. Thus, the fundamental discrete invariants of a minimal surface S of general type are the Chern numbers $c_1^2(S)$ and $c_2(S)$. They characterize a surface up to a finite number of families. Sometimes is more convenient to work with $c_1^2(S)$ and $\chi(S)$ as basic invariants.

The geometric genus $p_g(S)$ and the irregularity $q(S)$ are birational invariants for a smooth surface S, whereas the Chern numbers are not.

The values of the Chern invariants are restricted by Noether's congruence

$$c_1^2 + c_2 \equiv 0 \mod 12$$

and by the following inequalities:

$$c_1^2 > 0,\ \chi > 0,\ 2\chi - 6 \leq c_1^2 \leq 9\chi.$$

With the exception of the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$, or equivalently $c_1^2 \leq 9\chi$, all the rest are classical and elementary. Moreover, for irregular surfaces, *i.e.* $q > 0$, we have Debarre's inequality $c_1^2 \geq 2\chi$. For all these restrictions, see [7], chapter 7.

For every pair of positive integers a, b the coarse moduli space $\mathcal{M}_{a,b}$ of surfaces S with $c_1^2 = a, c_2 = b$, is known to be a quasiprojective variety. In other words, the surfaces S are parametrized by finitely many irreducible families, thus in principle a classification is possible. It is a very difficult problem to completely describe the moduli spaces $\mathcal{M}_{a,b}$, but already the geographical question, that is to decide whether $\mathcal{M}_{a,b}$ is non-empty for a given pair a, b, is highly non-trivial.

The fundamental group $\pi_1(S)$ is the main invariant of a surface S which distinguishes connected components of the moduli space $\mathcal{M}_{a,b}$. However, it is not easy to determine $\pi_1(S)$, hence most often one calculates just its abelianization $H_1(S)$.

2. ABELIAN COVERINGS OF THE PLANE RAMIFIED OVER AN ARRANGEMENT OF LINES

In this section, we recall the basic facts about abelian Galois coverings $Y \to \mathbb{P}^2$ of the projective plane $\mathbb{P}^2$ branched along a line arrangement $\mathcal{L}$ in $\mathbb{P}^2$. For all the details relevant to the case at hand, we refer to E. Hironaka's memoir [25], whereas for the general theory to M. Namba's monograph [34]. Following Hirzebruch [27] and Kulikov [29], we show how to obtain a resolution X of the singular points of Y in terms of the singular points of $\mathcal{L}$, and we discuss the Hirzebruch formulae for the Chern numbers of X. Finally, we describe a procedure to compute the irregularity $q(X)$, based on a formula of Sakuma [37] expressing the first Betti number of an abelian branched covering in terms of the characteristic varieties of rank one local systems on the complement of the ramification locus.

Let $\mathcal{L}$ be an arrangement of n lines $L_1, \ldots, L_n$ in $\mathbb{P}^2$. The singular points $\operatorname{Sing}\mathcal{L}$ of $\mathcal{L}$ are the intersection points among the lines. Denote by t_m the number of m-fold points of $\mathcal{L}$, with $m \geq 2$, that is the points $P_{i_1,\ldots,i_m} = L_{i_1} \cap \cdots \cap L_{i_m}$ on m lines of the arrangement.

The first homology group $H_1(\mathbb{P}^2 \setminus \mathcal{L})$ is free abelian of rank $n-1$ generated by meridian loops $\lambda_1, \ldots, \lambda_n$ around the lines of $\mathcal{L}$, satisfying the relation $\lambda_1 + \cdots + \lambda_n = 0$.

2.1. Chern numbers of abelian coverings

Let $G = \mathbb{Z}_p^r$ be an elementary abelian group for p a prime number and $k < n$ a positive integer. The Galois G-coverings $g : Y \to \mathbb{P}^2$ branched along $\mathcal{L}$

are described by the epimorphisms $\phi : H_1(\mathbb{P}^2 \setminus \mathcal{L}) \to G$, see for example [34].

By construction, Y is a normal surface with isolated singularities. The singular points of Y can appear only over the intersection points of $\mathcal{L}$. In order to resolve the singularities of Y, we consider the blow up $\sigma : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of all the points in $\operatorname{Sing}\mathcal{L}$. For such a point P let $E_P = \sigma^{-1}(P)$ be the exceptional curve. Denote by λ_P the meridian loop around E_P viewed as an element in $H_1(\widetilde{\mathbb{P}}^2 \setminus \sigma^{-1}(\mathcal{L})) \cong H_1(\mathbb{P}^2 \setminus \mathcal{L})$ as identified by σ_*. It is a standard fact that $\lambda_P = \lambda_{i_1} + \cdots + \lambda_{i_m}$ for $P = P_{i_1,\dots,i_m} = L_{i_1} \cap \cdots \cap L_{i_m}$. We say that $P \in \operatorname{Sing}\mathcal{L}$ is a *non-branch point for* ϕ if $\phi(\lambda_P) = 0$.

Let $f : X \to \widetilde{\mathbb{P}}^2$ be the covering associated to the epimorphism $\phi : H_1(\widetilde{\mathbb{P}}^2 \setminus \sigma^{-1}(\mathcal{L})) \to G$. We are going to make the following assumptions on ϕ, see Kulikov [29]:

- $r \geq 2$ and g is ramified over each line L_i, that is $\phi(\lambda_i) \neq 0$, for all $1 \leq i \leq n$.
- all the m-fold points $P = P_{i_1,\dots,i_m}$ of $\mathcal{L}$ with $m \geq 2$ are ϕ-good, that is either all the pairs $\phi(\lambda_P)$ and $\phi(\lambda_{i_j})$, $1 \leq j \leq m$ are $\mathbb{Z}_p$-linearly independent or $\phi(\lambda_P) = 0$.

Under these assumptions, the surface X is smooth, cf. [29]. We have that X gives a resolution ν of the singularities of Y that fits in a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ \widetilde{\mathbb{P}}^2 \arrow[r, "\sigma"'] & \mathbb{P}^2 \end{tikzcd}$$

Note that if for a double point $P = L_{i_1} \cap L_{i_2}$ the pair $\phi(\lambda_{i_1})$ and $\phi(\lambda_{i_2})$ is $\mathbb{Z}_p$-linearly independent then Y already is smooth along $g^{-1}(P)$. For that reason, it is enough in the blow up σ to only consider the double points in $\operatorname{Sing}\mathcal{L}$ that are non-branch points for ϕ.

