UNIFORM RADIUS AND EQUISINGULARITY

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In the present paper, we study a generalization of uniform radius of family of map germs with isolated singularity and relate this notion to equisingularity conditions of the canonical stratification associated to this family..

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1. INTRODUCTION

We look at deformations of germs of isolated singularities from $\mathbb{K}^n$ to $\mathbb{K}^k$ $(n \geqslant k)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and the relation with their natural stratification in some tame category (algebraic, analytic, semi-algebraic, subanalytic, polynomially bounded *o*-minimal structure (see [18, 20]). The word tame in what follows will refer to one of these categories.

We say that two map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ are topologically right equivalent if there exists a germ of homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $g \circ h = f$. Then the topological type of a germ represents its right equivalence class.

A family of map germs is the germ at $\{0\} \times \mathbb{R}^p$ of some continuous map $F : (\mathbb{R}^n \times \mathbb{R}^p, \{0\} \times \mathbb{R}^p) \to (\mathbb{R}^k, 0)$. We will usually denote a family of map germs by $f_t := (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ $t \in \mathbb{R}^p$ where $f_t(x) = F(t, x)$. (Notice that a family of map germs is more than just a germ $f_t := (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ for each $t \in \mathbb{R}^p$.) A family of map germs $F : (\mathbb{R}^n \times \mathbb{R}^p, \{0\} \times \mathbb{R}^p \to (\mathbb{R}^k, 0)$ is said to have no coalescing of critical points if there is a neighborhood U of $\{0\} \times \mathbb{R}^p$ in $\mathbb{R}^n \times \mathbb{R}^p$ and a representative F' of F so that F' restricted to $U \cap \{t\} \times (\mathbb{R}^n - \{0\})$ is a submersion for each $t \in \mathbb{R}^p$. Otherwise the family is said to coalesce.

Example 1.1. Consider the family $f_t : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, $f_t(x) = x^3 - 3tx^2$.

The points where f_t fails to be a submersion are $x = 0$ and $x = 2t$. Hence, f_t coalesces since any neighborhood of $\{0\} \times \mathbb{R}$ in $\mathbb{R} \times \mathbb{R}$ contains points of the line $x = 2t$ with $x \neq 0$.

Notice that $f_0 = x^3$ is topologically right equivalent to the identity but if $t \neq 0$ then f_t has either a maximum or a minimum at 0 so f_t could not possibly be topologically right equivalent to the identity.

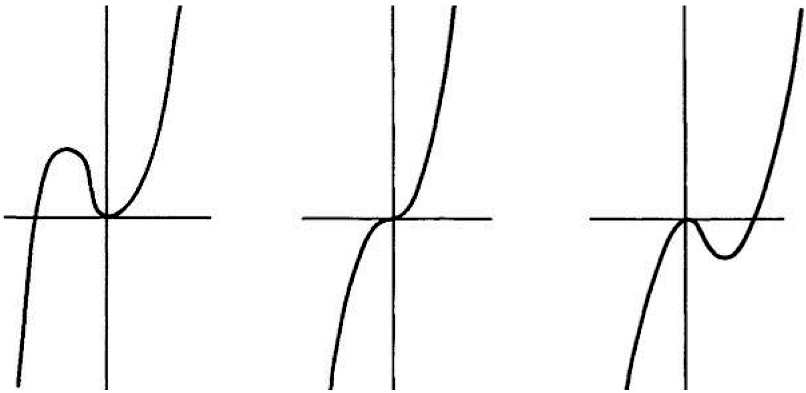


Fig. 1. Coalascing of critical points from $t < 0$ to $t > 0$.

Example 1.2. Let $f_t : \mathbb{K} \to \mathbb{K}$, $t \in \mathbb{K}$, $f_t(x) = x^2(x^4 + t^2)$.

$\{f_t\}$ has no coalescing if $\mathbb{K} = \mathbb{R}$ but coalesces if $\mathbb{K} = \mathbb{C}$.

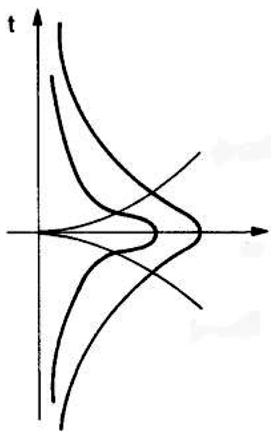
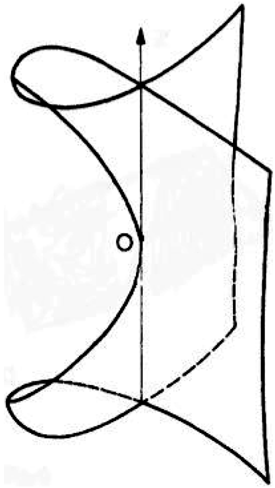


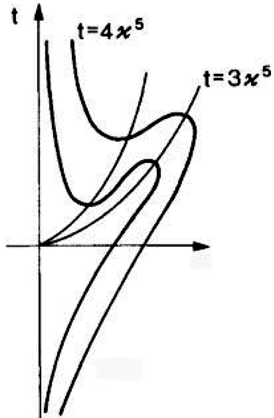
Fig. 2. No coalascing in $\mathbb{R}$, coalesces in $\mathbb{C}$.

Example 1.3. Let $f_t : \mathbb{R}^2 \to \mathbb{R}$, $t \in \mathbb{R}$, $f_t(x, y) = y^2 - t^2x^2 - x^3$.

The points where f_t fails to be a submersion are $y = x = 0$ and $y = 3x + 2t^2 = 0$. Hence, $\{f_t\}$ coalesces since any neighborhood of $\{0\} \times \mathbb{R}$ in $\mathbb{R}^2 \times \mathbb{R}$ contains points of the curve $x = -\frac{2}{3}t^2$ with $t \neq 0$.



