MORSE CLASSIFICATION OF LOW ORDER JET SPACES

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Following the work of Kuo and Paunescu [4], we look at the classical problem of classifying function germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. The aim is to find a condition which is finer than the classification by topological type, but which does not generate too many strata. This paper describes an altered version of Morse stability which is invariant under linear coordinate change. It also lists the classification under Morse stability of the low order jet spaces $J^4_{(2,1)}(\mathbb{C})$ and $J^5_{(2,1)}(\mathbb{C})$, and gives a brief comparison to the classical classification by topological type.

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1. INTRODUCTION

This work follows that of Kuo and Paunescu [4]. Morse stability is an equivalent relation for germs which is stronger than topological equivalence. In this paper we use a modified version of Morse stability to classify the jet space $J^5_{(2,1)}(\mathbb{C})$.

Under topological equivalence, two jets f and g are equivalent if there is a germ of homeomorphism h such that $f = g \circ h$. Morse equivalence requires a deformation $F(x, y, t)$ within the jet space such that $F(x, y, 0) = f(x, y)$, $F(x, y, 1) = g(x, y)$, that for each $t \in \mathbb{C}^2$ with $|t| \leq 1$, $f_t = F(x, y, t)$ is topological equivalent to f, and also places some equivalence conditions on the polars of f_t. Loosely speaking, that the structure of the polars of f is preserved when the polars are truncated at their respective order of contact with the roots of f.

The original definition uses the standard polar $f_x = \partial f/\partial x$. For the classification of jet spaces this is undesirable as the classification would not be preserved by linear coordinate change. For example, consider the jets $g(x, y) = x^3 + y^3$ and $h(x, y) = x^3 + xy^2$, $g_x = 3x^2$ and $h_x = 3x^2 + y^2$. These are not Morse equivalent as g has one polar $x = 0$ while h has two polars. However, they are diffeomorphically equivalent. If we use the generic polars, the roots of $f_x + cf_y$ for generic c, then they are equivalent.

The first section recalls the definition of Morse stability and gives our modified version. The second section contains some lemmas which are helpful in the classification. The third section quotes the classification of J^5. Its proof along with the classification of $J^6_{(2,1)}(\mathbb{C})$, will follow in the author's thesis. The classification is also compared to the classical classification by topological type. Compare Arnold [1] for the classical classification under topological and diffeomorphic equivalence.

2. DEFINITION OF MORSE EQUIVALENCE

Below we recall the definition of Morse equivalence as stated in the paper of Kuo and Paunescu [4].

Throughout this section, let $f(x, y)$ be an analytic map germ at 0 of a map $\mathbb{C}^2 \to \mathbb{C}$. Note that although we will use Puiseux series with potentially fractional exponents, we always choose the Puiseux expansion with order at least 1. For example using $x = y^2$ instead of $y = x^{\frac{1}{2}}$.

We start with some definitions. Let $\{r_i\}$ be the set of Newton-Puiseux roots of f, and λ a Newton-Puiseux arc in $\mathbb{C}^2$ at 0, $O(\lambda(y)) \geq 1$.

The *height* of λ relative to f is

$$h_f(\lambda) = max(O(\lambda - r_i)) \ ,$$

where $O(\lambda)$ denotes the order of λ.

The *truncation* of λ with respect to f, denoted by λ_f, is λ with all terms of order greater than $h_f(\lambda)$ deleted.

The *Lojasiewicz exponent* of an arc $L_f(\lambda)$ with respect to f is the order of $f(\lambda(y), y)$.

The *valuation* of λ at f, $val_f(\lambda)$, is the pair consisting of the coefficient and the exponent of the lowest order term in $f(\lambda)$.

Specifically, if $x = \lambda(y)$ is the Puiseux expansion of λ, (or $y = \lambda(x)$, but by applying a change of coordinates if necessary we assume all the roots will have expansions of the form $x = \lambda(y)$) then we can substitute this into the expression $f = u(x, y) \prod_i (x - r_i(y))$, where $u(0, 0) \neq 0$ and $r_i(y)$ are the Puiseux expansions of the roots. The pair consisting of coefficient and exponent of the lowest order term in this expansion is $val_f(\lambda)$.

We shall now define the tree model of $f = \prod_i (x - r_i(y))$, introduced by Kuo and Lu in [3]. To construct the tree of f, we first draw a vertical segment, called the *main trunk* and write next to it the multiplicity of f. Let $b_1 = min\{O(r_i - r_j\})$. We now draw a horizontal line, called a *bar*, touching the top of the trunk and mark it with the number b_1, which is the *height* of the bar. Now we divide the roots into groups which have order of contact greater than b_1. For each such group, we draw a vertical segment, called a *trunk*, and write next to it the number of members in that group. This is the *multiplicity* of the trunk. If the trunk is of multiplicity 1, it is called a *twig*, and we will omit its multiplicity. For each trunk T with multiplicity greater than 1, let $b_T = min\{O(r_i - r_j | r_i, r_j \in T\})$. We then draw a bar at the top of the trunk, and mark it b_T. This procedure is repeated until all new trunks are of multiplicity 1.

The result of this procedure is called the *tree model* of f. For more details see Kuo-Lu [3].

Let f_x be the partial derivative of f with respect to x, and define f_y similarly. The generic polars of f are the roots of $f_x + cf_y$, where c is a generic complex number.