The Chern numbers of the smooth surface X associated to the branched covering Y were calculated by Kulikov in [29], extending the results of Hirzebruch from [27]. It turns out that $c_1^2(X)$ and $c_2(X)$ depend only on combinatorial data extracted from the arrangement $\mathcal{L}$ and the epimorphism ϕ. More precisely, on the number of lines n, the degree p^r of the covering, and the numbers of branch and non-branch points in $\operatorname{Sing}\mathcal{L}$ relative to ϕ. Denote by t'_m the number of non-branch m-fold points, and by t''_m the number of branch m-fold points for ϕ.

THEOREM 2.1 (Kulikov). *The Chern numbers of the surface $X = X(\mathcal{L})$ associated to a $\mathbb{Z}_p^r$-covering of the plane ramified along an arrangement $\mathcal{L}$ of n*

lines are given by

$$
\begin{aligned}
c_1^2(X) =& p^{r-2}[(np-3p-n)^2 - \sum_{m\geq 2}(mp-p-m)^2 t'_m \\
& - \sum_{m\geq 3}(mp-2p-m+1)^2 t''_m]
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
c_2(X) =& p^{r-2}[(3p^2 - 2p^2 n + 2pn + p^2 \sum_{m\geq 2} t'_m \\
& - t''_2 + \sum_{m\geq 3}((m-1)(p-1)^2+1)t''_m]
\end{aligned}
\tag{2.2}
$$

2.2. **Irregularities of abelian coverings**

The first Betti number $b_1(X) = 2q(X)$ of X, or equivalently its irregularity, depends in principle on more complicated data than just the combinatorics of $\mathcal{L}$ and ϕ. In [37], Sakuma gave a formula for computing the first Betti numbers of finite abelian branched covers that we are going to state here in terms of the first twisted cohomology group of the arrangement complement $\mathbb{P}^2 \setminus \mathcal{L}$.

The rank one complex local systems on a finite CW complex M are parametrized by the character group $\mathbb{T}(M) := \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$. The *characteristic varieties* of M record the jumps in dimension of the twisted cohomology of characters, and can be calculated from Alexander matrices associated to $\pi_1(M)$, see Hironaka [25].

For $\tau : \pi_1(M) \to \mathbb{C}^* = \mathrm{Aut}(\mathbb{C})$, we let $H^1(M,\tau)$ be the cohomology of M with coefficients in the $\pi_1(M)$-module $\mathbb{C}_\tau$ on $\mathbb{C}$ with action $\gamma\cdot v = \tau(\gamma)v$, and we denote its dimension by $h^1(M,\tau) := \dim_{\mathbb{C}} H^1(M,\tau)$.

The d-th *characteristic variety* of M is the subvariety of $\mathbb{T}(M)$, defined by

$$\mathcal{V}_d(M) := \{\tau \in \mathbb{T}(M) \mid \dim_{\mathbb{C}} H^1(M,\tau) \geq d\}.$$

Now let the space M be the complement $M(\mathcal{A}) = \mathbb{P}^2 \setminus \mathcal{A}$ of an arrangement $\mathcal{A}$ of n lines in $\mathbb{P}^2$. The characteristic varieties $\mathcal{V}_d(\mathcal{A}) := \mathcal{V}_d(M(\mathcal{A}))$ of arrangement complements are known to be the union of subtori of the character torus $\mathbb{T}(M(\mathcal{A})) \cong (\mathbb{C}^*)^{n-1}$, cf. Arapura [1], possibly translated by torsion characters, see [4, 11, 20, 21]. Morever, these subtori intersect pairwise only in finitely many torsion points. In fact, the subtori structural property of $V_d(M)$ holds more generally, for $M = P \setminus D$ a smooth quasi-projective variety, the complement of a normal crossing divisor D in a smooth projective variety P, and it is related with the existence of pencils of hypersurfaces on M, as shown by Arapura [1], see [1, 4, 13, 19, 21].

There are a number of types of irreducible components in $\mathcal{V}_d(\mathcal{A})$, see [2, 19, 39]. The simplest ones, the *local* components, determined by the points in $\operatorname{Sing}\mathcal{A}$ of multiplicity $m \geq 3$. Then components that depend on subtler data, that take into account the relative position of those intersection points. The *coordinate* components, which are contained in a subtorus $\mathbb{T}_k = \{t_k = 1\}$ of $\mathbb{T}(M)$, or in a translated coordinate subtorus $\tau\mathbb{T}_k$, and the *global* or *essential* components, those that are not translated coordinate components.

Let τ be a character on $M(\mathcal{A})$. Denote by $\mathcal{A}_\tau$ the subarrangement of $\mathcal{A}$ consisting of the lines L so that $\tau(\lambda_L) \neq 1$. For a subarrangement $\mathcal{B} \subseteq \mathcal{A}$ we denote by $\tau_{|\mathcal{B}}$ the pull-back of τ to $\mathbb{T}(M(\mathcal{B}))$.

THEOREM 2.2 (Sakuma). *The first Betti number of the surface $X = X(\mathcal{L})$ associated to an abelian Galois covering of $\mathbb{P}^2$ defined by an epimorphism $\phi : H_1(\mathbb{P}^2 \setminus \mathcal{L}) \to G$ and ramified along an arrangement of lines $\mathcal{L}$ is given by*

$$b_1(X) = \sum_{1 \neq \rho \in \widehat{G}} h^1(M(\mathcal{L}_{\rho\phi}), \rho\phi_{|\mathcal{L}_{\rho\phi}}), \tag{2.3}$$

where $\widehat{G}$ is the set of characters $\rho : G \to \mathbb{C}^*$ of the abelian group G.

Note that the knowledge of the characteristic varieties $\mathcal{V}_d(\mathcal{A})$ for all the subarrangements $\mathcal{A}$ of $\mathcal{L}$ are in general needed to calculate the Betti number $b_1(X(\mathcal{L}))$.

The twisted cohomology groups $H^1(M(\mathcal{A}), \tau)$, and so the cohomology jumping loci $\mathcal{V}_d(\mathcal{A})$, can be calculated, by a standard method, using Fox calculus, from a presentation of the fundamental group $\pi_1(M(\mathcal{A}))$, see Suciu [39] for references and more details. The group $\pi := \pi_1(M(\mathcal{A}))$ admits a presentation $\langle x_1, \ldots, x_n \mid w_1, \ldots, w_m \rangle$ having all relators $w_1, \ldots, w_m$ commutators in the free group generated by $x_1, \ldots, x_n$. Explicit and efficient methods to derive such presentations, based on braid monodromy techniques, are available, see [2, 14, 17], and in the case of real complexified line arrangements, also [26].

3. A CLASS OF ALGEBRAIC SURFACES

Following the procedures outlined in Section 2, we construct branched covering surfaces associated to a class of line arrangements in $\mathbb{P}^2$, and we calculate their Chern numbers and their irregularities, or equivalently their first Betti numbers.

