To the memory of Professor Kenjiro Okubo

FLAT STRUCTURES WITHOUT POTENTIALS

MITSUO KATO, TOSHIYUKI MANO and JIRO SEKIGUCHI

Communicated by Laurentiu Paunescu

Flat structures are formulated by K. Saito in the course of the study of moduli spaces of isolated singularities. The purpose of this paper is to study flat structures without potentials, to formulate one of generalisations of ordinary differential equations of Okubo type to several variables case and to give examples of potential vector fields related with algebraic solutions of Painlevé VI, free divisors arising from 1-parameter deformations of singularities on plane curves and discriminants of complex reflection groups.

AMS 2010 Subject Classification: 34M56, 35N10, 32S25.

Key words: flat structure, WDVV equation, complex reflection group, Painlevé equation.

1. INTRODUCTION

The purpose of this paper is to report the recent progress on the study concerning free divisors, algebraic solutions of Painlevé VI and flat structures. Flat structure is established by K. Saito [17] when he investigated the structure of the moduli space of deformations of isolated singular points. Later B. Dubrovin [6] generalised it so that the theory contains the 2D topological field theory. In the formulation of B. Dubrovin the existence of potentials plays a central role in the construction of Frobenius algebra structure on the tangent space of the moduli space. C. Sabbah's book [15] is a nice introduction of flat structure = Frobenius structure. In spite that potentials are important in the theory developed by Dubrovin, Sabbah's book also treats the case of Saito structure without the existence of the potential called "*Saito structure without a metric*. Our original interest is the investigation of the relationship between algebraic solutions of Painlevé VI and certain kinds of holonomic systems of linear differential equations in three variables whose singularities are contained in free divisors. Recently we recognised the relationship between flat structure without a metric and our studies. Comparing our formulation with that of

Sabbah, we are led to develop the theory of *flat structure without a potential* which is same as the theory of Saito structure without a metric. Potential vector field plays an important role in our theory. We note that the notion of *potential vector field* is already introduced by Konishi and Minabe [12]. Potential vector field is the object corresponding to the gradient vector field of the potential in the theory of Saito structure with a metric. In the first part of this paper, we formulate flat structures without potentials and mention their basic results which are contained in [15] or its easy consequences. Then we introduce systems of differential equations of n variables of rank n which is one of generalisations of ordinary differential equations of Okubo type. The second half of this paper is to show examples of potential vector fields and free divisors derived from them. These examples show that there are non-trivial examples of systems of differential equations introduced in §4. Potential vector fields which we first treat are related to algebraic solutions of Painlevé VI. Then we introduce potentials which are related to real reflection groups of types A_3, B_3, H_3. They are already treated in [6]. Our next examples of potential vector fields are related with three of fourteen exceptional singularities called E_{12}, E_{13}, E_{14} in the sense of Arnol'd. The defining polynomials of free divisors naturally arising from two of these examples are nothing but discriminants of complex reflection groups of No. 24 and No. 27 in Shephard-Todd notation (cf. [23]). The last one is seeming new. It is known (cf. [6]) that discriminants of real reflection groups are obtained from corresponding potentials. This leads us to consider the case of discriminants of complex reflection groups. H. Terao [24] proved the freeness of the zero locus of discriminants of irreducible finite complex reflection groups. In §9, we discuss the existence of potential vector fields for the case of irreducible finite complex reflection groups.

In this paper, we only treat the case where a potential vector field consists of polynomial entries. It is possible to construct potential vector fields which consist of algebraic functions at least when we treat algebraic solutions of Painlevé VI. The detail of the subject treated in this paper will be published elsewhere [9].

2. DEFINITION OF A POTENTIAL VECTOR FIELD

In this section, we define a potential vector field. This notion is found in Konishi and Minabe [12].

Let $x = (x_1, x_2, \ldots, x_n)$ be a standard coordinate system of $\mathbf{C}^n$. We define an Euler vector field

$$E = \sum_{k=1}^{n} w_k x_k \partial_k,$$

where $\partial_k = \partial/\partial x_k$ $(k = 1, 2, \ldots, n)$. We assume a condition on w_k $(k = 1, 2, \ldots, n)$:

$$0 < w_1 < w_2 < \cdots < w_n.$$

We introduce weighted homogeneous polynomials $h_1(x), h_2(x), \ldots, h_n(x)$ such that

$$Eh_j = (w_j + w_n)h_j \quad (j = 1, 2, \ldots, n)$$

and that

$$h_j = \begin{cases} x_j x_n + h_j^{(0)}(x_1, \ldots, x_{n-1}) & (j = 1, 2, \ldots, n-1), \\ \frac{1}{2}x_n^2 + h_n^{(0)}(x_1, \ldots, x_{n-1}) & (j = n) \end{cases}$$

with polynomials $h_j^{(0)}(x_1, \ldots, x_{n-1})$ of $x' = (x_1, \ldots, x_{n-1})$. Using $h_j(x)$ $(j = 1, 2, \ldots, n)$, we define $\gamma_{ij} = \partial_i h_j$ and an $n \times n$ matrix $C = (\gamma_{ij})$. It is easy to see that $\gamma_{nj} = x_j$ $(j = 1, 2, \ldots, n)$. We define matrices

$$\tilde{B}^{(p)} = \partial_p C \quad (p = 1, 2, \ldots, n).$$

We denote by $b_{ij}^{(p)}$ the (i, j)-entry of $\tilde{B}^{(p)}$ and collect basic properties of $\tilde{B}^{(p)}$ $(p = 1, 2, \ldots, n)$:

1. $\partial_p \tilde{B}^{(q)} = \partial_q \tilde{B}^{(p)} \quad (\forall p,\ q)$,
2. $b_{pq}^{(r)} = b_{rq}^{(p)} \quad (\forall p,\ q,\ r)$,
3. $b_{nq}^{(p)} = \delta_{pq} \quad (\forall p,\ q)$,
4. $\tilde{B}^{(n)} = I_n$,
5. $\partial_n \tilde{B}^{(p)} = O \quad (p = 1, 2, \ldots, n-1)$,

where δ_{pq} is Kronecker's delta and I_n is the identity matrix.

Definition 1. If $\tilde{B}^{(p)}\tilde{B}^{(q)} = \tilde{B}^{(q)}\tilde{B}^{(p)}$ $(\forall p,\ q = 1, 2, \ldots, n)$, then $\vec{h} = (h_1, h_2, \ldots, h_n)$ is called a potential vector field.

Remark 1. Let J be an $n \times n$ matrix whose (i, j)-entry is $\delta_{i,n-j+1}$ for all i, j. If CJ is symmetric, there is a weighted homogeneous polynomial $P(x)$ such that

$$\partial_i P = h_{n-i+1} \quad (i = 1, 2, \ldots, n).$$

Then P is called a prepotential in [6] and a potential in [15]. In this case the commutativity of the matrices $\tilde{B}^{(p)}$ $(p = 1, 2, \cdots, n)$, namely, $\tilde{B}^{(p)}\tilde{B}^{(q)} = \tilde{B}^{(q)}\tilde{B}^{(p)}$ $(\forall p,\ q)$ implies a system of non-linear differential equations for the potential P. This system is called WDVV-equation (cf. [6]). When CJ is not symmetric, there is no potential. In spite of this fact, we find that the potential vector field is a solution of a certain system of non-linear differential equations arising from the commutativity of matrices $\tilde{B}^{(p)}$ $(p = 1, 2, \cdots, n)$. In this sense, $\tilde{B}^{(p)}\tilde{B}^{(q)} = \tilde{B}^{(q)}\tilde{B}^{(p)}$ $(\forall p,\ q)$ is called a generalised WDVV equation.

In this paper, a coordinate system $(x_1, x_2, \ldots, x_n)$ with a potential vector field $\vec{h} = (h_1, h_2, \ldots, h_n)$ is called a flat coordinate system. From $\vec{h}$, it is possible to define a Saito structure without a metric in the sense of Sabbah [15]. If the matrix C defined by $\vec{h}$ satisfies the condition that CJ is symmetric (cf. the notation in Remark 1), there is a potential $P(x_1, x_2, \ldots, x_n)$ such that $\partial_i P = h_{n-i+1}$ $(i = 1, 2, \ldots, n)$. In this case, P defines a Saito structure with a metric (cf. [15]). For this reason, we say that the latter is a flat structure with a potential and the former is a flat structure without a potential.

3. FREE DIVISORS CONSTRUCTED BY POTENTIAL VECTOR FIELDS

Let $\vec{h} = (h_1, h_2, \ldots, h_n)$ be a potential vector field. We define an $n \times n$ matrix $T = (T_{ij})$ by

$$T = \sum_{j=1}^{n} w_j x_j \partial_j C = \sum_{j=1}^{n} w_j x_j \tilde{B}^{(j)}$$

and $F(x) = \det T$. From the assumption, we find that

$$F(x) = c_0 x_n^n + p_1(x') x_n^{n-1} + \cdots + p_{n-1}(x') x_n + p_n(x')$$

for a constant $c_0 \neq 0$. We denote by V_i the vector field defined by

$$V_i = \sum_{j=1}^{n} T_{ij} \partial_j \quad (i = 1, 2, \ldots, n).$$

The purpose of this section is to show that if $F(x)$ is reduced, then the set $\mathcal{S}_F = \{x \in \mathbf{C}^n;\ F(x) = 0\}$ is a free divisor. For the definition of free divisors, we refer to [16]. This statement is already shown in [15]. Our proof employed here is to construct a set of generators of logarithmic vector fields along $\mathcal{S}_F$.

Proposition 1.

$$V_i F = w_n (\mathrm{tr} \tilde{B}^{(i)}) F \quad (i = 1, 2, \ldots, n).$$

Proof. We introduce the following notation. If A is an $n \times n$ matrix, we denote by A_{ij} the (i, j)-entry of A and by $A[i] = (A_{i1}, A_{i2}, \cdots, A_{in})$ the i-th column vector of A. Then

$$A = \begin{pmatrix} A[1] \\ A[2] \\ \cdot \\ \cdot \\ \cdot \\ A[n] \end{pmatrix}.$$

It follows from the definition that

$$T_{ij} = E\gamma_{ij} = E(\partial_i h_j) = (w_j + w_n - w_i)\partial_i h_j = (w_j + w_n - w_i)\gamma_{ij}.$$

Since $T_{nj} = w_j \partial_n h_j = w_j x_j$, it follows that $V_n = E$. We use the notation $u_{ij} = w_n + w_j - w_i$ for a moment. Then $T_{ij} = u_{ij}\gamma_{ij}$.