More specifically, we can assume $f(x,y)$ is mini-regular in x,

$$f(x,y) = unit \cdot \prod_i (x - r_i(y)), \quad O_y(r_i) \geq 1,$$

where $r_i(y)$ are fractional power series. Let us write

$$f(\lambda(y), y) = \begin{cases} ay^e + \cdots & \text{if } f(\lambda(y), y) \neq 0, \\ 0y^\infty & \text{if } f(\lambda(y), y) = 0, \end{cases}$$

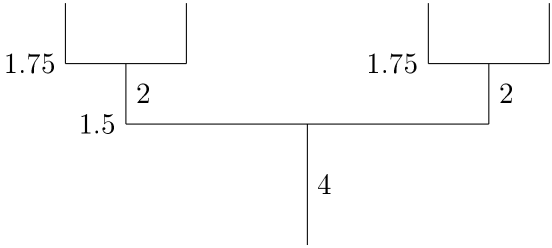
where $a \neq 0$, $e < \infty$, and $0y^\infty$ is a symbol. Then, by definition,

$$val_f(\lambda) = \begin{cases} ay^e & \text{in the former case,} \\ 0y^\infty & \text{in the latter case.} \end{cases}$$

Example 2.1. Consider the polynomial $f(x,y) = (x^2 - y^3)^2 + xy^5$. The four Newton-Puiseux roots are:

$$\begin{aligned} x &= y^{\frac{3}{2}} + \frac{1}{2} i y^{\frac{7}{4}} + \ldots \\ x &= y^{\frac{3}{2}} - \frac{1}{2} i y^{\frac{7}{4}} + \ldots \\ x &= -y^{\frac{3}{2}} + \frac{1}{2} y^{\frac{7}{4}} + \ldots \\ x &= -y^{\frac{3}{2}} - \frac{1}{2} y^{\frac{7}{4}} + \ldots . \end{aligned}$$

The tree model of f will have a bar at height $\frac{3}{2}$, with two trunks of multiplicity two on it. On top of both those trunks will be another bar at height $\frac{7}{4}$, both of which have two twigs. This tree is given in the diagram below.



The roots of the generic polar are:

$$x = \lambda_1(y) = \frac{1}{4}y^2 + \frac{1}{64}y^3 + \dots$$
$$x = \lambda_2(y) = y^{\frac{3}{2}} - \frac{1}{8}y^2 + \dots$$
$$x = \lambda_3(y) = -y^{\frac{3}{2}} - \frac{1}{8}y^2 + \dots .$$

The truncation of the roots is simply $x = 0$ and $x = \pm y^{\frac{3}{2}}$. Now:

$$f(\frac{1}{4}y^2 + \frac{1}{64}y^3 + \dots, y) = y^6 + \frac{1}{8}y^7 + H.O.T. .$$

Hence, $val_f(\lambda_1) = y^6$. Similarly, $val_f(\lambda_2) = y^{\frac{13}{2}}$, and $val_f(\lambda_3) = -y^{\frac{13}{2}}$.

The set of critical points of the valuation function at f is denoted by

$$Cr(f) = \{\gamma_i \mid \gamma_i \text{ a generic polar of } f\} .$$

Let $F(x, y, t)$ be an analytic function such that $F(x, y, 0) = f$. For fixed t we write f_t for $F(x, y, t)$. We can define the valuation function of F as the valuation functions at f_t for each t. Likewise if we denote the set of generic polars of f_t by $\{\gamma_{f_t,i}\}$, the set of critical points of the valuation function of F is: $Cr(F) = \{(\gamma_{f_t,i}, t)\}$.

Let $I_{\mathbb{C}}$ be the complex unit interval, $I_{\mathbb{C}} = \{t \in \mathbb{C} \bigm| |t| \leq 1\}$. We say F is an *almost Morse stable deformation* if there exists a homeomorphism (using the topology defined in Kuo-Paunescu [4])

$$\tau : Cr(f) \times I_{\mathbb{C}} \to \quad Cr(F), \qquad (\gamma_{f,i}, t) \to \big(\tau(\gamma_{f_t,i}, t), t\big) ,$$

which preserves the Lojasiewicz exponent of each element of $Cr(f)$.

Definition 2.2 (Morse Stable Deformation [4]). An almost Morse stable deformation F is Morse stable if it preserves the tree model and the critical points of the valuation functions at $f_t(x, y)$ for all t. Specifically, if γ is in $Cr(f)$ and γ' is another point in $Cr(f)$ which leaves the tree of f from the same bar then $f_t(\gamma)$ and $f_t(\gamma')$ will be on the same bar in the tree model of f_t for all t and additionally,

$$val_{f_t}(\gamma_t) = val_{f_t}(\gamma'_t) \text{ iff } val_f(\gamma) = val_f(\gamma') .$$

We say that two germs $f(x,y)$ and $g(x,y)$ are *Morse equivalent* $f \overset{M}{\sim} g$ if there is a Morse stable deformation $f_t(x,y)$ such that $f_0 = f$ and $f_1 = g$.

As Morse equivalency requires the bar spaces to be equivalent, if f and g are Morse equivalent they have the same tree and hence are topologically equivalent. Recall that two germs f_1, f_2 are *topologically (right) equivalent* if there is a germ of homeomorphism $h : \mathbb{C}^2 \to \mathbb{C}^2$ such that $f_1 = f_2 \circ h$. From Burau [2] (see also Zariski [6]) and Parusinski [5] two germs are topologically (right) equivalent if and only if their tree models coincide.

3. TECHNICAL LEMMAS

Consider the space $J^k_{(2,1)}$ of all k-jets at 0 of holomorphic functions $\mathbb{C}^2 \to \mathbb{C}$ with the natural vector space structure.

LEMMA 3.1. *The Lojasiewicz exponent for all polar curves are preserved under linear change of coordinates. In particular, $f(x,y)$ is Morse equivalent to $f(ax+by, cx+dy)$ whenever $ad - bc \neq 0$.*

Proof. Let $g(x,y) = f(ax+by, cx+dy)$. Writing f_x for $\frac{\partial}{\partial x} f(x,y)$ and f_y for $\frac{\partial}{\partial y} f(x,y)$, we have $g_x = af_x + cf_y$ and $g_y = bf_x + df_y$. We want to show that for generic k, $g_x + kg_y$ will have equivalent polars to $f_x + k_1 f_y$ for some k_1, and then show that the set of all k_1 corresponding to generic k forms a dense set in $\mathbb{C}^2$. In particular, k_1 is generic and so $f_x + k_1 f_y$ is a generic polar of f.

$$\begin{aligned} g_x + kg_y &= af_x + cf_y + k(bf_x + df_y) \\ &= (a+bk)f_x + (c+dk)f_y \\ &= (a+bk)\Big(f_x + \frac{c+dk}{a+bk} f_y\Big) \qquad \text{for } bk \neq -a \end{aligned}$$

Note that from the assumption $ad - bc \neq 0$ at least one of a and b must be non-zero. Hence, there will be at most one value of k for which $bk = -a$. For all other values of k we can set $k_1 = \frac{c+dk}{a+bk}$. Now this function $k \to \frac{c+dk}{a+bk}$ is defined on $\mathbb{C} - \{\frac{-a}{b}\} \to \mathbb{C}$ for $b \neq 0$ and $\mathbb{C} \to \mathbb{C}$ for $b = 0$ and maps bijectively onto either $\mathbb{C} - \{\frac{d}{b}\}$ (for $b \neq 0$) or $\mathbb{C}$ (for $b = 0$). So for almost all k, $k_1 = \frac{c+dk}{a+bk}$ will be in a dense subset of $\mathbb{C}$. Hence, $g(x,y)$ has the same generic polars as $f(x,y)$.