Example 1.4 (T. Kuo [10]). Let $f_t:\mathbb{R}\to\mathbb{R}$, $t\in\mathbb{R}$, $f_t(x)=x^2t^2-2x^7t+2x^{12}$.



The points where $\{f_t\}$ fails to be a submersion are on the points of the curves $t=4x^5$ and $t=3x^5$. Hence, $\{f_t\}$ coalesces.

Notice that, along each level curve $x^2t^2-2x^7t+2x^{12}=c$, $c\neq 0$, the values of t reach a local minimum on $t=4x^5$ and a local maximum on $t=3x^5$. $\{f_t\}$ is not topologically trivial along the t-axis.

Suppose however that $f_t:(\mathbb{R}^n,0)\to(\mathbb{R}^k,0)$, $t\in\mathbb{R}^p$ is a family of map germs with no coalescing of critical points. Can the topological types of the f_t change? In the complex case there is almost a complete result. From the non-splitting of vanishing cycle (see [7, 11, 12]), no coalescing for family of complex analytic germs is equivalent to the constancy of Milnor's number [15]; Lê and Ramanujam and combined with H.King have shown that if $f_t:(\mathbb{C}^n,0)\to(\mathbb{C},0)$

is a family of germs of analytic functions with no coalescing and $n \neq 3$ then the topological type of each f_t is the same. (The case $n = 3$ is unknown).

The corresponding real result is true for $n \leqslant 3$, (see [3, 8]), but it is far from true in large dimensions. In particular H. King gave an example for each $n > 5$ of a family of polynomial germs $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, with no coalescing of critical points so that for each $t \neq 0$, f_t is not topologically right equivalent to f_0. Thus, in the real case it is possible to have no coalescing and changing topological type, even for polynomials. However, this is a rather subtle phenomenon when it does occur.

THEOREM 1.5 (H. King [8]). *Let $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $t \in \mathbb{R}^p$ be a family of map germs with no coalescing of critical points and suppose there is a family of homeomorphism germs $g_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ $t \in \mathbb{R}^p$ so that the germ at 0 of each set $g_t \circ f_t^{-1}(0)$ is the germ of $f_0^{-1}(0)$. Then there is a family of homeomorphism germs $h_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ $t \in \mathbb{R}^p$ and a neighborhood V of 0 in $\mathbb{R}^p$ so that the germ at 0 of $f_t \circ h_t$ is the germ of f_0 for each $t \in V$.*

THEOREM 1.6 (Lê-Ramanujam [13], H. King [8]). *Suppose $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $t \in \mathbb{R}$ is a continuous family of complex analytic germs with no coalescing and $n \neq 3$. Then there is a continuous family of germs of homeomorphisms $h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ so that $f_0 = f_t \circ h_t$, for all $t \in \mathbb{C}^p$.*

2. UNIFORM RADIUS

Let $\rho : (\mathbb{R}^n, 0) \to (\mathbb{R}^+, 0)$ be a smooth and tame function (eg. a positive polynomial function) such that $\rho^{-1}(0) = \{0\}$.

An important example is given by the square of the distance function $\rho(x) = |x_1|^2 + \ldots + |x_n|^2$.

We can see, by using the curve selection lemma, that such control function is submersive in a neighborhood of 0.

Definition 2.1. Let $\rho : (\mathbb{K}^n, 0) \to \mathbb{R}^+$ be a germ of a tame submersion such that $\rho^{-1}(0) = \{0\}$.

For a smooth tame map germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ with an isolated critical point at 0, define $\rho(f)$, the ρ-radius of f, to be the smallest critical value of the "control function" $\rho(x)$ restricted to the smooth manifold $f^{-1}(0) - \{0\}$. If there are no critical values then $\rho(f) = \infty$.

Remark 2.2. If $k = 1$, $\rho(f) = \inf_{x \in C} \rho(x)$ where $C = \{x \in f^{-1}(0) - \{0\} \mid d_x f = \lambda d_x \bar{\rho}\}$.

Here we use the following notation: $df = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial t})$, $d_x f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ and $\|d_x f\|^2 = \sum_{i=1}^n \|\frac{\partial f}{\partial x_i}\|^2$.

Definition 2.3. We say that a family of tame map germs at $\{0\} \times \mathbb{K}^p$ of some continuous map $\{f_t\} : (\mathbb{K}^n \times \mathbb{K}^p, \{0\} \times \mathbb{K}^p) \to (\mathbb{K}^k, 0)$, has a ρ-uniform radius if there is a $\delta > 0$ so that $\rho(f_t) > \delta$ for all $t \in \mathbb{K}^p$.

The notion of uniform radius, used in this paper is a generalization of the "uniform Milnor radius", where ρ, is the squared distance function. See for example [9], where the uniform Milnor radius takes an important role in the proof of the blow-Nash triviality for a family of Nash set germs.

THEOREM 2.4. *Suppose* $f_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$, $t \in \mathbb{R}^p$ *is a smooth family of tame map germs with no coalescing and has a ρ-uniform radius. Then there is a continuous family of homeomorphism germs* $h_t : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ *so that* $f_0 = f_t \circ h_t$, *for all* $t \in \mathbb{R}^p$ *i.e. the family is topologically trivial.*

Proof. Let us set for $r > 0$, $S(r) := \{x \in \mathbb{K}^n : \rho(x) = r\}$ and $U(r) = \{x \in \mathbb{K}^n : \rho(x) < r\}$.

Let $F : (\mathbb{K}^n \times \mathbb{R}^p, \{0\} \times \mathbb{R}^p) \to (\mathbb{K}^k, 0)$, be a representative of $\{f_t\}$, $F(x,t) = f_t(x)$ and $G = (F, \rho)$.