We first introduce the arrangements $\mathcal{L}$ in $\mathbb{P}^2$ used to construct the algebraic surfaces $X(\mathcal{L})$ whose invariants we are going to determine. This family of arrangements $\mathcal{A}_t(n_1, n_2, n_3)$ depends on a triple of positive integers n_1, n_2, n_3 and another integer $t \geq 0$.

We start from three lines in general position intersecting in three points P_1, P_2, P_3, say the coordinate lines in $\mathbb{P}^2$ and the triangle they form. Then, for each i we add $n_i - 1$ lines through P_i to obtain an arrangement $\mathcal{L}$ of $n = n_1 + n_2 + n_3$ lines. The points appearing as intersections of the added $n-3$ lines are either 2-fold or 3-fold. We denote by t the number of those triple points. If $t = 0$ then clearly the number of double points is equal to $\sum_{i<j}(n_i - 1)(n_j - 1)$. It follows, by adding the $\sum_i (n_i - 1) = n - 1$ double points on the sides of the triangle $P_1 P_2 P_3$, that the arrangement $\mathcal{L}$ with $t = 0$ has a total of $s - n$ double points, where $s = \sum_{i<j} n_i n_j$. Observe that in general we have that $3t \leq s - n$.

We denote a line arrangement $\mathcal{L}$ constructed as above by $\mathcal{A}_t(n_1, n_2, n_3)$. The combinatorial characteristics of $\mathcal{A}_t(n_1, n_2, n_3)$ are as follows: $t_2 = s - n - 3t$ double points, $t \geq 0$ triple points not on the triangle and the points P_i of multiplicities $m_i = n_i + 1$.

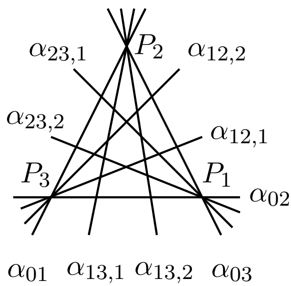

Fig. 1. The arrangement $\mathcal{A}_0(3, 3, 3)$.

Assume from now on that $n_i \geq 2$. We address the concrete realizability of these arrangements and their combinatorics, by linear equations. More precisely, when speaking about $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$, we are going to use the following defining equations

$$x_1 x_2 x_3 \prod_{1 \leq l < n_3} (x_1 - z_{12,l} x_2) \prod_{1 \leq l < n_2} (x_1 - z_{13,l} x_3) \prod_{1 \leq l < n_1} (x_2 - z_{23,l} x_3),$$

where $z_{ij,l}$ are all non-zero. Note that three lines $x_1 - z_{12} x_2 = 0$, $x_1 - z_{13} x_3 = 0$, and $x_2 - z_{23} x_3 = 0$ are concurent if and only if $z_{12} z_{23} = z_{13}$.

We are indexing the lines of $\mathcal{L}$, as well as the corresponding generators $\lambda_{0k}, \lambda_{ij,l}$ of $H_1(\mathbb{P}^2 \setminus \mathcal{L})$ and $\alpha_{0k}, \alpha_{ij,l}$ of $\pi_1(\mathbb{P}^2 \setminus \mathcal{L})$, in the following manner:

$$\{L_{0k} = \{x_k = 0\}\}_{1 \leq k \leq 3}, \{L_{ij,l} = \{x_i - z_{ij,l} x_j = 0\}\}_{1 \leq i < j \leq j \leq 3, 1 \leq l < n_k}.$$

We consider coverings associated to an epimorphism $\phi : H_1(\mathbb{P}^2 \setminus \mathcal{L}) \to \mathbb{Z}_p^r$ with p prime and $r \geq 2$, satisfying the conditions from subsection 2.1, that is

$\phi(\lambda_i) \neq 0$, for all $1 \leq i \leq n$, and all the intersection points of $\mathcal{L}$ are ϕ-good, and, in addition, the conditions that all double points and the points P_i are indeed branch points.

We are going to distinguish two types of epimorphisms ϕ, according to the kind of ϕ-goodness condition the triple points not on the triangle $P_1P_2P_3$ satisfy. More precisely, those ϕ for which all such triple points are branch points, which we call of *Hirzebruch type*, and respectively those ϕ for which they are all non-branch points, which we call of *Burniat type*. The difference is as follows. In the first case, we have a contribution of t to the number t_3'' of branch 3-fold points, whereas $t_3' = 0$. In the second case, we have a contribution of t to the number t_3' of non-branch 3-fold points, whereas $t_3'' = 0$ (unless some $n_i = 2$). In both cases, we are going to have $t_2' = 0$ and $t_2'' = s - n - 3t$, and a contribution of 1 to the branch m_i-fold points from P_i.

An immediate application of the Hirzebruch-Kulikov formulae from Theorem 2.1 gives the Chern numbers of the covering surface $X = X(\mathcal{A}_t(n_1, n_2, n_3))$ associated to an epimorphism ϕ of the two types.

PROPOSITION 3.1. *The Chern numbers of the surface $X = X(\mathcal{L})$ associated to a $\mathbb{Z}_p^r$-covering of Hirzebruch type of $\mathbb{P}^2$ ramified along an arrangement $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ of n lines are given by*

$$c_1^2(X) = p^{r-2}[p^2(2s - 4n + 6 - t) + p(4t - 4s + 4n) + (2s - 4t)] \tag{3.1}$$

$$c_2(X) = p^{r-2}[p^2(s - 2n + 3 - t) + p(2t - 2s + 2n) + (s + 3)] \tag{3.2}$$

PROPOSITION 3.2. *The Chern numbers of the surface $X = X(\mathcal{L})$ associated to a $\mathbb{Z}_p^r$-covering of Burniat type of $\mathbb{P}^2$ ramified along an arrangement $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ of n lines are given by*

$$c_1^2(X) = p^{r-2}[p^2(2s - 4n + 6 - 4t) + p(12t - 4s + 4n) + (2s - 9t)] \tag{3.3}$$

$$c_2(X) = p^{r-2}[p^2(s - 2n + 3 - 2t) + p(6t - 2s + 2n) + (s + 3 - 3t)] \tag{3.4}$$

The irregularity $q(X)$ of the surface $X = X(\mathcal{L})$ associated to a $\mathbb{Z}_p^r$-covering ϕ of $\mathbb{P}^2$ ramified along an arrangement $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ of n lines is more difficult to determine, as it depends on more than just the combinatorics of $\mathcal{L}$ and ϕ. We are going to focus here on the two extreme cases, namely $r = n - 1$ and $r = 2$.