Now we need to find a deformation. Firstly if $ad = 0$ then $bc \neq 0$ and at least one of a, d must be 0. If precisely one is we can use the following deformation to get to the case where $ad \neq 0$:
If $a = 0$, set $A(t) = kt$, where $|k| < |bc|/|d|$, and $B(t) = b, C(t) = c, D(t) = d$
if $d = 0$, set $D(t) = kt$, where $|k| < |bc|/|a|$, and $A(t) = a, B(t) = b, C(t) = c$
if both a and d are 0, set $A = kt$, $D(t) = t$ where $|k| < |bc|$, and $B(t) =$

$b, C(t) = c$

and consider the deformation $F(x,y,t) = f(A(t)x + B(t)y, C(t)x + D(t)y)$.

In all these cases $|A(t)D(t)| \leq |k| < |bc|$ for all t, which implies $A(t)D(t) - B(t)C(t) \neq 0$ for all t with $|t| \leq 1$. Hence, all of these deformations are Morse stable. From these deformations and the transitive property of equivalence relationships we may assume $ad \neq 0$.

Now consider the deformation $G(x,y,t) = f(A(t)x+B(t)y, C(t)x+D(t)y)$ where A, B, C, D are defined as follows:

$$\begin{aligned} A(t) &= & a \\ B(t) &= & b(1-t) \\ C(t) &= & c(1-t) \\ D(t) &= & d + bc/a(t^2 - 2t) \end{aligned}$$

In this case:

$$\begin{aligned} A(t)D(t) - B(t)C(t) &= & ad + bc(t^2 - 2t) - bc(1 - 2t + t^2) \\ &= & ad - bc \end{aligned}$$

Hence, this deformation is Morse stable and so $f(ax+by, cx+dy)$ is Morse equivalent to $f(ax, (d+bc/a)y)$. As this is true for any a, b, c, d with $ad-bc \neq 0$, $f(x,y)$ is Morse equivalent to $f(ax, dy)$ for any $a, d \neq 0$.

The remaining deformations can be constructed piecewise analytically, using the transitive property of an equivalence relation. $\square$

Lemma 3.2. *If $f(x,y)$ is a polynomial in x and y with a non-zero linear term, f is Morse equivalent to $g(x,y) = x$.*

Proof. For any polynomial $f(x,y)$ with non-zero linear term, the generic polar $f_x + cf_y$ will have non-zero constant term. Hence, there are no roots of the generic polar at 0 and so f will be trivially Morse equivalent to $g(x,y) = x$. $\square$

Lemma 3.3. *If $f(x,y) = H_k(x,y) + H_{k+1}(x,y) + \dots$, is mini-regular in x, i.e. $H_k(1,0) \neq 0$, and λ is a generic polar of f with Lojasiewicz exponent $L_f(\lambda) > k$ then the truncated polar λ_f is the same as if we had defined λ using the truncated roots of f_x.*

This is not the case for roots with Lojasiewicz exponent k. For example, the generic polars of $x^3 + y^3$ are the roots of $x^2 + cy^2$, while the standard polar is $x = 0$ (with multiplicity 2).

Proof. As f is mini-regular, $(k, 0)$ is on the Newton polygon of f. This implies the gradient of all edges of the Newton polygon of f are at least 1. Choose such a coordinate system and write $f(x,y) = \sum_{i,j} a_{i,j} x^i y^j$.

The Newton diagram of f_x can be obtained by deleting all dots (i, j) with $i < 1$, and shifting all other dots 1 unit to the left, i.e. $(i, j) \to (i-1, j)$. Similarly, the Newton diagram of f_y consists of $(i, j-1)$ with $j \geq 1$. The Newton polygon of f_x will have edges parallel to all the edges of $NP(f)$ for which the highest dot has x coordinate ≥ 1, shifted one unit left. Similarly, the Newton polygon of f_y will have edges parallel to all the edges of $NP(f)$ for which the lowest dot has y coordinate ≥ 1.

From this we can determine the Newton diagram, and hence Newton polygon, of $f_x + cf_y$. Note that if (i, j) is a dot on f_x and f_y then it will be a dot on $f_x + cf_y$ for all but one value of c. Hence, for generic c the Newton diagram of $f_x + cf_y$ will consist of all dots from the Newton diagram of f_x and f_y.

Note that there will always be a dot at $(k-1, 0)$ due to mini-regularity. In addition the order of f_x is $k-1$, and the order of f_y is at least $k-1$. Hence, $f_x + cf_y$ is mini-regular in x and so the edges of $NP(f_x + cf_y)$ have gradient ≤ -1.

Now for $i, j \geq 1$, if there is a dot at (i, j) on $NP(f)$, there will be dots at $(i-1, j)$ and $(i, j-1)$ on $NP(f_x + cf_y)$. So lot (p, q) be a dot on an edge E of $NP(f_x + cf_y)$ with $p \geq 1$. If (p, q) is also on the Newton polygon of f_y, then there is a dot $(p-1, q+1)$ on the Newton diagram of f_x. Now if E is the lowest edge of $NP(f_x + cf_y)$ and has gradient -1 this dot $(p-1, q+1)$ will also be on E, and E will be the same as the lowest edge of f_x . However, if E has gradient < -1, then this dot will be below E, which contradicts the convex nature of the Newton polygon. So if (p, q) is on $NP(f_x + cf_y)$, it can only be a dot on f_y if it is on the lowest edge E_1 and E_1 has gradient -1, or if it is on the highest edge E_h with gradient < -1.

