A RANGE CHARACTERIZATION OF TOPOLOGICAL RADON TRANSFORMS ON GRASSMANN MANIFOLDS

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In this paper, we characterize the ranges of topological Radon transformations on Grassmann manifolds by a system of topological integral equations.

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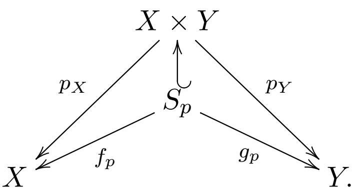
1. INTRODUCTION

The analytic Radon transformation is one of the most important integral transformations in mathematics, which is applied to the CT-scan, partial differential equations and so on. In this paper, we study its topological analogy. In our theory, the integral is based on the topological Euler characteristics of subanalytic subsets (see [8, 16] etc.). Recently, the topological Radon transformation is applied to class formulas for dual varieties in algebraic geometry, sensor networks in applied mathematics and so on (see [1], [10–12] etc.).

We denote the field of real numbers or that of complex numbers by $\mathbb{K}$ (i.e. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We also denote by $F_N(k)$ the Grassmann manifold consisting of k-dimensional linear subspaces in $\mathbb{K}^N$. Let p, q be positive integers satisfying $p < q$, set $X = F_N(p)$ and $Y = F_N(q)$. Let us also set $S_p = \{(x, y) \in X \times Y \mid x \subset y\}$, which is an incidence submanifold of $X \times Y$, and consider the diagram:



Here f_p and g_p are restrictions of natural projections p_X and p_Y to S_p respectively. By composing the inverse image by f_p and the direct image by g_p, for a constructible function φ on X we define the topological Radon transform $\mathcal{R}_{S_p}(\varphi)$ of φ, which is a constructible function on Y. See Section 2 for the precise definitions. In [9], we proved an inversion formula for $\mathcal{R}_{S_p}$. In particular, we explicitly construct a transformation $\widehat{\mathcal{R}}$ considered as a left inverse transformation of $\mathcal{R}_{S_p}$. See Section 3.1 for the review. In this paper, we will prove a range characterization of the topological Radon transformation $\mathcal{R}_{S_p}$. We characterize the images $\mathcal{R}_{S_p}(\varphi)$ of constructible functions φ on X by a system of topological integral equations and prove that the transformation $\widehat{\mathcal{R}}$ is also considered as a right inverse transformation of $\mathcal{R}_{S_p}$ for constructible functions satisfying the system of topological integral equations.

On the range characterizations of topological integral transformations, the following results are known in the previous studies. The range of the topological polar transformation of constructible functions on the Euclidean space is characterized by a condition on topological integrals by Bröcker [2]. It is similar to our result to characterize the range of a topological integral transformation by conditions on topological integrals. In [9], we obtained a partial characterization of some images of the topological Radon transformation $\mathcal{R}_{S_p}$ on the Grassmann manifold $F_N(p)$ by Young diagrams. In [10], we studied the microlocal images of the topological Radon transformation on the projective space by characteristic cycles of constructible functions.

In the analytic case, it is well-known that the range of the analytic d-dimensional Radon transformation of C^∞-functions on the Euclidean space is characterized by a system of second order ultrahyperbolic differential equations (see [5, 6] etc.). More generally, the range of the analytic Radon transformation of C^∞-functions on the Grassmann manifold is characterized by a system of invariant differential equations (see [7] etc.).

Finally, the author would like to greatly appreciate several useful comments of the referee.

2. PRELIMINARIES

2.1. Constructible functions

In this subsection, we recall the definition and basic properties of constructible functions. See [8] and [16] for more details. In the theory of o-minimal structures [15], we can define them in more general settings.

Definition 2.1. Let X be a real analytic manifold. We say that an integer-valued function $\varphi \colon X \longrightarrow \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}_{i \in I}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_{i \in I} c_i \mathbf{1}_{X_i} \qquad (c_i \in \mathbb{Z}).$$

Here $\mathbf{1}_{X_i}$ denotes the characteristic function of X_i. We denote the abelian group of constructible functions on X by $CF(X)$.

We define several operations on constructible functions in the following way.

Definition 2.2 ([8, 16]). Let X and Y be real analytic manifolds and $f\colon Y \longrightarrow X$ a real analytic map from Y to X.

(i) (The inverse image) For $\varphi \in CF(X)$, we define the inverse image $f^*\varphi \in CF(Y)$ of φ by f by

$$f^*\varphi(y) = \varphi(f(y)).$$

(ii) (The integral) Let $\varphi = \sum_i c_i \mathbf{1}_{X_i} \in CF(X)$ be a constructible function on X and assume that its support $\mathrm{supp}(\varphi)$ is compact. Then we define the topological (Euler) integral $\int_X \varphi \in \mathbb{Z}$ of φ by

$$\int_X \varphi = \sum_i c_i \cdot \chi(X_i),$$

where $\chi(X_i)$ is the topological Euler characteristic of X_i.

(iii) (The direct image) Let $\psi \in CF(Y)$ such that $f|_{\mathrm{supp}(\psi)}\colon \mathrm{supp}(\psi) \longrightarrow X$ is proper. Then we define the direct image $\int_f \psi \in CF(X)$ of ψ by f by

$$\left(\int_f \psi\right)(x) = \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

The group of constructible functions is isomorphic to the Grothendieck group of the derived category of $\mathbb{R}$-constructible sheaves via the local Euler-Poincaré index. We can easily see that the operations in Definition 2.2 are well-defined and satisfying functorial properties by this identification. See [8, 13] for more details. In particular, the following properties will be used later.