We are going to compute $V_i F$. Since $F = \begin{vmatrix} T[1] \\ T[2] \\ \vdots \\ T[n] \end{vmatrix}$, we have

$$V_i F = \begin{vmatrix} V_i T[1] \\ T[2] \\ \vdots \\ T[n] \end{vmatrix} + \begin{vmatrix} T[1] \\ V_i T[2] \\ \vdots \\ T[n] \end{vmatrix} + \cdots + \begin{vmatrix} T[1] \\ T[2] \\ \vdots \\ V_i T[n] \end{vmatrix}.$$

Since T is invertible on $\mathbf{C}^n - \mathcal{S}_F$, the vectors $T[1]$, $T[2]$, $\cdots, T[n]$ are linearly independent on $\mathbf{C}^n - \mathcal{S}_F$. As a consequence, $V_i T[j]$ is a linear combination of $T[1]$, $T[2]$, $\cdots, T[n]$, namely, there are rational functions S_{jk} such that

$$V_i T[j] = \sum_{k=1}^{n} S_{jk} T[k].$$

Let S be an $n \times n$ matrix whose (j,k)-entry is S_{jk}. Since

$$\begin{vmatrix} T[1] \\ \vdots \\ T[j-1] \\ V_i T[j] \\ T[j+1] \\ \vdots \\ T[n] \end{vmatrix} = \begin{vmatrix} T[1] \\ \vdots \\ T[j-1] \\ \sum_{k=1}^{n} S_{jk} T[k] \\ T[j+1] \\ \vdots \\ T[n] \end{vmatrix} = \begin{vmatrix} T[1] \\ \vdots \\ T[j-1] \\ S_{jj} T[j] \\ T[j+1] \\ \vdots \\ T[n] \end{vmatrix} = S_{jj} \begin{vmatrix} T[1] \\ \vdots \\ T[j-1] \\ T[j] \\ T[j+1] \\ \vdots \\ T[n] \end{vmatrix} = S_{jj} F,$$

it follows that

$$V_i F = \sum_{j=1}^{n} S_{jj} F = (\mathrm{tr} S) F.$$

On the other hand, it follows that

$$V_iT = \begin{pmatrix} V_iT[1] \\ \vdots \\ V_iT[j] \\ \vdots \\ V_iT[n] \end{pmatrix} = ST.$$

As a consequence, we have $S = (V_iT)T^{-1}$. Then we obtain a formula $\mathrm{tr}S = \mathrm{tr}((V_iT)T^{-1})$.

By the equations

$$\begin{aligned}(V_iT)_{pq} = V_iT_{pq} = u_{pq}V_i\gamma_{pq} = u_{pq}\textstyle\sum_k T_{ik}\partial_k\gamma_{pq} \\ = u_{pq}\textstyle\sum_k T_{ik}\partial_p\gamma_{kq} = u_{pq}\sum_k T_{ik}b^{(p)}_{kq} = u_{pq}(T\tilde{B}^{(p)})_{iq},\end{aligned}$$

$$T\tilde{B}^{(p)} = \tilde{B}^{(p)}T,$$

we have

$$(V_iT)_{pq} = u_{pq}(\tilde{B}^{(p)}T)_{iq}.$$

As a consequence,

$$\begin{aligned}\mathrm{tr}((V_iT)T^{-1}) &= \sum_{p,q}(V_iT)_{pq}(T^{-1})_{qp} = \sum_{p,q}u_{pq}(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp} \\ &= \sum_{p,q}(w_n - w_p + w_q)(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp} \\ &= \sum_{p,q}(w_n - w_p)(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp} + \sum_{p,q}w_q(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp}.\end{aligned}$$

We compute

$$U_1 = \sum_{p,q}(w_n - w_p)(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp}$$

and

$$U_2 = \sum_{p,q}w_q(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp},$$

separately. For this purpose, we introduce the diagonal matrix

$$D = \mathrm{diag}(w_1, w_2, \ldots, w_n).$$

On the one hand,

$$\begin{aligned} U_1 &= \sum_{p,q}(w_n - w_p)(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp} \\ &= \sum_{p}(w_n - w_p)\sum_{q}(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp} = \sum_{p}(w_n - w_p)((\tilde{B}^{(p)}T)T^{-1})_{ip} \\ &= \sum_{p}(w_n - w_p)b^{(p)}_{ip} \\ &= \sum_{p}(w_n - w_p)b^{(i)}_{pp} \\ &= \mathrm{tr}((w_n I_n - D)\tilde{B}^{(i)}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &U_2 \\ =& \sum_{p,q} w_q(\tilde{B}^{(p)}T)_{iq}(T^{-1})_{qp} = \sum_{p,q}(\tilde{B}^{(p)}TD)_{iq}(T^{-1})_{qp} \\ =& \sum_{p}\{\sum_{q}(\tilde{B}^{(p)}TD)_{iq}(T^{-1})_{qp}\} = \sum_{p}((\tilde{B}^{(p)}TD)T^{-1})_{ip} \\ =& \sum_{p}(\tilde{B}^{(p)}(TDT^{-1}))_{ip} = \sum_{p}\{\sum_{q} b^{(p)}_{iq}(TDT^{-1})_{qp}\} = \sum_{p,q} b^{(i)}_{pq}(TDT^{-1})_{qp} \\ =& \mathrm{tr}(\tilde{B}^{(i)}(TDT^{-1})) = \mathrm{tr}((\tilde{B}^{(i)}T)(DT^{-1})) = \mathrm{tr}((T\tilde{B}^{(i)})(DT^{-1})) \\ =& \mathrm{tr}(T(\tilde{B}^{(i)}D)T^{-1}) = \mathrm{tr}(\tilde{B}^{(i)}D). \end{aligned}$$

As a consequence,

$$\begin{aligned} & \mathrm{tr}((V_iT)T^{-1}) = U_1 + U_2 = \mathrm{tr}((w_n I_n - D)\tilde{B}^{(i)}) + \mathrm{tr}(\tilde{B}^{(i)}D) \\ =\ & \mathrm{tr}(w_n\tilde{B}^{(i)} - [D, \tilde{B}^{(i)}]) = w_n \mathrm{tr}\tilde{B}^{(i)}. \end{aligned}$$

Then we find that

$$\mathrm{tr}S = \mathrm{tr}((V_iT)T^{-1}) = w_n \mathrm{tr}\tilde{B}^{(i)}$$

and the required formula is shown. □

COROLLARY 1. *If $F(x)$ is reduced, $\mathcal{S}_F$ is free.*

Proof. Since $\mathrm{tr}\tilde{B}^{(i)}$is a polynomial of x, the equation

$$V_iF = w_n(\mathrm{tr}\tilde{B}^{(i)})F$$

means that V_i is logarithmic along $\mathcal{S}_F$. Therefore by the criterion shown by K. Saito [16], $\mathcal{S}_F$ is free. □

4. A GENERALISATION OF ORDINARY DIFFERENTIAL EQUATIONS OF OKUBO TYPE

In this section we freely use the notation in previous sections without any comment. The matrix T is defined by C and E (cf. §3). We always assume in this section that $(h_1, h_2, \ldots, h_n)$ is a potential vector field. Then by definition, $\tilde{B}^{(p)}\tilde{B}^{(q)} = \tilde{B}^{(q)}\tilde{B}^{(p)}$ for all p, q.

We define a diagonal matrix $B_\infty^{(n)}$ by

$$B_\infty^{(n)} = \mathrm{diag}(r + w_1, r + w_2, \ldots, r + w_n).$$

for some constant $r \in \mathbf{C}$ and show a formula for derivations of T.

Lemma 1. $$\partial_p T = w_n \tilde{B}^{(p)} - [B_\infty^{(n)},\ \tilde{B}^{(p)}] \quad (p = 1, 2, \ldots, n).$$

Proof. The proof of the lemma is accomplished by comparing the (i, j)-entry of both sides for all i, j. Since $EC = T$ and the (i, j)-entry of C is γ_{ij}, it follows that the (i, j)-entry of $\partial_p T$ is

$$\partial_p(E\gamma_{ij}) = (w_n - w_i + w_j)\partial_p \gamma_{ij} = (w_n - w_i + w_j) b_{ij}^{(p)}.$$

On the other hand, the (i, j)-entry of the matrix $w_n \tilde{B}^{(p)} - [B_\infty^{(n)},\ \tilde{B}^{(p)}]$ is

$$w_n b_{ij}^{(p)} - (w_i - w_j) b_{ij}^{(p)} = (w_n - w_i + w_j) b_{ij}^{(p)}.$$

Then the (i, j)-entry of the matrix $\partial_p T$ coincides with that of $w_n \tilde{B}^{(p)} - [B_\infty^{(n)}, \tilde{B}^{(p)}]$ and the lemma follows. □

We define $n \times n$ matrices $B^{(p)}$ $(p = 1, 2, \ldots, n)$ by

(2) $$B^{(p)} = -T^{-1}\tilde{B}^{(p)} B_\infty^{(n)}$$

and a system of differential equations

(3) $$\partial_p Y = B^{(p)} Y \quad (p = 1, 2, \ldots, n).$$

Using the 1-form Ω defined by

$$\Omega = \sum_{p=1}^{n} B^{(p)} dx_p,$$

the system (3) is rewritten by

(4) $$\mathrm{d}Y = \Omega Y.$$

Theorem 1. *The system (4) is integrable.*

Proof. The integrability condition for (4) is

(5) $$\partial_p B^{(q)} - \partial_q B^{(p)} - [B^{(p)},\ B^{(q)}] = O \quad (p, q = 1, 2, \ldots, n).$$

