FURTHER STUDIES OF STRONGLY AMENABLE $*$-REPRESENTATIONS OF LAU $*$-ALGEBRAS

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The new notion of strong amenability for a $*$-representation of a Lau $*$-algebra A on a Hilbert space, was recently introduced and studied by the second author. The strong amenability has several operator theoretic characterizations which unify the notion of amenability and of inner amenability for locally compact groups. In this work, we continue our study to give further properties of strongly amenable $*$-representations of Lau $*$-algebras.

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1. INTRODUCTION

By a Lau $*$-algebra, we shall mean a Banach $*$-algebra which is also a Lau algebra, a Banach algebra A whose dual space has a W^*-algebra structure and the identity element u_A of A^* is a multiplicative linear functional on A. Such a W^*-algebra structure on A^* is not necessarily unique; we will endow the dual space A^* of A with the structure of a fixed W^*-algebra whose identity element is a multiplicative linear functional on A.

The large family of Lau $*$-algebras include the group algebra, the measure algebra and the Fourier algebra of a locally compact group; see Lau [9]. The Fourier-Stieltjes algebra of a topological group is also a Lau $*$-algebra; see Lau and Ludwig [13]. Finally, the predual of a Hopf von-Neumann algebra is another example of Lau $*$-algebra.

The subject of Lau algebras originated with a paper published in 1983 by Lau [9] in which he referred to them as F-algebras. Later on, in his useful monograph, Pier [19] introduced the name Lau algebra. Analysis on Lau algebras have been widely studied by several authors; see for example [5, 7, 8, 10–12] and [16].

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The second author [15] has recently introduced and investigated a new notion of amenability, called strong amenability, for an arbitrary $*$-representation of a Lau $*$-algebra on a Hilbert space. The strong amenability is a strengthening of amenability of unitary representations of a locally compact group G introduced by Bekka [1] as an answer to a question raised by Eymard [6]. For a nice improvement of the subject of amenability for unitary representations of G see Chou and Lau [3]; see also Buneci [2].

In the case of the left regular representation of G, amenability and strong amenability are equivalent, and characterize amenability of G; moreover, in the case of the inner regular representation of G, amenability and strong amenability are equivalent, and characterize inner amenability of G; however, each one dimensional representation is amenable but is, not in general, strongly amenable; see [15] and [18].

The purpose of this paper is to give some further properties of strong amenability for $*$-representations of a Lau $*$-algebra A on a Hilbert space. After some elementary considerations, we consider the projective tensor products $A_1\widehat{\otimes}A_2$ of two Lau $*$-algebras A_1 and A_2, to investigate the interaction between strong amenability of $*$-representations π_1 and π_2 of A_1 and A_2, respectively, and the tensor product $*$-representation $\pi_1 \otimes \pi_2$ of $A_1\widehat{\otimes}A_2$. We then investigate strong amenability of $*$-representations of the unitization $A \oplus \mathbb{C}$ of A to show that a $*$-representation of $A \oplus \mathbb{C}$ is strongly amenable if and only if its restriction to A is strongly amenable. We finally study strong amenability of $*$-representations of subalgebras, ideals and the second dual of A and their relationships with strong amenability of $*$-representations of A.

2. PRELIMINARIES

Let A be a Lau $*$-algebra and let us recall that a $*$-representation of A on Hilbert space H is a $*$-homomorphism from A into $B(H)$, the von Neumann algebra of all bounded linear operators on H with identity I_H.

A state on $B(H)$ is a linear functional β with $\beta(I_H) = \|\beta\| = 1$; the set of all states on $B(H)$ is denoted by $S(B(H))$. Also, for any subset E of $B(H)$, denote by $C^*(E)$ the C^*-algebra generated by E in $B(H)$.

We commence the section by the main concept of the work.

Definition 2.1. Let A be a Lau $*$-algebra and let π be a $*$-representation of A on a Hilbert space H. We say that π is *strongly amenable* if there exists a state β on $B(H)$ such that

$$\beta(\pi(a)) = u_A(a)$$

for all $a \in A$; that is, the following diagram commutes

$$
\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[d, "\beta"'] & B(H) \arrow[dl, "u_A"] \\
\mathbb{C} &
\end{tikzcd}
$$

we call β a π-*stable state*. We denote by $S_\pi(A)$ the set of all π-stable states in $S(B(H))$.