PROPOSITION 2.3. *Let X, Y, Z be real analytic manifolds and $f\colon Y \longrightarrow X$, $g\colon Z \longrightarrow Y$ real analytic proper maps.*

(i) *(Functoriality* [14]*) For $\varphi \in CF(X)$, $\psi \in CF(Z)$, we have*

$$g^*(f^*\varphi) = (f \circ g)^*\varphi,$$

$$\int_f \left(\int_g \psi\right) = \int_{f\circ g} \psi.$$

(ii) *(Base change formula* [14]*) Let W be also a real analytic manifold. As-*

sume that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{t} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{f} & X \end{array}$$

is a Cartesian square of real analytic maps. Then for $\psi \in CF(Y)$ *we have*

$$s^*\left(\int_f \psi\right) = \int_t (g^*\psi).$$

(iii) *(Projection formula* [12]*) For* $\varphi \in CF(X)$, $\psi \in CF(Y)$, *we have*

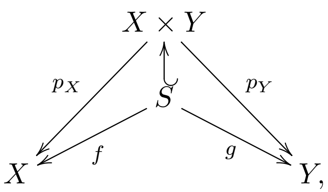
$$\int_f (f^*\varphi \cdot \psi) = \varphi \cdot \int_f \psi.$$

2.2. **Topological Radon transforms**

In this subsection, let us define our main subject: the topological Radon transforms. See [9, 14] for more details.

Let X and Y be real analytic manifolds and S a locally closed subanalytic subset of $X \times Y$. For simplicity, we assume that X and Y are compact. Consider the diagram:

(2.1)



where p_X and p_Y are natural projections and f and g are restrictions of p_X and p_Y to S respectively. Note that we can generalize the definitions of the inverse image and the direct image to f and g. In the situation above, we define the topological Radon transforms for constructible function as follows.

Definition 2.4. For a constructible function $\varphi \in CF(X)$ on X, we define the topological Radon transform $\mathcal{R}_S(\varphi) \in CF(Y)$ of φ by

$$\mathcal{R}_S(\varphi) = \int_g f^*\varphi = \int_{p_Y} \mathbf{1}_S \cdot p_X^*\varphi.$$

Similarly, for a constructible function $\psi \in CF(Y)$ on Y, we define the transposed transform ${}^t\mathcal{R}_S(\psi) \in CF(X)$ of ψ by

$$^t\mathcal{R}_S(\psi) = \int_f g^*\psi = \int_{p_X} \mathbf{1}_S \cdot p_Y^*\psi.$$

We consider $\mathbf{1}_S$ as the kernel function of transformations $\mathcal{R}_S$ and ${}^t\mathcal{R}_S$.

In the theory of integral transformations, it is important to study the following problems: (1) inversion formula (2) range characterization (3) support theorem. In this paper, we study the second problem of topological Radon transforms on Grassmann manifolds. Note that the first and third problems for them were studied in [9] and [10] respectively. Note also that the second problem of that was partially studied in [3, 9] and [10].

By Proposition 2.3, we have the following property, which will be used in Section 3.2.

PROPOSITION 2.5 ([12]). *For $\varphi \in CF(X)$ and $\psi \in CF(Y)$, we have*

$$\int_Y \psi \cdot \mathcal{R}_S(\varphi) = \int_X {}^t\mathcal{R}_S(\psi) \cdot \varphi.$$

Proof. Let $\pi_X \colon X \longrightarrow \{\mathrm{pt}\}$ and $\pi_Y \colon Y \longrightarrow \{\mathrm{pt}\}$ be natural projections. By Proposition 2.3, we have

$$\begin{aligned}&\int_{\pi_Y} \psi \cdot \int_g f^*\varphi \\ &= \int_{\pi_Y} \int_g (g^*\psi) \cdot (f^*\varphi) = \int_{\pi_X} \int_f (g^*\psi) \cdot (f^*\varphi) = \int_{\pi_X} \varphi \cdot \int_f g^*\psi. \quad \square\end{aligned}$$

2.3. GRASSMANN MANIFOLDS AND SCHUBERT VARIETIES

In this subsection, we recall the definition of Grassmann manifolds and the Euler characteristic of special Schubert varieties, which will play an important role in Section 3. See [4, 9] for the detail.

Let N be a positive integer. We denote the field of real numbers or that of complex numbers by $\mathbb{K}$ (*i.e.* $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

Definition 2.6. For $k = 0, 1, 2 \ldots, N$, we denote by $F_N(k)$ the Grassmann manifold consisting of k-dimensional linear subspaces $L \simeq \mathbb{K}^k$ in $\mathbb{K}^N$. That is, we set

$F_N(k) = \{L \mid L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{K}^N \text{ (through the origin)}\}.$

In the case $k = 0$, $F_N(0) = \{0\}$. And in the case $k = 1$, $F_N(1)$ is nothing but the $(N-1)$-dimensional projective space $\mathbb{P}_{N-1}$. By projectivizing each linear subspace, we could identify $F_N(k)$ with the set of all $(k-1)$-dimensional linear subspaces in $\mathbb{P}_{N-1}$. In this paper we do not use this identification. For simplicity, we set $F_N(k) = \emptyset$ unless $0 \leq k \leq N$.

Let us explain a cell decomposition of the Grassmann manifold $F_N(k)$.

Definition 2.7. Let k be a positive integer satisfying $1 \leq k \leq N$.

(i) We call a sequence of integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ a Young diagram with at most k rows and $N-k$ columns if σ satisfies

$$N-k \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0.$$

We denote by $I(N,k)$ the set of all Young diagrams with at most k rows and $N-k$ columns.

(ii) We fix a complete flag (i.e. a sequence of linear subspaces)

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_N = \mathbb{K}^N, \qquad \dim V_i = i \quad (i = 1, 2, \ldots, N)$$

in $\mathbb{K}^N$. For a Young diagram $\sigma = (\sigma_1, \ldots, \sigma_k) \in I(N,k)$, we define the Schubert cell corresponding to σ by

$$\Omega_\sigma^\circ = \left\{ L \in F_N(k) \;\middle|\; \begin{array}{l} \dim(L \cap V_{N-k-\sigma_i+i}) = i, \\ \dim(L \cap V_{N-k-\sigma_i+i-1}) = i-1 \end{array} \quad (i = 1, 2, \ldots, k) \right\}.$$

Note that we often identify a Young diagram with a collection of boxes arranged in left justified rows, with a weakly decreasing number of boxes in each row.