In the latter case, consider $NP(f_x + cf_y)$ compared to $NP(f)$ shifted one unit to the left: as we are sliding for a root $x = \gamma(y)$, the x coordinates of the dots on the Newton diagram of f will all be integers (the y coordinates will be in $\mathbb{Q}$). Hence, the x coordinate of the lowest dot of the highest edge of $NP(f)$ will be at least 1. In particular, the Newton polygon of both f_x and $f_x + cf_y$ will be identical to $NP(f)$ shifted one unit to the left up except for the highest edge of this. Now the corresponding edges of $NP(f_x + cf_y)$ will have lower gradient than the highest edge of $NP(f)$. So the contact order between the roots of $f_x + cf_y$ and f will be equal to the negative of the gradient of the highest edge of $NP(f)$. Hence, sliding along these higher edges of $NP(f_x + cf_y)$ will not change the truncated polars.

Now consider sliding towards a root of $f_x + cf_y$. Let $x = a_1 y^{\alpha_1}$ be the first approximation to the polar. If $\alpha_1 = 1$ and $x = a_1 y$ is a multiple root of multiplicity m of $f(x, y) = 0$ then $x = a_1 y$ is a root of multiplicity $m-1$ of both f_x and f_y, and we may factorise $f_x + cf_y$ to see that $x = a_1 y$ is a multiple root of multiplicity $m-1$ (for c generic). In this case $L_f(\lambda) > k$ If $x = a_1 y$ is not a root of f then $x = a_1 y$ is the truncated generic polar and by considering $f(a_1 y, y)$ the Lojasiewicz exponent L_f of $x = a_1 y$ is k.

So for all polars with Lojasiewicz exponent larger than k the truncated polar is the same with the generic polar as simply using f_x. $\square$

This may be restated as:

COROLLARY 3.4. *Let f be mini-regular in x, and let $\{r_i\}$ be the set of Puiseux roots of f. If the contact order of a generic polar λ with $\{r_i\}$ is greater than one (it leaves the tree of f at height greater than one), then the truncation λ_f is also a truncated root of f_x.*

We now quote the truncation theorem from Kuo and Paunescu [4].

Definition 3.5 (Puiseux Root Truncation). Let $f(x,y) = \prod_i (x - r_i(y))^{m_i}$ be mini-regular in x. For each i let $e_i = max_{j \neq i}\{O_y(r_i - r_j)\}$ and let $\hat{r}_i$ be r_i with all terms of order greater than e_i deleted. These are the truncated roots of f. If there is only one root, $f(x,y) = (x - r(y))^m$, then let $\hat{r} = r$.

The Puiseux root truncation of f is $\hat{f}_{root}(x,y) = \prod_i (x - \hat{r}_i(y))^{m_i}$.

THEOREM 3.6 (Truncation Theorem (Kuo-Paunescu [4])). *$f(x,y)$ is Morse equivalent to $\hat{f}_{root}(x,y)$.*

An immediate consequence of this is the following lemma:

LEMMA 3.7. *If $f(x,y)$ has no multiple roots, and we let $l = max\{L(\gamma) \mid \gamma \in Cr(f)\}$, then f is l-sufficient in the sense that adding terms of order higher than l will not change the Morse type of f.*

Proof. Note that the multiplicity of f must be less than or equal to l.

We may assume $f(x,y)$ is mini-regular in x. Let $g(x,y) = f(x,y) + H_{l+1}(x,y) + H_{l+2}(x,y) + H.O.T.$. We want to show that the Puiseux root truncations of f and g are identical.

Now consider the process of sliding for a root r of $g(x,y)$: Assume we have an approximation $x = \lambda_{i-1} = \sum_{j=1}^{i-1} a_j y^{\alpha_j}$, we construct the Newton polygon of f relative to λ_{i-1}: $f(x + \lambda_{i-1}(y), y)$ and choose an edge E_i of gradient $-\alpha_i$ (where $1 \leq \alpha_1 < \alpha_2 < \dots$. We find a root a_i of the associated polynomial of E_i. Our new approximation is $x = \lambda_i = \sum_{j=1}^{i} a_j y^{\alpha_j}$. As the exponents α_i are increasing, each edge E_i will be higher than the previous one. So in particular, if the highest edge E_h corresponding to the highest order term $a_h y^{\alpha_h}$ in the truncated root $\hat{r} = \lambda_h$ is below the line $x + y = l + 1$ in the Newton diagram of $f(x + \lambda_h(y), y)$, then all other edges E_i will also be below that line.

Now the Lojasiewicz exponent of an arc λ is simply the y-coordinate of the lowest dot on the y-axis in $f(x+\lambda(y), y)$. So the highest dot on E_h is below the line $x + y = l + 1$ (since f was assumed to be mini-regular, the slope of E_h is less than -1). So all edges that we slide along to find r are below the line $x + y = l + 1$. Also, since $1 \leq \alpha_1 < \alpha_2 < \dots$, all dots resulting from the additional terms $H_{l+1}(x, y) + H_{l+2}(x, y) + H.O.T.$ in $g(x, y)$ will be on or above this line. Hence, the edges of the Newton polygon of $f(x + \lambda_i(y), y)$ will be the same as the edges of $g(x + \lambda_i(y), y)$ for all approximations λ_i to a root r. Hence, the truncated polars of f and g will be the same, and so by the truncation theorem f is Morse equivalent to g. (Using the deformation $G(x, y, t) = f(x, y) + t \times H.O.T.$.) □

LEMMA 3.8. *If $f(x, y)$ is a jet with quadratic first term such that the Lojasiewicz exponent of the polar of f is l, then f is Morse equivalent to $x^2 + y^l$ in $J^w_{(2,1)}(\mathbb{C})$ for all $w \geq l$.*

Proof. First note that as a quadratic function has only one generic polar, the additional conditions for an almost Morse stable deformation to be Morse stable are trivially true. So when constructing a deformation we only have to consider the Lojasiewicz exponent of the generic polar.

