SPECIAL GENERIC MAPS OF 5-DIMENSIONAL MANIFOLDS

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A smooth map of a closed n-dimensional manifold into $\mathbb{R}^p$, $n > p$, is called a special generic map if it has only definite fold points as its singularities. Special generic maps were first defined by Burlet and de Rham for $(n, p) = (3, 2)$. Then they were extended to general (n, p) and studied by Porto, Furuya, Sakuma, Saeki, et al. In this paper, we give a new topological restriction on the source manifold of special generic maps for $(n, p) = (2k + 1, k + 2)$, $k > 1$, and completely determine the diffeomorphism types of those simply connected closed 5-dimensional manifolds which admit special generic maps into $\mathbb{R}^p$ for $1 \leq p \leq 5$.

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1. INTRODUCTION

Let M be a closed connected smooth n-dimensional manifold and let $f : M \to \mathbb{R}^p$ be a smooth map with $1 \leq p \leq n$. A point q in M is called a *definite fold point* if the map f can be represented by the normal form:

$$y_i \circ f = x_i, \quad 1 \leq i \leq p - 1,$$
$$y_p \circ f = x_p^2 + x_{p+1}^2 + \cdots + x_n^2$$

for some local coordinates $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_p)$ centered at q and $f(q)$, respectively. The map f is called a *special generic map* if all the singular points of f are definite fold points.

Special generic maps were first defined by Burlet and de Rham [3], who studied such maps of 3-dimensional manifolds into the plane. Porto and Furuya [5] studied special generic maps of n-dimensional manifolds into the plane, $n > 3$. Èliašberg [4] studied special generic maps of n-dimensional manifolds into $\mathbb{R}^n$.

We note that a special generic map of an n-dimensional manifold into the line is a Morse function with critical points of indices 0 or n. It is a result of Reeb [6] that closed connected smooth n-dimensional manifolds which admit special generic maps into the line are homeomorphic to the n-dimensional sphere.

In this paper, we study the topology of closed 5-dimensional manifolds which admit special generic maps into $\mathbb{R}^p$ for arbitrary integers p with $1 \leq p \leq 5$, and we give the complete list of the diffeomorphism types of simply connected closed 5-dimensional manifolds which admit such special generic maps (see Theorem 3.3).

Note that for $p \neq 4$, this can be directly obtained by using results of Reeb [6], Porto and Furuya [5], Saeki [8], Èliašberg [4] and Barden [1]; however, for $p = 4$, this is a new result as far as the author knows.

The paper is organized as follows. In Section 2, we review several results about special generic maps and closed simply connected 5-dimensional manifolds. In Section 3, we prove that if there exists a special generic map $f : M \to \mathbb{R}^{k+2}$ of a closed orientable $(2k+1)$-dimensional manifold, $k \geq 2$, with $H_1(M;\mathbb{Z}) = 0$, then the homology group $H_k(M;\mathbb{Z})$ is free abelian. As a corollary, we prove that for a closed simply connected 5-dimensional manifold M, there exists a special generic map $f : M \to \mathbb{R}^4$ if and only if M is diffeomorphic to a connected sum of S^3-bundles over S^2.

Throughout this paper, all manifolds and maps are of class C^∞, unless otherwise indicated; all homology and cohomology groups have integer coefficients unless otherwise specified; for groups G_1 and G_2, "$G_1 \cong G_2$" means that they are isomorphic; for smooth manifolds M_1 and M_2, "$M_1 \cong M_2$" means that they are diffeomorphic; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; D^n denotes the closed unit disk in $\mathbb{R}^n$; and S^n denotes the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

2. SPECIAL GENERIC MAPS

In this section, we review several results on special generic maps and simply connected 5-dimensional manifolds, which will be important in the proofs of our results.

Let us first give some examples of special generic maps.

Example 2.1. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^p$, $n \geq p$, be the standard projection. Then its restriction to the unit sphere $f = \pi|_{S^n} : S^n \to \mathbb{R}^p$ is a special generic map.