In this paper, we use the generalized binomial coefficient defined by

$$\left(\begin{pmatrix} n \\ m \end{pmatrix}\right) = \begin{cases} \binom{n}{m} & (0 \leq m \leq n), \\ 0 & (\text{otherwise}). \end{cases}$$

Proposition 2.8. *Let k be a positive integer satisfying $1 \leq k \leq N$.*

(i) *For a Young diagram $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in I(N,k)$, we have*

$$\Omega_\sigma^\circ \simeq \mathbb{K}^{k(N-k)-|\sigma|},$$

where we set $|\sigma| = \sigma_1 + \sigma_2 + \cdots + \sigma_k$.

(ii) *The Grassmann manifold $F_N(k)$ has the following cell decomposition*

$$F_N(k) = \bigsqcup_{\sigma \in I(N,k)} \Omega_\sigma^\circ.$$

(iii) *The Euler characteristic of the Grassmann manifold $F_N(k)$ is computed as follows.*

(a) *In the case $\mathbb{K} = \mathbb{C}$, we have*

$$\chi(F_N(k)) = \left(\begin{pmatrix} N \\ k \end{pmatrix}\right).$$

(b) *In the case $\mathbb{K} = \mathbb{R}$, we have*

$$\chi(F_N(k)) = \begin{cases} 0 & (k(N-k) \text{ is odd}), \\ \left(\begin{pmatrix} [\frac{N}{2}] \\ [\frac{k}{2}] \end{pmatrix}\right) & (\text{otherwise}). \end{cases}$$

Here $[\cdot]$ *is the floor function. For a real number* $t \in \mathbb{R}$, $[t]$ *is the largest integer not greater than* t.

For the proof of Proposition 2.8 (iii), see for example ([9], Appendix).

In Section 3, the following special subvariety $\Omega(N,k,l,m)$ of the Grassmann manifold $F_N(k)$ plays an important role.

Definition 2.9. For $l = 0,1,\dots,N$, let us fix an l-dimensional linear subspace $V_l \simeq \mathbb{K}^l$ in $\mathbb{K}^N$. For $m = 0,1,\dots,N$, we set

$$\Omega(N,k,l,m) = \{L \in F_N(k) \mid \dim(L \cap V_l) = m\}.$$

Note that $\Omega(N,k,l,m)$ might be the empty set in some cases.

By using Young diagrams, the Euler characteristic of $\Omega(N,k,l,m)$ is computed as follows.

Proposition 2.10. (i) *In the case* $\mathbb{K} = \mathbb{C}$, *we have*

$$\chi(\Omega(N,k,l,m)) = \chi(F_{N-l}(k-m))\chi(F_l(m)) = \binom{N-l}{k-m}\binom{l}{m}.$$

(ii) *In the case* $\mathbb{K} = \mathbb{R}$, *we have*

$$\begin{aligned}
&\chi(\Omega(N,k,l,m)) \\
&= (-1)^{(k+m)(l+m)}\chi(F_{N-l}(k-m))\chi(F_l(m)) \\
&= \begin{cases} 0 & ((k-m)(N-l-k+m) \text{ or } m(l-m) \text{ is odd}), \\ (-1)^{k(l-m)}\binom{\left[\frac{N-l}{2}\right]}{\left[\frac{k-m}{2}\right]}\binom{\left[\frac{l}{2}\right]}{\left[\frac{m}{2}\right]} & (\text{otherwise}). \end{cases}
\end{aligned}$$

Proof. It is enough to consider the case where $m \le k-1$, $m \le l$, $N-l \ge k-m$. Let us use the following two special Young diagrams

$$\sigma_1 = (\overbrace{N-k-l+m,\dots,N-k-l+m}^{m \text{ times}}, \overbrace{0,\dots,0}^{(k-m) \text{ times}}),$$

$$\sigma_2 = (\overbrace{N-k-l+m+1,\dots,N-k-l+m+1}^{(m+1) \text{ times}}, \overbrace{0,\dots,0}^{(k-m-1) \text{ times}}).$$

Moreover, we set

$$I(\sigma_1,\sigma_2) = \{\sigma \in I(N,k) \mid \sigma_1 \subset \sigma,\ \sigma_2 \not\subset \sigma\}.$$

Then we have a cell decomposition of $\Omega(N,k,l,m)$

$$\Omega(N,k,l,m) = \bigsqcup_{\sigma \in I(\sigma_1,\sigma_2)} \Omega_\sigma^\circ.$$

In order to compute $\chi(\Omega(N,k,l,m))$, let us consider the shortest ways that connect from P_1 to P_2 through P_3 in the Fig. 1 below.

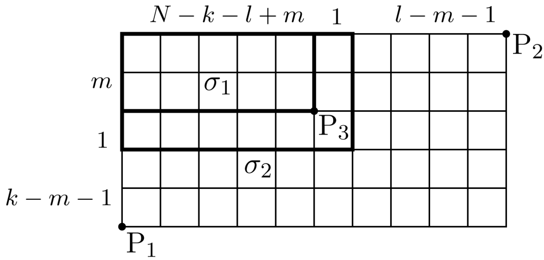


Fig. 1

(i) In the case $\mathbb{K} = \mathbb{C}$, since $\chi(\Omega_\sigma^\circ) = 1$ for $\sigma \in I(N, k)$, we have

$$\chi(\Omega(N, k, l, m)) = \#I(\sigma_1, \sigma_2) = \binom{N-l}{k-m}\binom{l}{m}.$$

(ii) In the case $\mathbb{K} = \mathbb{R}$, since $\chi(\Omega_\sigma^\circ) = (-1)^{k(N-k)-|\sigma|}$ for $\sigma \in I(N, k)$, we have

$$\begin{aligned}&\chi(\Omega(N, k, l, m))\\ &= (-1)^{k(N-k)}\{\#\{\sigma \in I(\sigma_1, \sigma_2) \mid |\sigma| \text{ is even}\}\\ &\qquad\qquad - \#\{\sigma \in I(\sigma_1, \sigma_2) \mid |\sigma| \text{ is odd}\}\}.\end{aligned}$$

