DUALITY FOR BINARY QUARTICS AND TERNARY CUBICS ARISING FROM THEIR MILNOR ALGEBRAS

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Let $\mathcal{Q}_n^d$ be the vector space of forms of degree $d \geq 3$ on $\mathbb{C}^n$, with $n \geq 2$. This note concerns the map Φ, introduced by J. Alper, M. Eastwood and the author, that assigns every nondegenerate form $f \in \mathcal{Q}_n^d$ the so-called associated form, which is an element of $\mathcal{Q}_n^{n(d-2)}$ derived from the Milnor algebra of the isolated singularity of the zero set of f at the origin. We concentrate on two cases: those of binary quartics ($n = 2$, $d = 4$) and ternary cubics ($n = 3$, $d = 3$), and show that in these situations the map Φ induces a rational equivariant involution on the projectivized space $\mathbb{P}(\mathcal{Q}_n^d)$. In particular, there exists a natural duality for elliptic curves with nonvanishing j-invariant.

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1. INTRODUCTION

Let $\mathcal{Q}_n^d$ be the vector space of forms of degree d on $\mathbb{C}^n$, where $n \geq 2$, $d \geq 3$. Assuming that the discriminant of $f \in \mathcal{Q}_n^d$ does not vanish, define $M_f := \mathbb{C}[z_1, \ldots, z_n]/(f_{z_1}, \ldots, f_{z_n})$ to be the Milnor algebra of the isolated hypersurface singularity at the origin of the zero set of f. Let $\mathfrak{m}$ be the maximal ideal of M_f. One can then introduce a form defined on the n-dimensional quotient $\mathfrak{m}/\mathfrak{m}^2$ with values in the one-dimensional socle $\mathrm{Soc}(M_f)$ of M_f as follows:

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathrm{Soc}(M_f),$$
$$x \mapsto y^{n(d-2)},$$

where y is any element of $\mathfrak{m}$ that projects to $x \in \mathfrak{m}/\mathfrak{m}^2$. There is a canonical isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}^n$ and, since the Hessian of f generates the socle, there is also a canonical isomorphism $\mathrm{Soc}(M_f) \cong \mathbb{C}$. Hence, one obtains a form $\mathbf{f}$ of degree $n(d-2)$ on $\mathbb{C}^n$ (*i.e.*, an element of $\mathcal{Q}_n^{n(d-2)}$), which is called the *associated form* of f (see Section 2 for more detail on this definition).

We study the morphism

$$\Phi : X_n^d \to \mathcal{Q}_n^{n(d-2)}, \quad f \mapsto \mathbf{f}$$

of affine algebraic varieties, where X_n^d is the variety of forms in $\mathcal{Q}_n^d$ with nonzero discriminant. One of the reasons for our interest in Φ is the following conjecture proposed in [1] (see also [5]):

CONJECTURE 1.1. *For every regular* $\mathrm{GL}_n(\mathbb{C})$*-invariant function* S *on* X_n^d *there exists a rational* $\mathrm{GL}_n(\mathbb{C})$*-invariant function* R *on* $\mathcal{Q}_n^{n(d-2)}$ *defined at all points of the set* $\Phi(X_n^d) \subset \mathcal{Q}_n^{n(d-2)}$ *such that* $R \circ \Phi = S$.

In [5], Conjecture 1.1 was shown to hold for binary forms (*i.e.*, for $n = 2$) of degrees $3 \le d \le 6$, and in [1] its weaker variant was established for arbitrary n and d. Furthermore, in [2] the conjecture was confirmed for binary forms of any degree. While Conjecture 1.1 is rather interesting from the purely invariant-theoretic viewpoint, it has an important implication for singularity theory. Namely, as explained in detail in [1, 2], if this conjecture is established, it will provide a solution, in the homogeneous case, to the so-called *reconstruction problem*, which is the question of finding a constructive proof of the well-known Mather-Yau theorem (see [16, 20]). Settling Conjecture 1.1 is part of our program to solve the reconstruction problem for quasihomogeneous isolated hypersurface singularities. This amounts to showing that a certain system of invariants introduced in [5] is complete, and Conjecture 1.1 implies completeness in the homogeneous case.