We will use Sakuma's formula 2.3 to derive the Betti numbers in these two cases. The results will be a consequence of an explicit description of the characteristic varieties of $M = M(\mathcal{L})$ for the arrangements $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ of n lines.

First we identify the character torus $\mathbb{T}(M) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \cong (\mathbb{C}^*)^{n-1}$ with a subset of $(\mathbb{C}^*)^n$ as follows

$$\mathbb{T}(M) \cong \{t_{01}t_{02}t_{03}t_{12,1}\dots t_{12,n_3-1}t_{13,1}\dots t_{13,n_2-1}t_{23,1}\dots t_{23,n_1-1} = 1\}.$$

We denote the local component in $\mathcal{V}_1(M(\mathcal{L}))$ coming from a point $P \in \operatorname{Sing}\mathcal{L}$ by C_P. Thus, we have in $\mathbb{T}(M)$ the following local components for $\mathcal{V}_1(M)$:

$$\begin{aligned} C_{P_k} = \{t_{0i}t_{0j}t_{ij,1}\dots t_{ij,n_k-1} = t_{0k} = t_{ik,1} = \cdots = t_{ik,n_j-1} = \\ = t_{jk,1} = \cdots = t_{jk,n_i-1} = 1\}, \end{aligned}$$

for the m_k-fold point $P_k = L_{0i} \cap L_{0j} \cap L_{ij,1} \cap \dots \cap L_{ij,n_k-1}$, and

$$\begin{aligned} C_Q = \{t_{12,l_3}t_{13,l_2}t_{23,l_1} = 1, t_{01} = t_{02} = t_{03} = 1, t_{ij,l} = 1, \\ \text{for all } 1 \le i < j \le 3, l \neq l_k\}, \end{aligned}$$

for a triple point $Q = L_{12,l_3} \cap L_{13,l_2} \cap L_{23,l_1}$.

For $\pi_1(M(\mathcal{L}))$ we employ presentations inspired by those derived in [3, 16], where our situation in which $t = 0$ it is a particular case. We use Terada type meridian generators which satisfy commutator relators among them: α_{0k}, $1 \le k \le 3$ around the lines L_{0k}, and $\alpha_{ij,l}$, $1 \le i < j \le j \le 3$, $1 \le l < n_k$ around the lines $L_{ij,l}$. We take advantage of the nice form of this presentation for $\pi_1(M(\mathcal{L}))$ to write an explicit formula for the boundary maps in dimensions up to 2 in the chain complex of the universal abelian cover of $M(\mathcal{L})$, by means of the Fox free calculus, see [39] for the outline of the procedure. We then calculate the dimension $h^1(M(\mathcal{L}), \tau)$ of the twisted cohomology of $M(\mathcal{L})$ arriving at the following statement, whose detailed proof will be given elsewhere.

THEOREM 3.3. *The characteristic variety $\mathcal{V}_1(\mathcal{L})$ of $M = M(\mathcal{L})$ is the following union of subtori pairwise intersecting only in the unit 1 of the character torus:*

$$\mathcal{V}_1(\mathcal{L}) = C_{P_1} \cup C_{P_2} \cup C_{P_3} \cup C_{Q_1} \cup \dots \cup C_{Q_t} \cup D_1 \cup \dots \cup D_\beta,$$

where $\beta = \beta(\mathcal{L})$ is the number of type $\mathcal{A}_q$ complex reflection subarrangements of $\mathcal{L}$. The dimension of the cohomology $H^1(M, \tau)$ of M with coefficients in the rank one local system $\tau \neq 1$ is then determined by the components of $\mathcal{V}_1(\mathcal{L})$ as follows:

$$h^1(M,\tau) = \begin{cases} 0 & \text{if } \tau \notin \mathcal{V}_1(\mathcal{L}) \\ 1 & \text{if } \tau \in \mathcal{V}_1(\mathcal{L}) \setminus (C_{P_1} \cup C_{P_2} \cup C_{P_3}) \\ 2 & \text{if } \tau \in D_k \cap D_l \\ n_i & \text{if } \tau \in C_{P_i} \end{cases}$$

There are two facts that facilitate the computation of the first Betti numbers of the surfaces $X = X(\mathcal{L})$. The first, and most important, is that $\mathcal{A}_t(n_1, n_2, n_3)$ may be taken to be subarrangements of certain reflection arrangements $\mathcal{B}_q$ described below, or of their deformations. The second is that sometimes $\mathcal{A}_t(n_1, n_2, n_3)$ may be realized by linear equations with real coefficients.

Example 3.4. Let $\mathcal{B}_q$ be the family of full monomial arrangements, see Orlik-Terao [35], corresponding to the complex reflection groups of type $\mathbb{G}(q, 1, 3), q \geq 1$, and defined by the polynomials

$$x_1x_2x_3(x_1^q - x_2^q)(x_1^q - x_3^q)(x_2^q - x_3^q).$$

Presentations for $\pi_1(M(\mathcal{B}_q))$ are determined by D. Cohen in [15], see also [33].

Notice that $\mathcal{B}_q$ is an arrangement $\mathcal{A}_t(q+1, q+1, q+1)$ in our class, with $t = q^2$. We also need the monomial arrangements $\mathcal{C}_q$, which the subarrangements of $\mathcal{B}_q$ corresponding to the irreducible complex reflection groups $\mathbb{G}(q, q, 3), q \geq 2$, and defined by the polynomials

$$(x_1^q - x_2^q)(x_1^q - x_3^q)(x_2^q - x_3^q).$$

Note that $\mathcal{B}_1$ is isomorphic to $\mathcal{C}_2$, and that $\mathcal{B}_1, \mathcal{B}_2$ are real reflection arrangements.

The irreducible components of the characteristic varieties $\mathcal{V}_1(M(\mathcal{A}_q))$ of the monomial arrangements $\mathcal{A}_q = \mathcal{B}_q, \mathcal{C}_q$ are discussed extensively in Cohen-Suciu [17]. The non-local ones turn out to be in correspondence with the monomial subarrangements of $\mathcal{A}_q$, and they are coordinate components if the subarrangement is proper, and essential otherwise. Moreover, all of them are subtori of dimension 2, passing through the unit 1 of $\mathbb{T}(M(\mathcal{A}_q))$, and intersecting in finitely many torsion points. With exception of the case $q = 3$, where there are more of them, there is a unique essential component in $\mathcal{V}_1(M(\mathcal{A}_q))$, cf. [17].