Let us set

$$\begin{aligned}e_N(k) &= \#\{\sigma \in I(N, k) \mid |\sigma| \text{ is even}\},\\ o_N(k) &= \#\{\sigma \in I(N, k) \mid |\sigma| \text{ is odd}\}.\end{aligned}$$

(ii-a) In the case where $|\sigma_1| = m(N-k-l+m)$ is even, since we have

$$\begin{aligned}\#\{\sigma \in I(\sigma_1, \sigma_2) \mid |\sigma| \text{ is even}\} &= e_{N-l}(k-m)e_l(m) + o_{N-l}(k-m)o_l(m),\\ \#\{\sigma \in I(\sigma_1, \sigma_2) \mid |\sigma| \text{ is odd}\} &= e_{N-l}(k-m)o_l(m) + o_{N-l}(k-m)e_l(m),\end{aligned}$$

we have

$$\begin{aligned}&\chi(\Omega(N, k, l, m))\\ &= (-1)^{k(N-k)}(e_{N-l}(k-m) - o_{N-l}(k-m))(e_l(m) - o_l(m))\\ &= (-1)^{k(N-k)} \cdot (-1)^{(k-m)(N-l-k+m)}\chi(F_{N-l}(k-m)) \cdot (-1)^{m(l-m)}\chi(F_l(m)).\end{aligned}$$

(ii-b) In the case where $|\sigma_1| = m(N-k-l+m)$ is odd, since we have

$$\begin{aligned}\#\{\sigma \in I(\sigma_1, \sigma_2) \mid |\sigma| \text{ is even}\} &= e_{N-l}(k-m)o_l(m) + o_{N-l}(k-m)e_l(m),\\ \#\{\sigma \in I(\sigma_1, \sigma_2) \mid |\sigma| \text{ is odd}\} &= e_{N-l}(k-m)e_l(m) + o_{N-l}(k-m)o_l(m),\end{aligned}$$

we have

$$\begin{aligned}\chi(\Omega(N, k, l, m)) &= (-1)^{k(N-k)+1}(e_{N-l}(k-m) - o_{N-l}(k-m))(e_l(m) - o_l(m))\\ &= (-1)^{k(N-k)+1} \cdot (-1)^{(k-m)(N-l-k+m)}\chi(F_{N-l}(k-m)) \cdot (-1)^{m(l-m)}\chi(F_l(m)).\end{aligned}$$

This completes the proof. □

Remark 2.11. The referee pointed out that the elements in $\Omega(N,k,l,m)$ for each fixed m-dimensional linear subspace of V_l form a topological space homotopic to $F_{N-l}(k-m)$ and we can give more geometrical proof for Proposition 2.10.

3. TOPOLOGICAL RADON TRANSFORMS ON GRASSMANN MANIFOLDS

Let p, q be positive integers satisfying $1 \leq p \leq q \leq N$. Set $X = F_N(p)$ and $Y = F_N(q)$. For $r = 0, \dots, p$, we also set

$$S_r = \{(x,y) \in X \times Y \mid \dim(x \cap y) = r\}.$$

Let us consider the diagram (2.1) for $X = F_N(p)$, $Y = F_N(q)$ and $S = S_r$. We denote the restrictions of p_X (resp. p_Y) to S_r by f_r (resp. g_r). Then we define the topological Radon transformation by

$$\mathcal{R}_{S_r} = \int_{g_r} {f_r}^* = \int_{p_Y} \mathbf{1}_{S_r} \cdot p_X^* \colon CF(X) \longrightarrow CF(Y)$$

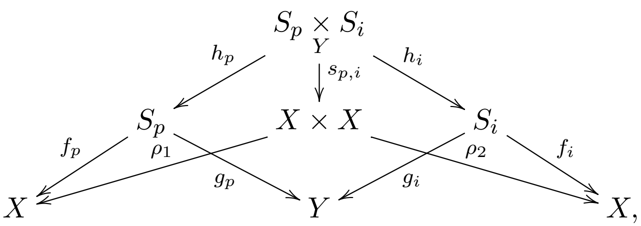
and its transposed transformation by

$${}^t\mathcal{R}_{S_r} = \int_{f_r} {g_r}^* = \int_{p_X} \mathbf{1}_{S_r} \cdot p_Y^* \colon CF(Y) \longrightarrow CF(X).$$

In general case, we do not expect that the transposed transformation ${}^t\mathcal{R}_{S_r}$ is a left inverse transformation of $\mathcal{R}_{S_r}$. In [9], we proved an inversion formula for $\mathcal{R}_{S_p}$. The aim of this paper is to characterize the range of $\mathcal{R}_{S_p}$ in $CF(F_N(q))$.

3.1. A review of an inversion formula for topological Radon transforms on Grassmann manifolds

For the reader's convenience, let us recall briefly our explicit construction of a left inverse transformation of $\mathcal{R}_{S_p}$ in [9]. For $i = 0, \dots, p$, let us consider the following commutative diagram:



where ρ_1, ρ_2 (resp. h_p, h_i) are the natural projections from $X \times X$ to each X (resp. from $S_p \underset{Y}{\times} S_i$ to S_p and S_i respectively) and $s_{p,i} \colon S_p \underset{Y}{\times} S_i \longrightarrow X \times X$

is the natural projection from $S_p \underset{Y}{\times} S_i$ to $X \times X$. Then by Proposition 2.3 for $\varphi \in CF(X)$ we have

$$ {}^t\mathcal{R}_{S_i} \circ \mathcal{R}_{S_p}(\varphi) = \int_{f_i} g_i^* \int_{g_p} f_p^* \varphi = \int_{\rho_2} \left(\int_{s_{p,i}} \mathbf{1}_{S_p \underset{Y}{\times} S_i} \right) \rho_1^* \varphi. \tag{3.1} $$