Note that $S_\pi(A)$ is a weak* closed and convex subset of $S(B(H))$. It follows that if $S_\pi(A)$ has at least two elements, then it has infinitely many.

It is easily seen that any $*$-representation of A which contains a strongly amenable $*$-subrepresentation of A is strongly amenable. Moreover, any $*$-representation of A which is equivalent to a strongly amenable $*$-representation of A is strongly amenable.

Recall that a positive functional p on A is called π-*representable* if there is a vector $h \in H$ such that for each $a \in A$,

$$p(a) = \langle \pi(a)h, h\rangle;$$

so, each representable positive functional on A is self-adjoint. This implies that the following assertions are equivalent.

(i) A has a strongly amenable $*$-representation.

(ii) u_A is a representable positive functional on A.

(iii) u_A is a $*$-representation of A on $\mathbb{C}$.

The following theorem is the main result of the work [15] by the second author which provides a number of equivalent assertions characterizing strong amenability of π. First, let us denote by $I_0(A)$ the maximal closed ideal $u_A^{-1}(0)$ of A; that is,

$$I_0(A) = \{a \in A : \; u_A(a) = 0\};$$

moreover, let $P_1(A)$ denote the set of all elements of A that induces a positive functional on the dual W^*-algebra A^*; in fact,

$$P_1(A) = \{a \in A : \; \|a\| = u_A(a) = 1\}.$$

Theorem 2.2. *Let A be a Lau $*$-algebra and let π be a $*$-representation of A on a Hilbert space H. Then the following assertions are equivalent.*

(a) *π is strongly amenable.*

(b) *$\|I_H + \pi(a_1 + \cdots + a_m)\| = 1 + m$ for all $a_1, \ldots, a_m \in P_1(A)$.*

(c) *There is a net (h_γ) in H with $\|h_\gamma\| = 1$ for all γ such that*

$$\|\pi(a)h_\gamma - h_\gamma\| \to 0 \textit{ for all } a \in P_1(A).$$

(d) *There is a net (p_γ) of π-representable positive functionals on A such that*

$$p_\gamma(a) \to 1 \textit{ for all } a \in P_1(A).$$

(e) *There is a state* $\beta \in S(B(H))$ *such that* $\beta(\pi(a_0)) = 1$ *for some* $a_0 \in P_1(A)$ *and*

$$C^*(\pi(I_0(A))) = C^*(\pi(A)) \cap \beta^{-1}(0).$$

Example 2.3. Let $\mathbb{T}$ be the circle group and let $\ell^1(\mathbb{T})$ be corresponding group algebra endowed with the convolution product

$$(a * b)(s) = \sum_{t \in \mathbb{T}} a(t)\, b(s/t)$$

for all $a, b \in \ell^1(\mathbb{T})$ and $s \in \mathbb{T}$. Then $\ell^1(\mathbb{T})$ with the involution

$$a^*(s) = \overline{a(1/s)}$$

for all $a \in \ell^1(\mathbb{T})$ and $s \in \mathbb{T}$ is a Lau $*$-algebra. For each $k \in \mathbb{Z}$ and each Hilbert space H, the mappings

$$\pi_k : \ell^1(\mathbb{T}) \to B(H)$$

defined by

$$\pi_k(a) = \sum_{s \in \mathbb{T}} \frac{a(s)}{s^k}\, I_H$$

are $*$-representations of $\ell^1(\mathbb{T})$. Moreover, for any $\beta \in S(B(H))$ and $a \in \ell^1(\mathbb{T})$ we have

$$\beta(\pi_k(a)) = \sum_{s \in \mathbb{T}} \frac{a(s)}{s^k}$$

whereas

$$u_{\ell^1(\mathbb{T})}(a) = \sum_{s \in \mathbb{T}} a(s).$$

It follows that

$$S_{\pi_0}(\ell^1(\mathbb{T})) = S(B(H))$$

and

$$S_{\pi_k}(\ell^1(\mathbb{T})) = \emptyset$$

for all $k \in \mathbb{Z}$ with $k \neq 0$; in particular, π_0 is strongly amenable, but π_k is not strongly amenable when $k \in \mathbb{Z}$ and $k \neq 0$.