The global component of $V_1(B_q)$ is the following 2-dimensional subtorus of the character torus $\mathbb{T}(M(\mathcal{B}_q)) \subset (\mathbb{C}^*)^{3q+3}$

$$D = \{t_{0k} = t_{ij,1}^q,\ t_{ij,1} = \ldots = t_{ij,q},\ t_{ij,1}t_{ik,1}t_{jk,1} = 1,\ 1 \leq i < j \leq 3, k \neq i, j\}. \tag{3.5}$$

The global component of $V_1(C_q)$ is the following 2-dimensional subtorus of the character torus $\mathbb{T}(M(\mathcal{C}_q)) \subset (\mathbb{C}^*)^{3q}$

$$D = \{t_{ij,1} = \cdots = t_{ij,q},\ t_{ij,1}t_{ik,1}t_{jk,1} = 1,\ 1 \leq i < j \leq 3, k \neq i, j\}. \tag{3.6}$$

The covering associated to $\phi : H_1(\mathbb{P}^2 \setminus \mathcal{L}) \to H_1(\mathbb{P}^2 \setminus \mathcal{L}, \mathbb{Z}_p) \cong \mathbb{Z}_p^{n-1}$ is known as the congruence covering, see [25, 37, 39]. Notice that ϕ is a Hirzebruch type epimorphism, as all the multiple points of $\mathcal{L}$ are branching points for ϕ.

Proposition 3.5. *The first Betti number of the surface $X = X(\mathcal{L})$ associated to the $\mathbb{Z}_p^{n-1}$ congruence covering of $\mathbb{P}^2$ ramified along an arrangement $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ of n lines is given by*

$$b_1(X) = \sum_{m \geq 3} t_m b(p, m) + b(p, 3)\beta(\mathcal{L}), \tag{3.7}$$

with $\beta(\mathcal{L}) = \sum_{q \geq 1} \beta_q(\mathcal{L})$, where $\beta_q(\mathcal{L})$ is the number of subarrangements of $\mathcal{L}$ isomorphic to type $\mathcal{A}_q$ monomial arrangements, and

$$b(p, m) = (p - 1) \left[(m - 2)p^{m-2} - 2(p^{m-3} + \cdots + p + 1)\right].$$

We note that $b(p, m)$ is the first Betti number of $X(\mathcal{P})$, the surface associated to the $\mathbb{Z}_p^{m-1}$ congruence covering of $\mathbb{P}^2$ ramified along a pencil of m lines $\mathcal{P}$ through a point, cf. [40]. In that paper, Tayama shows that, for an arbitrary line arrangement $\mathcal{L}$ in $\mathbb{P}^2$, the right hand side of (3.7), truncated to $\sum_{m \geq 3} t_m b(p, m) + b(p, 3)\beta_1(\mathcal{L})$, is a lower bound for the Betti number $b_1(X)$ of the surface corresponding to the congruence covering.

Proposition 3.6. *The first Betti number of the surface $X = X(\mathcal{L})$ associated to the $\mathbb{Z}_p^2$-covering of Hirzebruch type of $\mathbb{P}^2$ ramified along an arrangement $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ can take the following values*

$$b_1(X) = \begin{cases} 0 & \text{if } p = 2 \\ \text{either } 0 \text{ or } p - 1 & \text{if } p \text{ odd prime} \end{cases} \tag{3.8}$$

The first Betti number of the surface $X = X(\mathcal{L})$ associated to the $\mathbb{Z}_p^2$-covering of Burniat type of $\mathbb{P}^2$ ramified along an arrangement $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$ is given by

$$b_1(X) = 0. \tag{3.9}$$

With Theorem 3.3 in hand, the calculation of the Betti number of the congruence coverings from Proposition 3.5 is immediate.

Proof of Proposition 3.5. As observed by Suciu in [39], the calculation of $b_1(X)$ in the case of congruence $\mathbb{Z}_p^{n-1}$ coverings reduces to a counting of p-torsion points:

$$b_1(X) = \sum_{\tau \in \mathrm{Tors}_p \mathbb{T}(M)} h^1(M(\mathcal{L}_\tau), \tau_{|\mathcal{L}_\tau}),$$

A character $\tau \neq 1$ will contribute a non-zero h^1 if and only if $\tau_{|\mathcal{L}_\tau}$ it is a torsion point of order p lying on some essential global component of $\mathcal{V}_1(\mathcal{L}_t)$. One may verify, following arguments similar to Cohen-Suciu [17], that the only non-local components of $\mathcal{V}_1(\mathcal{L})$ are those induced from the essential components in $\mathcal{V}_1(\mathcal{A})$ of the monomial subarrangements $\mathcal{A}$ of $\mathcal{L}$. Thus, we only have to add up, for each monomial and pencil subarrangement, the number of p-torsion points that do not lie on any coordinate subtorus. For a component of dimension d will there are $b(p, d+1)$ such p-torsion points.

This count will be enough as long as the points τ that potentially may contribute a $h^1 = 2$ are contained in some coordinate subtorus of $\mathbb{T}(M(\mathcal{L}_\tau))$. Indeed, suppose τ is a p-torsion point in the intersection of two non-local components $D' \cap D''$ of $\mathcal{V}_1(\mathcal{L})$. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the monomial subarrangements of $\mathcal{L}$ for which the pull-backs of D' and D'' are essential global components. Then clearly $\mathcal{L}_\tau$ is a proper subarrangement of either $\mathcal{A}'$ and $\mathcal{A}''$, and using again [17], one may check that $\mathcal{L}_\tau$ is never a monomial or a pencil subarrangement, so it does not have a global component in $\mathcal{V}_1$.

It remains to record the dimensions of the components in $\mathcal{V}_1(\mathcal{L})$ and use their correspondence with the the points in $\operatorname{Sing}\mathcal{L}$ and to the monomial subarrangements of $\mathcal{L}$. $\square$

We turn now to the calculation of the first Betti number of the surfaces $X = X(\mathcal{L})$ associated to the $\mathbb{Z}_p^2$-coverings of $\mathbb{P}^2$ ramified over $\mathcal{L} = \mathcal{A}_t(n_1, n_2, n_3)$.