Example 2.2. Let $f : M \to \mathbb{R}^n$ be a special generic map and let Q be a closed k-dimensional submanifold of $\mathbb{R}^{n+k}$ having a trivial normal bundle. Then the composition

$$Q \times M \xrightarrow{\mathrm{id}\times f} Q \times \mathbb{R}^n \to \mathbb{R}^{n+k}$$

is a special generic map, where id is the identity map, and the last map is the composition of a trivialization of the open tubular neighborhood of Q in $\mathbb{R}^{n+k}$ with the inclusion map.

Now, we prove the following lemma, which will be used in the proof of our main theorem.

LEMMA 2.3. *Let M be the non-trivial S^3-bundle over S^2, $p = 3, 4$. Then there exists a special generic map of M into $\mathbb{R}^p$.*

Proof. When $p = 3$, the above lemma is already proved in [7]. So we will prove it only for $p = 4$. For real numbers t with $0 \leq t \leq 2\pi$, we define the map $h_t : S^3 \to S^3$ by

$$h_t(x_1, x_2, x_3, x_4) = (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, x_4).$$

By using the above map, we define the diffeomorphism $\Phi : S^3 \times \partial D^2 \to S^3 \times \partial D^2$ by

$$\Phi(x, (\cos t, \sin t)) = (h_t(x), (\cos t, \sin t)).$$

Pasting $S^3 \times D^2$ and its copy along the boundary by Φ, we obtain the closed 5-dimensional manifold X. We see easily that X is diffeomorphic to M.

Now, we define the special generic map $f : S^3 \to \mathbb{R}^2$ by

$$f(x_1, x_2, x_3, x_4) = (x_3, x_4).$$

Then we have

$$(f \times \mathrm{id}) \circ \Phi = f \times \mathrm{id} : S^3 \times \partial D^2 \to \mathbb{R}^2 \times \partial D^2.$$

Therefore, the map

$$(f \times \mathrm{id}) \cup (f \times \mathrm{id}) : X = (S^3 \times D^2) \cup_\Phi (S^3 \times D^2) \to (\mathbb{R}^2 \times D^2) \cup (\mathbb{R}^2 \times D^2)$$

is well-defined, where $(\mathbb{R}^2 \times D^2) \cup (\mathbb{R}^2 \times D^2)$ is the space obtained by pasting $\mathbb{R}^2 \times D^2$ and its copy along the boundary by the identity map. So we obtain the composition map

$$g : X \longrightarrow \mathbb{R}^2 \times S^2 \to \mathbb{R}^4,$$

where the last map is the composition of a trivialization of the open tubular neighborhood of S^2 in $\mathbb{R}^4$ with the inclusion map. We see easily that the map $g : X \to \mathbb{R}^4$ is a special generic map. $\square$

Definition 2.4. Let M be a closed n-dimensional manifold and $f : M \to \mathbb{R}^p$ a smooth map, $n \geq p$. The set

$$S(f) = \{x \in M \mid \mathrm{rank}(df_x) < p\} \tag{2.1}$$

is called the *singular point set* of f, whose points are called *singular points*.

Definition 2.5. Let $f : X \to Y$ be a continuous map between two topological spaces. For two points x_1 and x_2 in X, we define $x_1 \sim x_2$ if x_1 and x_2 are in the same connected component of the pre-image $f^{-1}(y)$ for a point y in Y. This relation "$\sim$" is an equivalence relation, and therefore we can consider the quotient space W_f and the quotient map $q_f : X \to W_f$ with respect to this relation. Then it is not difficult to prove that there exists a unique continuous map $\bar{f} : W_f \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q_f"'] & & Y \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd} \tag{2.2}$$

The diagram (2.2) or the space W_f is called the *Stein factorization* of f.

The quotient space W_f is not always a manifold for a smooth map $f : M \to \mathbb{R}^p$ of a manifold; however, if f is a special generic map, then we have the following.