We will prove (5) assuming that $B^{(p)}$ is defined by (2). We need some preparation to show the theorem. By direct computation we have

$$\begin{aligned}\partial_q B^{(p)} &= -\partial_q(T^{-1})\tilde{B}^{(p)}B_\infty^{(n)} - T^{-1}(\partial_q\tilde{B}^{(p)})B_\infty^{(n)}\\ &= T^{-1}(\partial_q T)T^{-1}\tilde{B}^{(p)}B_\infty^{(n)} - T^{-1}(\partial_q\tilde{B}^{(p)})B_\infty^{(n)}.\end{aligned}$$

Noting that $T\tilde{B}^{(p)} = \tilde{B}^{(p)}T\,(p = 1, 2, \ldots, n)$ and that $\partial_q\tilde{B}^{(p)} = \partial_p\tilde{B}^{(q)}$, we have

$$\begin{aligned}&\partial_p B^{(q)} - \partial_q B^{(p)}\\ =\ &(T^{-1}(\partial_p T)T^{-1}\tilde{B}^{(q)}B_\infty^{(n)} - T^{-1}(\partial_p\tilde{B}^{(q)})B_\infty^{(n)})\\ &-(T^{-1}(\partial_q T)T^{-1}\tilde{B}^{(p)}B_\infty^{(n)} - T^{-1}(\partial_q\tilde{B}^{(p)})B_\infty^{(n)})\\ =\ &-T^{-1}(\partial_p\tilde{B}^{(q)} - \partial_q\tilde{B}^{(p)})B_\infty^{(n)} + T^{-1}((\partial_p T)\tilde{B}^{(q)} - (\partial_q T)\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}\\ =\ &T^{-1}((\partial_p T)\tilde{B}^{(q)} - (\partial_q T)\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}.\end{aligned}$$

Furthermore we have

$$[B^{(p)},\ B^{(q)}] = T^{-1}(\tilde{B}^{(p)}B_\infty^{(n)}\tilde{B}^{(q)} - \tilde{B}^{(q)}B_\infty^{(n)}\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}.$$

We first treat the case $q = n$ of (5). In this case, noting that $\tilde{B}^{(n)} = I_n,\ \partial_n T = w_n I_n$, we have

$$\begin{aligned}&\partial_p B^{(n)} - \partial_n B^{(p)} - [B^{(p)},\ B^{(n)}]\\ =\ &T^{-1}((\partial_p T)\tilde{B}^{(n)} - (\partial_n T)\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}\\ &-T^{-1}(\tilde{B}^{(p)}B_\infty^{(n)}\tilde{B}^{(n)} - \tilde{B}^{(n)}B_\infty^{(n)}\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}\\ =\ &T^{-1}\{(\partial_p T) - w_n\tilde{B}^{(p)}) - [\tilde{B}^{(p)},\ B_\infty^{(n)}]\}T^{-1}B_\infty^{(n)}.\end{aligned}$$

Then it follows from Lemma 1 that

$$\partial_p B^{(n)} - \partial_n B^{(p)} - [B^{(p)},\ B^{(n)}] = O.$$

We next treat the case $p < n,\ q < n$ of (5). In this case

$$\begin{aligned}&\partial_p B^{(q)} - \partial_q B^{(p)} - [B^{(p)},\ B^{(q)}]\\ =\ &T^{-1}((\partial_p T)\tilde{B}^{(q)} - (\partial_q T)\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}\\ &-T^{-1}(\tilde{B}^{(p)}B_\infty^{(n)}\tilde{B}^{(q)} - \tilde{B}^{(q)}B_\infty^{(n)}\tilde{B}^{(p)})T^{-1}B_\infty^{(n)}\\ =\ &T^{-1}\{(\partial_p T - \tilde{B}^{(p)}B_\infty^{(n)})\tilde{B}^{(q)} - (\partial_q T - \tilde{B}^{(q)}B_\infty^{(n)})\tilde{B}^{(p)}\}T^{-1}B_\infty^{(n)}.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}&(\partial_p T - \tilde{B}^{(p)}B_\infty^{(n)})\tilde{B}^{(q)} - (\partial_q T - \tilde{B}^{(q)}B_\infty^{(n)})\tilde{B}^{(p)}\\ =\ &(w_n\tilde{B}^{(p)} - B_\infty^{(n)}\tilde{B}^{(p)})\tilde{B}^{(q)} - (w_n\tilde{B}^{(q)} - B_\infty^{(n)}\tilde{B}^{(q)})\tilde{B}^{(p)}\\ =\ &(w_n I_n - B_\infty^{(n)})[\tilde{B}^{(p)},\ \tilde{B}^{(q)}]\\ =\ &O.\end{aligned}$$

As a consequence,

$$\partial_p B^{(q)} - \partial_q B^{(p)} - [B^{(p)},\ B^{(q)}] = O.$$

We have thus proved Theorem 1. □

Remark 2. The 1-form $\tilde{\Omega}$ defined by

$$\tilde{\Omega} = \sum_{p=1}^{n} \tilde{B}^{(p)} dx_p$$

is nothing but the Higgs field introduced in [15].

Remark 3. We put $T_0 = x_n I_n - \frac{1}{w_n} T$. Since $T - w_n x_n I_n$ does not depend on x_n and since $B^{(n)} = -T^{-1} B_\infty^{(n)}$, the differential equation

$$\partial_n Y = B^{(n)} Y \tag{6}$$

turns out to be

$$(x_n I_n - T_0)\partial_n Y = -\frac{1}{w_n} B_\infty^{(n)} Y. \tag{7}$$

Regarding (7) as an ordinary differential equation with respect to the variable x_n, (7) is called an ordinary differential equation of Okubo type. In this sense, the system (4) (or (3)) is one of generalisations of Okubo type ordinary differential equation to several variables case.

Remark 4. We already constructed the systems (4) for some of discriminant sets of complex reflection groups (cf. [8, 10]). Other examples related with algebraic solutions to Painlevé VI will be given in a paper under preparation (cf. [9]).

5. FREE DIVISORS AND POTENTIAL VECTOR FIELDS

In this section, we produce some of potentials and potential vector fields without potentials in three dimensional case. Related with real reflection groups of rank three, there are three kinds of potentials introduced in [6]:

$$\begin{aligned}
A_3 \text{ case}: \quad P &= \tfrac{x_1 x_3^2 + x_2^2 x_3}{2} + \tfrac{x_1^2 x_2^2}{4} + \tfrac{x_1^5}{60}, \\
B_3 \text{ case}: \quad P &= \tfrac{x_1 x_3^2 + x_2^2 x_3}{2} + \tfrac{x_1 x_2^3}{6} + \tfrac{x_1^3 x_2^2}{6} + \tfrac{x_1^7}{210}, \\
H_3 \text{ case}: \quad P &= \tfrac{x_1 x_3^2 + x_2^2 x_3}{2} + \tfrac{x_1^2 x_2^3}{6} + \tfrac{x_1^5 x_2^2}{20} + \tfrac{x_1^{11}}{3960}.
\end{aligned}$$

In these cases, the polynomials defining free divisors are discriminants corresponding to real reflection groups. Concrete forms are given as follows.

$$\begin{aligned}
\Delta_{A_3} &= -8x_1^6 + 56x_1^3x_2^2 + 27x_2^4 - 16x_1^4x_3 - 144x_1x_2^2x_3 + 32x_1^2x_3^2 + 64x_3^3, \\
\Delta_{B_3} &= -(x_1^3 - 3x_1x_2 + 3x_3)(x_1^6 + 6x_1^4x_2 + 3x_1^2x_2^2 + 8x_2^3 - 18x_1x_2x_3 - 9x_3^2), \\
\Delta_{H_3} &= \tfrac{1}{8}(-x_1^{15} - 10x_1^{12}x_2 + 80x_1^9x_2^2 + 20x_1^6x_2^3 + 920x_1^3x_2^4 + 216x_2^5 - 10x_1^{10}x_3 \\
&\qquad -1200x_1^4x_2^2x_3 - 1800x_1x_2^3x_3 + 100x_1^5x_3^2 + 1000x_1^2x_2x_3^2 + 1000x_3^3).
\end{aligned}$$

It is known (and easy to see) that each of Δ_{A_3}, Δ_{B_3}, Δ_{H_3} is weighted homogeneous.

It is underlined here that polynomials of the right hand sides of

$$\begin{aligned}\Delta_{A_3}|_{x_1=0} &= 27x_2^4 + 64x_3^3,\\ \Delta_{B_3}|_{x_1=0} &= 3x_3(-8x_2^3 + 9x_3^2),\\ \Delta_{H_3}|_{x_1=0} &= 27x_2^5 + 125x_3^3.\end{aligned}$$

are defining polynomials of curve singularities of types E_6, E_7, E_8.

This observation suggests the existence of the relationship between free divisors and 1-parameter deformation of singularities on plane curves and naturally leads us to the following consideration. Among the fourteen exceptional singularities in the sense of Arnol'd, we consider

$$E_{12} : x^7 + y^3,\ E_{13} : y(x^5 + y^2),\ E_{14} : x^8 + y^3.$$

Corresponding to these polynomials, we introduce potential vector fields and weighted homogeneous polynomials which define free divisors as follows.

(I) E_{12} case

In this case, we put

$$\begin{aligned}h_1 &= \tfrac{1}{3}(-x_1^3x_2 + 9x_2^3 + 3x_1x_3),\\ h_2 &= \tfrac{1}{45}(x_1^5 + 45x_1^2x_2^2 + 45x_2x_3),\\ h_3 &= \tfrac{1}{126}(-4x_1^7 + 189x_1^4x_2^2 + 1134x_1x_2^4 + 63x_3^2).\end{aligned}$$

Then (h_1, h_2, h_3) is a potential vector field and $F(x) = \det T$ defines a free divisor, where

$$\begin{aligned}F(x) = \tfrac{1}{3087}(&1344x_1^9x_2 - 3843x_1^6x_2^3 + 260820x_1^3x_2^5 + 157464x_2^7 + 448x_1^7x_3\\ &-22491x_1^4x_2^2x_3 - 142884x_1x_2^4x_3 + 3087x_1^2x_2x_3^2 + 3087x_3^3).\end{aligned}$$

It is clear from the definition that $F(0, x_2, x_3) = \frac{17496}{343}x_2^7 + x_3^3$. On the other hand, $F(x)$ is regarded as the discriminant of the complex reflection group No. 24 in [23].