The morphism Φ is quite natural and deserves attention regardless of Conjecture 1.1. In fact, this map is interesting even for small values of n and d. In what follows, we look at Φ in two situations: those of binary quartics ($n = 2$, $d = 4$) and ternary cubics ($n = 3$, $d = 3$). These are the only choices of n, d for which Φ preserves the form's degree. In this note, we observe that, curiously, in each of the two cases the projectivization $\mathbb{\Phi}$ of Φ induces an equivariant involution on the image $\mathbb{X}_n^d$ of X_n^d in the projective space $\mathbb{P}(\mathcal{Q}_n^d)$, with one $\mathrm{SL}_n(\mathbb{C})$-orbit removed (see Propositions 3.1 and 3.2 in Section 3). In particular, $\mathbb{\Phi}$ yields an equivariant involution on the space of elliptic curves with nonvanishing j-invariant, which appears to have never been mentioned in the extensive literature on elliptic curves. The duality induced by $\mathbb{\Phi}$ will be studied in detail in our forthcoming paper joint with J. Alper and N. Kruzhilin.

2. PRELIMINARIES

Let $\mathcal{Q}_n^d$ be the vector space of forms of degree d on $\mathbb{C}^n$ where $n \ge 2$. The standard action of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$ induces an action on $\mathcal{Q}_n^d$ as follows:

$$(C \cdot f)(z) := f\left(z\, C^{-T}\right)$$

for $C \in \mathrm{GL}_n$, $f \in \mathcal{Q}_n^d$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Two forms that lie in the same GL_n-orbit are called linearly equivalent. Below we will be mostly concerned with the induced action of $\mathrm{SL}_n = \mathrm{SL}_n(\mathbb{C})$.

To every nonzero $f \in \mathcal{Q}_n^d$ we associate the hypersurface

$$V_f := \{z \in \mathbb{C}^n : f(z) = 0\}$$

and consider it as a complex space with the structure sheaf induced by f. The singular set of V_f is then the critical set of f. In particular, if $d \geq 2$ the hypersurface V_f has a singularity at the origin. We are interested in the situation when this singularity is isolated, or, equivalently, when V_f is smooth away from 0. This occurs if and only if f is nondegenerate, *i.e.*, $\Delta(f) \neq 0$, where Δ is the discriminant (see Chapter 13 in [9]).

For $d \geq 3$ define

$$X_n^d := \{f \in \mathcal{Q}_n^d : \Delta(f) \neq 0\}.$$

Observe that GL_n acts on the affine variety X_n^d and note that every $f \in X_n^d$ is stable with respect to this action, *i.e.*, the orbit of f is closed in X_n^d and has dimension n^2 (see, *e.g.*, Corollary 5.24 in [17]).

Fix $f \in X_n^d$ and consider the Milnor algebra of the singularity of V_f, which is the complex local algebra

$$M_f := \mathbb{C}[[z_1, \dots, z_n]]/(f_1, \dots, f_n),$$

where $\mathbb{C}[[z_1, \dots, z_n]]$ is the algebra of formal power series in $z_1, \dots, z_n$ with complex coefficients and $f_j := \partial f / \partial z_j$, $j = 1, \dots, n$. Since the singularity of V_f is isolated, the algebra M_f is Artinian, *i.e.*, $\dim_{\mathbb{C}} M_f < \infty$ (see Proposition 1.70 in [10]). Therefore, $f_1, \dots, f_n$ is a system of parameters in $\mathbb{C}[[z_1, \dots, z_n]]$. Since $\mathbb{C}[[z_1, \dots, z_n]]$ is a regular local ring, $f_1, \dots, f_n$ is a regular sequence in $\mathbb{C}[[z_1, \dots, z_n]]$. This yields that M_f is a complete intersection.

It is convenient to utilize another realization of the Milnor algebra. Namely, we can write

$$M_f = \mathbb{C}[z_1, \dots, z_n]/(f_1, \dots, f_n).$$

Let $\mathfrak{m}$ denote the maximal ideal of M_f, which consists of all elements represented by polynomials in $\mathbb{C}[z_1, \dots, z_n]$ vanishing at the origin. The maximal ideal is nilpotent and we let $\nu := \max\{\eta \in \mathbb{N} \mid \mathfrak{m}^\eta \neq 0\}$ be the socle degree of M_f.