THEOREM 2.6 ([3]). *Let $f : M \to \mathbb{R}^p$ be a special generic map of a closed connected n-dimensional manifold M into $\mathbb{R}^p$, $n > p$. Then the following holds.*

(1) *The quotient space W_f has the structure of a smooth p-dimensional manifold with non-empty boundary.*
(2) *The map $\bar{f} : W_f \to \mathbb{R}^p$ is an immersion.*
(3) *The singular point set $S(f)$ is a closed $(p-1)$-dimensional submanifold of M, and the restriction of q_f to $S(f)$ is a diffeomorphism onto ∂W_f.*
(4) *The induced map $(q_f)_* : \pi_1(M) \to \pi_1(W_f)$ is a group isomorphism.*
(5) *We have $q_f(M - S(f)) = \operatorname{Int} W_f$ and $q_f|_{M-S(f)} : M - S(f) \to \operatorname{Int} W_f$ is a smooth S^{n-p}-bundle over $\operatorname{Int} W_f$.*

The above theorem is essentially proved for $(n, p) = (3, 2)$ in [3]. We can prove it for general (n, p), $n > p$, by using the same method.

The following result gives us a method of reconstructing all manifolds admitting special generic maps through the Stein factorizations, up to homeomorphism.

THEOREM 2.7 ([7]). *Let $f : M \to \mathbb{R}^p$ be a special generic map of a closed n-dimensional manifold with $n > p$. Then there exists a topological D^{n-p+1}-bundle E over W_f with M homeomorphic to ∂E.*

For special generic maps of closed orientable manifolds into the Euclidean space of the same dimensions, Èliašberg [4] proved the following result.

THEOREM 2.8 ([4]). *Let M be a closed orientable n-dimensional manifold. Then M admits a special generic map into $\mathbb{R}^n$ if and only if it is stably parallelizable.*

The following result is useful for constructing special generic maps of a connected sum of manifolds into $\mathbb{R}^p$.

PROPOSITION 2.9 ([7]). *Let M_1 and M_2 be closed orientable n-dimensional manifolds. If there exist special generic maps $f_i : M_i \to \mathbb{R}^p$, $i = 1, 2$, then there exists a special generic map $f : M_1 \sharp M_2 \to \mathbb{R}^p$, where $M_1 \sharp M_2$ denotes the connected sum of M_1 and M_2.*

Now let us recall several results of Barden [1]. In the following, $w_2(M) \in H^2(M; \mathbb{Z}_2)$ denotes the second Stiefel-Whitney class of a manifold M.

In [1], Barden constructed the simply connected closed 5-dimensional manifolds X_j and M_k; here j is either an integer greater than -2 or the symbol ∞, and k is either an integer greater than 1 or the symbol ∞. The following results hold for these 5-dimensional manifolds X_j and M_k.

PROPOSITION 2.10 ([1]). *We have the following:*
(1) $X_0 = S^5$, $M_\infty = S^3 \times S^2$.
(2) X_∞ *is the non-trivial* S^3*-bundle over* S^2.
(3) $H_2(M_\infty) \cong H_2(X_\infty) \cong \mathbb{Z}$, $H_2(X_0) = 0$, $H_2(X_{-1}) \cong \mathbb{Z}_2$.
(4) *For* $2 \leq k < \infty$, $H_2(M_k) \cong \mathbb{Z}_k \oplus \mathbb{Z}_k$.
(5) *For* $1 \leq j < \infty$, $H_2(X_j) \cong \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$.
(6) $w_2(X_0) = 0$.
(7) *For all* k, $w_2(M_k) = 0$.
(8) *For* $j \neq 0$, $w_2(X_j) \neq 0$.

THEOREM 2.11 ([1]). *Let M be a simply connected closed 5-dimensional manifold. Then M is diffeomorphic to one of the following manifolds:*

$$X_j \sharp M_{k_1} \sharp M_{k_2} \sharp \cdots \sharp M_{k_s}, \tag{2.3}$$

where $-1 \leq j \leq \infty$, $s \geq 0$, $2 \leq k_i \leq \infty$, and $k_{i+1} = \infty$ or k_i divides k_{i+1} when $i = 1, 2, \ldots, s-1$. When $s = 0$, we think of the part $M_{k_1} \sharp M_{k_2} \sharp \cdots \sharp M_{k_s}$ as S^5. Furthermore, this representation is unique, that is, if

$$X_j \sharp M_{k_1} \sharp M_{k_2} \sharp \cdots \sharp M_{k_s} \cong X_\ell \sharp M_{m_1} \sharp M_{m_2} \sharp \cdots \sharp M_{m_t},$$

then we have $j = \ell$, $s = t$, $k_1 = m_1$, $k_2 = m_2, \ldots, k_s = m_s$.