3. REPRESENTATIONS OF TENSOR PRODUCTS OF LAU $*$-ALGEBRAS

Let H_1 and H_2 be Hilbert spaces, and let $H_1 \overline{\otimes} H_2$ be their Hilbert space tensor product. Moreover, let A_1 and A_2 be Lau $*$-algebras and let $A_1 \widehat{\otimes} A_2$ be their projective tensor product endowed with the multiplication

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$$

and the natural involution

$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*$$

for all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. For $*$-representations $\pi_1 : A_1 \to B(H_1)$ and $\pi_2 : A_2 \to B(H_2)$, the tensor product map

$$\pi_1 \otimes \pi_2 : A_1 \widehat{\otimes} A_2 \to B(H_1 \overline{\otimes} H_2)$$

defined by

$$(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) := \pi_1(a_1) \otimes \pi_2(a_2)$$

for all $a_1 \in A_1$ and $a_2 \in A_2$ is a $*$-representation; for more details see [17], Theorems 10.1.32 and 10.1.33. Finally, the formula

$$(u_{A_1} \otimes u_{A_2})(a_1 \otimes a_2) := u_{A_1}(a_1) u_{A_2}(a_2)$$

for all $a_1 \in A_1$ and $a_2 \in A_2$ defines a multiplicative linear functional $u_{A_1} \otimes u_{A_2}$ on $A_1 \widehat{\otimes} A_2$.

Lemma 3.1. *Let A_1 and A_2 be Lau $*$-algebras such that $(A_1 \widehat{\otimes} A_2)^*$ has a W^*-algebra structure with identity $u_{A_1} \otimes u_{A_2}$ and let π_1 and π_2 be $*$-representations of A_1 and A_2 on Hilbert spaces H_1 and H_2, respectively. If π_1 and π_2 are strongly amenable, then $\pi_1 \otimes \pi_2$ is strongly amenable.*

Proof. First, note that for each $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$, the element $T_1 \otimes T_2$ of the tensor product $B(H_1) \otimes B(H_2)$ of $B(H_1)$ and $B(H_2)$ can be considered as an element of $B(H_1 \overline{\otimes} H_2)$ by the formula

$$(T_1 \otimes T_2)(h_1 \otimes h_2) := T_1(h_1) \otimes T_2(h_2)$$

for all $h_1 \in H_1$ and $h_2 \in H_2$; this defines an injective $*$-homomorphism of $B(H_1) \otimes B(H_2)$ into $B(H_1 \overline{\otimes} H_2)$ satisfying

$$\|T_1 \otimes T_2\|_{B(H_1 \overline{\otimes} H_2)} = \|T_1\|_{B(H_1)} \|T_2\|_{B(H_2)}$$

for all $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$. Under this map, the image of $B(H_1) \otimes B(H_2)$ is norm dense in $B(H_1 \overline{\otimes} H_2)$, and the operator norm of $B(H_1 \overline{\otimes} H_2)$ restricted on the image of $B(H_1) \otimes B(H_2)$ is the unique C^*-norm on it.

Now, let β_1 be a π_1-stable state and β_2 be a π_2-stable state, and define the functional $\beta_1 \otimes \beta_2$ on $B(H_1) \otimes B(H_2)$ by

$$(\beta_1 \otimes \beta_2)(T_1 \otimes T_2) := \beta_1(T_1)\, \beta_2(T_2)$$

for all $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$. Then $\beta_1 \otimes \beta_2$ can be extended to a state β on $B(H_1 \overline{\otimes} H_2)$; see for example [14], Theorem 6.4.6. Now, we have

$$\begin{aligned} \beta((\pi_1 \otimes \pi_2)(a_1 \otimes a_2)) &= \beta(\pi_1(a_1) \otimes \pi_2(a_2)) \\ &= (\beta_1 \otimes \beta_2)(\pi_1(a_1) \otimes \pi_2(a_2)) \end{aligned}$$

$$
\begin{aligned}
&= \beta_1(\pi_1(a_1))\, \beta_2(\pi_2(a_2)) \\
&= u_{A_1}(a_1)\, u_{A_2}(a_2) \\
&= (u_{A_1} \otimes u_{A_2})(a_1 \otimes a_2).
\end{aligned}
$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. Thus, β is a $\pi_1 \otimes \pi_2$-stable state. $\square$

We now apply Lemma 3.1 to characterize strong amenability of the tensor product $*$-representations under an additional hypothesis.