Since M_f is a complete intersection, by [3] it is a Gorenstein algebra. This means that the socle of M_f, defined as

$$\mathrm{Soc}(M_f) := \{x \in \mathfrak{m} : x\,\mathfrak{m} = 0\},$$

is a one-dimensional vector space over $\mathbb{C}$ (see, *e.g.*, Theorem 5.3 in [12]). We then have $\mathrm{Soc}(M_f) = \mathfrak{m}^\nu$. Furthermore, $\mathrm{Soc}(M_f)$ is spanned by the element of

M_f represented by the Hessian $H(f)$ of f (see, *e.g.*, Lemma 3.3 in [18]). Since $H(f)$ is a form in $\mathcal{Q}_n^{n(d-2)}$, it follows that $\nu = n(d-2)$ (see [1, 2] for details).

Let $\omega\colon \operatorname{Soc}(M_f) \to \mathbb{C}$ be the linear isomorphism defined by the condition $\omega(H(f)) = 1$ (with $H(f)$ viewed as an element of M_f). Introduce $\mathbf{f} \in \mathcal{Q}_n^{n(d-2)}$ by the formula

$$\mathbf{f}(z) := \omega\left((z_1\mathbf{z}_1 + \cdots + z_n\mathbf{z}_n)^{n(d-2)}\right),$$

where $\mathbf{z}_j$ is the element of the algebra M_f represented by the coordinate function $z_j \in \mathbb{C}[z_1, \ldots, z_n]$ (which is not to be confused with the jth component of the vector $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$). We call $\mathbf{f}$ the *associated form* of f. Observe that $\mathbf{f}$ is a coordinate representation of the following $\operatorname{Soc}(M_f)$-valued function on the quotient $\mathfrak{m}/\mathfrak{m}^2$:

$$x \mapsto y^{n(d-2)},$$

where y is any element of $\mathfrak{m}$ that projects to $x \in \mathfrak{m}/\mathfrak{m}^2$.

To give an expanded expression for $\mathbf{f}$, observe that if $i_1, \ldots, i_n$ are nonnegative integers such that $i_1 + \cdots + i_n = n(d-2)$, the product $\mathbf{z}_1^{i_1} \cdots \mathbf{z}_n^{i_n}$ lies in $\operatorname{Soc}(M_f)$, hence we have

$$\mathbf{z}_1^{i_1} \cdots \mathbf{z}_n^{i_n} = \mu_{i_1,\ldots,i_n}(f)H(f)$$

for some $\mu_{i_1,\ldots,i_n}(f) \in \mathbb{C}$. In terms of the coefficients $\mu_{i_1,\ldots,i_n}(f)$ the form $\mathbf{f}$ is written as

$$\mathbf{f}(z) = \sum_{i_1+\cdots+i_n=n(d-2)} \frac{(n(d-2))!}{i_1! \cdots i_n!}\mu_{i_1,\ldots,i_n}(f)z_1^{i_1} \cdots z_n^{i_n}.$$

It is not hard to observe that each $\mu_{i_1,\ldots,i_n}$ is a regular function on X_n^d, therefore

$$\mu_{i_1,\ldots,i_n} = \frac{P_{i_1,\ldots,i_n}}{\Delta^{p_{i_1,\ldots,i_n}}},$$

for some $P_{i_1,\ldots,i_n} \in \mathbb{C}[\mathcal{Q}_n^d]$ and nonnegative integer $p_{i_1,\ldots,i_n}$.

Consider the morphism

$$\Phi\colon X_n^d \to \mathcal{Q}_n^{n(d-2)}, \quad f \mapsto \mathbf{f}$$

of affine varieties. This map is rather natural; in particular, the following equivariance property holds:

Proposition 2.1 ([1]). *For every $f \in X_n^d$ and $C \in \mathrm{GL}_n$ one has*

$$\Phi(C \cdot f) = (\det C)^2 \left(C^{-T} \cdot \Phi(f)\right).$$

The present note concerns two situations: the case of binary quartics and that of ternary cubics. In the next section we will give an explicit description of the morphism Φ in these situations and state our results.