Proof of Proposition 3.6. By Sakuma's formula (2.3), the first Betti number of the surface $X = X(\mathcal{L})$ associated to the epimorphism $\phi : H_1(\mathbb{P}^2 \setminus \mathcal{L}) \to G$ is calculated from the twisted cohomology $H^1(M(\mathcal{A}), \tau)$ of certain subarrangements $\mathcal{A} = \mathcal{L}_{\rho\phi}$ in $\mathcal{L}$, constructed from ϕ and the set of characters $\rho : G \to \mathbb{C}^*$ of the abelian group $G = \mathbb{Z}_p^2$, as follows

$$b_1(X) = \sum_{1 \neq \rho \in \widehat{G}} h^1(M(\mathcal{L}_{\rho\phi}), \rho\phi_{|\mathcal{L}_{\rho\phi}}).$$

We shall take advantage of two key facts: all the points in $\operatorname{Sing}\mathcal{L}$ are required to be ϕ-good, and the subarrangements $\mathcal{A}$ of $\mathcal{L}$ are again in the same class, thus their twisted cohomology is known from Theorem 3.3. The first thing to point out regarding the latter fact is that there are only two types of subarrangements $\mathcal{A}$ of $\mathcal{L}$ that support global components in their characteristic varieties, namely either $\mathcal{A} = \mathcal{P}$ a pencil of lines through a point of multiplicity at least 3, or $\mathcal{A} = \mathcal{B}$ a non-pencil monomial arrangement. The second feature is that we may ignore the torsion characters in V_2, as their support is neither pencil nor monomial, as a consequence of the results in [17], thus they do not in fact contribute to the Betti number.

Fix a_1, a_2 generators for $G = \mathbb{Z}_p^2$, and let ϕ be defined by $\phi(\alpha_{0k}) = a_1^{u_{0k}} a_2^{v_{0k}}$ and $\phi(\alpha_{ij,l}) = a_1^{u_{ij,l}} a_2^{v_{ij,l}}$, where the $u's$ and the $v's$ are residues modulo p. Then let ρ be a non-trivial character of G, and set $\xi_1 := \rho(a_1)$ and $\xi_2 := \rho(a_2)$, where $\xi_1 = \xi^{\mu_1}$ and $\xi_2 = \xi^{\mu_2}$ for some fixed primitive root of unity ξ of order p. Note that if for some α we have $\phi(\alpha) = a_1^u a_2^v \neq 1$, and $\tau(\alpha) = \xi_1^u \xi_2^v = 1$ then

u, v must satisfy $\mu_1 u + \mu_2 v \equiv 0$. If $\xi_1 = 1$ then $v = 0$, hence $\phi(\alpha) = a_1^u$, whereas if $\xi_2 = 1$ then $u = 0$, hence $\phi(\alpha) = a_2^v$. If neither ξ_1, ξ_2 equals 1, then $v \equiv -\mu_1\mu_2^{-1}u$, and $\phi(\alpha) = (a_1 a_2^{-\mu_1\mu_2^{-1}})^u$. In all three cases, for fixed ξ_1, ξ_2, we obtain that all images $\phi(\alpha)$ of elements α in $\ker\tau$ belong to the same cyclic subgroup of $G = \mathbb{Z}_p^2$. This implies that two lines L, L' in $\mathcal{L}$ intersecting in a double point P cannot be both out of $\mathcal{L}_\tau$. If not, their meridians α, α' would be in $\ker\tau$, and by the previous remark, would be linearly dependent over $\mathbb{Z}_p$, thus violating the ϕ-good condition for P.

Suppose $\tau = \rho\phi$ is a character in $\mathbb{T}(M)$ that contributes to $b_1(X)$ a positive value $h^1(\tau) := h^1(M(\mathcal{L}_{\rho\phi}), \rho\phi_{|\mathcal{L}_{\rho\phi}})$. If $\mathcal{L}_\tau = \mathcal{L}$, that is no coordinate of τ is equal to 1, then $\mathcal{V}_1(M(\mathcal{L}))$ must contain an irreducible component that is global, but this only happens if $\mathcal{L}$ is the monomial arrangement $\mathcal{B}_q$. Then τ must satisfy the equations (3.5) defining the global component. That yields words in $\ker\tau$, and as discussed before their images generate a cyclic subgroup of $\mathbb{Z}_p^2$, say $A = \langle a \rangle$. If ϕ is Hirzebruch type then we may assume $\phi(\alpha_{ij,1}\alpha_{ik,1}\alpha_{jk,1}) = a$. Then one must have $\phi(\alpha_{01}\alpha_{02}\alpha_{03}) = \phi(\alpha_{ij,1}\alpha_{ik,1}\alpha_{jk,1}) = a^q$ from (3.5). We end up with $a^{2q} = 1$, which implies $p = 2$ and $q = 1$. We reach a contradiction, since $\mathcal{L}_\tau = \mathcal{L}$ forces $\tau = -1$, but $\tau_{01}\tau_{02}\tau_{03}$ should be 1. If ϕ is Burniat type, then $\phi(\alpha_{ij,1}\alpha_{ik,1}\alpha_{jk,1}) = 1$, which forces in turn $\phi(\alpha_{0i}\alpha_{0j}\alpha_{ij,1}^q) = \phi(\alpha_{01}\alpha_{02}\alpha_{03}) = \phi(\alpha_{ij,1}\alpha_{ik,1}\alpha_{jk,1})^q = 1$, a contradiction, as P_k are branch points for ϕ.

Now suppose that $\mathcal{L}_\tau$ is a proper subarrangement of $\mathcal{L}$, that is some coordinates of τ are equal to 1. If $\mathcal{L}_\tau$ is a non-pencil monomial arrangement then either we repeat the argument using the equations (3.5) if $\mathcal{L}_\tau$ is of type $\mathcal{B}_q$, or we argue similarly using the equations (3.6) if $\mathcal{L}_\tau$ is of type $\mathcal{C}_q$. We remain with the case $\mathcal{L}_\tau = \mathcal{P}$ is a pencil of lines, that is only the multiple points in $\mathcal{L}$ may contribute positively to the Betti number.

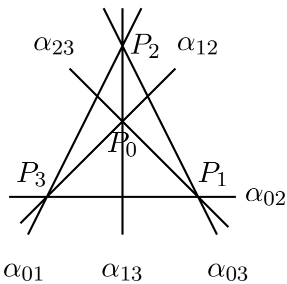

Fig. 2. The braid arrangement $\mathcal{A}_1(2, 2, 2)$.

First note that at most one pencil subarrangement $\mathcal{P}$ of $\mathcal{L}$ may in fact contribute, as τ uniquely determines $\mathcal{L}_\tau$. Then if $\mathcal{P}$ does contribute, then its contribution is precisely $p - 1$. Finally, for the Burniat type no contributions really occur. □

The number $\beta = \beta(\mathcal{L})$ is defined in terms of the monomial arrangements, the simplest of which is the braid arrangement, which occurs as $\mathcal{A} = \mathcal{A}_1(2,2,2)$ in our family. We discuss next this arrangement and the non-local component D that appears in $\mathcal{V}_1(\mathcal{A})$.