THEOREM 3.2. *Let A_1 and A_2 be Lau $*$-algebras with bounded left approximate identities $(e_\alpha)_{\alpha\in\Lambda}$ and $(f_\gamma)_{\gamma\in\Gamma}$ respectively, and let $(A_1\widehat{\otimes}A_2)^*$ has a W^*-algebra structure with identity $u_{A_1} \otimes u_{A_2}$. Suppose that π_1 and π_2 are $*$-representations of A_1 and A_2 on Hilbert spaces H_1 and H_2 respectively, with*

$$\pi_1(e_\alpha) \to I_{H_1} \quad \textit{and} \quad \pi_2(f_\gamma) \to I_{H_2}.$$

in the weak operator topology of $B(H)$. Then π_1 and π_2 are strongly amenable if and only if $\pi_1 \otimes \pi_2$ is strongly amenable.

Proof. By Lemma 3.1, we only need to prove that π_1 and π_2 are strongly amenable if $\pi_1 \otimes \pi_2$ is strongly amenable. For this end, let β be a $\pi_1 \otimes \pi_2$-stable state and replace $(f_\gamma)_{\gamma\in\Gamma}$ by

$$(f_\gamma + {f_\gamma}^* - {f_\gamma}^* f_\gamma)_{\gamma\in\Gamma}$$

if necessary, to assume that $(f_\gamma)_{\gamma\in\Gamma}$ is a self-adjoint bounded two-sided approximate identity with

$$\pi_2(f_\gamma) \to I_{H_2}.$$

For $\gamma \in \Gamma$, let $\beta_{1,\gamma}$ be the linear functional on $B(H_1)$ defined by

$$\beta_{1,\gamma}(T) := \beta(T \otimes \pi_2(f_\gamma))$$

for all $T \in B(H_1)$ and note that $(\beta_{1,\gamma})$ is a bounded net in $B(H_1)^*$. Then there exists a subnet of $(\beta_{1,\gamma})_{\gamma\in\Gamma}$ convergent to an element β_1 of $B(H_1)^*$ in the weak* topology. Without less of generality, we may assume that β_1 is the weak*-limit of the net $(\beta_{1,\gamma})_{\gamma\in\Gamma}$. It follows that

$$
\begin{aligned}
\beta_1(I_{H_1}) &= \lim_\gamma \beta_{1,\gamma}(I_{H_1}) \\
&= \lim_\gamma \beta(I_{H_1} \otimes \pi_2(f_\gamma)) \\
&= \beta(I_{H_1} \otimes I_{H_2}) \\
&= \beta(I_{H_1 \overline{\otimes} H_2}) \\
&= 1.
\end{aligned}
$$

Moreover, for each $T \in B(H_1)$,

$$\beta_1(TT^*) = \lim_\gamma \beta_{1,\gamma}(TT^*)$$

$$\begin{aligned} &= \lim_\gamma \beta(TT^* \otimes \pi_2(f_\gamma)) \\ &= \lim_\gamma \beta(TT^* \otimes \pi_2(f_\gamma)\pi_2(f_\gamma^*)) \\ &= \lim_\gamma \beta((T \otimes \pi_2(f_\gamma))(T \otimes \pi_2(f_\gamma))^*) \\ &\geq 0. \end{aligned}$$

Consequently, $\beta_1 \in S(B(H))$. Since

$$\lim_\gamma u_{A_2}(f_\gamma) = 1,$$

we obtain that

$$\begin{aligned} \beta_1(\pi(a_1)) &= \lim_\gamma \beta(\pi_1(a_1) \otimes \pi_2(f_\gamma)) \\ &= \lim_\gamma \beta((\pi_1 \otimes \pi_2)(a_1 \otimes f_\gamma)) \\ &= \lim_\gamma\, (u_{A_1} \otimes u_{A_2})(a_1 \otimes f_\gamma) \\ &= \lim_\gamma u_{A_1}(a_1)\, u_{A_2}(f_\gamma) \\ &= u_{A_1}(a_1) \end{aligned}$$

for all $a_1 \in A_1$. That is, β_1 is a π_2-stable state; similarly, there is a π_1-stable state, and the proof is complete. $\square$

Let A_1 and A_2 be Lau $*$-algebras with identities e_1 and e_2, respectively, and let π be a $*$-representation of $A_1 \widehat{\otimes} A_2$ on a Hilbert space H. Then the maps

$$\pi_{A_1} : a_1 \to \pi(a_1 \otimes e_2) \text{ and } \pi_{A_2} : a_2 \to \pi(e_1 \otimes a_2)$$

are $*$-representations of A_1 and A_2 on H, respectively.