3. DUALITY FOR BINARY QUARTICS AND TERNARY CUBICS

We will now projectivize the setup of Section 2 and replace the action of GL_n with that of SL_n. Namely, let $\mathbb{P}(\mathcal{Q}_n^d)$ be the projectivization of $\mathcal{Q}_n^d$, *i.e.*, $\mathbb{P}(\mathcal{Q}_n^d) := (\mathcal{Q}_n^d \setminus \{0\})/\mathbb{C}^*$. In what follows we often write elements of $\mathbb{P}(\mathcal{Q}_n^d)$ as forms meaning that they are considered up to scale. The action of SL_n on $\mathcal{Q}_n^d$ induces an SL_n-action on $\mathbb{P}(\mathcal{Q}_n^d)$, and for $f \in \mathbb{P}(\mathcal{Q}_n^d)$ we denote its orbit $\mathrm{SL}_n \cdot f$ by $O(f)$. Further, define $\mathbb{X}_n^d \subset \mathbb{P}(\mathcal{Q}_n^d)$ to be the image of X_n^d under the quotient morphism $\mathcal{Q}_n^d \setminus \{0\} \to \mathbb{P}(\mathcal{Q}_n^d)$. Clearly, for $f \in \mathbb{X}_n^d$ the orbit $O(f)$ is closed in $\mathbb{X}_n^d$ and has dimension $n^2 - 1$.

The map Φ descends to a morphism

$$\mathbb{\Phi}\colon \mathbb{X}_n^d \to \mathbb{P}(\mathcal{Q}_n^{n(d-2)}).$$

By Proposition 2.1, the morphism $\mathbb{\Phi}$ is equivariant in the following sense:

$$\mathbb{\Phi}(C \cdot f) = C^{-T} \cdot \mathbb{\Phi}(f), \quad f \in \mathbb{X}_n^d,\ C \in \mathrm{SL}_n\,.$$

Hence, in the case when $\mathbb{\Phi}$ maps the variety $\mathbb{X}_n^d$ into the semistable locus $\mathbb{P}(\mathcal{Q}_n^{n(d-2)})^{\mathrm{ss}}$ of $\mathbb{P}(\mathcal{Q}_n^{n(d-2)})$, it gives rise to a morphism ϕ of good GIT quotients for which the following diagram commutes:

$$\begin{array}{ccc} \mathbb{X}_n^d & \xrightarrow{\mathbb{\Phi}} & \mathbb{P}(\mathcal{Q}_n^{n(d-2)})^{\mathrm{ss}} \\ \downarrow & & \downarrow \\ \mathbb{X}_n^d /\!/ \mathrm{SL}_n & \xrightarrow{\phi} & \mathbb{P}(\mathcal{Q}_n^{n(d-2)})^{\mathrm{ss}} /\!/ \mathrm{SL}_n\,. \end{array}$$

In the diagram, the quotient on the left is affine and geometric, and the one on the right is projective. Furthermore, $\mathbb{X}_n^d$ is a Zariski open subset of the stable locus $\mathbb{P}(\mathcal{Q}_n^d)^{\mathrm{s}}$, hence the affine quotient $\mathbb{X}_n^d \to \mathbb{X}_n^d /\!/ \mathrm{SL}_n$ is a restriction of the projective quotient $\mathbb{P}(\mathcal{Q}_n^d)^{\mathrm{ss}} \to \mathbb{P}(\mathcal{Q}_n^d)^{\mathrm{ss}} /\!/ \mathrm{SL}_n$. Observe that the situation $n = 2$, $d = 3$ is trivial and can be excluded from consideration. Indeed, since all nondegenerate binary cubics are pairwise linearly equivalent, $\mathbb{X}_2^3 = \mathbb{P}(\mathcal{Q}_2^3)^{\mathrm{ss}} = \mathbb{P}(\mathcal{Q}_2^3)^{\mathrm{s}}$ is a single orbit and $\mathbb{X}_2^3 /\!/ \mathrm{SL}_2$ is a point. For elementary introductions to GIT quotients and various notions of stability we refer the reader to [17] and Chapter 9 in [14].