By the assumption on $\{f_t\}$, there exists r_0 such that for $0 < r \leqslant r_0$, $(0, r)$ is a regular value of the restriction of G to $(U(r) - \{0\}) \times \mathbb{R}^p$ then for each $0 < r \leqslant r_0$, $S(r)$ is a smooth manifold diffeomorphic to the $n-1$ unit sphere, $G^{-1}(0, r) = F^{-1}(0) \cap (S(r) \times \mathbb{R}^p)$ is a smooth manifold and the restriction of the projection on the second factor, $\pi : G^{-1}(0, r) \to \mathbb{R}^p$ has no critical points.

Let $\Sigma(F, r) :=$ the set of singular points of the restriction of F to $S(r) \times \mathbb{R}^p$. Let $d(r)$ be a continuous function on $]0, r_0]$ such that

$$0 < d(r) < \text{ the distance of } \Sigma(F, r) \text{ to } F^{-1}(0) \cap (S(r) \times \mathbb{R}^p).$$

Such function exists because for any $0 < r \leqslant r_0$,

$$\Sigma(F, r) \cap F^{-1}(0) \cap (S(r) \times \mathbb{R}^p) = \varnothing.$$

Let

$$U_1 = \left\{(x,t) : 0 < \rho(x) < r_0,\ dist((x,t), \Sigma(F, \rho(x))) < \frac{d(\rho(x))}{2}\right\},$$

$$U_2 = \left\{(x,t) : 0 < \rho(x) < r_0,\ dist((x,t), \Sigma(F, \rho(x))) > \frac{d(\rho(x))}{2}\right\}.$$

Then $\{U_1, U_2\}$ is an open covering of $(U(r_0) - \{0\}) \times \mathbb{R}^p$. Since F has no coalescing of critical points, we may suppose that the restriction of (F, π) to $(U(r_0) - \{0\}) \times \mathbb{R}^p$ is a submersion on $\mathbb{K}^k \times \mathbb{R}^p$. Let ξ_1^i, $1 \leqslant i \leqslant p$, be a smooth lift by (F, π) of the vector field $\frac{\partial}{\partial t_i}$.

Since U_2 is disjoint from the singular points of G, $\Sigma(G)$, and ρ does not depend on the parameter t, then the restriction of (G, π) to

$(U(r_0) - \{0\}) \times \mathbb{R}^p - \Sigma(G)$ is a submersion, and hence on U_2. Let $\xi_2^i, 1 \leqslant i \leqslant p$, be a smooth lift by (G, π) of the vector field $\frac{\partial}{\partial t_i}$.

Now, let $\{\varphi_1, \varphi_2\}$ be a partition of unity associated to the covering $\{U_1, U_2\}$. We define, for $1 \leqslant i \leqslant p$, the vector field ξ^i on $(U(r_0) - \{0\}) \times \mathbb{R}^p$, by

$$\xi^i = \varphi_1 \xi_1^i + \varphi_2 \xi_2^i.$$

By construction ξ^i is smooth tangent to the level surface of F, and controlled by ρ. We extend, for $1 \leqslant i \leqslant p$, ξ^i to $U(r_0) \times \mathbb{R}^p$, by $\xi^i(0,t) = \frac{\partial}{\partial t_i}$. By construction, ξ^i is controlled and locally integrable, then integrable (see [6]). For $(x,t) \in U(r_0) \times \mathbb{R}^p$, let $\phi^i_{(x,t)}(s)$ be the integral curve of ξ^i satisfying the initial condition $\phi^i_{(x,t)}(t) = (x,t)$. For small $0 < \tilde{r} < r_0$, we define a homeomorphism $h^i : U_{\tilde{r}} \times \mathbb{R}^p \to U$ by $h^i(x,t) = \phi^i_{(x,0)}(t)$. Let $h(x,t) = h^1(h^2(\ldots(h^p(x,t),t))\ldots)$. Then h is a homeomorphism and we have $F \circ h(x,t) = f_0(x)$. □

Remark 2.5. We may suppose in theorem 2.4, that the family is only continuous with respect to the parameter, by applying theorem 1 of H. King [8].

2.1. **Sufficient conditions for uniform radius and no coalescing**

Definition 2.6. Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ $(n \geqslant k)$ be a smooth tame map germ. Let $\rho : (\mathbb{K}^n, 0) \to \mathbb{R}^+$ be a germ of a tame submersion such that $\rho^{-1}(0) = \{0\}$.

We denote by $J_\rho(f)$ the Fukuda's ideal associated to (f, ρ) *i.e.* the jacobian ideal of the map germ (f, ρ) generated by $f_1, \ldots, f_p$ and $(k+1) \times (k+1)$ minors of the jacobian matrix

$$\frac{D(f,\rho)}{D(x_1,\ldots,x_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \\ \frac{\partial \rho}{\partial x_1} & \cdots & \frac{\partial \rho}{\partial x_n} \end{pmatrix}.$$

THEOREM 2.7 (T. Fukuda [4]). *Suppose $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}^p$ is a continuous family of analytic function germs and let $\rho : (\mathbb{K}^n, 0) \to \mathbb{R}^+$ be a germ of an analytic submersion in a punctured neighborhood of* $0 \in \mathbb{R}^n$ *and such that $\rho^{-1}(0) = \{0\}$.*

Suppose that:

$$\begin{cases} \mu(f_t) = \mu(f_0) < \infty, \quad t \in \mathbb{R}^p \text{ and} \\ \dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_t)} = \dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_0)} < \infty \qquad t \in \mathbb{R}^p \end{cases}$$

Then the family $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}^p$ is with no coalescing and a ρ-uniform radius, therefore it is topologically trivial.

Proof. The condition $\mu(f_t) = \mu(f_0) < \infty$, $t \in \mathbb{R}^p$ implies that $\{f_t\}$ is with no coalescing and $\dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_t)} = \dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_0)} < \infty$, $t \in \mathbb{R}^p$, implies that $\{f_t\}$ has ρ-uniform radius. □

THEOREM 2.8 (T. Fukuda [4]). *Suppose $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $t \in \mathbb{R}^p$ $(n > k)$ is a continuous family of analytic map germs and let $\rho : (\mathbb{K}^n, 0) \to \mathbb{R}^+$ be a germ of an analytic submersion such that $\rho^{-1}(0) = \{0\}$. Suppose that:*

$$\dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_t)} = \dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_0)} < \infty \qquad t \in \mathbb{R}^p$$

Then the family of zero sets $\{f_t^{-1}(0)\}_{t \in \mathbb{R}^p}$ is topologically trivial.