(II) E_{13} case

In this case, we put

$$\begin{aligned}h_1 &= \tfrac{1}{1215}(-4x_1^6 - 270x_1^4x_2 + 1215x_1^2x_2^2 + 3645x_2^3 + 1215x_1x_3),\\ h_2 &= \tfrac{1}{2187}(40x_1^7 + 108x_1^5x_2 + 2430x_1^3x_2^2 - 3645x_1x_2^3 + 2187x_2x_3),\\ h_3 &= \tfrac{1}{21870}(-560x_1^{10} + 4800x_1^8x_2 + 37800x_1^6x_2^2 + 382725x_1^2x_2^4\\ &\quad -137781x_2^5 + 10935x_3^2).\end{aligned}$$

Then (h_1, h_2, h_3) is a potential vector field and $F(x) = \det T$ defines a free divisor, where

$$
\begin{aligned}
&F(x)\\
&= \tfrac{1}{66430125}(1019200x_1^{15} + 26568000x_1^{13}x_2 - 119070000x_1^{11}x_2^2 + 242028000x_1^9x_2^3\\
&\quad +2552885100x_1^7x_2^4 + 9786308868x_1^5x_2^5 - 5933538765x_1^3x_2^6\\
&\quad +16070775840x_1x_2^7 + 7873200x_1^{10}x_3 - 69984000x_1^8x_2x_3\\
&\quad -570807000x_1^6x_2^2x_3 - 6178001625x_1^2x_2^4x_3 + 2295825120x_2^5x_3\\
&\quad +1968300x_1^5x_3^2 + 88573500x_1^3x_2x_3^2 - 199290375x_1x_2^2x_3^2 + 66430125x_3^3).
\end{aligned}
$$

It is clear from the definition that $F(0,x_2,x_3) = \frac{864}{25}x_2^5x_3 + x_3^3$. On the other hand, $F(x)$ is regarded as the discriminant of the complex reflection group No. 27 in [23].

(III) E_{14} case

In this case, we put

$$
\begin{aligned}
h_1 &= \tfrac{1}{315}(10x_1^9 - 252x_1^6x_2 - 945x_1^3x_2^2 + 945x_2^3 + 315x_1x_3),\\
h_2 &= \tfrac{1}{1155}(560x_1^{11} - 990x_1^8x_2 + 8316x_1^5x_2^2 + 10395x_1^2x_2^3 + 1155x_2x_3),\\
h_3 &= \tfrac{1}{95550}(-382720x_1^{16} - 2593920x_1^{13}x_2 + 7023744x_1^{10}x_2^2 - 786240x_1^7x_2^3\\
&\qquad +20638800x_1^4x_2^4 + 6191640x_1x_2^5 + 47775x_3^2).
\end{aligned}
$$

Then (h_1,h_2,h_3) is a potential vector field and $F(x) = \det T$ defines a free divisor, where

$$
\begin{aligned}
&F(x)\\
&= \tfrac{1}{8232000}(-408608000x_1^{24} + 3486336000x_1^{21}x_2 + 11964637440x_1^{18}x_2^2\\
&\quad +24377746176x_1^{15}x_2^3 - 68291566560x_1^{12}x_2^4 + 84987403200x_1^9x_2^5\\
&\quad +31905997200x_1^6x_2^6 + 55510434000x_1^3x_2^7 + 2531725875x_2^8\\
&\quad +154560000x_1^{16}x_3 + 1089446400x_1^{13}x_2x_3 - 3063432960x_1^{10}x_2^2x_3\\
&\quad +355622400x_1^7x_2^3x_3 - 9668484000x_1^4x_2^4x_3 - 3000564000x_1x_2^5x_3\\
&\quad -4704000x_1^8x_3^2 + 79027200x_1^5x_2x_3^2 + 148176000x_1^2x_2^2x_3^2 + 8232000x_3^3).
\end{aligned}
$$

It is clear from the definition that $F(0,x_2,x_3) = \frac{19683}{64}x_2^8 + x_3^3$. There is no complex reflection group of rank three whose discriminant coincides with $F(x)$.

Remark 5. The polynomials corresponding to the singularities E_{12}, E_{13}, E_{14} given above are obtained by the method explained in [19]. By the method there one of the authors (J.S) obtained more than sixty defining polynomials of free divisors corresponding to the singularities E_{12}, E_{13}, E_{14}. Among such polynomials only the three given above and two others are those which are constructed by potential vector fields.

We explain how to find potential vector fields by taking E_{12} case as an example. Since $(w_1,w_2,w_3) = (2,3,7)$ in this case, we define

$$\begin{aligned} h_1 &= s_{11}x_1^3x_2 + s_{12}x_2^3 + x_1x_3, \\ h_2 &= s_{21}x_1^5 + s_{22}x_1^2x_2^2 + x_2x_3, \\ h_3 &= s_{31}x_1^7 + s_{32}x_1^4x_2^2 + s_{33}x_1x_2^4 + \tfrac{1}{2}x_3^2, \end{aligned}$$

and put $\vec{h} = (h_1, h_2, h_3)$ for some constants s_{ij}. Using $\vec{h}$, we define C and $\tilde{B}^{(p)}$ $(p = 1, 2, 3)$ as in §2. Then the condition that $\vec{h}$ is a potential vctor field is equivalent to $\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}$. If $s_{22} = 0$, we don't obtain a free divisor and $\vec{h}$ is not a potential vector field. On the other hand, if $s_{22} \neq 0$, we obtain

$$\begin{aligned} h_1 &= -\tfrac{1}{3}s_{22}x_1^3x_2 + s_{12}x_2^3 + x_1x_3, \\ h_2 &= \tfrac{s_{22}^2}{15s_{12}}x_1^5 + s_{22}x_1^2x_2^2 + x_2x_3, \\ h_3 &= -\tfrac{2s_{22}^3}{21s_{12}}x_1^7 + \tfrac{3s_{22}^2}{2}x_1^4x_2^2 + 3s_{12}s_{22}x_1x_2^4 + \tfrac{1}{2}x_3^2. \end{aligned}$$

If $s_{22}s_{12} \neq 0$, then $\vec{h}$ is a potential vector field. If we choose $s_{22} = 1, s_{12} = 3$, then the potential vector field in this case is obtained.

6. ALGEBRAIC SOLUTIONS OF PAINLEVÉ VI AND FLAT STRUCTURES

In this section, we mention the relationship between the system of differential equations (4) and algebraic solutions of Painlevé VI. Algebraic solutions of Painlevé VI are studied by many authors. The readers who are interested in this subject, refer to [3–7, 11, 13] and references there. For our purpose we assume that $n = 3$ and $w_3 = 1$. The condition $w_3 = 1$ is satisfied if we change w_j by w_j/w_3 $(j = 1, 2, 3)$. Since $F = \det T$ is a cubic polynomial of x_3, let $z_j(x')$ $(j = 1, 2, 3)$ be defined by

$$F(x) = \prod_{i=1}^{3}(x_3 - z_i(x'))$$

and let

$$\delta_F(x') = \prod_{i\neq j}(z_i(x') - z_j(x'))$$

be the discriminant of F as a polynomial of x_3, where $x' = (x_1, x_2)$. We consider $B^{(3)} = -T^{-1}\tilde{B}^{(3)}B_\infty^{(3)}$ as before. It follows from the definition that if $i \neq j$, the (i, j)-entry of $FB^{(3)}$ is a linear function of x_3. Noting this, we define $z_{ij}(x')$ $(i \neq j)$ by the condition that $x_3 = z_{ij}(x')$ is the zero of the (i, j)-entry of $FB^{(3)}$. If each of the diagonal entries of $B_\infty^{(3)}$ is not zero, that is, $r \neq -w_k$ $(k = 1, 2, 3)$, then by an easy computation, we have

$$z_{ij} = \left.\frac{\det(T)(T^{-1})_{ij}}{T_{ij}}\right|_{x_3=0}.$$

This formula holds for $j \neq k$ in the case $r = -w_k$. It can be shown that if the (k, k)-entry of the diagonal matrix $B_\infty^{(3)}$ is zero, namely, if $r = -w_k$ and $j \neq k$, then

$$w_{ij} = \frac{z_{ij}(x') - z_1(x')}{z_2(x') - z_1(x')}$$

is an algebraic solution of Painlevé VI as a function of

$$t = \frac{z_3(x') - z_1(x')}{z_2(x') - z_1(x')}.$$

7. SOME EXAMPLES OF POTENTIAL VECTOR FIELDS FOR GIVEN WEIGHT SYSTEMS

In this section, we treat the case $n = 3$ and give some examples of potential vector fields which consist of polynomials by determining the commutativity condition on $\tilde{B}^{(1)}$, $\tilde{B}^{(2)}$.

7.1. The case $w_1 = 1,\ w_2 = 2,\ w_3 = 5$

We first treat the case where $w_1 = 1$, $w_2 = 2$, $w_3 = 5$. Then the components of potential vector field $\vec{h} = (h_1, h_2, h_3)$ take the forms

$$\begin{aligned} h_1 &= s_{11}x_1^6 + s_{12}x_1^4x_2 + s_{13}x_1^2x_2^2 + s_{14}x_2^3 + x_1x_3, \\ h_2 &= s_{21}x_1^7 + s_{22}x_1^5x_2 + s_{23}x_1^3x_2^2 + s_{24}x_1x_2^3 + x_2x_3, \\ h_3 &= s_{31}x_1^{10} + s_{32}x_1^8x_2 + s_{33}x_1^6x_2^2 + s_{34}x_1^4x_2^3 + s_{35}x_1^2x_2^4 + s_{36}x_2^5 + x_3^2/2, \end{aligned}$$

where s_{ij} are constants to be determined. The matrix C is defined as before, namely, the (i, j)-entry of C is $\partial_i h_j$ and $\tilde{B}^{(p)} = \partial_p C$ $(p = 1, 2, 3)$. In this case $\tilde{B}^{(3)} = I_3$ and the commutativity condition is

$$\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}. \tag{8}$$

Then by direct computation, we obtain at least two non-trivial solutions to (8), namely solutions of generalised WDVV-equation. Both have two parameters.