For $j = 0, \ldots, p$, we set

$$ Z_j = \{(x_1, x_2) \in X \times X \mid \dim(x_1 \cap x_2) = j\}. $$

For $(x_1, x_2) \in Z_j$, by considering the condition in the quotient space $\mathbb{K}^N/x_1$ we have

$$ \begin{aligned} s_{p,i}^{-1}(x_1, x_2) &= \{y \in F_N(q) \mid x_1 \subset y,\ \dim(x_2 \cap y) = i\} \\ &\simeq \{y \in F_{N-p}(q-p) \mid \dim(x_3 \cap y) = i - j\} \\ &= \Omega(N-p, q-p, p-j, i-j), \end{aligned} $$

where x_3 is a $(p-j)$-dimensional linear subspace. Thus, $\chi(s_{p,i}^{-1}(x_1, x_2))$ is constant on Z_j, which we set $a_{ij} = \chi(s_{p,i}^{-1}(x_1, x_2))$. By Proposition 2.10, a_{ij} is explicitly computed as follows.

(i) In the case $\mathbb{K} = \mathbb{C}$, we have

$$ a_{ij} = \binom{N-2p+j}{q-p-i+j} \binom{p-j}{i-j}. \tag{3.2} $$

(ii) In the case $\mathbb{K} = \mathbb{R}$, we have

$$ a_{ij} = \begin{cases} 0 & \begin{pmatrix} (q-p-i+j)(N-p-q+i) \\ \text{or } (i-j)(p-i) \text{ is odd} \end{pmatrix}, \\ (-1)^{(q-p)(p-i)} \begin{pmatrix} \left[\frac{N-2p+j}{2}\right] \\ \left[\frac{q-p-i+j}{2}\right] \end{pmatrix} \begin{pmatrix} \left[\frac{p-j}{2}\right] \\ \left[\frac{i-j}{2}\right] \end{pmatrix} & \text{(otherwise)}. \end{cases} \tag{3.3} $$

By using a_{ij}, we have

$$ \int_{s_{p,i}} \mathbf{1}_{S_p \underset{Y}{\times} S_i} = \sum_{j=0}^{p} a_{ij} \mathbf{1}_{Z_j} \qquad (i = 0, 1, \ldots, p). \tag{3.4} $$

Therefore by (3.1) and (3.4) we obtain

$$ {}^t\mathcal{R}_{S_i} \circ \mathcal{R}_{S_p}(\varphi) = \sum_{j=0}^{p} a_{ij} \left(\int_{\rho_2} \mathbf{1}_{Z_j} \cdot \rho_1^* \varphi \right). \tag{3.5} $$

By the computation above, let us construct a left inverse transformation of $\mathcal{R}_{S_p}$ as follows. Let us set $A = (a_{ij})_{0 \le i,j \le p}$. Note that the $(p+1) \times (p+1)$ matrix A is a lower triangular one. We denote its determinant by $\lambda_{p,q} = \det A$ and its

$(k+1, p+1)$-cofactor by d_k $(k = 0, 1, \ldots, p)$. Then we define a constructible function $K_{p,q} \in CF(X \times Y)$ by

$$K_{p,q} = \sum_{k=0}^{p} d_k \mathbf{1}_{S_k}$$

and a new transform $\widehat{\mathcal{R}}(\psi)$ of $\psi \in CF(Y)$ by

$$\widehat{\mathcal{R}}(\psi) = \int_{p_X} K_{p,q} \cdot p_Y^* \psi = \sum_{k=0}^{p} d_k \, {}^t\mathcal{R}_{S_k}(\psi). \tag{3.6}$$

By (3.5) and the equality

$$\int_{\rho_2} \mathbf{1}_{Z_p} \cdot \rho_1^* \varphi = \varphi,$$

we obtain the following inversion formula for $\mathcal{R}_{S_p}$ in [9].

Theorem 3.1 ([9]). (i) *For $\varphi \in CF(X)$, we have*

$$\widehat{\mathcal{R}} \circ \mathcal{R}_{S_p}(\varphi) = \lambda_{p,q} \cdot \varphi.$$

(ii) *If one of the following conditions*

(a) *$\mathbb{K} = \mathbb{C}$ and $p + q \leq N$,*

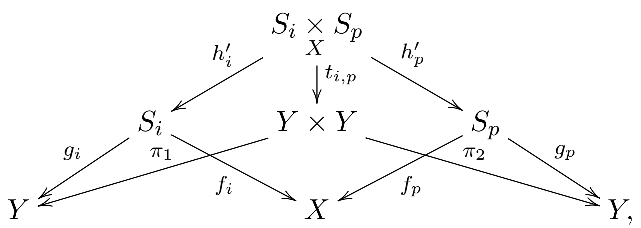
(b) *$\mathbb{K} = \mathbb{R}$, $p + q \leq N$ and $q - p$ is even,*

are satisfied, then $\lambda_{p,q}$ does not vanish.

By Theorem 3.1, under the condition of (ii) we can completely reconstruct the original function $\varphi \in CF(X)$ from its topological Radon transform $\mathcal{R}_{S_p}(\varphi)$. In this meaning, we see $\widehat{\mathcal{R}}$ as a left inverse transformation of $\mathcal{R}_{S_p}$. Hereafter, we always assume the condition of Theorem 3.1 (ii). Therefore, we assume that there exists the left inverse transformation $\widehat{\mathcal{R}}$ of $\mathcal{R}_{S_p}$ by our method.