We focus on the morphism $\mathbb{\Phi}$ in two cases. Indeed, notice that for all pairs n, d (excluding the trivial situation $n = 2$, $d = 3$) one has $n(d-2) \geq d$, and the equality holds precisely for two pairs: $n = 2$, $d = 4$ and $n = 3$, $d = 3$. We will now explain that in each of these two cases $\mathbb{\Phi}$ maps $\mathbb{X}_n^d$ to $\mathbb{P}(\mathcal{Q}_n^d)^{\mathrm{ss}}$ and induces an equivariant involution on the variety $\mathbb{X}_n^d$ with one orbit removed. Some of the facts that follow can be extracted from articles [4, 5].

Let $n=2$, $d=4$. It is a classical result that every nondegenerate binary quartic is linearly equivalent to a quartic of the form

$$q_t(z_1,z_2) := z_1^4 + tz_1^2z_2^2 + z_2^4, \quad t \neq \pm 2 \tag{3.1}$$

(see pp. 277–279 in [7]). A straightforward calculation yields that the associated form of q_t is

$$\mathbf{q}_t(z_1,z_2) := \frac{1}{72(t^2-4)}(tz_1^4 - 12z_1^2z_2^2 + tz_2^4). \tag{3.2}$$

For $t \neq 0, \pm 6$ the quartic $\mathbf{q}_t$ is nondegenerate, and in this case the associated form of $\mathbf{q}_t$ is proportional to q_t, hence $\Phi^2(q_t) = q_t$. As explained below, the exceptional quartics q_0, q_6, q_{-6}, are pairwise linearly equivalent.

It is easy to show that $\mathbb{P}(\mathcal{Q}_2^4)^{\rm ss}$ is the union of $\mathbb{X}_2^4$ (which coincides with $\mathbb{P}(\mathcal{Q}_2^4)^{\rm s}$) and two orbits that consist of strictly semistable forms: $O_1 := O(z_1^2z_2^2)$, $O_2 := O(z_1^2(z_1^2+z_2^2))$, of dimensions 2 and 3, respectively. Notice that O_1 is closed in $\mathbb{P}(\mathcal{Q}_2^4)^{\rm ss}$ and is contained in the closure of O_2. We then observe that Φ maps $\mathbb{X}_2^4$ onto $\mathbb{P}(\mathcal{Q}_2^4)^{\rm ss} \setminus (O_2 \cup O_3)$, where $O_3 := O(q_0)$ (as we will see shortly, O_3 contains the other exceptional quartics q_6, q_{-6} as well). Also, notice that Φ maps the 3-dimensional orbit O_3 onto the 2-dimensional orbit O_1.

Thus, we obtain:

PROPOSITION 3.1. *The morphism Φ restricts to an equivariant involutive automorphism of $\mathbb{X}_2^4 \setminus O_3$, which for $t \neq 0, \pm 6$ establishes a duality between the quartics $C \cdot q_t$ and $C^{-T} \cdot q_{-12/t}$ with $C \in \mathrm{SL}_2$, hence between the orbits $O(q_t)$ and $O(q_{-12/t})$.*

In order to understand the induced map ϕ of GIT quotients, we note that the algebra of invariants $\mathbb{C}[\mathcal{Q}_2^4]^{\mathrm{SL}_2}$ is generated by a pair of elements I_2, I_3 (the latter invariant is called the catalecticant), where the subscripts indicate their degrees (see, *e.g.*, pp. 41, 101–102 in [7]). One has

$$\Delta = I_2^3 - 27\, I_3^2, \tag{3.3}$$

and for a binary quartic of the form

$$f(z_1,z_2) = az_1^4 + 6bz_1^2z_2^2 + cz_2^4$$

the values of I_2 and I_3 are computed as

$$I_2(f) = ac + 3b^2, \quad I_3(f) = abc - b^3. \tag{3.4}$$

It then follows that the algebra $\mathbb{C}[X_2^4]^{\mathrm{GL}_2} \simeq \mathbb{C}[\mathbb{X}_2^4]^{\mathrm{SL}_2}$ is generated by the invariant

$$J := \frac{I_2^3}{\Delta}. \tag{3.5}$$

Therefore, the quotient $X_2^4/\!/\,\mathrm{GL}_2 \simeq \mathbb{X}_2^4/\!/\,\mathrm{SL}_2$ is the affine space $\mathbb{C}$, and $\mathbb{P}(\mathcal{Q}_2^4)^{\mathrm{ss}}/\!/\,\mathrm{SL}_2$ can be identified with $\mathbb{P}^1$, where both O_1 and O_2 project to the point at infinity in $\mathbb{P}^1$.