Proof. In fact, the conditon $\dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_t)} = \dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_0)} < \infty$ for $t \in \mathbb{R}^p$, implies that there exists a neighborhood U of 0 in $\mathbb{R}^n$ such that $U \cap \{f_t = J_\rho(f_t) = 0\} = \{0\}$ for all t, *i.e.* $\{f_t\}$ has ρ-uniform radius. □

Let us fix a vector $w = (w_1, \ldots, w_n) \in \mathbb{N}^n - \{0\}$. We will usually refer to w as the *vector of weights.* Let $h \in \mathcal{O}_n$, $h \neq 0$, the *degree of h with respect to w*, or *w-degree* of h, is defined as

$$d_w(h) = \min\{\langle k, w\rangle : k \in \operatorname{supp}(h)\},$$

where $\langle\,,\rangle$ stands for the usual scalar product. In particular, if $x_1, \ldots, x_n$ denote a system of coordinates in $\mathbb{K}^n$ and $x_1^{k_1} \ldots x_n^{k_n}$ is a monomial in $\mathcal{O}_n$, then $d_w(x_1^{k_1} \ldots x_n^{k_n}) = w_1 k_1 + \cdots + w_n k_n$. By convention, we set $d_w(0) = +\infty$. If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ is the Taylor expansion of h at the origin, then we define the *principal part of h with respect to w* as the polynomial given by the sum of those terms $a_k x^k$ such that $\langle k, w\rangle = d_w(h)$. We denote this polynomial by $p_w(h)$.

Definition 2.9. We say that a function $h \in \mathcal{O}_n$ is *weighted homogeneous of degree d with respect to w* (or of type $(w; d)$) if $\langle k, w\rangle = d$, for all $k \in \operatorname{supp}(h)$. The function h is said to be *semi-weighted homogeneous of degree d with respect to w* when $p_w(h)$ has an isolated singularity at the origin.

It is well-known, see [1], that if h is a semi-weighted homogeneous function, then h has an isolated singularity at the origin and that h and $p_w(h)$ have the same Milnor number given by:

$$\mu(h) = \frac{(d - w_1)(d - w_2) \ldots (d - w_n)}{w_1 . w_2 . \ldots w_n}$$

Let $g = (g_1, \ldots, g_n) : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ be an analytic map germ, and let us denote the map $(p_w(g_1), \ldots, p_w(g_n))$ by $p_w(g)$. The map g is said to be *semi-weighted homogeneous with respect to w* when $(p_w(g))^{-1}(0) = \{0\}$.

Remark 2.10. It is not difficult to see that a family of weighted homogeneous functions with isolated singularity has a ρ-uniform radius with $\rho = \sum_{i=1}^{n} |x_i|^2$. In fact, if $\{f_t\}$ is a weighted homogeneous family of type $(w_1, \ldots, w_n; d)$, the "Euler formula" gives: $\sum_{i=1}^{n} w_i x_i \frac{\partial f_t}{\partial x_i} = d.f_t$. And now if f has a kink at $p = (x, t)$ then there exists $\lambda \in \mathbb{R}^*$ such that $d_x f(p) = \lambda d_x \rho(p)$ *i.e.* $\frac{\partial f_t}{\partial x_i} = \lambda x_i$. Since $p \in f^{-1}(0)$ the Euler formula gives $\lambda \sum_{i=1}^{n} w_i |x_i|^2 = 0$ which implies $p \in \{0\} \times \mathbb{R}^p$ so it is not in the smooth part. Thus, $\rho(f_t) = +\infty$.

It has also ρ-uniform radius, for ρ a weighted homogeneous control function with respect to the system of weights, for example:

$$\rho_w(x_1, \ldots, x_n) = \sum_{i=1}^{n} |x_i|^{\frac{2d}{w_i}}.$$

It is no worth to mention the following paper [5], for reference concerning, singular metric in weighted homogeneous setting.

Proposition 2.11. *Suppose* $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $t \in \mathbb{R}^p$ *is a continuous family of weighted homogeneous polynomial germs with no coalescing. Then there is a continuous family of homeomorphism germs* $h_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *so that* $f_0 = f_t \circ h_t$ *all* $t \in \mathbb{R}^p$.

Proof. In this case $\rho(f_t) = +\infty$, we can apply Theorem 2.4. □

Proposition 2.12. *Let* $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $t \in \mathbb{R}^p$ *be a continuous family and* $\rho : (\mathbb{K}^n, 0) \to \mathbb{R}^+$ *weighted homogeneous of degree* d *with respect to* w.

If $\dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_t)} < \infty$, $t \in \mathbb{R}^p$, *then* $\{f_t^{-1}(0)\}_{t \in \mathbb{R}^p}$ *is topologically trivial.*

Proof. In this case the Fukuda's ideal is weighted homogeneous with the same weights, then $\dim_{\mathbb{R}} \frac{\mathcal{O}_n}{J_\rho(f_t)} < \infty$, for all t, implies that for any neighborhood U of 0 in $\mathbb{R}^n$, $U \cap \{f_t = J_\rho(f_t) = 0\} = \{0\}$ for all t. i.e $\{f_t\}$ has ρ-uniform radius. □

In the semi-weighted homogeneous case, M. Oka show that the family of complexe polynomials $f_t(x, y, z) = z^5 + ty^6 z + (x+y+z)y^7 + (x+y+z)^{15}$ is semi-weighted homogeneous in $\mathbb{C}^3$ with constant Milnor number but do not have a ρ-uniform radius with respect to the standard distance function $\rho(x, y, z) = |x|^2 + |y|^2 + |z|^2$. Writing it this way $f_t(x, y, z) = z^5 + ty^6 z + xy^7 + x^{15} + \{(y + z)y^7 + \sum_{k=1}^{15} \binom{n}{k} x^{15-k} (y + z)^k\}$. We see that f_t is semi-weighted homogeneous of type $(1, 2, 3; 15)$ and with constant Milnor number $\mu(f_t) = 364$.