3.2. A range characterization of $\mathcal{R}_{S_p}$ (1)

In this subsection, we observe that for $\varphi \in CF(X)$ the topological Radon transform $\mathcal{R}_{S_p}(\varphi)$ satisfies a system of topological integral equations. First, by similar computation to obtain (3.5), for $\psi \in CF(Y)$ let us compute $\mathcal{R}_{S_p} \circ {}^t\mathcal{R}_{S_i}(\psi)$ as follows. For $i = 0, 1, \ldots, p$, let us consider the following commutative diagram:



where π_1, π_2 (resp. h'_i, h'_p) are the natural projections from $Y \times Y$ to each Y (resp. from $S_i \underset{X}{\times} S_p$ to S_i and S_p respectively) and $t_{i,p}\colon S_i \underset{X}{\times} S_p \longrightarrow Y \times Y$ is the natural projection from $S_i \underset{X}{\times} S_p$ to $Y \times Y$. Then for $\psi \in CF(Y)$ we have

$$\mathcal{R}_{S_p} \circ {}^t\mathcal{R}_{S_i}(\psi) = \int_{g_p} f_p^* \int_{f_i} g_i^* \psi = \int_{\pi_2} \left(\int_{t_{i,p}} \mathbf{1}_{S_i \underset{X}{\times} S_p} \right) \pi_1^* \psi. \tag{3.7}$$

For $j = 0, \ldots, q$, we set

$$W_j = \{(y_1, y_2) \in Y \times Y \mid \dim(y_1 \cap y_2) = j\}.$$

For $(y_1, y_2) \in W_j$, by considering the condition in $y_2 \simeq \mathbb{K}^q$ we have

$$\begin{aligned} t_{i,p}^{-1}(y_1, y_2) &= \{x \in F_N(p) \mid \dim(x \cap y_1) = i,\ x \subset y_2\} \\ &\simeq \{x \in F_q(p) \mid \dim(x \cap y_3) = i\} \\ &= \Omega(q, p, j, i), \end{aligned}$$

where $y_3 = y_1 \cap y_2$ is a j-dimensional linear subspace. Thus, $\chi(t_{i,p}^{-1}(y_1, y_2))$ is constant on W_j, which we set $b_{ij} = \chi(t_{i,p}^{-1}(y_1, y_2))$ and $B = (b_{ij})_{\substack{0 \le i \le p \\ 0 \le j \le q}}$. By Proposition 2.10, b_{ij} is explicitly computed as follows.

(i) In the case $\mathbb{K} = \mathbb{C}$, we have

$$b_{ij} = \binom{q-j}{p-i} \binom{j}{i}. \tag{3.8}$$

(ii) In the case $\mathbb{K} = \mathbb{R}$, we have

$$b_{ij} = \begin{cases} 0 & ((q-p-j+i)(p-i) \text{ or } i(j-i) \text{ is odd}), \\ (-1)^{p(j-i)} \binom{\left[\frac{q-j}{2}\right]}{\left[\frac{p-i}{2}\right]} \binom{\left[\frac{j}{2}\right]}{\left[\frac{i}{2}\right]} & (\text{otherwise}). \end{cases} \tag{3.9}$$

By using b_{ij}, we have

$$\int_{t_{i,p}} \mathbf{1}_{S_i \underset{X}{\times} S_p} = \sum_{j=0}^{q} b_{ij} \mathbf{1}_{W_j} \qquad (i = 0, 1, \ldots, p). \tag{3.10}$$

Therefore by (3.7) and (3.10) we have

$$\mathcal{R}_{S_p} \circ {}^t\mathcal{R}_{S_i}(\psi) = \sum_{j=0}^{q} b_{ij} \left(\int_{\pi_2} \mathbf{1}_{W_j} \cdot \pi_1^* \psi \right).$$

By (3.6), we obtain

$$\mathcal{R}_{S_p} \circ \widehat{\mathcal{R}}(\psi) = \sum_{k=0}^{p} d_k \mathcal{R}_{S_p} \circ {}^t\mathcal{R}_{S_k}(\psi) = \sum_{k=0}^{p} \sum_{j=0}^{q} d_k b_{kj} \int_{\pi_2} \mathbf{1}_{W_j} \cdot \pi_1^* \psi. \tag{3.11}$$

Now, for $\varphi \in CF(X)$ let us consider the case $\psi = \mathcal{R}_{S_p}(\varphi)$. By Proposition 2.5, for $j = 0, 1, \ldots, q$ and $y_2 \in Y$ we have

$$\left(\int_{\pi_2} \mathbf{1}_{W_j} \cdot \pi_1^* \psi\right)(y_2) = \int_Y \mathbf{1}_{W_j}(\cdot, y_2) \cdot \mathcal{R}_{S_p}(\varphi) = \int_X {}^t\mathcal{R}_{S_p}(\mathbf{1}_{W_j}(\cdot, y_2)) \cdot \varphi. \tag{3.12}$$

For $l = 0, 1, \ldots, p$ and $(x, y_2) \in S_l$, by considering the condition in the quotient space $\mathbb{K}^N/x$ we have

$$\begin{aligned} &\{y \in F_N(q) \mid x \subset y,\ \dim(y \cap y_2) = j\} \\ &\quad \simeq \{y \in F_{N-p}(q-p) \mid \dim(y \cap y_3) = j - l\} \\ &\quad = \Omega(N-p, q-p, q-l, j-l), \end{aligned}$$

where y_3 is a $(q-l)$-dimensional linear subspace. Thus, the Euler characteristic of the set $\{y \in F_N(q) \mid x \subset y,\ \dim(y \cap y_2) = j\}$ is constant on S_l, we set it c_{jl} and $C = (c_{ij})_{\substack{0 \le i \le q \\ 0 \le j \le p}}$. By Proposition 2.10, c_{jl} is explicitly computed as follows.