The first one is given by the following polynomials.

$$\begin{aligned} h_1 &= -(4s_{13}^3x_1^6 + 90s_{13}^2s_{14}x_1^4x_2 - 135s_{13}s_{14}^2x_1^2x_2^2 - 135s_{14}^3x_2^3 - 135s_{14}^2x_1x_3) \\ &\quad /(135s_{14}^2), \\ h_2 &= -(-40s_{13}^4x_1^7 - 36s_{13}^3s_{14}x_1^5x_2 - 270s_{13}^2s_{14}^2x_1^3x_2^2 + 135s_{13}s_{14}^3x_1x_2^3 \\ &\quad -81s_{14}^3x_2x_3)/(81s_{14}^3), \\ h_3 &= (-560s_{13}^6x_1^{10} + 1600s_{13}^5s_{14}x_1^8x_2 + 4200s_{13}^4s_{14}^2x_1^6x_2^2 + 4725s_{13}^2s_{14}^4x_1^2x_2^4 \\ &\qquad -567s_{13}s_{14}^5x_2^5 + 135s_{14}^4x_3^2)/(270s_{14}^4). \end{aligned}$$

These polynomials contain parameters s_{13}, s_{14}. If $s_{13} = 1$, $s_{14} = 3$, then these polynomials coincide with those introduced in (II) E_{13} case of §5.

The second one is given by the following polynomials.

$$\begin{aligned} h_1 &= (2s_{13}^3x_1^6 + 135s_{13}s_{14}^2x_1^2x_2^2 + 135s_{14}^3x_2^3 + 135s_{14}^2x_1x_3)/(135s_{14}^2),\\ h_2 &= x_2(2s_{13}^3x_1^5 + 10s_{13}^2s_{14}x_1^3x_2 + 15s_{13}s_{14}^2x_1x_2^2 + 15s_{14}^2x_3)/(15s_{14}^2),\\ h_3 &= (4s_{13}^6x_1^{10} + 540s_{13}^4s_{14}^2x_1^6x_2^2 + 1620s_{13}^3s_{14}^3x_1^4x_2^3 + 3645s_{13}^2s_{14}^4x_1^2x_2^4\\ &\quad +729s_{13}s_{14}^5x_2^5 + 1215s_{14}^4x_3^2)/(2430s_{14}^4). \end{aligned}$$

These polynomials contain parameters s_{13}, s_{14} as in the previous case. In this case we obtain an algebraic solution of Painlevé VI:

$$t = -\frac{(s+2)^2(s-3)^3}{(s+3)^3(s-2)^2}, \quad w = \frac{3(s+2)(s-3)}{(s+3)^2(s-2)},$$

which is first obtained by Kitaev [11]. The free divisor is isomorphic to the hypersurface of $\mathbf{C}^3$ defined as the zero locus of the polynomial

$$\begin{aligned} f(y_1,y_2,y_3) = y_3(&729y_1^{10} - 2430y_1^8y_2 + 3105y_1^6y_2^2 - 1900y_1^4y_2^3 + 560y_1^2y_2^4\\ &-64y_2^5 - 54y_1^5y_3 + 90y_1^3y_2y_3 - 40y_1y_2^2y_3 + y_3^2). \end{aligned}$$

This means that the choice of the parameters s_{13}, s_{14} is not essential as far as we treat free divisors.

7.2. The case $w_1 = 1$, $w_2 = 2$, $w_3 = 6$

In this case, there are at least two potential vector fields.

The first one is given by

$$\begin{aligned} h_1 &= x_1(s_1x_1^4x_2 + s_2x_2^3 + x_3),\\ h_2 &= (25s_1^2x_1^8 - 63s_2^2x_2^4 + 42s_2x_2x_3)/(42s_2),\\ h_3 &= (1250s_1^3x_1^{12} + 12375s_1^2s_2x_1^8x_2^2 + 2079s_2^3x_2^6 + 495s_2x_3^2)/(990s_2). \end{aligned}$$

The polynomial defined as $\det(T)$ coincides with

$$\begin{aligned} &(-5s_1x_1^4x_2 + s_2x_2^3 + x_3)\\ &\times(250s_1^3x_1^{12} + 225s_1^2s_2x_1^8x_2^2 + 1890s_1s_2^2x_1^4x_2^4 + 189s_2^3x_2^6 - 270s_1s_2x_1^4x_2x_3\\ &\quad +162s_2^2x_2^3x_3 - 27s_2x_3^2) \end{aligned}$$

up to a constant factor. In this case, we obtain an algebraic solution of Painlevé VI:

$$t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}, \quad w = \frac{(s-1)(s+2)}{s(s+1)},$$

which is first obtained by Dubrovin [6].

The second one is given by

$$\begin{aligned} h_1 &= x_1(s_1x_1^6 - 2s_2x_2^3 + x_3),\\ h_2 &= -x_2(14s_1x_1^6 - s_2x_2^3 - x_3),\\ h_3 &= (8820s_1^2x_1^{12} + 27720s_1s_2x_1^6x_2^3 + 396s_2^2x_2^6 + 55x_3^2)/110. \end{aligned}$$

The polynomial defined as $\det(T)$ coincides with

$$(21s_1x_1^6 + 6s_2x_2^3 + x_3)$$
$$\times(196s_1^2x_1^{12} - 1064s_1s_2x_1^6x_2^3 + 4s_2^2x_2^6 - 28s_1x_1^6x_3 - 4s_2x_2^3x_3 + x_3^2).$$

In this case, we obtain an algebraic solution of Painlevé VI: $w^2 = t$ (cf. [13], Solution II).

7.3. The case $w_1 = 1$, $w_2 = 2$, $w_3 = 7$

In this case, there is at least one potential vector field.

The polynomials of the potential vector field are

$$h_1 = (5s_2^4x_1^8 + 56s_1s_2^3x_1^6x_2 - 630s_1^2s_2^2x_1^4x_2^2 + 945s_1^4x_2^4 + 945s_1^3x_1x_3)/(945s_1^3),$$
$$h_2 = (-175s_2^5x_1^9 + 540s_1s_2^4x_1^7x_2 + 1134s_1^2s_2^3x_1^5x_2^2 + 11340s_1^3s_2^2x_1^3x_2^3 + 8505s_1^4s_2x_1x_2^4 + 8505s_1^4x_2x_3)/(8505s_1^4),$$
$$h_3 = (-400s_2^8x_1^{14} + 167440s_1s_2^7x_1^{12}x_2 - 91728s_1^2s_2^6x_1^{10}x_2^2 - 851760s_1^3s_2^5x_1^8x_2^3 + 3865680s_1^4s_2^4x_1^6x_2^4 + 2948400s_1^5s_2^3x_1^4x_2^5 + 8845200s_1^6s_2^2x_1^2x_2^6 + 1263600s_1^7s_2x_2^7 + 552825s_1^6x_3^2)/(1105650s_1^6)$$

and the polynomial $F = \det(T)$ is given by

$$F = -\frac{7}{273375s_1^9}(-20s_2^4x_1^7 - 132s_1s_2^3x_1^5x_2 + 180s_1^2s_2^2x_1^3x_2^2 - 540s_1^3s_2x_1x_2^3 + 135s_1^3x_3) \times (-5200s_2^8x_1^{14} + 30240s_1s_2^7x_1^{12}x_2 - 214704s_1^2s_2^6x_1^{10}x_2^2 - 504000s_1^3s_2^5x_1^8x_2^3 + 398160s_1^4s_2^4x_1^6x_2^4 + 3175200s_1^5s_2^3x_1^4x_2^5 + 1134000s_1^6s_2^2x_1^2x_2^6 + 777600s_1^7s_2x_2^7 - 25200s_1^3s_2^4x_1^7x_3 - 158760s_1^4s_2^3x_1^5x_2x_3 - 793800s_1^6s_2x_1x_2^3x_3 - 99225s_1^6x_3^2).$$

The algebraic solution of Painlevé VI corresponding to this case is equivalent to Klein solution obtained by Boalch [3] (and Solution 8 of [13]). It is underlined here that the algebraic solution corresponding to the potential vector field shown in the E_{12}-singularity (cf. §5) is also Klein solution (= Solution 8 of [13]).

8. FROM FREE DIVISORS TO FLAT STRUCTURES WITHOUT POTENTIALS

In this section, we continue the study on the construction of potential vector fields which consist of polynomials in the three dimensional case.

8.1. The case $F_{B,6}$

The polynomial

(9) $$F_{B,6} = 9x_1x_2^4 + 6x_1^2x_2^2x_3 - 4x_2^3x_3 + x_1^3x_3^2 - 12x_1x_2x_3^2 + 4x_3^3$$

is introduced in [20]. It is clear from the definition that $F_{B,6}$ is weighted homogeneous with the weight system $(w_1, w_2, w_3) = (1, 2, 3)$ same as the discriminant of the real reflection group of type B_3. The zero locus of $F_{B,6}$ is free because of the existence of Saito matrix

$$M = \begin{pmatrix} x_1 & 2x_2 & 3x_3 \\ 2x_2 & 3x_1x_2 + \frac{5}{2}x_3 & \frac{9}{2}x_2^2 + \frac{15}{2}x_1x_3 \\ 3x_3 & \frac{3}{4}(15x_2^2 + x_1x_3) & 18x_2x_3 \end{pmatrix}.$$

As is shown in [22], there are three holonomic systems of rank two. One of them is defined by

$$(10) \qquad \begin{cases} V_1u &= r_1u, \\ V_2u &= \frac{1}{12}(20r_1 - 7)x_1u, \\ V_3V_3u &= -\frac{1}{16}(20r_1 - 7)\{2(10r_1 + 1)x_2^2 - 3x_1x_3\}u \\ & \qquad +10(r_1 + 1)x_2V_3u, \end{cases}$$

where V_1, V_2, V_3 are vector fields defined by

$$^t(V_1, V_2, V_3) = M^t(\partial_1, \partial_2, \partial_3).$$

It is possible to obtain a differential equation of u from (10) of the form

$$(11) \qquad \partial_3^2 u + \frac{P_1(x)}{(x_3 - a_s)F_{B,6}}\partial_3 u + \frac{P_2(x)}{(x_3 - a_s)F_{B,6}{}^2}u = 0,$$

where $P_1(x)$, $P_2(x)$ are polynomials of x_3 and are rational functions of $x' = (x_1, x_2)$ and $a_s = a_s(x')$ is a rational function of x'. Putting $u = F_{B,6}{}^{\alpha}y$ for a constant α, we obtain a differential equation for y from (11). In this case, if $\alpha = \frac{5r_1-4}{45}$, then the equation for y takes the form

$$(12) \quad y'' + \left((1-\beta)\frac{F'_{B,6}}{F_{B,6}} - \frac{1}{x_3 - a_s}\right)y' + \left(\frac{c_0x_3^2 + c_1x_3 + c_2}{F_{B,6}} - \frac{c_0}{x_3 - a_s}\right)y = 0,$$

where

$$y' = \partial_3 y, \; y'' = \partial_3^2 y, \; F'_{B,6} = \partial_3 F_{B,6},$$

β is a constant and c_0, c_1, c_2 are rational functions of x'. By regarding (14) as an ordinary differential equation with respect to the variable x_3, there are four regular singular points $x_3 = z_1, z_2, z_3, \infty$ and an apparent singular point $x_3 = a_s$, where z_1, z_2, z_3 are roots of the equation $F_{B,6} = 0$ as a cubic polynomial of x_3. Then

$$w = \frac{a_s - z_1}{z_2 - z_1}$$

is an algebraic solution of Painlevé VI with respect to the variable $t = \frac{z_3 - z_1}{z_2 - z_1}$. By direct computation we find that the solution (w, t) is equivalent to solution 27 in [4] (= Solution 13 in [13]).