See [16, page 207], for the proof of the fact that this family has non uniform radius with respect to $\rho(x, y, z) = |x|^2 + |y|^2 + |z|^2$.

Remark 2.13. This family is topologically trivial, since $F(x, y, z, t) = G \circ H(x, y, z, t)$ where H is the automorphism $H(x, y, z, t) = (x + y + z, y, z, t)$

and $G(x,y,z,t) = g_t(x,y,z) = z^5 + ty^6z + xy^7 + x^{15}$ is the Briançon-Speder example (see [2]), which is weighted homogeneous of type $(1,2,3;15)$ with isolated singularity.

But we do have

THEOREM 2.14. *Suppose $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $t \in \mathbb{R}^p$ is a continuous family of semi-weighted homogeneous tame germs with no coalescing. Then there is a continuous family of homeomorphism germs $h_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ so that $f_0 = f_t \circ h_t$ all $t \in \mathbb{R}^p$.*

Proof. Let $(w;d)$ be the type of f_t, then f_t has a ρ_w-uniform radius, with $\rho_w(x_1,\ldots,x_n) = \sum_{i=1}^n |x_i|^{2\frac{d}{w_i}}$. In fact $\rho_w(f_t) = +\infty$ therefore the statement follows from Theorem 2.4. □

3. VANISHING FOLDS, MILNOR NUMBER AND WHITNEY CONDITION

Following D. O'shea [17], we say that a point $p \in f^{-1}(0)$ is a **ρ-kink** of $f^{-1}(0)$ if p is non singular point of f and if p is a critical point of ρ restricted to the manifold of smooth points of $f^{-1}(0)$.

Remark 3.1. For $k = 1$, an easy computation shows that a nonsingular $p \in f^{-1}(0)$ is a ρ-kink if and only if $df(p) = \lambda d_x\bar{\rho}(p)$ for some λ in $\mathbb{K} - \{0\}$.

We suppose that for every $t \in \mathbb{K}^p$, $f_t(0) = 0$ and 0 is an isolated critical point of f_t. Let $\gamma : [0,\epsilon] \to \mathbb{K}^n \times [0,1]$ be a real analytic path $\gamma(s) = (x(s), t(s))$ such that:

1) $\gamma(0) = (0,0)$
2) $|x(s)| > 0$ and $|t(s)| > 0$ for all $0 < s < \epsilon$, and
3) $f(x(s), t(s)) = 0$ for all $0 \leqslant s \leqslant \epsilon$.

Remark 3.2. It is easy (remark 2.10) to see that a weighted homogeneous functions with isolated singularity cannot have a ρ-kink, with $\rho = \sum_{i=1}^n |x_i|^2$.

Definition 3.3. The path γ will be called a **ρ-vanishing fold** of f (centered at 0) if $x(s)$ is a ρ-kink of $f_{t(s)}^{-1}(0)$ for every $s \in (0,\epsilon]$.

Remark 3.4. It is easy to see that: $\{f_t\}$ has a ρ-uniform radius if and only if it has no ρ-vanishing fold in $U(\epsilon_0) = \{x \in \mathbb{K}^n : \rho(x) < \epsilon_0\}$ for some $\epsilon_0 > 0$.

For a family of function germs with isolated singularities

$$F : (\mathbb{K}^n \times \mathbb{K}^p, \{0\} \times \mathbb{K}^p) \to (\mathbb{K}, 0),$$

we associate the canonical stratification of $\mathbb{K}^n \times \mathbb{K}^p$ given by the partition

$$\mathcal{S}_F = \{\mathbb{K}^n \times \mathbb{K}^p \backslash F^{-1}(0), F^{-1}(0)\backslash\{0\} \times \mathbb{K}^p, \{0\} \times \mathbb{K}^p\}$$

We shall denote by π the projection on the second factor, $V = F^{-1}(0)$, $Y = \{0\} \times \mathbb{K}^p$, $X = V - Y$ and $X_t = \{x \in \mathbb{K}^n \,|F(x,t) = 0\}$.

Since X_t has an isolated singularity at $(0,t)$, the critical set of the restriction of π to V is Y.

Then X is a smooth manifold, and for each point $(x,t) \in X$, we have

$$T_{(x,t)}X = \{(u,v) \in \mathbb{C}^n \times \mathbb{C}| \sum_{i=1}^{n} u_i \frac{\partial F}{\partial x_i}(x,t) + \sum_{j=1}^{p} v_j \frac{\partial F}{\partial t_j}(x,t) = 0\} = (\mathbb{R}dF)^{\perp}.$$

Here we use the following notation: $dF = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t})$, $d_xF = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$ and $\|d_xF\|^2 = \sum_{i=1}^{n} \|\frac{\partial F}{\partial x_i}\|^2$.

The property that the canonical stratification associated to a family of germs with isolated singularities be (a) (resp. (b)) regular (see [22]), can be made more practically by using the following form:

Definition 3.5. We say that F is Whitney regular at 0 if its canonical stratification $\mathcal{S}_f$ is Whitney regular and this is equivalent to: the following conditions are satisfied

condition (a):

$$\lim_{\substack{(x,t)\to 0\\(x,t)\in X}} \left(\frac{\frac{\partial F}{\partial t_j}(x,t)}{\|d_xF(x,t)\|} \right) = 0 \qquad \text{for each } 1 \leqslant j \leqslant p.$$

condition (b'):

$$\lim_{\substack{(x,t)\to 0\\(x,t)\in X}} \left(\frac{\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x,t)}{\|x\|\|d_xF(x,t)\|} \right) = 0.$$

Remark 3.6. (1) It is known that condition $a + b'$ is equivalent to Whitney condition (b) see [14, 19, 21].