Now let f be a jet of order 2 with generic polar γ and $L_f(\gamma) = l$. Note that in this case l will always be an integer, so $\lceil l \rceil = l$. As the $x^2 + y^2$ case is covered as part of the general quadratic case in the next chapter, we can without loss of generality assume that the quadratic term in f is x^2, i.e. $f = x^2 + R(x, y)$, where the order of R is at least 3. Write $val_f(\gamma) = ay^l$. Consider the deformation $f_t = x^2 + R(x, ty) + (1 - t^l)ay^l$.

$$\frac{\partial}{\partial x} f_t + c\frac{\partial}{\partial y} f_t = 2x + R_x(x, ty) + ctR_y(x, ty) + acl(1 - t^l)y^{l-1}.$$

For $t \neq 0$ the only possible difference between the Newton polygon of $\frac{\partial}{\partial x} f_t + c\frac{\partial}{\partial y} f_t(x, y)$ and the Newton polygon of $\frac{\partial}{\partial x} f + c\frac{\partial}{\partial y} f(x, ty)$ will be the dot at $(0, l - 1)$. Hence, the generic polar of f_t, $\gamma_t(y)$ will be the same as $\gamma(ty)$ up to the term of order $l - 1$.

As $l > 2$, we may ignore the term in y^{l-1} and all higher order terms, as the truncated generic polar will be $\gamma_{f_t}(y) = \gamma_f(ty)$.

Now consider $val_{f_t}(\gamma_t)$: the valuation of $\gamma_{f_t}(y) = \gamma_f(ty)$ in $f_t(x, y)$ is $at^l y^l$. As $f_t = f(x, ty) + (1 - t^l)ay^l$, the valuation of $\gamma_{f_t}(y)$ in f_t will be

$$val_{f_t}(\gamma_{f_t}) = at^l y^l + a(1 - t^l)y^l = ay^l.$$

Hence, the Lojasiewicz exponent is preserved and so f is Morse equivalent to $x^2 + R(x, 0) + y^l$, which is trivially Morse equivalent to $x^2 + y^l$. As this holds for any such f, all jets of order 2 are Morse equivalent in J^w. □

This also shows that in sufficiently large jet spaces, the classification of jets with quadratic first term is the same as the classification of those jets under topological triviality. But as we will see in the next lemma, the classifications differ in lower order jet spaces.

LEMMA 3.9. *The jets $x^2 + xy^{n-1}$ and $(x + y^2)^2 + x^{n-1}$ are not equivalent in J^n but are equivalent in J^{n+1}.*

Proof. Assume there is a $f_t(x, y) = F(x, y, t)$ such that $f_0(x, y) = x^2 + xy^{n-1}$ and $f_1 = (x+y^2)^2+x^{n-1}$. Write $F(x, y, t) = (x-a_2(t)y^2)^2+\sum_{i,j} b_{i,j}(t)x^iy^j$, with $b_{1,n-1}(0) = 1$, and $b_{i,j}(0) = 0$ for all other i, j. Now let $x = \gamma(y, t) = a_2(t)y^2 + a_3(t)y^3 + \dots$, be the generic polar. Note that $a_i(0) = 0$ for $i < n-1$.

Now assume that by restricting t to some subset A of $I_{\mathbb{C}}$, where 0 is a limit point of A, that there is k such that for $|t|$ sufficiently small $|a_k(t)| \geq |a_j(t)|$ for all j, $2 \leq j \leq n-2$. (Note that from now on we assume $t \in A$.) Consider the term in y^{n+k-1} in the above expansion. This is:

$$b_{1,n-1}(t)a_k(t) + \sum_i^{n-2} a_i(t)b_{1,n+k-1-i} + \sum_{i,j|i\geq 2} \sum (a_2^{m_{i,j,2}} a_3^{m_{i,j,3}} ...)b_{i,j}(t) ,$$

where the nested sum is over all possible sets of indices $m_{i,j,l}$ such that $\sum_l m_{i,j,l} = i$ and $j + 2m_{i,j,2} + 3m_{i,j,3} + \cdots = n + k - 1$. i.e. this is the sum of all additional terms contributing to the dot at $(0, n + k - 1)$.

So for $t \neq 0$:

$$b_{1,n-1}(t)a_k(t) = -\sum_i^{n-2} a_i(t)b_{1,n+k-1-i} - \sum_{i,j|i\geq 2} \sum (a_2^{m_{i,j,2}} a_3^{m_{i,j,3}} ...)b_{i,j}(t) .$$

By assumption, this must be 0 for all t with $|t| \leq 1$. Now consider the magnitude of each of the terms: as F is smooth, by continuity for every $\epsilon > 0$ there exists δ such that if $|t| < \delta$, $|b_{1,n-1}(t) - 1| < \epsilon$, $|b_{1,n-1-i}(t)| < \epsilon$, and $a_i(t) < \epsilon$ when $i < n-1$. Hence, for such t and $\epsilon < \frac{1}{n}$,

$$|\sum_i^{n-2} a_i(t)b_{1,n+k-1-i}| < \frac{\epsilon^2}{n} < \epsilon .$$

So near $t = 0$ we must have:

$$\left|b_{1,n-1}(t)a_k(t) + \sum_{i,j|i\geq 2} \sum (a_2^{m_{i,j,2}} a_3^{m_{i,j,3}} ...)b_{i,j}(t)\right| \leq \epsilon ,$$

hence

$$\left|b_{1,n-1}(t)a_k(t)\right| < \left| \sum_{i,j|i\geq 2} \sum (a_2^{m_{i,j,2}} a_3^{m_{i,j,3}} ...)b_{i,j}(t)\right| + \epsilon .$$