(i) In the case $\mathbb{K} = \mathbb{C}$, we have

$$c_{jl} = \binom{N-p-q+l}{q-p-j+l}\binom{q-l}{j-l}. \tag{3.13}$$

(ii) In the case $\mathbb{K} = \mathbb{R}$, we have

$$c_{jl} = \begin{cases} 0 & \left(\begin{array}{l}(q-p-j+l)(N-2q+j) \text{ or} \\ (j-l)(q-j) \text{ is odd}\end{array}\right), \\ (-1)^{(q-p)(q-j)} \dbinom{\left[\frac{N-p-q+l}{2}\right]}{\left[\frac{q-p-j+l}{2}\right]} \dbinom{\left[\frac{q-l}{2}\right]}{\left[\frac{j-l}{2}\right]} & \text{otherwise.} \end{cases} \tag{3.14}$$

By using c_{jl}, we have

$$ {}^t\mathcal{R}_{S_p}(\mathbf{1}_{W_j}(\cdot, y_2))(x) = \chi(\{y \in F_N(q) \mid x \subset y,\ \dim(y \cap y_2) = j\}) \tag{3.15}$$

$$= \sum_{l=0}^{p} c_{jl} \mathbf{1}_{S_l}(x, y_2). \tag{3.16}$$

By (3.12), (3.16), we have the relations of $(p+1)$ transforms $\mathcal{R}_{S_0}(\varphi), \ldots, \mathcal{R}_{S_p}(\varphi)$:

$$\int_{\pi_2} \mathbf{1}_{W_j} \cdot \pi_1^* \mathcal{R}_{S_p}(\varphi) = \sum_{l=0}^{p} c_{jl} \mathcal{R}_{S_l}(\varphi) \qquad (j = 0, 1, \ldots, q). \tag{3.17}$$

Let us set $C_1 = (c_{ij})_{\substack{0 \le i \le q-p-1 \\ 0 \le j \le p}}$ and $C_2 = (c_{ij})_{\substack{q-p \le i \le q \\ 0 \le j \le p}}$. Then the relations

(3.17) are equivalent to the following systems of linear equations:

$$\begin{pmatrix} \int_{\pi_2} \mathbf{1}_{W_0} \cdot \pi_1^* \mathcal{R}_{S_p}(\varphi) \\ \vdots \\ \int_{\pi_2} \mathbf{1}_{W_{q-p-1}} \cdot \pi_1^* \mathcal{R}_{S_p}(\varphi) \end{pmatrix} = C_1 \begin{pmatrix} \mathcal{R}_{S_0}(\varphi) \\ \vdots \\ \mathcal{R}_{S_p}(\varphi) \end{pmatrix},$$

$$\begin{pmatrix} \int_{\pi_2} \mathbf{1}_{W_{q-p}} \cdot \pi_1^* \mathcal{R}_{S_p}(\varphi) \\ \vdots \\ \int_{\pi_2} \mathbf{1}_{W_q} \cdot \pi_1^* \mathcal{R}_{S_p}(\varphi) \end{pmatrix} = C_2 \begin{pmatrix} \mathcal{R}_{S_0}(\varphi) \\ \vdots \\ \mathcal{R}_{S_p}(\varphi) \end{pmatrix}.$$

By (3.13) and (3.14), the $(p+1)\times(p+1)$ matrix C_2 is upper triangular and regular under the assumption of Theorem 3.1 (ii) (In this section, we always assume the condition of Theorem 3.1 (ii)). We denote by μ_{ij} the $(i+1, j-q+p+1)$ component of the matrix $C_1 C_2^{-1}$. That is, $(\mu_{ij})_{\substack{0\le i\le q-p-1 \\ q-p\le j\le q}} = C_1 C_2^{-1}$. Note that μ_{ij} is explicitly computable by Proposition 2.10. Therefore, we obtain the following theorem.

Theorem 3.2. *In the situation above, assume the condition of Theorem 3.1 (ii). Then for $\varphi \in CF(X)$ the topological Radon transform $\psi = \mathcal{R}_{S_p}(\varphi)$ of φ satisfies the following system of topological integral equations:*

$$\int_{\pi_2} \mathbf{1}_{W_i} \cdot \pi_1^* \psi = \sum_{j=q-p}^{q} \mu_{ij} \cdot \int_{\pi_2} \mathbf{1}_{W_j} \cdot \pi_1^* \psi \qquad (i = 0, 1, \ldots, q-p-1). \tag{3.18}$$

Definition 3.3. We denote by $\widetilde{CF(F_N(q))}$ the set of all constructible functions $\psi \in CF(Y)$ satisfying the condition (3.18).

Note that $\widetilde{CF(F_N(q))}$ is an Abelian subgroup of $CF(F_N(q))$.

3.3. A range characterization of $\mathcal{R}_{S_p}$ (2)

In this subsection, we prove the following theorem.

Theorem 3.4. *Assume the condition of Theorem 3.1 (ii). Then we have*

$$\mathcal{R}_{S_p} \circ \widehat{\mathcal{R}}(\psi) = \lambda_{p,q} \cdot \psi$$

for $\psi \in \widetilde{CF(F_N(q))}$. Here $\widehat{\mathcal{R}}$ and $\lambda_{p,q}$ are defined in Section 3.1.

Proof. First, note that we rewrite the condition (3.18) as the matrix equation

$$\begin{pmatrix} \int_{\pi_2} \mathbf{1}_{W_0} \cdot \pi_1^*\psi \\ \vdots \\ \int_{\pi_2} \mathbf{1}_{W_q} \cdot \pi_1^*\psi \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} C_2^{-1} \begin{pmatrix} \int_{\pi_2} \mathbf{1}_{W_{q-p}} \cdot \pi_1^*\psi \\ \vdots \\ \int_{\pi_2} \mathbf{1}_{W_q} \cdot \pi_1^*\psi \end{pmatrix}. \tag{3.19}$$

For $\psi \in \widetilde{CF(F_N(q))}$, by (3.11) and (3.19) we have

$$\begin{aligned} \mathcal{R}_{S_p} \circ \widehat{\mathcal{R}}(\psi) &= \sum_{k=0}^{p} \sum_{j=0}^{q} d_k b_{kj} \int_{\pi_2} \mathbf{1}_{W_j} \cdot \pi_1^*\psi \\ &= \begin{pmatrix} d_0 & d_1 & \cdots & d_p \end{pmatrix} B C C_2^{-1} \begin{pmatrix} \int_{\pi_2} \mathbf{1}_{W_{q-p}} \cdot \pi_1^*\psi \\ \vdots \\ \int_{\pi_2} \mathbf{1}_{W_q} \cdot \pi_1^*\psi \end{pmatrix}. \end{aligned}$$