When M is a simply connected closed 5-dimensional manifold, M is stably parallelizable if and only if $w_2(M) = 0$ (see [9]). Therefore, Theorem 2.11 and Proposition 2.10 lead to the following corollary.

Corollary 2.12. *Let M be a closed simply connected 5-dimensional manifold. Then $H_2(M)$ is free abelian if and only if M is diffeomorphic either to S^5 or to a connected sum of S^3-bundles over S^2. Furthermore, M is stably parallelizable if and only if M is diffeomorphic to $M_{k_1}\sharp M_{k_2}\sharp\cdots\sharp M_{k_s}$ for some $s\geq 0$ and $k_i\geq 2$.*

3. MAIN THEOREM

We need the following to prove our main theorem.

Lemma 3.1. *Let W be a compact connected $(k+2)$-dimensional manifold with non-empty boundary, $k\geq 1$, $H_1(W)=0$, and let $F_1, F_2, \ldots, F_\ell$ be the connected components of the boundary ∂W. Then the homology group $H_{k+1}(W)$ is free abelian of rank $\ell-1$ and it has the homology classes $[F_i]$ represented by F_i $(i=1,2,\ldots,\ell-1)$ as a basis. Furthermore, the group $H_k(W)$ is also free abelian.*

Proof. Since $H_1(W)=0$, W is orientable. By Poincaré-Lefschetz duality and the universal coefficient theorem, we have

$$\begin{aligned} H_{k+1}(W,\partial W) &\cong H^1(W) \\ &\cong \mathrm{Hom}(H_1(W),\mathbb{Z})\oplus \mathrm{Ext}(H_0(W);\mathbb{Z}) \\ &= 0. \end{aligned}$$

Consider the homology exact sequence

$$H_{k+2}(W,\partial W)\xrightarrow{\partial} H_{k+1}(\partial W)\to H_{k+1}(W)\to 0$$

for the pair $(W,\partial W)$. If we give ∂W the orientation induced by an orientation of W, then we have

$$\partial[W,\partial W]=[F_1]+[F_2]+\cdots+[F_\ell],$$

where $[W,\partial W]\in H_{k+2}(W,\partial W)$ is the fundamental class of $(W,\partial W)$. So we have

$$\begin{aligned} H_{k+1}(W) &\cong H_{k+1}(\partial W)/\mathrm{Im}\,\partial \\ &\cong \mathbb{Z}^{\ell-1}, \end{aligned}$$

and we can take $[F_1]$, $[F_2]$, $\ldots$, $[F_{\ell-1}]$ as a basis for $H_{k+1}(W)$. Now consider the homology exact sequence

$$H_1(W)\to H_1(W,\partial W)\to \tilde{H}_0(\partial W)\to \tilde{H}_0(W)$$

for the pair $(W,\partial W)$. Since the both ends are zero, $H_1(W,\partial W)\cong \tilde{H}_0(\partial W)$ is free abelian. Therefore, we have

$$H_k(W)\cong H^2(W,\partial W)$$

$$\cong \operatorname{Hom}(H_2(W, \partial W), \mathbb{Z}) \oplus \operatorname{Ext}(H_1(W, \partial W); \mathbb{Z})$$
$$\cong \operatorname{Hom}(H_2(W, \partial W), \mathbb{Z}),$$

and $H_k(W)$ is also free abelian. □

LEMMA 3.2. *Let M be a connected closed $(2k+1)$-dimensional manifold, $k \geq 2$, with $H_1(M) = 0$. If $f : M \to \mathbb{R}^{k+2}$ is a special generic map, then we have*

$$H_k(M) \cong H^{k+1}(W_f) \oplus H_k(W_f),$$

and $H_k(M)$ is a free abelian group of rank $b + c - 1$, where b is the number of connected components of $S(f)$ and c is the k-th Betti number of W_f.

The above lemma implies that there exists no special generic map $f : M \to \mathbb{R}^{k+2}$ if $H_k(M)$ has a non-trivial torsion element.