Next, from formulas (3.1), (3.3), (3.4), (3.5) we calculate

$$J(q_t) = \frac{(t^2+12)^3}{108(t^2-4)^2} \quad \text{for all } t \neq \pm 2. \tag{3.6}$$

Clearly, (3.6) yields

$$J(q_0) = J(q_6) = J(q_{-6}) = 1, \tag{3.7}$$

which implies that q_0, q_6, q_{-6} are indeed pairwise linearly equivalent as claimed above and that the orbit O_3 is described by the condition $J = 1$.

Using (3.2), (3.6) one obtains

$$J(\mathbf{q}_t) = \frac{J(q_t)}{J(q_t)-1} \quad \text{for all } t \neq 0, \pm 6.$$

This shows that the map ϕ extends to the automorphism $\widetilde{\phi}$ of $\mathbb{P}^1$ given by

$$\zeta \mapsto \frac{\zeta}{\zeta - 1}.$$

Clearly, one has $\widetilde{\phi}^2 = \mathrm{id}$, that is, $\widetilde{\phi}$ is an involution. It preserves $\mathbb{P}^1 \setminus \{1, \infty\}$, which corresponds to the duality between the orbits $O(q_t)$ and $O(q_{-12/t})$ for $t \neq 0, \pm 6$ noted above. Further, $\widetilde{\phi}(1) = \infty$, which agrees with (3.7) and the fact that O_3 is mapped onto O_1. We also have $\widetilde{\phi}(\infty) = 1$, but this identity has no interpretation at the level of orbits. Indeed, Φ cannot be equivariantly extended to an involution $\mathbb{P}(\mathcal{Q}_2^4)^{\mathrm{ss}} \to \mathbb{P}(\mathcal{Q}_2^4)^{\mathrm{ss}}$ as the fiber of the quotient $\mathbb{P}(\mathcal{Q}_2^4)^{\mathrm{ss}}/\!/\,\mathrm{SL}_2$ over the point at infinity contains O_1, which cannot be mapped onto O_3 since $\dim O_1 < \dim O_3$.

Let $n = 3$, $d = 3$. Every nondegenerate ternary cubic is linearly equivalent to a cubic of the form

$$c_t(z_1, z_2, z_3) := z_1^3 + z_2^3 + z_3^3 + t z_1 z_2 z_3, \quad t^3 \neq -27 \tag{3.8}$$

(see, *e.g.*, Theorem 1.3.2.16 in [19]). The associated form of c_t is easily found to be

$$\mathbf{c}_t(z_1, z_2, z_3) := -\frac{1}{24(t^3+27)}(t z_1^3 + t z_2^3 + t z_3^3 - 18 z_1 z_2 z_3). \tag{3.9}$$

For $t \neq 0$, $t^3 \neq 216$ the cubic $\mathbf{c}_t$ is nondegenerate, and in this case the associated form of $\mathbf{c}_t$ is proportional to c_t, hence $\Phi^2(c_t) = c_t$. Below we will see that the exceptional cubics c_0, $c_{6\tau}$, with $\tau^3 = 1$, are pairwise linearly equivalent.

It is well-known (see, *e.g.*, Theorem 1.3.2.16 in [19]) that $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}}$ is the union of $\mathbb{X}_3^3$ (which coincides with $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{s}}$) and the following three orbits that consist of strictly semistable forms: $\mathrm{O}_1 := O(z_1z_2z_3)$, $\mathrm{O}_2 := O(z_1z_2z_3 + z_3^3)$, $\mathrm{O}_3 := O(z_1^3 + z_1^2z_3 + z_2^2z_3)$ (the cubics lying in O_3 are called nodal). The dimensions of the orbits are 6, 7 and 8, respectively. Observe that O_1 is closed in $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}}$ and is contained in the closures of each of O_2, O_3. We then see that Φ maps $\mathbb{X}_3^3$ onto $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}} \setminus (\mathrm{O}_2 \cup \mathrm{O}_3 \cup \mathrm{O}_4)$, where $\mathrm{O}_4 := O(c_0)$ (as explained below, O_4 also contains the other exceptional cubics $c_{6\tau}$, with $\tau^3 = 1$). Further, note that the 8-dimensional orbit O_4 is mapped by Φ onto the 6-dimensional orbit O_1.