The question to be answered is how to construct a flat structure related to the algebraic solution derived above manner. For this purpose, we start with the polynomials h_1, h_2, h_3 of x_1, x_2, x_3 with coefficients s_{ij} to be determined:

$$\begin{aligned}
h_1 &= x_1(s_{11}x_1^{15} + s_{12}x_1^{10}x_2 + s_{13}x_1^5x_2^2 + s_{14}x_2^3) + x_1x_3,\\
h_2 &= s_{21}x_1^{20} + s_{22}x_1^{15}x_2 + s_{23}x_1^{10}x_2^2 + s_{24}x_1^5x_2^3 + s_{25}x_2^4 + x_2x_3,\\
h_3 &= s_{31}x_1^{30} + s_{32}x_1^{25}x_2 + s_{33}x_1^{20}x_2^2 + s_{34}x_1^{15}x_2^3 + s_{35}x_1^{10}x_2^4 + s_{36}x_1^5x_2^5\\
&\quad +s_{37}x_2^6 + \tfrac{1}{2}x_3^2.
\end{aligned}$$

The weight system for these polynomials is $(w_1, w_2, w_3) = (1, 5, 15)$. The matrix C is defined as before, namely, the (i, j)-entry of C is $\partial_i h_j$ and $\tilde{B}^{(p)} = \partial_p C\,(p = 1, 2, 3)$. By the condition

$$\tilde{B}^{(1)}\tilde{B}^{(2)} = \tilde{B}^{(2)}\tilde{B}^{(1)}, \tag{13}$$

we obtain at least four non-trivial potential vector fields. Among others we treat the case:

$$\begin{aligned}
h_1 &= -\tfrac{2}{33}s_{23}x_1^{11}x_2 - \tfrac{4s_{25}}{3}x_1x_2^3 + x_1x_3,\\
h_2 &= -\tfrac{5s_{23}^2}{684s_{25}}x_1^{20} + s_{23}x_1^{10}x_2^2 + s_{25}x_2^4 + x_2x_3,\\
h_3 &= \tfrac{20s_{23}^3}{2349s_{25}}x_1^{30} + \tfrac{8}{9}s_{23}^2x_1^{20}x_2^2 - 16s_{23}s_{25}x_1^{10}x_2^4 + \tfrac{32s_{25}^2}{15}x_2^6 + \tfrac{1}{2}x_3^2.
\end{aligned}$$

In this case $F = \det(T)$ is defined by

(14)

$$\begin{aligned}
F = \tfrac{25}{27s_{25}}&(-400s_{23}^4x_1^{40}x_2 - 4480s_{23}^3s_{25}x_1^{30}x_2^3 - 399744s_{23}^2s_{25}^2x_1^{20}x_2^5\\
&+1313280s_{23}s_{25}^3x_1^{10}x_2^7 + 34560s_{25}^4x_2^9 - 150s_{23}^3x_1^{30}x_3 - 16740s_{23}^2s_{25}x_1^{20}x_2^2x_3\\
&+320760s_{23}s_{25}^2x_1^{10}x_2^4x_3 - 45360s_{25}^3x_2^6x_3 + 4860s_{23}s_{25}x_1^{10}x_2x_3^2\\
&+9720s_{25}^2x_2^3x_3^2 + 3645s_{25}x_3^3).
\end{aligned}$$

Remark 6. By the substitution $(x_1, x_2, x_3) = (y_1^{1/10}, y_2, y_3)$, the polynomial (14) becomes a polynomial $\tilde{F}(y_1, y_2, y_3)$ of (y_1, y_2, y_3). Then by an appropriate coordinate transformation $y_1 = u_1x_2 + u_2x_1^2$, $y_2 = u_3x_1$, $y_3 = x_3 + u_4x_2x_1 + u_5x_1^3$, $\tilde{F}(y_1, y_2, y_3)$ turns out to be $F_{B,6}$ (cf. (11)) up to a non-zero constant factor.

Remark 7. We explain the reason why each component of the potential vector field in this case is a weighted homogenous polynomial of weight system (1,5,15). The argument of finding a flat structure related with the algebraic solution obtained above starts with constructing a system of differential equations of rank three from the system (10) by middle convolution. By analysis of the new system of rank three, we find the existence of a flat coordinate system of weight system (1,5,15) in this case.

8.2. The case $F_{H,2}$

The polynomial

(15) $$F_{H,2}=100x_1^3x_2^4+x_2^5+40x_1^4x_2^2x_3-10x_1x_2^3x_3+4x_1^5x_3^2-15x_1^2x_2x_3^2+x_3^3$$

is also introduced in [20]. Clearly $F_{H,2}$ is weighted homogeneous of type $(w_1,w_2,w_3)=(1,3,5)$ same as the discriminant of the real reflection group of type H_3. The zero locus of $F_{H,2}$ is free because of the existence of its Saito matrix

$$M=\begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ 3x_2 & 36x_1^2x_2+6x_3 & 90x_1x_2^2+90x_1^2x_3 \\ 5x_3 & -\frac{10}{3}(12x_1^3-55x_2)x_1x_2 & -\frac{50}{3}(6x_1^3x_2^2-x_2^3+6x_1^4x_3-18x_1x_2x_3) \end{pmatrix}.$$

In this case, there are two holonomic systems of rank two (cf. [22]). One of them is given by

(16) $$\begin{cases} V_1u &= r_1u, \\ V_2u &= 3(4r_1-1)x_1^2u, \\ V_3V_3u &= -\frac{25}{9}\begin{pmatrix} 4(-1+4r_1)(17+4r_1)x_1^8 \\ -12(-27+100r_1+40r_1^2)x_1^5x_2 \\ +15(-7+20r_1+60r_1^2)x_1^2x_2^2 \\ +8(12r_1-1)x_1^3x_3-9(10r_1-1)x_2x_3 \end{pmatrix}u \\ & \quad -\frac{20}{3}(2+r_1)x_1(4x_1^3-15x_2)V_3u, \end{cases}$$

where V_1, V_2, V_3 are vector fields defined by

$$^t(V_1,V_2,V_3)=M\,^t(\partial_1,\partial_2,\partial_3).$$

It is possible to show by an argument similar to the case (10) that there is an algebraic solution of Painlevé VI derived from (16) equivalent to solution 29 in [4] (= Solution 18 in [13]).

To find a potential vector field which is related to solution 29 in [4], we consider the three polynomials with coefficients s_{ij} to be determined:

$$\begin{aligned} h_1 &= x_1(s_{11}x_1^{10}+s_{12}x_1^8x_2+s_{13}x_1^6x_2^2+s_{14}x_1^4x_2^3+s_{15}x_1^2x_2^4+s_{16}x_2^5)+x_1x_3, \\ h_2 &= s_{21}x_1^{12}+s_{22}x_1^{10}x_2+s_{23}x_1^8x_2^2+s_{24}x_1^6x_2^3+s_{25}x_1^4x_2^4+s_{26}x_1^2x_2^5+s_{27}x_2^6+x_2x_3, \\ h_3 &= s_{31}x_1^{20}+s_{32}x_1^{18}x_2+s_{33}x_1^{16}x_2^2+s_{34}x_1^{14}x_2^3+s_{35}x_1^{12}x_2^4+s_{36}x_1^{10}x_2^5+s_{37}x_1^8x_2^6 \\ &\quad +s_{38}x_1^6x_2^7+s_{39}x_1^4x_2^8+s_{3,10}x_1^2x_2^9+s_{3,11}x_2^{10}+x_3^2/2, \end{aligned}$$

so that $\vec{h}=(h_1,h_2,h_3)$ is a potential vector field. In this case, h_1,h_2,h_3 are weighted homogeneous of weight system $(w_1,w_2,w_3)=(1,2,10)$. The matrix C is defined as before, namely, the (i,j)-entry of C is ∂_ih_j and $\tilde{B}^{(p)}=\partial_pC$ $(p=1,2,3)$. By the condition

(17) $$\tilde{B}^{(1)}\tilde{B}^{(2)}=\tilde{B}^{(2)}\tilde{B}^{(1)},$$

we obtain at least two non-trivial potential vector fields. We treat one of them defined by the polynomials below:

$$h_1 = -\frac{3s_{24}}{10}x_1^7x_2^2 - \frac{21s_{27}}{5}x_1x_2^5 + x_1x_3,$$
$$h_2 = -\frac{s_{24}^2}{275s_{27}}x_1^{12} + s_{24}x_1^6x_2^3 + s_{27}x_2^6 + x_2x_3,$$
$$h_3 = \frac{2s_{24}^3}{125s_{27}}x_1^{18}x_2 + \frac{51s_{24}^2}{25}x_1^{12}x_2^4 - 51s_{24}s_{27}x_1^6x_2^7 + \frac{119s_{27}^2}{10}x_2^{10} + \frac{1}{2}x_3^2.$$

In this case, $F = \det(T)$ is given by

(18)
$$\begin{aligned}F = 4(&-108s_{24}^5x_1^{30} - 205200s_{24}^4s_{27}x_1^{24}x_2^3 - 1876500s_{24}^3s_{27}^2x_1^{18}x_2^6\\ &-507195000s_{24}^2s_{27}^3x_1^{12}x_2^9 + 1748503125s_{24}s_{27}^4x_1^6x_2^{12} + 140568750s_{27}^5x_2^{15}\\ &-67500s_{24}^3s_{27}x_1^{18}x_2x_3 - 8775000s_{24}^2s_{27}^2x_1^{12}x_2^4x_3 + 223593750s_{24}s_{27}^3x_1^6x_2^7x_3\\ &-53156250s_{27}^4x_2^{10}x_3 + 703125s_{24}s_{27}^2x_1^6x_2^2x_3^2 + 1406250s_{27}^3x_2^5x_3^2\\ &+781250s_{27}^2x_3^3)/(3125s_{27}^2).\end{aligned}$$

Moreover, the algebraic solution of Painlevé VI corresponding to the potential vector field defined by the polynomials h_1, h_2, h_3 is solution 29 in [4].