Example 3.7. The arrangement $\mathcal{A} = \mathcal{A}_1(2,2,2)$ in Figure 3 is known under various names, braid arrangement, complete quadrilater, Ceva arrangement, type $\mathbb{A}_2$ Coxeter reflection arrangement, etc., see [17, 39]. It has $n = 6$ lines, $t_2 = 3$ double points, and $t_3 = 4$ triple points. As defining equations for $\mathcal{A}$, we take

$$x_1x_2x_3(x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

For the fundamental group $\pi_1(M)$ of the complement $M = M(\mathcal{A})$ we use a presentation due to Terada [41], with generators $\alpha_{ij}, 0 \le i < j \le 3$ satisfying $\alpha_{01}\alpha_{02}\alpha_{12}\alpha_{03}\alpha_{13}\alpha_{23} = 1$ and for all $i < j < k$ the commutator relators

$$[\alpha_{ij}\alpha_{ik}\alpha_{jk}, \alpha_{ij}] = 1, [\alpha_{ij}\alpha_{ik}\alpha_{jk}, \alpha_{ij}] = 1, [\alpha_{ij}\alpha_{ik}\alpha_{jk}, \alpha_{ij}] = 1.$$

The characteristic variety $\mathcal{V}_1(\mathcal{A})$ consists of 5 irreducible components, 2-dimensional subtori C_0, C_1, C_2, C_3, D of the torus $\mathbb{T}(M) = \{t_{01}t_{02}t_{03}t_{12}t_{13}t_{23} = 1\} \cong (\mathbb{C}^*)^5 \subset (\mathbb{C}^*)^6$. Four of the components are local, determined by the 4 triple points $P_i = L_{jk} \cap L_{jl} \cap L_{kl}$

$$C_i := C_{P_i} = \{t_{jk}t_{jl}t_{kl} = t_{ij} = t_{ik} = t_{il} = 1\}, 0 \le i \le 3, \{i,j,k,l\} = \{0,1,2,3\}.$$

The remaining component D is supported on the entire arrangement

$$D = \{t_{01} = t_{23}, t_{02} = t_{13}, t_{03} = t_{12}, t_{01}t_{02}t_{03} = 1\}.$$

The appearance of this non-local essential component is due to the presence of a pencil of conics in $\mathbb{P}^2$ whose special fibers form a partition of the lines of $\mathcal{A}$, see [19, 22, 32] for the whole story.

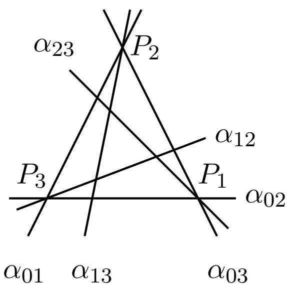

Fig. 3. The Arvola arrangement $\mathcal{A}_0(2,2,2)$.

The twisted cohomology $H^1(M,\tau)$ of M for $1 \neq \tau \in \mathbb{T}(M)$ is then given by

$$h^1(M,\tau) = \begin{cases} 0 & \text{if } \tau \notin \mathcal{V}_1(\mathcal{A}) \\ 1 & \text{if } \tau \in \mathcal{V}_1(\mathcal{A}) \setminus \{1\}. \end{cases}$$

An application of Sakuma's formula gives, as explained in the proof of Proposition 3.5, $b_1(X_p) = 5(p-1)(p-2)$ for the Hirzebruch type surface X_p associated to the $\mathbb{Z}_p^5$ congruence covering, see [25, 40, 39]. From Proposition 3.1, the Chern numbers of X_p are equal to $c_1^2(X_p) = 5p^3(p-2)^2$ and $c_2(X_p) = p^3(2p^2-10p+15)$, see [27, 25, 39]. For $p > 2$ the surface X_p is of general type, and X_2 it is a $K3$-surface, cf. [27]. The most interesting case is the one for $p = 5$, where $c_1^2 = 3c_2$.

We see now to the Hirzebruch type surface $X_{p,2}$ associated to a $\mathbb{Z}_p^2$-covering. We ignore the case $p = 2$ as $X_{2,2}$ is not of general type. Using Sakuma's formula we obtain that either $b_1(X_{p,2}) = 0$ or $b_1(X_{p,2}) = p-1$, recovering, for $p = 5$, the computations of Bauer-Catanese [10] and Ishida [28]. By Proposition 3.1, the Chern numbers of $X_{p,2}$ are $c_1^2(X_{p,2}) = 5(p-2)^2$ and $c_2(X_{p,2}) = 2p^2 - 10p + 15$. Again for $p = 5$, we get $c_1^2 = 3c_2$.

Finally, for the Burniat type surfaces $X_{p,2}$ we have that $b_1(X_{p,2}) = 0$, and $c_1^2(X_p) = 5p^2 - 12p + 15$, $c_2(X_p) = p^2 - 6p + 12$, but these surfaces are not very interesting.

Example 3.8. The arrangement $\mathcal{A} = \mathcal{A}_0(2,2,2)$ in Figure 3.7 is a deformation of the braid arrangement, first considered by Arvola, and which enjoys interesting homotopy properties, see [3]. It has $n = 6$ lines, $t_2 = 6$ double points, and $t_3 = 3$ triple points. Note that $t = 0$, thus there is no difference between the Hirzebruch and Burniat types. As defining equations for $\mathcal{A}$, we take

$$x_1x_2x_3(x_1-x_2)(x_1-x_3)(x_2-zx_3), z \neq 0, 1.$$

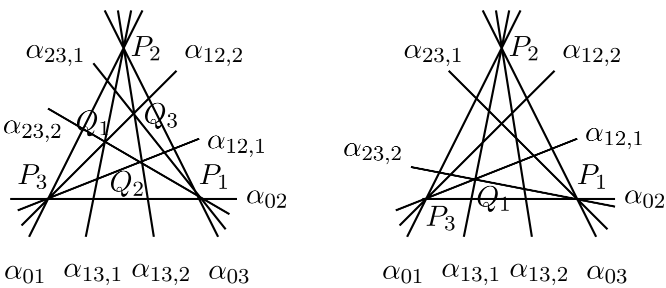

Fig. 4. The Burniat arrangements $\mathcal{A}_t(3,3,3)$, $t = 3$ and $t = 1$.