Proof. By Theorem 2.7, there exists a topological D^k-bundle $\pi : E \to W_f$ over W_f such that M is homeomorphic to ∂E. Note that W_f is homotopy equivalent to E, and that E is a connected compact orientable $(2k+2)$-dimensional topological manifold with boundary. We take an orientation on E and give ∂E the orientation induced from E.

By assumption, we have $2k \geq k+2$; since W_f is a $(k+2)$-dimensional manifold with non-empty boundary, by Poincaré-Lefschetz duality, we have

$$H_2(E, \partial E) \cong H^{2k}(E) \cong H^{2k}(W_f) = 0,$$
$$H_1(E, \partial E) \cong H^{2k+1}(E) \cong H^{2k+1}(W_f) = 0.$$

Therefore, for the homology exact sequence

$$H_2(E, \partial E) \to H_1(\partial E) \to H_1(E) \to H_1(E, \partial E)$$

associated with the pair $(E, \partial E)$, the both ends are 0, and we get

$$H_1(W_f) \cong H_1(E) \cong H_1(\partial E) \cong H_1(M) = 0.$$

Therefore, by Lemma 3.1, $H_{k+1}(W_f)$ is a free abelian group of rank $\ell - 1$ with a basis consisting of $[F_j] \in H_{k+1}(W_f)$, $j = 1, 2, \ldots, \ell - 1$, and $H_k(W_f)$ is also free abelian, where $F_1, F_2, \ldots, F_\ell$ are the connected components of ∂W_f.

Let $i^* : H^{k+1}(E, \partial E) \to H^{k+1}(E)$ be the homomorphism induced by the inclusion $i : (E, \emptyset) \to (E, \partial E)$ and let $h : H^{k+1}(E) \to \operatorname{Hom}(H_{k+1}(E), \mathbb{Z})$ be the homomorphism defined by $h(\alpha)(\beta) = \langle \alpha, \beta \rangle$, where $\langle \alpha, \beta \rangle \in \mathbb{Z}$ is the Kronecker product of $\alpha \in H^{k+1}(E)$ and $\beta \in H_{k+1}(E)$. Since $\operatorname{Ext}(H_k(E), \mathrm{Z}) \cong \operatorname{Ext}(H_k(W_f), \mathrm{Z}) = 0$, by the universal coefficient theorem, we have that h is an isomorphism. Let $D : H^{k+1}(E, \partial E) \to H_{k+1}(E)$ be the Poincaré-Lefschetz duality isomorphism defined by $D(\alpha) = \alpha \cap [E, \partial E]$, where $[E, \partial E] \in H_{2k+2}(E, \partial E)$ is the fundamental class of $(E, \partial E)$ and "$\cap$" is the cap product. Furthermore, let $D^* : \operatorname{Hom}(H_{k+1}(E), \mathbb{Z}) \to \operatorname{Hom}(H^{k+1}(E, \partial E); \mathbb{Z})$ be the isomorphism given by

$\varphi \mapsto \varphi \circ D$. Let us show that i^* vanishes. It suffices to show that the composition $\Psi = D^* \circ h \circ i^*$ vanishes, since D^* and h are isomorphisms. The homomorphism Ψ sends $\alpha \in H^{k+1}(E, \partial E)$ to the homomorphism $H^{k+1}(E, \partial E) \to \mathbb{Z}$ given by

$$\beta \mapsto \langle i^*(\alpha), \beta \cap [E, \partial E] \rangle = \langle i^*(\alpha) \cup \beta, [E, \partial E] \rangle,$$

where "$\cup$" is the cup product. We can consider the space W_f as the "zero section" of the D^k-bundle E, and then we see that the intersection number of F_j and F_k always vanishes for each j and k. On the other hand, if A and B are oriented $(k+1)$-dimensional closed submanifolds of E which intersect transversely at finitely many points, then the intersection number $A \cdot B$ of A and B in E coincides with $\langle [A]^* \cup [B]^*, [E, \partial E] \rangle$, where $[A]^*$ and $[B]^*$ are the Poincaré-Lefschetz duals of the homology classes represented by A and B, respectively (see, for example, Theorem 11.9 of Chapter VI in [2]). Therefore, for each j and k, we have

$$\Psi([F_j]^*)([F_k]^*) = \langle i^*[F_j]^* \cup [F_k]^*, [E, \partial E] \rangle = F_j \cdot F_k = 0.$$

Since $H^{k+1}(E, \partial E)$ is generated by $[F_j]^*$, $j = 1, 2, \ldots, \ell - 1$, we see that Ψ vanishes, and consequently so does i^*.