(2) A Whitney regular family of map germs is topologically trivial, so that the topological type is constant in such a family.

Definition 3.7. We say that F is μ-constant deformation if $\mu(F(.,t)) = \mu(F(.,0))$ for any t.

Remark 3.8. In general ρ-uniform radius for some "control function" ρ for a family of germs is weaker than Whitney regularity. For example, take the Briançon-Speder family $F(x,y,z,t) = z^5 + ty^6z + y^7x + x^{15}$; it is a family of weighted homogeneous polynomials of type $(1,2,3;15)$ and of constant Milnor number $\mu(f_t) = 364$. The family $\{f_t\}$ is μ-constant but for a generic hyperplane

H in $\mathbb{R}^3$, of equation $z = ax + by$ with $a, b \in \mathbb{R} - \{0\}$, $g_t = f_t|_H$ is a family of semi weighted homogeneous polynomials with $\mu(g_t) = 26 < \mu(g_0) = 28$. Since the Milnor number jumps, this family must have a ρ-vanishing fold (see 3.12).

Let F be an analytic function from $\mathbb{K}^n \times \mathbb{K}^p$ to $\mathbb{K}$, in a neighborhood of 0

$$\begin{array}{rccc} F: & \mathbb{K}^n \times \mathbb{K}^p, 0 & \to & \mathbb{K}, 0 \\ & (x,t) & \mapsto & F(x,t) \end{array}$$

with $F(0,t) = 0$. We denote by π the projection on the second factor, $V = F^{-1}(0)$, $Y = \{0\} \times \mathbb{K}^p$ and $X_t = \{x \in \mathbb{K}^n : F(x,t) = 0\}$. We suppose X_t has an isolated singularity at $(0,t)$ *i.e.* the critical set of the restriction of π to V is Y. Then $X = V - Y$ is an analytic manifold of dimension n, and for each point $(x,t) \in X$ we have

$$T_{(x,t)}X = \{(u,v) \in \mathbb{K}^n \times \mathbb{K}^p : \sum_{i=1}^{n} u_i \frac{\partial F}{\partial x_i}(x,t) + \sum_{j=1}^{p} v_j \frac{\partial F}{\partial t}(x,t) = 0\} = \left(\mathbb{K} d\bar{F}\right)^{\perp}.$$

where $dF = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t})$, $d_x F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$.

Let $\mathcal{G}$ be the set of analytic map germs from $\mathbb{K}^n \times \mathbb{K}^p, 0$ to $\mathbb{K}^n \times \mathbb{K}^p, 0$ of the following type: $\Phi(y,\tau) = (\Psi(y,\tau), \lambda(\tau)) = (x,t)$, where Ψ for small τ is a germ of automorphisms of $(\mathbb{K}^n, 0)$ (*i.e.* $\det\left(\frac{\partial \Psi}{\partial y}\right) \neq 0$ and $\Psi(0,\tau) = 0$).

We suppose given $F : \mathbb{K}^n \times \mathbb{K}^p, 0 \to \mathbb{K}, 0$ an analytic deformation of $f = f_0$ such that $F(0,t) = \frac{\partial F}{\partial x_1}(0,t) = \ldots = \frac{\partial F}{\partial x_n}(0,t) = 0$,

$$X = F^{-1}(0) \backslash \{0\} \times \mathbb{K}^p, \;\; X_t = f_t^{-1}(0) \quad \text{and} \quad Y = \{0\} \times \mathbb{K}^p.$$

The following theorem shows that Whitney regularity is equivalent to the stability of the ρ-uniform radius property with respect to families of linear change of variable in x.

THEOREM 3.9. *Let F be a μ-constant deformation. The following conditions are equivalent*

(i) *F is Whitney regular*

(ii) *For any $\Phi \in \mathcal{G}$, $F \circ \Phi$ has no ρ-vanishing fold with respect to the square of the distance function.*

Proof. (i) $\Rightarrow$ (ii)

Since having a vanishing fold $\gamma(s) = (x(s), t(s))$, implies that $\frac{\sum_{i=1}^{n} x_i(s) \frac{\partial F}{\partial x_i}(\gamma(s))}{\|x(s)\| \|d_x F(\gamma(s))\|} = 1$; therefore if a deformation is Whitney regular then it has no vanishing folds. We have then only to show that if F is Whitney then so is $F \circ \Phi$ for all $\Phi \in \mathcal{G}$.

By definition $F \circ \Phi(y,\tau) = F(\Psi(y,\tau), \lambda(\tau))$; this suggests doing it in the following two steps: Firstly, for $\lambda = Id_{\mathbb{K}^p}$, $\Phi_1(y,\tau) = (\Psi(y,\tau), \tau)$, is then an analytic diffeomorphism of $\mathbb{K}^{n+p}$, and since Whitney's conditions are invariant by diffeomorphism, if F is Whitney regular, so is $F \circ \Phi_1$, where $\Phi_1(y,\tau) = (\Psi(y,\tau),\tau)$.

Secondly, if F is Whitney regular, then so is $F \circ \Phi_2$, where $\Phi_2(y,\tau) = (y,\lambda(\tau))$ and $\lambda : \mathbb{K}^p, 0 \to \mathbb{K}^p, 0$.

In fact, the condition (b') is trivially satisfied by $F \circ \Phi$ since it does not make use of the partial derivative relative to the parameter τ.