As $|a_k(t)| \geq |a_j(t)|$ for all j, $2 \leq j \leq n-2$,

$$\begin{aligned} |\sum_{i,j|i\geq 2} \sum (a_2^{m_{i,j,2}} a_3^{m_{i,j,3}} \ldots) b_{i,j}(t)| \leq & \quad |\sum_{i,j|i\geq 2} \sum (a_k^{m_{i,j,2}} a_k^{m_{i,j,3}} \ldots) b_{i,j}(t)| \\ \leq & \quad |a_k^2| \sum_{i,j|i\geq 2} M_{i,j} |b_{i,j}(t)| \ , \end{aligned}$$

hence,

$$\begin{aligned} |b_{1,n-1}(t) a_k(t)| \leq & \quad |a_k^2| \sum_{i,j|i\geq 2} M_{i,j} |b_{i,j}(t)| + \epsilon \\ (1-\epsilon)|a_k| \leq & \quad |a_k|^2 \sum_{i,j|i\geq 2} M_{i,j} |b_{i,j}(t)| + \epsilon \\ 1 \leq & \quad |a_k| \sum_{i,j|i\geq 2} M_{i,j} |b_{i,j}(t)| + 2\epsilon \ . \end{aligned}$$

Here $M_{i,j}$ is the number of different sets of indices $m_{i,j,l}$. Now as each $M_{i,j}$ is bounded, $a_k \to 0$, and there are a finite number of combinations i, j, we have that at least one of the $b_{i,j}(t)$ must approach infinity as t approaches zero. Hence, F is not smooth, which is a contradiction, and so $x^2 + xy^{n-1}$ and $(x+y^2)^2 + x^{n-1}$ are not equivalent in J^n.

Now to show these jets are equivalent in J^{n+1}, consider the deformation in J^{n+1} defined by $F(x,y,t) = (x + t/2y^2)^2 + xy^{n-1} + t/2y^{n+1} + x^{n-1}$ for small t: For $t = 0$, the truncated polar is $x = -1/2y^{n-1}$, which has valuation $-1/4y^{2n-2}$. For $t \neq 0$ the truncated polar is $x = -t/2y^2 - 1/2y^{n-1}$, which has valuation $(-1/4 + (t/2)^{n-1})y^{2n-2}$ for $t \neq 2\sqrt[n-1]{1/4}$. As $n \geq 4$, $|2\sqrt[n-1]{1/4}| > 1$, and so this deformation is Morse stable. $\square$

Consider the space $J^n_{(2,1)}(\mathbb{C})$. This has a natural vector space representation in $\mathbb{C}^{\frac{n^2+3n}{2}}$ constructed in the following way:
Let $f(x,y) = \sum_{i,j} a_{i,j} x^i y^j$. We associate f with the point
$(a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, \ldots) \in \mathbb{C}^{\frac{n^2+3n}{2}}$.

LEMMA 3.10. *Consider the subspace of $J^n_{(2,1)}$ consisting of the homogeneous jets of order n (plus the origin). This is the subspace given by $a_{i,j} = 0$ for $i + j < n$. For $H_n(x,y)$ in this plane, write $H_n = \prod_i^k (a_i x + b_i y)^{m_i}$ where $a_i/b_i = a_j/b_j$ if and only if $i = j$. The classification of H_n under Morse equivalency is entirely determined by the set of multiplicities $\{m_i\}$. This is also the classification by topological type, with a stratum for each partition of n. (A partition of n is an unordered set of positive integers m_i such that $\sum_i m_i = n$.)*

Proof. First note that if two jets have differing sets of indices $\{m_i\}$ then they have different topological type, and so are not Morse equivalent. Hence,

that many classes are necessary, we now prove that is sufficient:
We may write $f_x = R_x(x,y)\prod_i^k(a_ix+b_iy)^{m_i-1}$ and $f_y = R_y(x,y)\prod_i^k(a_ix+b_iy)^{m_i-1}$, for some polynomials R_x and R_y. Note that these will both be homogeneous of order $k-1$, and have no shared roots with each other or $f(x,y)$.

Now $f_x + cf_y = (R_x(x,y) + cR_y(x,y)\prod_i^k(a_ix+b_iy)^{m_i-1}$. Hence, the generic polars consist of: the root of $f(x,y)$; $a_ix+b_iy=0$ for each $i=1,\dots,k$ with multiplicity m_i-1 (when $m_i \geq 2$) and Lojasiewicz exponent ∞ and the roots of $R_x(x,y)+cR_y(x,y)=0$. As c is generic and R_x and R_y share no roots with each other or with $f(x,y)$, the equation $R_x(x,y)+cR_y(x,y)=0$ has $k-1$ roots of multiplicity 1 and Lojasiewicz exponent n, all of which have distinct valuation. Hence, two homogeneous polynomial germs have the same Morse type if and only if they have the same set of multiplicities. It remains to construct a deformation, which can easily be done piecewise, using the transitive property of equivalence relations. □

LEMMA 3.11. *Consider the subspace of $J^n_{(2,1)}$ consisting of polynomials of the form $H_{n-1}+H_n$ (including the trivial polynomial). This corresponds to a subspace in the natural vector space representation of $J^n_{(2,1)}$, given by $a_{i,j}=0$ for $i+j<n-1$. For $f(x,y)=H_{n-1}+H_n$, in this plane, write $H_{n-1}=\prod_i(a_ix+b_iy)^{m_i}$ where $a_i/b_i=a_j/b_j \Rightarrow i=j$. We can now write f as:*

$$f = \prod_i(a_ix+b_iy)^{m_i} + \prod_i(a_ix+b_iy)^{n_i} \times R(x,y)\ ,$$

where $R(x,y)$ is the polynomial such that $H_n=\prod_i(a_ix+b_iy)^{n_i}\times R(x,y)$ and $R(x,y)$ shares no roots with H_{n-1}. Note that some or all of the n_i may be zero. The Morse classification of f is entirely determined by the set of multiplicities $\{m_i\}$ and $\{n_i\}$ and is also the same as classification by topological type.

Proof. Note that by using a suitable coordinate change we may assume $a_i=1$ for all i.

We will first slide for the roots of $f(x,y)$ to show topological type is determined by the multiplicities m_i and n_i. Now in the Newton polygon of f, the dots from H_{n-1} will form an edge. The solutions of this edge will be $x=-b_iy$. (Note that this will be the only edge unless one of the b_i is zero, but in that case we can still use $x=-b_iy$, it will simply have no effect on the Newton polygon. In addition, only one of the b_i can be zero.)