We consider the cohomology exact sequence

$$H^{k+1}(E, \partial E) \xrightarrow{i^*} H^{k+1}(E) \to H^{k+1}(\partial E) \to H^{k+2}(E, \partial E) \to H^{k+2}(E).$$

By Poincaré-Lefschetz duality, we have

$$\begin{gathered} H^{k+1}(E) \cong H^{k+1}(W_f), \\ H^{k+1}(\partial E) \cong H^{k+1}(M) \cong H_k(M), \\ H^{k+2}(E, \partial E) \cong H_k(E) \cong H_k(W_f), \\ H^{k+2}(E) \cong H^{k+2}(W_f) = 0. \end{gathered}$$

Therefore, since i^* vanishes, we get the short exact sequence

$$0 \to H^{k+1}(W_f) \to H_k(M) \to H_k(W_f) \to 0.$$

Since $H_k(W_f)$ is free abelian, the above sequence splits, so we have

$$H_k(M) \cong H^{k+1}(W_f) \oplus H_k(W_f).$$

By using an argument in the proof of Lemma 3.1, we have

$$H^{k+1}(W_f) \cong H_1(W_f, \partial W_f) \cong \tilde{H}_0(\partial W_f),$$

which implies that $H^{k+1}(W_f)$ is a free abelian group of rank $b - 1$, since ∂W_f is diffeomorphic to $S(f)$. Therefore, $H_k(M)$ is a free abelian group of rank $b + c - 1$, since $H_k(W_f)$ is free abelian of rank c, which is the k-th Betti number of W_f. This completes the proof of Lemma 3.2. $\square$

Now, we give the complete list of the diffeomorphism types of simply connected closed 5-dimensional manifolds which admit special generic maps, by using Lemmas 3.1 and 3.2.

THEOREM 3.3. *Let M be a closed simply connected 5-dimensional manifold, and $1 \leq p \leq 5$. Then there exists a special generic map $f : M \to \mathbb{R}^p$ if and only if one of the following conditions holds:*

(1) $M \cong S^5$, and $p = 1, 2$,

(2) $M \cong \sharp^k M_\infty \sharp (\sharp^\ell X_\infty)$ for some $k \geq 0$ and $\ell \geq 0$, and $p = 3, 4$,

(3) $M \cong M_{k_1} \sharp M_{k_2} \sharp \cdots \sharp M_{k_s}$ for some $s \geq 0$ and $k_i \geq 2$, and $p = 5$.

Proof. Suppose that there exists a special generic map $f : M \to \mathbb{R}^p$.

When $p < 5$, by Theorem 2.7, there exists a topological D^{6-p}-bundle E over W_f such that E is a compact orientable 6-dimensional manifold with non-empty boundary, and M is homeomorphic to ∂E. Note that E is homotopy equivalent to W_f and W_f is a compact p-dimensional manifold with non-empty boundary.

When $p \leq 3$, for $i = 1, 2, 3$, since $6 - i \geq p$, we have

$$H_i(E, \partial E) \cong H^{6-i}(E) \cong H^{6-i}(W_f) = 0.$$

In the homology exact sequence

$$H_3(E, \partial E) \to H_2(\partial E) \to H_2(E) \to H_2(E, \partial E)$$

for the pair $(E, \partial E)$, since the both ends are zero, we have

$$H_2(M) \cong H_2(\partial E) \cong H_2(E) \cong H_2(W_f).$$

When $p = 1, 2$, since $H_2(W_f) = 0$, we have $H_2(M) = 0$. By Corollary 2.12 and Proposition 2.10, we have $M \cong S^5$.

When $p = 3$, in the homology exact sequence

$$H_2(E, \partial E) \to H_1(\partial E) \to H_1(E) \to H_1(E, \partial E)$$

for the pair $(E, \partial E)$, since the both ends are zero, we have

$$H_1(W_f) \cong H_1(E) \cong H_1(\partial E) = H_1(M) = 0.$$

By Lemma 3.1, the group $H_2(W_f)$ is free abelian, and hence so is $H_2(M)$. Then by Corollary 2.12, M is diffeomorphic either to S^5 or to a connected sum of S^3-bundles over S^2.