To check the (a) condition, we compute

$$\frac{\partial F \circ \Phi_2}{\partial t_j}(y,\lambda(\tau)) = \sum_{m=1}^{p} \frac{\partial \lambda_m}{\partial t_j}(\tau) \frac{\partial F}{\partial \lambda_m}(y,\lambda(\tau)).$$

Since F satisfies condition (a), we have

$$\lim_{\substack{(y,\tau)\to 0\\(x,y)\in X-Y}} \left(\frac{\frac{\partial F\circ\Phi_2}{\partial t_j}(y,\tau)}{\|d_x F \circ \Phi_2(y,\tau)\|} \right) = \lim_{\substack{(y,\tau)\to 0\\(x,y)\in X-Y}} \sum_{m=1}^{p} \frac{\partial \lambda_m}{\partial t_j}(\tau) \left(\frac{\frac{\partial F\circ\Phi_2}{\partial t_m}(y,\tau)}{\|d_x F \circ \Phi_2(y,\tau)\|} \right) = 0.$$

Therefore $F \circ \Phi$ satisfies condition (a). Then it follows from these facts that for any $\Phi \in \mathcal{G}$, $F \circ \Phi$ is Whitney regular, so that it has no vanishing folds.

(ii) $\Rightarrow$ (i)

Firstly, since F is a μ-constant deformation in a neighborhood of 0, it satisfies the (a) regularity condition (in fact we have more, μ-constant implies "good stratification" in the sense of Thom.

Let us suppose that (b) fails, which in turn implies that (b') fails, since (a) holds.

Let $\Delta(z,\tau) = \frac{\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(z,\tau)}{\|x\|\|grad_x F(z,\tau)\|}$ where $(z,\tau) \in X - Y$. Then there exists a real analytic curve $\gamma : [0,\epsilon] \to X$, $\gamma(s) = (x(s), t(s))$ and $\delta_0 > 0$ such that:

1) $\gamma(0) = (0,0)$,
2) $f(z(s),\tau(s)) = 0$ for all $0 \leqslant s \leqslant \epsilon$, and
3) $\lim_{s\to 0} \Delta(z(s),\tau(s)) = l \neq 0$.

Let us denote by v the valuation of $\mathcal{O}_{\mathbb{K},0}$ associated to γ.

We will use the following notations:

$$v(z) = \inf_{1\leqslant i\leqslant n} v(z_i) \text{ for } z \in \mathbb{K}^n, \quad v\left(\frac{\partial f}{\partial x}\right) = \inf_{1\leqslant i\leqslant n} v\left(\frac{\partial f}{\partial x_i}\right).$$

In these conditions, if we denote $v(z) = p$ and $v(\frac{\partial f}{\partial x}) = q$, we can suppose (change the order of variables if necessary) that $v(z_1) = p$.

Since $\Delta \circ \gamma(s)$ has a non zero limit when s tends to 0, we may conclude that $v(< z, \overline{\frac{\partial f}{\partial x}}(z,\tau) >) = v(z) + v(\frac{\partial f}{\partial x}(z,\tau)) = p + q$.

Let us denote $\gamma(s) = (p_1(s), \dots, p_n(s), \lambda(s))$, $\overline{\frac{\partial f}{\partial x}} \circ \gamma(s) = (q_1(s), \dots, q_n(s))$ and $v(< z, \overline{\frac{\partial f}{\partial x}} > \circ \gamma(s)) = u(s)$.

We now define $\Phi : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^n \times \mathbb{K}, 0)$ by:

$$\Phi(y_1, \dots, y_n, s) = (\Psi(y, \tau), \lambda(s))$$

where $\Psi(y,s) = (y_1 - \frac{p_1}{u}h, y_2 + \frac{p_2}{p_1}y_1 - \frac{p_2}{u}h, \dots, y_n + \frac{p_n}{p_1}y_1 - \frac{p_n}{u}h)$ and $h = q_2y_2 + q_3y_3 + \ldots + q_ny_n$.

We may first check that $\Phi \in \mathcal{G}$:

1) Φ is analytic.
   We use for this the valuation along γ.
   If $j \neq 1$, then for $y_j + \frac{p_j}{p_1}y_1 - \frac{p_j}{u}h$, we have by hypothesis $v(\frac{p_j}{p_1}y_1) \geqslant v(p_1) + v(y_1) - v(p_1) \geqslant 0$ and $v(\frac{p_j}{u}h) = v(p_j) + v(h) - v(u) \geqslant (p+q) - (p+q) = 0$.
   If $j = 1$ for $y_1 - \frac{p_1}{u}h$, we have $v(\frac{p_1}{u}h) = v(p_1) + v(h) - v(u) = (p+q) - (p+q) = 0$.
2) $\Psi(0,s) = 0$
3) The jacobian of Ψ is invertible in a neighborhood of 0.

For this we compute the determinant of this jacobian and show it equals 1.

Let $\Phi_1(y,\tau) = y_1 - \frac{p_1}{u}h$ and $\Phi_j(y,\tau) = y_j + \frac{p_j}{p_1}y_1 - \frac{p_j}{u}h$ for $2 \leqslant j \leqslant n$. Then $\frac{\partial \Phi_1}{\partial y_1} = 1$ and $\forall j \geqslant 2$, $\frac{\partial \Phi_1}{\partial y_j} = -\frac{p_1}{u}q_j$.

$\forall i,j \geqslant 2$, $i \neq j$, $\frac{\partial \Phi_j}{\partial y_i} = -\frac{p_j}{u}q_i$

$\forall i \geqslant 2$, $\frac{\partial \Phi_i}{\partial y_i} = 1 - \frac{p_i}{u}q_i$ and $\frac{\partial \Phi_i}{\partial y_1} = \frac{p_i}{p_1}$.

$$\det\left(\frac{\partial \Psi}{\partial y}(y,\tau)\right) = \begin{vmatrix} 1 & -\frac{p_1}{u}q_2 & \cdots & -\frac{p_1}{u}q_i & \cdots & -\frac{p_1}{u}q_n \\ \frac{p_2}{p_1} & 1 - \frac{p_2}{u}q_2 & & \vdots & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & 1 - \frac{p_i}{u}q_i & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & \ddots & \vdots \\ \frac{p_n}{p_1} & -\frac{p_n}{u}q_2 & \cdots & -\frac{p_n}{u}q_i & \cdots & 1 - \frac{p_n}{u}q_n \end{vmatrix} \tag{3.1}$$

If we denote by C_j the jth row, and apply the transformation $C_j = C_j + \frac{p_iq_i}{u}C_1$ for $j = 1, \cdots, m$.