Remark 8. As in the case of $F_{B,6}$, by the substitution $(x_1, x_2, x_3) = (y_1^{1/6}, y_2, y_3)$, the polynomial (18) becomes a polynomial $\tilde{F}(y_1, y_2, y_3)$ of (y_1, y_2, y_3). Then by an appropriate coordinate transformation $y_1 = u_1x_2 + u_2x_1^2$, $y_2 = u_3x_1$, $y_3 = x_3 + u_4x_2x_1^2 + u_5x_1^5$, $\tilde{F}(y_1, y_2, y_3)$ turns out to be $F_{H,2}$ (cf. (15)) up to a non-zero constant factor.

9. POTENTIAL VECTOR FIELDS FOR DISCRIMINANTS OF COMPLEX REFLECTION GROUPS

The notion of free divisors is formulated by K. Saito (cf. [16]) and the freeness of the discriminant sets of irreducible finite real reflection groups in the quotient space by the group action is firstly shown in [16]. In the case of discriminant sets of irreducible finite complex reflection groups, the freeness is shown by Terao [24] (see also Orlik and Terao [14]). Flat structure for the case of finite real reflection groups is firstly treated in [18] and its existence in this case is shown in [17]. In this section we discuss the existence of potential vector fields for the case of finite complex reflection groups. Flat structure without potential is constructed by a potential vector field.

Irreducible finite complex reflection groups are classified by Shephard-Todd [23]: There is an infinite family $G(de, d, r)$, plus 34 exceptional groups $G_4, G_5, \ldots, G_{37}$. In this section we only treat the exceptional and rank>2 case. Among the 34 groups, the groups G_k $(4 \le k \le 22)$ have rank 2 and the remaining groups G_k $(23 \le k \le 37)$ have rank>2. Real reflection groups are also contained in these groups. In fact, G_{23}, G_{28}, G_{30}, G_{35}, G_{36}, G_{37} are real reflection groups of types H_3, F_4, H_4, E_6, E_7, E_8, respectively. Taking our attention to discriminants, those of G_{25}, G_{26}, G_{32} are same as those of $W(A_3)$, $W(B_3)$, $W(A_4)$, respectively. For this reason, we concentrate our attention to the groups G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}.

In general, let G be an irreducible finite complex reflection group of rank n. If V is a standard representation space of G, there are basic invariants x_k $(k = 1, 2, \ldots, n)$ of the ring of G-invariant polynomials on V. Let w_k be the degree of x_k. Then we may assume that $0 < w_1 < w_2 < \cdots < w_n$ since we treat the groups G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34} and the assumption $0 < w_1 < w_2 < \cdots < w_n$ is satisfied for all these groups. The discriminant for G is G-invariant and as a consequence, it is regarded as a polynomial of $x_1, x_2, \ldots, x_n$. For this reason we write $\Delta_G(x_1, x_2, \ldots, x_n)$ for the discriminant. As a polynomial of x_n, the degree of Δ_G is equal or greater than n. It is known (cf. [14], [2]) that $\deg_{x_n} \Delta_G = n$ for that case $G = G_{24}$, G_{27}, G_{29}, G_{33}, G_{34} and $\deg_{x_n} \Delta_G = n + 1$ for the case $G = G_{31}$. To construct the system of the form (4) we need the condition $\deg_{x_n} \Delta_G = n$. As a consequence, in the following we treat the five groups G_{24}, G_{27}, G_{29}, G_{33}, G_{34} but exclude group G_{31}.

ST	Rank	Degrees
24	3	4,6,14
27	3	6,12,30
29	4	4,8,12,20
33	5	4,6,10,12,18
34	6	6,12,18,24,30,42

The argument to derive potential vector fields related to the groups G_{24}, G_{27}, G_{29}, G_{33}, G_{34} is as follows. Let $\vec{h} = (h_1, h_2, \ldots, h_n)$ be a potential vector field and let C be the $n \times n$ matrix whose (i, j)-entry is $\partial_i h_j$. Then it is sufficient to solve the condition that $\partial_p C$ $(p = 1, 2, \ldots, n)$ commute each other for the existence of h_p $(p = 1, 2, \ldots, n)$.

Let G be one of these groups. Orlik and Terao [14] obtained basic derivations of the discriminant for the case G_{24} and Bessis and Michel [2] did for the cases G_{27}, G_{29}, G_{33}. Using the results of [14] and [2], we compute potential vector fields for these cases. Potential vector fields for the cases G_{24}, G_{27} are already introduced in E_{12} case and E_{13} case of §5 and those for the three cases G_{29}, G_{33} are given below. As to the case G_{34}, we find neither reference on the concrete form of the discriminant nor that on its basic derivations. Therefore, assuming the existence of a potential vector field, we determine polynomials $h_1, h_2, \ldots, h_6$ given below by direct computation. It is provable that $\vec{h} = (h_1, h_2, \ldots, h_6)$ is a potential vector field. In fact, define the 6×6 matrix

C whose (i,j)-entry is $\partial_i h_j$. Then the matrices $\tilde{B}^{(i)} = \partial_i C$ $(i = 1, 2, \ldots, 6)$ commute each other. This implies that $T = x_1\tilde{B}^{(1)} + \cdots + 5x_5\tilde{B}^{(5)} + 7x_6\tilde{B}^{(6)}$ is a Saito matrix of the polynomial $F = \det(T)$ and $F = 0$ is a free divisor in $\mathbf{C}^6$.

It is plausible that F is the discriminant of the group G_{34} if $x_1, x_2, \ldots, x_6$ are identified with appropriate basic invariants of the ring of G_{34}-invariant polynomials. As an evidence to support this observation we have shown that if G is an irreducible finite complex reflection group and it is well-generated in the sense of [1], there is a unique flat structure for the discriminant of G.

(1) G_{29} case

$$
\begin{aligned}
h_1 &= (-3x_1^2x_2^2 + 4x_2^3 + 2x_1^3x_3 + 6x_1x_2x_3 + 3x_3^2 + 6x_1x_4)/6,\\
h_2 &= (-x_1^7 + 7x_1^5x_2 + 7x_1^3x_2^2 + 14x_1x_2^3 + 7x_1^4x_3 - 14x_1^2x_2x_3 - 14x_2^2x_3 + 7x_1x_3^2\\
&\quad +14x_2x_4)/14,\\
h_3 &= (x_1^8 + 2x_1^6x_2 + 8x_1^4x_2^2 - 6x_1^2x_2^3 - 4x_2^4 + 6x_1^5x_3 + 12x_1^3x_2x_3 + 8x_1x_2^2x_3\\
&\quad +8x_1^2x_3^2 + 2x_2x_3^2 + 4x_3x_4)/4,\\
h_4 &= (13x_1^{10} + 70x_1^8x_2 - 70x_1^6x_2^2 + 200x_1^4x_2^3 + 120x_1^2x_2^4 + 56x_2^5 + 20x_1^7x_3\\
&\quad +100x_1^5x_2x_3 + 160x_1^3x_2^2x_3 - 200x_1x_2^3x_3 + 130x_1^4x_3^2 + 120x_1^2x_2x_3^2 + 60x_2^2x_3^2\\
&\quad +40x_1x_3^3 + 20x_4^2)/40.
\end{aligned}
$$

(2) G_{33} case

$$
\begin{aligned}
h_1 &= (-2x_1x_2^3 + 2x_1^3x_3 + 3x_2^2x_3 + 6x_1x_2x_4 + 3x_3x_4 + 3x_1x_5)/3,\\
h_2 &= (-4x_1^6 + 60x_1^3x_2^2 + 15x_2^4 + 30x_1x_3^2 + 60x_1^3x_4 - 30x_2^2x_4 + 15x_4^2 + 30x_2x_5)/30,\\
h_3 &= (4x_1^7 + 12x_1^4x_2^2 + 9x_1x_2^4 + 36x_1^3x_2x_3 - 2x_2^3x_3 + 6x_1^2x_3^2 + 12x_1^4x_4\\
&\quad +18x_1x_2^2x_4 + 6x_2x_3x_4 + 9x_1x_4^2 + 3x_3x_5)/3,\\
h_4 &= (20x_1^6x_2 + 24x_1^3x_2^3 - 3x_2^5 + 12x_1^5x_3 + 18x_1^2x_2^2x_3 + 12x_1x_2x_3^2 + x_3^3\\
&\quad +24x_1^3x_2x_4 + 6x_2^3x_4 + 18x_1^2x_3x_4 + 3x_2x_4^2 + 3x_4x_5)/3,\\
h_5 &= (64x_1^9 + 576x_1^6x_2^2 + 48x_2^6 + 288x_1^5x_2x_3 + 432x_1^2x_2^3x_3 + 144x_1^4x_3^2 + 24x_2x_3^3\\
&\quad +48x_1^6x_4 + 144x_1^3x_2^2x_4 - 36x_2^4x_4 + 432x_1^2x_2x_3x_4 + 72x_1x_3^2x_4 + 144x_1^3x_4^2\\
&\quad +72x_2^2x_4^2 + 12x_4^3 + 9x_5^2)/18.
\end{aligned}
$$