When $p = 4$, By Lemma 3.2, $H_2(M)$ is free abelian, therefore we have the same conclusion as that for $p = 3$.

When $p = 5$, by Theorem 2.8, M is stably parallelizable. By Corollary 2.12, M is a connected sum of some M_k's.

Conversely, suppose that M and p are a 5-dimensional manifold and an integer, respectively, as in (1), (2) or (3). Then we can construct a special generic map of M into $\mathbb{R}^p$ by using Examples 2.1 and 2.2, Lemma 2.3, Theorem 2.8 and Proposition 2.9. □

Let M be a simply connected closed 5-dimensional manifold. By Theorem 2.11, we have

$$M \cong X_j \sharp (M_{k_1} \sharp M_{k_2} \sharp \cdots \sharp M_{k_t}) \sharp (\sharp^u (S^3 \times S^2)),$$

where $-1 \le j \le \infty$, $t \ge 0$, $u \ge 0$, $2 \le k_i < \infty$ and k_i divides k_{i+1}. Then we put $j(M) = j$, $t(M) = t$, and $u(M) = u$. Note that $j(M)$ is either an integer greater than -2 or ∞, and $t(M)$ and $u(M)$ are non-negative integers. Now, we put

$$\mathcal{S}(M) = \{1 \le p \le 5 \mid M \text{ admits a special generic map into } \mathbb{R}^p\}.$$

Then Theorem 3.3 means that if M is a simply connected closed 5-dimensional manifold, then the following holds:

(1) $1 \in \mathcal{S}(M) \Leftrightarrow 2 \in \mathcal{S}(M) \Leftrightarrow j(M) = t(M) = u(M) = 0$,
(2) $3 \in \mathcal{S}(M) \Leftrightarrow 4 \in \mathcal{S}(M) \Leftrightarrow j(M) \in \{0, \infty\}$, $t(M) = 0$, and
(3) $5 \in \mathcal{S}(M) \Leftrightarrow j(M) = 0$.

In [1], Barden proved the following (see Lemma 2.4, Corollary 2.4.1 and Theorem 2.5 in [1]). In the following, for a smooth manifold M and a positive integer p, "$M \hookrightarrow \mathbb{R}^p$" means that M can be embedded in $\mathbb{R}^p$, and "$M \looparrowright \mathbb{R}^p$" means that M can be immersed in $\mathbb{R}^p$.

PROPOSITION 3.4. *Let M be a simply connected closed 5-dimensional manifold. Then we have the following.*

(1) *We always have $M \hookrightarrow \mathbb{R}^9$ and $M \looparrowright \mathbb{R}^8$.*
(2) $M \hookrightarrow \mathbb{R}^8 \Leftrightarrow M \looparrowright \mathbb{R}^7 \Leftrightarrow j(M) \in \{0, \infty\}$.
(3) $M \hookrightarrow \mathbb{R}^7 \Leftrightarrow M \hookrightarrow \mathbb{R}^6 \Leftrightarrow M \looparrowright \mathbb{R}^6 \Leftrightarrow j(M) = 0$.

By the above proposition and Theorem 3.3, we have the following.

COROLLARY 3.5. *Let M be a simply connected closed 5-dimensional manifold. Then we have the following.*

(1) $1 \in \mathcal{S}(M) \Leftrightarrow 2 \in \mathcal{S}(M) \Rightarrow M \hookrightarrow \mathbb{R}^6$,
(2) $3 \in \mathcal{S}(M) \Leftrightarrow 4 \in \mathcal{S}(M) \Rightarrow M \hookrightarrow \mathbb{R}^8 \Leftrightarrow M \looparrowright \mathbb{R}^7$,
(3) $5 \in \mathcal{S}(M) \Leftrightarrow M \hookrightarrow \mathbb{R}^7 \Leftrightarrow M \hookrightarrow \mathbb{R}^6 \Leftrightarrow M \looparrowright \mathbb{R}^6$.

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