Hence, we obtain:

Proposition 3.2. *The morphism Φ restricts to an equivariant involutive automorphism of $\mathbb{X}_3^3 \setminus \mathrm{O}_4$, which for $t \neq 0$, $t^3 \neq 216$ establishes a duality between the cubics $C \cdot c_t$ and $C^{-T} \cdot c_{-18/t}$ with $C \in \mathrm{SL}_3$, therefore between the orbits $O(c_t)$ and $O(c_{-18/t})$.*

To determine the induced map ϕ of GIT quotients, we recall that the algebra of invariants $\mathbb{C}[\mathcal{Q}_3^3]^{\mathrm{SL}_3}$ is generated by the two Aronhold invariants I_4, I_6, where, as before, the subscripts indicate the degrees (see pp. 381–389 in [7]). One has

$$\Delta = \mathrm{I}_6^2 + 64\,\mathrm{I}_4^3, \tag{3.10}$$

and for a ternary cubic of the form

$$f(z_1, z_2, z_3) = az_1^3 + bz_2^3 + cz_3^3 + 6dz_1z_2z_3$$

the values of I_4 and I_6 are calculated as

$$\mathrm{I}_4(f) = abcd - d^4, \quad \mathrm{I}_6(f) = a^2b^2c^2 - 20abcd^3 - 8d^6. \tag{3.11}$$

It then follows that the algebra $\mathbb{C}[X_3^3]^{\mathrm{GL}_3} \simeq \mathbb{C}[\mathbb{X}_3^3]^{\mathrm{SL}_3}$ is generated by the invariant

$$\mathrm{J} := \frac{64\,\mathrm{I}_4^3}{\Delta}. \tag{3.12}$$

Hence, the quotient $X_3^3 /\!/ \mathrm{GL}_3 \simeq \mathbb{X}_3^3 /\!/ \mathrm{SL}_3$ is the affine space $\mathbb{C}$, and $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}} /\!/ \mathrm{SL}_3$ is identified with $\mathbb{P}^1$, where O_1, O_2, O_3 project to the point at infinity in $\mathbb{P}^1$.

Further, from formulas (3.8), (3.10), (3.11), (3.12) we find

$$\mathrm{J}(c_t) = -\frac{t^3(t^3 - 216)^3}{2^6 3^3 (t^3 + 27)^3} \quad \text{for all } t \text{ with } t^3 \neq -27. \tag{3.13}$$

From identity (3.13) one obtains

$$\mathrm{J}(c_0) = \mathrm{J}(c_{6\tau}) = 0 \quad \text{for } \tau^3 = 1, \tag{3.14}$$

which implies that the orbit O_4 is given by the condition $\mathrm{J} = 0$ and that the four cubics c_0, $c_{6\tau}$ are indeed pairwise linearly equivalent.

Using (3.9), (3.13) we see

$$\mathrm{J}(\mathbf{c}_t) = \frac{1}{\mathrm{J}(c_t)} \quad \text{for all } t \neq 0 \text{ with } t^3 \neq 216.$$

This shows that the map ϕ extends to the involutive automorphism $\widetilde{\phi}$ of $\mathbb{P}^1$ given by

$$\zeta \mapsto \frac{1}{\zeta}.$$

The involution $\widetilde{\phi}$ preserves $\mathbb{P}^1 \setminus \{0, \infty\}$, which agrees with the duality between the orbits $O(c_t)$ and $O(c_{-18/t})$ for $t \neq 0$, $t^3 \neq 216$ established above. Next, $\widetilde{\phi}(0) = \infty$, which corresponds to (3.14) and the fact that O_4 is mapped onto O_1. Also, one has $\widetilde{\phi}(\infty) = 0$, but this identity cannot be illustrated by a correspondence between orbits. Indeed, Φ cannot be equivariantly extended to an involution $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}} \to \mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}}$ as the fiber of the quotient $\mathbb{P}(\mathcal{Q}_3^3)^{\mathrm{ss}} /\!/ \mathrm{SL}_2$ over the point at infinity contains O_1, which cannot be mapped onto O_4 since $\dim \mathrm{O}_1 < \dim \mathrm{O}_4$.