Now consider the Newton polygon of $f(x-b_iy,y)$: from evaluating $H_{n-1}(x-b_iy,y)$ there will be a line between $(n-1,0)$ and $(m_i,n-m_i-1)$. In addition, by evaluating $H_n(x-b_iy,y)$ there will be a dot at $(n_i,n-n_i)$, and potentially other dots on the line between $(n,0)$ and $(n_i,n-n_i)$. These are all of the dots on the Newton polygon of $f(x-b_iy,y)$. Hence, the leading edge will be between the dot $(m_i, n-m_i-1)$ and the dot $(n_i, n-n_i)$. There will be no additional dots on this edge, and no dots to the left of this edge. We will write the equation for this edge as $E(x,y) = a_{m_i,n-m_i-1}x^{m_i}a_{n_i,n-n_i}x^{n_i}y^{n-n_i}$ So $x=-b_iy$ is a root of multiplicity n_i and there are m_i-n_i additional roots:

$$x = r(y) = -b_iy + \omega_{m_i-n_i} \sqrt[(m_i-n_i)]{-\frac{a_{n_i,n-n_i}}{a_{m_i,n-m_i-1}}}y^{\frac{m_i-n_i+1}{m_i-n_i}} + H.O.T. \quad ,$$

where $\omega_{m_i-n_i}$ is one of the (m_i-n_i)-th complex roots of 1. As these roots all have multiplicity one we have finished sliding. Hence, the tree of $f(x,y)$ is determined by $\{m_i\}$ and $\{n_i\}$.

Now consider the Newton polygon of $f_x(x-b_iy,y)+cf_y(x-b_iy,y)$. This will have dots at $(m_i-1, n-m_i-1), (n_i-1, n-n_i), (m_i, n-m_i-2)$ and $(n_i, n-n_i-1)$. The second dot will only exist if $n_i \geq 1$. In this case the leading edge of this will be between $(m_i-1, n-m_i-1)$ and $(n_i-1, n-n_i)$. Otherwise, the leading edge will be between $(m_i-1, n-m_i-1)$ and $(0, n-1)$.

So for $n_i \geq 1$, the polars are $x=-b_iy$ with multiplicity n_i-1 and m_i-n_i additional polars

$$x = \gamma(y) = -b_iy + \omega_{m_i-n_i} \sqrt[(m_i-n_i)]{-\frac{n_ia_{n_i,n-n_i}}{m_ia_{m_i,n-m_i-1}}}y^{\frac{m_i-n_i+1}{m_i-n_i}} + H.O.T. \quad .$$

These polars are clearly distinct from each other and from the roots. Hence, in this case we have found the truncated polars. By substituting these into f the Lojasiewicz exponent of each of these is:

$$n - m_i - 1 + m_i\frac{m_i-n_i+1}{m_i-n_i}$$

By substituting these polars into the leading edge of $f(x-b_iy,y)$ we can also see that the relationship between the valuations will be the same for any jet with the same set of multiplicities.

If $n_i = 0$, there cannot be a dot at $(n_i-1, n-n_i)$ in the Newton polygon of $f_x(x-b_iy,y)+cf_y(x-b_iy,y)$, so the leading edge of this will have gradient $\frac{m_i}{1-m_i}$. Hence, the polars found by sliding along this edge will be of the form:

$$x = -b_iy + O(y^{\frac{m_i}{m_i-1}}) + H.O.T. \quad .$$

But the leading edge of $f(x-b_iy,y)$ are will be between $(m_i, n-m_i-i)$ and $(0,n)$. Hence, the roots found by sliding along this edge will be of the form

$$x = r(y) = -b_iy + +O(y^{\frac{m_i+1}{m_i}}) + H.O.T. \quad .$$

As $\frac{m_i}{m_i-1} > \frac{m_i+1}{m_i}$ and the term of order $y^{\frac{m_i+1}{m_i}}$ exists, the truncated polar is $x=-b_iy$ of multiplicity m_i-1, and Lojasiewicz exponent n.

Hence, the multiplicities $\{m_i\}$ and $\{n_i\}$ determine the Morse type of f. A deformation can again be constructed in a piecewise manner. □

4. RESULTS

The Morse classifications of $J^1_{(2,1)}(\mathbb{C})$, $J^2_{(2,1)}(\mathbb{C})$ and $J^3_{(2,1)}(\mathbb{C})$ are the same as the classification by topological type.

The Morse classifications of $J^4_{(2,1)}(\mathbb{C})$ and $J^5_{(2,1)}(\mathbb{C})$ are listed in the following tables. Each strata is defined using a reference element, the Lojasiewicz exponents of the polar(s) is also given. Partly for simplicity and partly by convention we will not always use mini-regular expressions for the strata. For example, x^2y+y^4 is the standard expression for the simple singularity D_5 in two complex dimensions. It could however be represented using the mini-regular jet $x^2(x+y)+(x+y)^4$. Similar coordinate changes can be used for the other strata to obtain mini-regular expressions.

The results will simply be quoted here, the proof and a more detailed discussion will follow in the author's PhD thesis.

THEOREM 4.1. *The Morse classification for $J^4_{(2,1)}(\mathbb{C})$ is:*

Strata Containing	***Lojasiewicz Exponent(s)***
0	n/a
x	n/a
x^2+y^2	2
x^2+y^3	3
x^2+y^4	4
$(x+y^2)^2+x^2y$	5
x^2+xy^3	6
$(x+y^2)^2+x^3$	6
$(x+y^2)^2+x^3y$	7
$(x+y^2)^2+x^4$	8
x^2	∞
x^3+y^3	$(3,3)$
x^2y+y^4	$(3,4)$
x^2y+xy^3	$(3,5)$
x^2y	$(3,\infty)$
x^3+y^4	$(4,4)$
x^3+xy^3	$(4.5,4.5)$
$x^3+x^2y^2$	$(6,\infty)$
x^3	∞
x^4+y^4	$(4,4,4)$
$x^4+x^2y^2$	$(4,4,\infty)$
x^4+x^3y	$(4,\infty)$
$x^4+2x^2y^2+y^4$	$(4,\infty,\infty)$
x^4	∞

This classification is almost identical to the classification under topological equivalence with the exception of the strata containing $x^2 + xy^3$ and the strata containing $(x + y^2)^2 + x^3$, which by lemma 3.9 are disjoint in J^4 but are connected under topological equivalence, and indeed under Morse equivalence in all higher order jet spaces.