We see that

$$\det\left(\frac{\partial\Psi}{\partial y}(y,s)\right)=\begin{vmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \frac{p_2}{p_1} & 1 & & \vdots & & \vdots \\ \vdots & 0 & \ddots & 0 & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & \ddots & 0 \\ \frac{p_n}{p_1} & 0 & \dots & \dots & 0 & 1 \end{vmatrix}=1 \tag{3.2}$$

We can now conclude that $\Phi\in\mathcal{G}$.
Moreover, by construction we have $\Phi(p_1(s),0\dots,0,s)=\gamma(s)$.
The computation gives us for $i\geqslant 2$

$$\begin{aligned}\frac{\partial F\circ\Phi}{\partial y_i}(p_1(s),0\dots,0,s) &= \sum_{j=1}^{n}\frac{\partial F}{\partial x_i}(\gamma(s))\frac{\partial\Phi_j}{\partial y_i}(p_1(s),0\dots,0,s)\\ &= \sum_{j=1}^{n} q_j(s)\frac{\partial\Phi_j}{\partial y_i}(p_1(s),0\dots,0,s)\\ = -\frac{q_1(s)q_i(s)p_1(s)}{u}+q_i(s)\left(1-\frac{p_i(s)q_i(s)}{u}\right) &- \sum_{\substack{j\neq 1\\ j\neq i}}\frac{q_j(s)q_i(s)p_j(s)}{u}\\ = -\frac{q_i(s)}{u}\left(p_1(s)q_1(s)+p_i(s)q_i(s)+\sum_{\substack{j\neq 1\\ j\neq i}}p_j(s)q_j(s)\right) &+ q_i(s)\\ &= -\frac{q_i(s)}{u}.u+q_i(s)=0.\end{aligned}$$

If $i=1$, $I=q_1(s)+\sum_{j=2}^{n}q_j(s)\dfrac{p_i(s)}{p_1(s)}=\dfrac{u}{p_1}$. Then, we obtain that $\frac{\partial F\circ\Phi}{\partial y_i}(p_1(s),$ $0\dots,0,s)=\lambda(p_1(s),0,\dots,0)$ with $\lambda=\frac{u}{|p_1|^2}$ *i.e.* $F\circ\Phi$ has a vanishing fold. □

Remark 3.10. 1) This theorem is to be compared with the results (see [14, 19, 21]), on the characterization of the Whitney condition (b) by tubular neighborhood.

2) In this theorem we can replace $\mathcal{G}$ by the set $\mathcal{G}_l=\{\Phi=(\Psi,\lambda):\mathbb{K}^n\times\mathbb{K},0\to\mathbb{K}^n\times\mathbb{K},0$ such that $\Psi(.,s)\in Gl(\mathbb{K}^n)\}$ the proof is essentially the same.

3) The theorem and its proof continue to hold if we replace, the squared distance function by a squared distance function with respect to any affine metric in $\mathbb{C}^n$.

Example 3.11. From the theorem, Whitney faults are detected by vanishing folds. So to find a vanishing fold, it suffices to find an arc along which the Whitney regularity fails.

Let $F(x,y,z,t) = z^5 + ty^6z + y^7x + x^{15}$, Briançon and Speder in [2] shows that the Whitney condition fails at 0 along an arc $\gamma(s)$

$$\begin{cases} x(s) = 4s^5 + O(s^6) \\ y(s) = s^5 \\ z(s) = s^8 \\ t(s) = -5s^2. \end{cases}$$

From the construction (in the proof), the family of automorphisms $\Phi : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ is of this type $\Phi(X,Y,Z,s) = (\Psi(X,Y,Z,s), t(s))$ where $\Psi(X,Y,Z,s) = (X - \frac{p_1}{u}h, Y + \frac{p_2}{p_1}X - \frac{p_2}{u}h, Z + \frac{p_3}{p_1}X - \frac{p_3}{u}h)$ and $h = q_2Y + q_3Z$. After the computations, we obtain $\Psi(X,Y,Z,s) = (X - \frac{1}{2}(Y + s^3Z) + \ldots, 2Y - 2X + s^2Z) + \ldots, Z)$, the dots is for higher terms. Finally $F \circ \Phi(X,Y,Z,s)$, has a vanishing fold along the curve $\gamma(s) = (q_1, q_2, q_3, t(s)) = (\overline{\frac{\partial f}{\partial x}} \circ \gamma(s), t(s))$. Now the construction of the analytic family of automorphisms in the above theorem shows that with respect to $\rho = |X|^2 + |Y|^2 + |Z|^2$, the family $\{f_t\}$, has a ρ-vanishing fold *i.e.* it does not have a ρ-uniform radius.

It is worth to mention that, in the complex analytic case, we have a generalization of a theorem of O'Shea [17]; which relates the jump of Milnor numbers to the existence of vanishing folds.

THEOREM 3.12. *Let* $F : (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$ *be a family of holomorphic function germs with isolated singularities and* $X_t = \{f_t^{-1}(0)\}$ *the corresponding family of hypersurfaces. Let* $\mu_t \equiv \mu$ *be the Milnor number of* f_t *at the origin and suppose that* $\mu_t = \mu$ *is constant for* $0 < t \leqslant 1$ *and* $\mu < \mu_0$.

Then, the family $\{f_t^{-1}(0)\}$ *admits a* ρ*-vanishing fold centered at* 0 *or it has an analytic critical arc.*

The path γ is called a **critical arc** of f (centered at 0) if $x(s)$ is a singular point of $f_{t(s)}^{-1}(0)$ for every $s \in (0, \epsilon]$.

The proof will appear in a forthcoming paper.

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