Remark 3.3. We note that a cubic proportional to (3.9) previously appeared in [8] (see p. 405 therein) as a Macaulay inverse system for the Milnor algebra M_{c_t}, but it has never been studied systematically. In fact, we now know (see Corollary 3.3 in [1]) that the associated form of any $f \in X_n^d$ is an inverse system for M_f. This result has been instrumental in our recent work on the morphism Φ including the progress on Conjecture 1.1. For details on inverse systems we refer the reader to [8, 13, 15] (the brief survey given in [6] is also helpful).

If we regard $\mathbb{X}_3^3$ as the space of elliptic curves, the invariant J of ternary cubics translates into the j-invariant, and one obtains an equivariant involution on the locus of elliptic curves with nonvanishing j-invariant. It is well-known that every elliptic curve can be realized as a double cover of $\mathbb{P}^1$ branched over four points (see, *e.g.*, Exercise 22.37 and Proposition 22.38 in [11]). Therefore, it is not surprising that the cases of binary quartics and ternary cubics considered above have many similarities. The duality obtained in these situations in Propositions 3.1 and 3.2 will be studied in detail in our forthcoming article joint with J. Alper and N. Kruzhilin.

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REFERENCES

[1] J. Alper and A. Isaev, *Associated forms in classical invariant theory and their applications to hypersurface singularities*. Math. Ann. **360** (2014), 799–823.

[2] J. Alper and A. Isaev, *Associated forms and hypersurface singularities: the binary case*. Preprint, available from http://arxiv.org/abs/1407.6838.

[3] H. Bass, *On the ubiquity of Gorenstein rings*. Math. Z. **82** (1963), 8–28.

[4] M.G. Eastwood, *Moduli of isolated hypersurface singularities*. Asian J. Math. **8** (2004), 305–313.

[5] M.G. Eastwood and A.V. Isaev, *Extracting invariants of isolated hypersurface singularities from their moduli algebras*. Math. Ann. **356** (2013), 73–98.

[6] J. Elias and M.E. Rossi, *Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system*. Trans. Amer. Math. Soc. **364** (2012), 4589–4604.

[7] E.B. Elliott, *An Introduction to the Algebra of Quantics*. Clarendon Press, Oxford, 1895.

[8] J. Emsalem, *Géométrie des points épais*. Bull. Soc. Math. France **106** (1978), 399–416.

[9] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*. Modern Birkhäuser Classics. Birkhüser Boston, Inc., Boston, MA, 2008.

[10] G.-M. Greuel, C. Lossen and E. Shustin, *Introduction to Singularities and Deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[11] J. Harris, *Algebraic Geometry. A First Course*. Graduate Texts in Mathematics **133** (1992), Springer-Varlag, New York.

[12] C. Huneke, *Hyman Bass and ubiquity: Gorenstein rings*. In: *Algebra, K-theory, Groups, and Education.* New York, 1997, pp. 55–78. Contemp. Math. **243** (1999), Amer. Math. Soc., Providence, RI.

[13] A. Iarrobino, *Associated graded algebra of a Gorenstein Artin algebra*. Mem. Amer. Math. Soc. **514** (1994).

[14] V. Lakshmibai and K.N. Raghavan, *Standard Monomial Theory. Invariant Theoretic Approach.* Springer, Berlin, 2008.

[15] F.S. Macaulay, *The Algebraic Theory of Modular Systems. Reprint of the 1916 original*. Cambridge University Press, Cambridge, 1994.

[16] J. Mather and S.S.-T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras.* Invent. Math. **69** (1982), 243–251.

[17] S. Mukai, *An Introduction to Invariants and Moduli.* Cambridge Studies in Advanced Mathematics **81** (2003), Cambridge University Press, Cambridge.

[18] K. Saito, *Einfach-elliptische Singularitäten.* Invent. Math. **23** (1974), 289–325.

[19] A.H.W. Schmitt, *Geometric Invariant Theory and Decorated Principal Bundles*. Zurich Lectures in Advanced Mathematics. European Mathematical Society, Zürich, 2008.

[20] A.N. Shoshitaĭshvili, *Functions with isomorphic Jacobian ideals*. Funct. Anal. Appl. **10** (1976), 128–133.

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