For $\pi_1(M(\mathcal{A}))$ we employ a presentation due to Cohen-Falk-Randell [16], which we rewrite using Terada type generators $\alpha_{ij}, 0 \leq i < j \leq 3$ satisfying

$\alpha_{01}\alpha_{02}\alpha_{12}\alpha_{03}\alpha_{13}\alpha_{23} = 1$ and the commutator relators

$$[\alpha_{ij}\alpha_{ik}\alpha_{jk}, \alpha_{ij}] = 1, [\alpha_{ij}\alpha_{ik}\alpha_{jk}, \alpha_{ij}] = 1, [\alpha_{ij}\alpha_{ik}\alpha_{jk}, \alpha_{ij}] = 1,$$

for all $i < j < k$, except $i = 1, j = 2, k = 3$, and

$$[\alpha_{12}, \alpha_{13}] = 1, [\alpha_{12}, \alpha_{23}] = 1, [\alpha_{13}, \alpha_{23}] = 1,$$

$$[\alpha_{01}, \alpha_{23}] = 1, [\alpha_{02}, \alpha_{13}] = 1, [\alpha_{03}, \alpha_{12}] = 1.$$

The characteristic variety $\mathcal{V}_1(\mathcal{A})$ consists of 3 irreducible components, 2-dimensional subtori C_1, C_2, C_3 of the torus $\mathbb{T}(M) = \{t_{01}t_{02}t_{03}t_{12}t_{13}t_{23} = 1\} \cong (\mathbb{C}^*)^5 \subset (\mathbb{C}^*)^6$. All components are local, determined by the 3 triple points $P_i = L_{jk} \cap L_{0j} \cap L_{0k}$

$$C_i := C_{P_i} = \{t_{jk}t_{0j}t_{0k} = t_{ij} = t_{ik} = t_{0i} = 1\}, 1 \le i \le 3, \{i, j, k\} = \{1, 2, 3\}.$$

From Proposition 3.5, we have $b_1(X_p) = 3(p-1)(p-2)$ for the surface X_p associated to the $\mathbb{Z}_p^5$ congruence covering, see [25, 40, 39]. From Proposition 3.1, $c_1^2(X_p) = 6p^3(p-2)^2$ and $c_2(X_p) = 3p^3(p^2 - 4p + 5)$, see [25, 39].

The surface $X_{p,2}$ associated to a $\mathbb{Z}_p^2$-covering is regular, as we have $b_1(X_{p,2}) = 0$. From Proposition 3.1, $c_1^2(X_p) = 6(p-2)^2$ and $c_2(X_p) = 3(p^2 - 4p + 5)$. The surface $X_{3,2}$ is the most interesting as it has $c_1^2 = c_2 = 6$, hence vanishing geometric genus $p_g = 0$. We can show that $\pi = \pi_1(X_{3,2})$ is the abelian extension of its abelianization

$$0 \to \mathbb{Z}^6 = [\pi, \pi] \to \pi \to \mathbb{Z}_3^3 \to 0,$$

recovering, in particular, the computation of $\pi/[\pi, \pi] = H_1(X_{3,2}) = \mathbb{Z}_3^3$ by Kulikov [29].

Example 3.9. The arrangements $\mathcal{A} = \mathcal{A}_t(3, 3, 3)$, with $0 \le t \le 4$ were first considered by P. Burniat [12]. It has $n = 9$ lines, $t_2 = 18 - 3t$ double points, $t_3 = t$ triple points, and $t_4 = 3$ quadruple points. In Figures 1 and 4 we see the cases $t = 0, 1, 3$.As defining equations for $\mathcal{A}$, we take

$$x_1x_2x_3 \prod_{1 \le i < j \le 3} (x_i - z_{ij,1}x_j)(x_i - z_{ij,2}x_j),$$

with $z_{ij,l} \ne 0$ and $z_{ij,1} \ne z_{ij,2}$, for all $1 \le i < j \le 3$ and $l = 1, 2$, and so that the coefficients $\{z_{ij,l}\}$ satisfy precisely t collinearity relations. The case $t = 4$ is realized by the monomial arrangement $\mathcal{B}_2$. The case $t = 2$ has two non-isomorphic realizations, according as the two triple points lie on a line of $\mathcal{A}$ or not, see Figure 5.

The number of monomial subarrangements of $\mathcal{A}$ takes the values $\beta = 0, 1, 6, 12$, corresponding to $t = 0, 1, 3, 4$, whereas for $t = 2$ two values may occur $\beta = 4$ or $\beta = 2$.

It follows from Proposition 3.5 that $b_1(X_p) = 3b(p,4) + (\beta+t)b(p,3)$, for the Hirzebruch congruence covering surface X_p, thus $b_1(X_p) = (p-1)[6p^2+(\beta+t-2)p-2\beta-2t-6]$. From Proposition 3.1, the Chern numbers are $c_1^2(X_p) = p^6[p^2(24-t)+p(4t-72)+54-4t]$ and $c_2(X_p) = p^6[p^2(12-t)+p(2t-36)+30]$.

We now turn to $\mathbb{Z}_p^2$-coverings. As before, if $t=0$, there is no difference between the Hirzebruch and Burniat covering types.

The Hirzebruch type surface $X_{p,2}$ is regular $b_1(X_{p,2}) = 0$. The Chern numbers of $X_{p,2}$ are $c_1^2 = p^2(24-t)+p(4t-72)+54-4t$ and $c_2 = p^2(12-t)+p(2t-36)+30$. For $p=2$, we get $c_1^2 = c_2 = 6$, no matter the value of t. We thus obtain a surface $X_{2,2}$ with vanishing geometric genus and irregularity $p_g = q = 0$, which seems to be new.

The Burniat type surface $X_{p,2}$ is also regular $b_1(X_{p,2}) = 0$. The Chern numbers of $X_{p,2}$ are $c_1^2 = p^2(24-4t)+p(12t-72)+54-9t$ and $c_2 = p^2(12-2t)+p(6t-36)+30-30t$. For $p=2$, we get $c_1^2 = 6-t$ and $c_2 = 6+t$, hence again $p_g = q = 0$. The surfaces $X_{2,2}$ for $0 \le t \le 4$ are the classical Burniat surfaces [12]. They were analyzed in detail via modern techniques by Peters [36], Kulikov [29], and more recently by Bauer-Catanese [9] using a different approach.

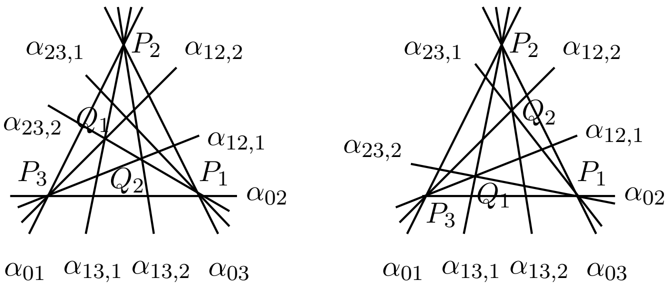

Fig. 5. The Burniat arrangements $\mathcal{A}_2(3,3,3)$.

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