By (3.13) and (3.14), the $(p+1)$-th row vector of C_2^{-1} is $\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$. By Proposition 3.7 below, we have

$$\begin{aligned} \mathcal{R}_{S_p} \circ \widehat{\mathcal{R}}(\psi) &= \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda_{p,q} \end{pmatrix} C_2^{-1} \begin{pmatrix} \int_{\pi_2} \mathbf{1}_{W_{q-p}} \cdot \pi_1^*\psi \\ \vdots \\ \int_{\pi_2} \mathbf{1}_{W_q} \cdot \pi_1^*\psi \end{pmatrix} \\ &= \lambda_{p,q} \int_{\pi_2} \mathbf{1}_{W_q} \cdot \pi_1^*\psi \\ &= \lambda_{p,q} \cdot \psi. \end{aligned}$$

This proves Theorem 3.4. □

Remark 3.5. In the special case $p+q = N$, we can easily show $\widetilde{CF(F_N(q))} = CF(F_N(q))$ since $q - p \leq \dim(y_1 \cap y_2) \leq q$ for any two q-dimensional linear subspaces $y_1, y_2 \in F_N(q)$. In this case, Theorem 3.4 was proved in ([9], Theorem 4.1).

On $\widetilde{CF(F_N(q))}$, we consider $\widehat{\mathcal{R}}$ as not only a left inverse transformation of $\mathcal{R}_{S_p}$ but also a right inverse one. By Theorems 3.2 and 3.4, we obtain the following result.

Corollary 3.6. *We have the inclusions:*

$$\lambda_{p,q} \cdot \widetilde{CF(F_N(q))} \subset \mathrm{Image}(\mathcal{R}_{S_p}) \, (= \mathcal{R}_{S_p}(CF(F_N(p)))) \subset \widetilde{CF(F_N(q))}$$

in $CF(F_N(q))$.

We may consider Corollary 3.6 as a range characterization theorem of the topological Radon transformation $\mathcal{R}_{S_p}$. Note that by introducing the theory of the $\mathbb{Q}$ (or $\mathbb{R}$)-valued constructible functions we could obtain more explicit range characterization. Equivalently, we have the equality:

$$\mathrm{Image}(\mathcal{R}_{S_p}) \otimes_{\mathbb{Z}} \mathbb{Q} = \widetilde{CF(F_N(q))} \otimes_{\mathbb{Z}} \mathbb{Q}$$

in $CF(F_N(q)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Finally, in order to complete the proof of Theorem 3.4 above, let us prove the following proposition.

Proposition 3.7. *In the situation of Theorem 3.4, we have the following equality of matrices*

$$\begin{pmatrix} d_0 & d_1 & \cdots & d_p \end{pmatrix} BC = \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda_{p,q} \end{pmatrix}. \tag{3.20}$$

Proof. Note that for $i, k = 0, 1, \ldots, p$ we have

$$\sum_{j=0}^{q} b_{ij} c_{jk} = \sum_{l=0}^{k} a_{il} b_{lk}. \tag{3.21}$$

Since d_i is the $(i+1, p+1)$-cofactor of $A = (a_{ij})_{0 \le i,j \le p}$ $(i = 0, 1, \ldots, p)$ and $\lambda_{p,q} = \det A$, we have

$$\begin{aligned} &\sum_{i=0}^{p} d_i \sum_{j=0}^{q} b_{ij} c_{jk} \\ &= \sum_{i=0}^{p} d_i \sum_{l=0}^{k} a_{il} b_{lk} = \sum_{l=0}^{k} b_{lk} \sum_{i=0}^{p} d_i a_{il} = \sum_{l=0}^{k} b_{lk} \delta_{l,p} \lambda_{p,q} = \delta_{k,p} \lambda_{p,q}. \end{aligned}$$

Here $\delta_{l,p}$ denotes the Kronecker's delta. Therefore we obtain the equality (3.20).

Let us explain the outline of the proof of (3.21). In computations, the following formula, proved by induction on k, is useful:

$$\sum_{j=0}^{q} \binom{q-p}{j-i} \binom{j}{k} \binom{N-p-q+k}{q-p-j+k} = \sum_{l=0}^{k} \binom{q-p}{k-l} \binom{i}{l} \binom{N-2p+l}{q-p-i+l}. \tag{3.22}$$

In the case $\mathbb{K} = \mathbb{C}$, by (3.2), (3.8), (3.13) and (3.22) we have

$$\sum_{j=0}^{q} b_{ij} c_{jk} = \sum_{j=0}^{q} \binom{q-j}{p-i} \binom{j}{i} \binom{N-p-q+k}{q-p-j+k} \binom{q-k}{j-k}$$

$$
\begin{aligned}
&= \frac{(q-k)!k!}{(p-i)!i!(q-p)!} \sum_{j=0}^{q} \binom{q-p}{j-i} \binom{j}{k} \binom{N-p-q+k}{q-p-j+k} \\
&= \frac{(q-k)!k!}{(p-i)!i!(q-p)!} \sum_{l=0}^{k} \binom{q-p}{k-l} \binom{i}{l} \binom{N-2p+l}{q-p-i+l} \\
&= \sum_{l=0}^{k} \binom{N-2p+l}{q-p-i+l} \binom{p-l}{i-l} \binom{q-k}{p-l} \binom{k}{l} \\
&= \sum_{l=0}^{k} a_{il} b_{lk}.
\end{aligned}
$$

In the case $\mathbb{K} = \mathbb{R}$, let us consider the following four cases: (ii-1) p, q, N are odd, (ii-2) p, q are odd and N is even, (ii-3) p, q, N are even, (ii-4) p, q are even and N is odd. Note that we assume that $q - p$ is even here. In each case, we compute the both sides of (3.21) by (3.3), (3.9) and (3.14). By using the equality (3.22), we could obtain (3.21). Since the computations are similar to the case $\mathbb{K} = \mathbb{C}$, we omit the details. □

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