THEOREM 4.2. *The classification of $J^5_{(2,1)}(\mathbb{C})$ is given in the following tables:*

Strata	***Lojasiewicz Exponent(s)***
Trivial Strata	
0	n/a
x	n/a
Homogeneous Quintics	
$x^5 + y^5$	$(5,5,5,5)$
$x^5 + x^2y^3$	$(5,5,5,\infty)$
$x(x^2 + y^2)^2$	$(5,5,\infty,\infty)$
$x^5 + x^3y^2$	$(5,5,\infty)$
x^3y^2	$(5,\infty,\infty)$
x^4y	$(5,\infty)$
x^5	(∞)
Quartic First Term	
x^4	(∞)
$x^4 + y^4$	$(4,4,4)$
$x^4 + y^5$	(5)
$x^4 + xy^4$	$(5\frac{1}{3}, 5\frac{1}{3}, 5\frac{1}{3})$
$x^4 + x^2y^3$	$(6,6,\infty)$
$x^4 + x^3y^2$	$(8,\infty)$
x^3y	$(4,\infty)$
$x^3y + y^5$	$(4,5)$
$x^3y + xy^4$	$(4,5.5,5.5)$
$x^3y + x^2y^3$	$(4,7,\infty)$
x^2y^2	$(4,\infty,\infty)$
$x^2y^2 + (x+y)^5$	$(4,5,5)$
$x^2y^2 + x(x+y)^4$	$(4,5,6)$
$x^2y^2 + x^5$	$(4,5,\infty)$
$x^2y^2 + xy(x+y)^3$	$(4,6,6)$
$x^2y^2 + x^4y$	$(4,6,\infty)$
$x^4 + x^2y^2$	$(4,4,\infty)$
$x^4 + x^2y^2 + y^5$	$(4,4,5)$
$x^4 + x^2y^2 + xy^4$	$(4,4,6)$

Strata	***Lojasiewicz Exponent(s)***
Cubic First Term	
x^3	(∞)
x^3+y^3	$(3,3)$
x^2y	$(3,\infty)$
x^2y+y^4	$(3,4)$
x^2y+y^5	$(3,5)$
$y(x+y^2)^2+x^3$	$(3,6)$
x^2y+xy^4	$(3,7)$
$y(x+y^2)^2+x^3y$	$(3,7)$
$y(x+y^2)^2+x^4$	$(3,8)$
$y(x+y^2)^2+x^4y$	$(3,9)$
$y(x+y^2)^2+x^5$	$(3,10)$
$y(x+y^2)^2+x^3(x+y^2)$	$(3,11)$
x^3+y^4	(4)
x^3+xy^3	$(4.5,4.5)$
x^3+y^5	(5)
$x^3+3x^2y^2+3xy^4$	(6)
x^3+xy^4	$(6,6)$
$x(x+y^2)^2+x^2y^3$	$(6,7)$
$x(x+y^2)^2+x^4$	$(6,8)$
$x(x+y^2)^2+x^4y$	$(6,9)$
$x(x+y^2)^2+x^5$	$(6,10)$
$x^3+x^2y^2$	$(6,\infty)$
$x(x+y^2)^2$	$(6,\infty)$
$x^3+x^2y^3$	$(9,\infty)$
Quadratic First Term	
x^2	∞
x^2+y^2	2
x^2+y^3	3
x^2+y^4	4
x^2+y^5	5
x^2+xy^3	6
$(x+y^2)^2+x^3y$	7
$(x+y^2)^2+x^4$	8
x^2+xy^4	8
$(x+y^2)^2+x^4y$	9
$(x+y^2)^2+x^5$	10
$(x+y^2)^2+2(x^2y+x^3)(x+y^2)-x^5;$	11
$(x+y^2)^2+x^4+x^3y^2$	12
$(1+y)(x+y^2)^2+(x^2y-\frac{1}{2}x^3)(x+y^2)-\frac{1}{4}x^5$	12
$(x+y^2)^2+2x^2y^3+2x^3y-x^5$	13

There are 5 additional strata compared to the classification by topological type, 2 quadratic and 3 cubic. These are all pairs of strata that are topologically equivalent. Four of the pairs are distinct due to the deformation condition, in all these cases deformations may be constructed in J^6. Specifically, these pairs are:

$$\begin{aligned}
x^2+xy^4 &\overset{M}{\not\approx} (x+y^2)^2+x^4\\
(x+y^2)^2+x^4+x^3y^2 &\overset{M}{\not\approx} (1+y)(x+y^2)^2+(x^2y-\frac{1}{2}x^3)(x+y^2)-\frac{1}{4}x^5\\
x^2y+xy^4 &\overset{M}{\not\approx} y(x+y^2)^2+x^3y\\
x^3+x^2y^3 &\overset{M}{\not\approx} x(x+y^2)^2
\end{aligned}$$

The cubic strata x^3+xy^4 and $x^3+3x^2y^2+3xy^4$ are not Morse equivalent in any higher order jet space as x^3+xy^4 has two polars, while $x^3+3x^2y^2+3xy^4$ has one polar of multiplicity two. They are topologically equivalent as they have the same tree model. The other distinct strata are all due to the deformation condition, and are equivalent to the corresponding strata in J^6.

Example 4.3. In J^6 the jets $x^4+x^2y^4$ and $x^4+x^2y^4+x^2y^4$ are not Morse equivalent, as the two non-zero polars of $x^4+x^2y^4$ have the same valuation, while the two non-zero polars of $x^4+x^2y^4+x^2y^4$ have different valuations. These jets are topologically equivalent. (N.B. there are no examples of this type in J^5.)

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