(3) G_{34} case (Conjecture)

$$
\begin{aligned}
h_1 &= (20x_1^4x_2^2 + 120x_1^2x_2^3 - 60x_2^4 - 12x_1^5x_3 + 60x_1^3x_2x_3 - 180x_1x_2^2x_3 + 135x_1^2x_3^2\\
&\quad +135x_2x_3^2 + 120x_1^4x_4 - 180x_1^2x_2x_4 + 540x_2^2x_4 + 270x_1x_3x_4 + 405x_4^2\\
&\quad +180x_1^3x_5 + 540x_1x_2x_5 + 405x_3x_5 + 405x_1x_6)/405,\\
h_2 &= (64x_1^9 - 288x_1^7x_2 - 1728x_1^5x_2^2 + 1728x_1^3x_2^3 - 2592x_1x_2^4 + 432x_1^6x_3\\
&\quad +3888x_1^4x_2x_3 + 3888x_1^2x_2^2x_3 + 1944x_2^3x_3 + 972x_1^3x_3^2 + 729x_3^3 - 1296x_1^5x_4\\
&\quad +7776x_1^3x_2x_4 + 8748x_1^2x_3x_4 - 2916x_2x_3x_4 - 5832x_1x_4^2 + 3888x_1^4x_5\\
&\quad -2916x_1^2x_2x_5 - 2916x_2^2x_5 + 4374x_1x_3x_5 + 4374x_4x_5 + 4374x_2x_6)/4374,\\
h_3 &= (64x_1^{10} + 3456x_1^8x_2 + 6624x_1^6x_2^2 + 2160x_1^4x_2^3 + 3888x_2^5 + 7776x_1^7x_3\\
&\quad +19440x_1^5x_2x_3 + 32400x_1^3x_2^2x_3 + 7776x_1x_2^3x_3 + 19440x_1^4x_3^2\\
&\quad +26244x_1^2x_2x_3^2 + 2916x_2^2x_3^2 + 7290x_1x_3^3 + 9504x_1^6x_4 + 38880x_1^4x_2x_4
\end{aligned}
$$

$$
\begin{aligned}
&-11664x_1^2x_2^2x_4 - 3888x_2^3x_4 + 25272x_1^3x_3x_4 + 34992x_1x_2x_3x_4 + 8748x_3^2x_4\\
&+5832x_1^2x_4^2 + 8748x_2x_4^2 + 11664x_1^5x_5 + 25272x_1^3x_2x_5 + 17496x_1x_2^2x_5\\
&+26244x_1^2x_3x_5 + 8748x_2x_3x_5 + 26244x_1x_4x_5 + 6561x_5^2 + 6561x_3x_6)/6561,\\
h_4 = {}&(1152x_1^{11} + 832x_1^9x_2 + 23616x_1^7x_2^2 - 13824x_1^5x_2^3 + 34560x_1^3x_2^4 + 7776x_1^8x_3\\
&+42768x_1^6x_2x_3 + 16200x_1^4x_2^2x_3 + 7776x_1^2x_2^3x_3 - 5832x_2^4x_3 + 17496x_1^5x_3^2\\
&+27216x_1^3x_2x_3^2 + 23328x_1x_2^2x_3^2 + 13122x_1^2x_3^3 + 2916x_2x_3^3 + 12960x_1^7x_4\\
&-25920x_1^5x_2x_4 + 103680x_1^3x_2^2x_4 - 31104x_1x_2^3x_4 + 19440x_1^4x_3x_4\\
&+69984x_1^2x_2x_3x_4 - 5832x_2^2x_3x_4 + 17496x_1x_3^2x_4 + 38880x_1^3x_4^2\\
&-46656x_1x_2x_4^2 - 4374x_3x_4^2 + 2592x_1^6x_5 + 21384x_1^4x_2x_5 - 5832x_1^2x_2^2x_5\\
&+11664x_2^3x_5 + 26244x_1^3x_3x_5 + 17496x_1x_2x_3x_5 + 6561x_3^2x_5 - 8748x_1^2x_4x_5\\
&+17496x_2x_4x_5 + 13122x_1x_5^2 + 13122x_4x_6)/13122,\\
h_5 = {}&(10496x_1^{12} + 70656x_1^{10}x_2 + 86976x_1^8x_2^2 + 233856x_1^6x_2^3 - 25920x_1^4x_2^4\\
&-20736x_2^6 + 71808x_1^9x_3 + 264384x_1^7x_2x_3 + 393984x_1^5x_2^2x_3 + 129600x_1^3x_2^3x_3\\
&+93312x_1x_2^4x_3 + 165888x_1^6x_3^2 + 408240x_1^4x_2x_3^2 + 221616x_1^2x_2^2x_3^2\\
&+134136x_1^3x_3^3 + 87480x_1x_2x_3^3 + 10935x_3^4 + 22464x_1^8x_4 + 209088x_1^6x_2x_4\\
&+38880x_1^4x_2^2x_4 + 233280x_1^2x_2^3x_4 + 23328x_2^4x_4 + 241056x_1^5x_3x_4\\
&+272160x_1^3x_2x_3x_4 + 139968x_1x_2^2x_3x_4 + 209952x_1^2x_3^2x_4 + 87480x_2x_3^2x_4\\
&+19440x_1^4x_4^2 + 349920x_1^2x_2x_4^2 - 69984x_2^2x_4^2 + 139968x_1x_3x_4^2 - 17496x_4^3\\
&+10368x_1^7x_5 + 38880x_1^5x_2x_5 + 93312x_1^3x_2^2x_5 - 23328x_1x_2^3x_5\\
&+134136x_1^4x_3x_5 + 227448x_1^2x_2x_3x_5 + 52488x_2^2x_3x_5 + 104976x_1x_3^2x_5\\
&+198288x_1^3x_4x_5 + 104976x_1x_2x_4x_5 + 78732x_3x_4x_5 + 78732x_1^2x_5^2\\
&+26244x_2x_5^2 + 39366x_5x_6)/39366,\\
h_6 = {}&(109056x_1^{14} + 433664x_1^{12}x_2 + 1983744x_1^{10}x_2^2 - 400512x_1^8x_2^3 + 2784768x_1^6x_2^4\\
&-282240x_1^4x_2^5 + 967680x_1^2x_2^6 + 207360x_2^7 + 403200x_1^{11}x_3\\
&+2395008x_1^9x_2x_3 + 3709440x_1^7x_2^2x_3 + 6096384x_1^5x_2^3x_3 - 846720x_1^3x_2^4x_3\\
&-725760x_1x_2^5x_3 + 1611792x_1^8x_3^2 + 4445280x_1^6x_2x_3^2 + 3492720x_1^4x_2^2x_3^2\\
&+1360800x_1^2x_2^3x_3^2 + 462672x_2^4x_3^2 + 1496880x_1^5x_3^3 + 2857680x_1^3x_2x_3^3\\
&+734832x_1x_2^2x_3^3 + 489888x_1^2x_3^4 + 91854x_2x_3^4 + 177408x_1^{10}x_4\\
&+80640x_1^8x_2x_4 + 3499776x_1^6x_2^2x_4 - 3870720x_1^4x_2^3x_4 + 2540160x_1^2x_2^4x_4\\
&-653184x_2^5x_4 + 1439424x_1^7x_3x_4 + 7039872x_1^5x_2x_3x_4 + 3991680x_1^3x_2^2x_3x_4\\
&+1741824x_1x_2^3x_3x_4 + 2721600x_1^4x_3^2x_4 + 1388016x_1^2x_2x_3^2x_4\\
&+244944x_2^2x_3^2x_4 + 857304x_1x_3^3x_4 + 1874880x_1^6x_4^2 - 3175200x_1^4x_2x_4^2\\
&+5225472x_1^2x_2^2x_4^2 + 925344x_1^3x_3x_4^2 + 2939328x_1x_2x_3x_4^2 + 306180x_3^2x_4^2\\
&+1469664x_1^2x_4^3 - 816480x_2x_4^3 - 8064x_1^9x_5 + 254016x_1^7x_2x_5 + 217728x_1^5x_2^2x_5\\
&+362880x_1^3x_2^3x_5 + 544320x_1x_2^4x_5 + 707616x_1^6x_3x_5 + 1632960x_1^4x_2x_3x_5\\
&+2122848x_1^2x_2^2x_3x_5 - 326592x_2^3x_3x_5 + 1714608x_1^3x_3^2x_5\\
&+1224720x_1x_2x_3^2x_5 + 183708x_3^3x_5 + 816480x_1^5x_4x_5 + 4572288x_1^3x_2x_4x_5\\
&-1959552x_1x_2^2x_4x_5 + 2694384x_1^2x_3x_4x_5 + 979776x_2x_3x_4x_5\\
&-244944x_1x_4^2x_5 + 1102248x_1^4x_5^2 + 979776x_1^2x_2x_5^2 + 489888x_2^2x_5^2\\
&+734832x_1x_3x_5^2 + 367416x_4x_5^2 + 137781x_6^2)/275562.
\end{aligned}
$$

Remark 9. (1) As for the case G_{31}, you can find basic derivations of the discriminant in [2] and a system of differential equations in [21].

(2) D. Bessis [1] discussed the existence of flat structures of discriminants of complex reflection groups. The definition of flat structure in [1] is seemingly weaker than the original Saito's definition.

Acknowledgments. The second author (T.M.) is partially supported by JSPS Grant-in-Aid for Young Scientists (B) (No. 25800082). The third auhtor (J.S.) is partially supported by JSPS Grant-in-Aid for Scientific Research (C) (No. 26400111).

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Received 30 September 2014

University of Ryukyus,
Colledge of Educations,
Department of Mathematics,
Nishihara-cho, Okinawa 903-0213,
Japan
mkato@edu.u-ryukyu.ac.jp

University of Ryukyus,
Faculty of Sciences,
Department of Mathematical Sciences,
Nishihara-cho, Okinawa 903-0213,
Japan
tmano@math.u-ryukyu.ac.jp

Tokyo University of Agriculture and Technology,
Faculty of Engineering,
Department of Mathematics,
Koganei, Tokyo 184-8588,
Japan
sekiguti@cc.tuat.ac.jp