Corollary 3.3. *Let A_1 and A_2 be Lau $*$-algebras with identities e_1 and e_2, respectively, and let $(A_1 \widehat{\otimes} A_2)^*$ has a W^*-algebra structure with identity $u_{A_1} \otimes u_{A_2}$. Suppose that π is a $*$-representation of $A_1 \widehat{\otimes} A_2$ on a Hilbert space H. If π_{A_1} and π_{A_1} are strongly amenable, then π is strongly amenable.*

Proof. On the one hand, note that the tensor product $*$-representation $\pi_{A_1} \otimes \pi_{A_2}$ is strongly amenable by Theorem 3.2. On the other hand,

$$\begin{aligned} (\pi_{A_1} \otimes \pi_{A_2})(a_1 \otimes a_2) &= \pi_{A_1}(a_1)\pi_{A_2}(a_2) \\ &= \pi(a_1 \otimes e_2) \otimes \pi(e_1 \otimes a_2) \\ &= (\pi \otimes \pi)((a_1 \otimes e_2) \otimes (e_1 \otimes a_2)) \\ &= (\pi \otimes \pi)(a_1 \otimes a_2) \end{aligned}$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. Therefore, $\pi \otimes \pi$ is strongly amenable, and so π is strongly amenable by another application of Theorem 3.2. $\square$

4. REPRESENTATIONS OF SUBALGEBRAS

Let A be a Lau $*$-algebra and recall from Lau [9] that the unitization $A \oplus \mathbb{C}$ of A is a Lau algebra; we endow $A \oplus \mathbb{C}$ with the natural involution

$$(a,t)^* = a^* + \bar{t}$$

for all $(a,t) \in A \oplus \mathbb{C}$. Then $A \oplus \mathbb{C}$ is a Lau $*$-algebra and the identity element $u_{A\oplus\mathbb{C}}$ of the W^*-algebra $(A \oplus \mathbb{C})^*$ is the unique extension of u_A to a multiplicative linear functional on $A \oplus \mathbb{C}$ defined by

$$u_{A\oplus\mathbb{C}}(a,t) := u_A(a) + t$$

for all $(a,t) \in A \oplus \mathbb{C}$.

Every $*$-representation of $A\oplus\mathbb{C}$ on a Hilbert space H is of the form $(\pi, u_{\mathbb{C}})$ for some $*$-representation π of A which $(\pi, u_{\mathbb{C}})$ is an extension of π from A into $A \oplus \mathbb{C}$ defined by

$$(\pi, u_{\mathbb{C}})(a,t) := \pi(a) + tI_H$$

for all $(a,t) \in A \oplus \mathbb{C}$.

PROPOSITION 4.1. *Let A be a Lau $*$-algebra and let π be a $*$-representation of A on a Hilbert space H. Then $(\pi, u_{\mathbb{C}})$ is strongly amenable if and only if π is strongly amenable. In particular, $(0, u_{\mathbb{C}})$ is never strongly amenable.*

Proof. Suppose that $(\pi, u_{\mathbb{C}})$ is strongly amenable and let β_0 be a $(\pi, u_{\mathbb{C}})$-stable state. Then

$$\begin{aligned} \beta_0(\pi(a)) &= \beta_0((\pi, u_{\mathbb{C}})(a,0)) \\ &= u_{A\oplus\mathbb{C}}(a,0) \\ &= u_A(a) \end{aligned}$$

for all $a \in A$. Therefore, β_0 is a π-stable state and hence π is strongly amenable.

Conversely, suppose that π is strongly amenable and let β is a π-stable state. Then

$$\begin{aligned} \beta((\pi, u_{\mathbb{C}})(a,t)) &= \beta(\pi(a)) + t\beta(I_H) \\ &= u_A(a) + t \\ &= u_{A\oplus\mathbb{C}}(a,t) \end{aligned}$$

for all $(a,t) \in A\oplus\mathbb{C}$. Hence β is a $(\pi, u_{\mathbb{C}})$-stable state and so $(\pi, u_{\mathbb{C}})$ is strongly amenable. $\square$

Recall that a closed subspace S of a Lau $*$-algebra A is said to be *topological complemented* if there exists a continuous projection Q_S on A with

$$Q_S(A) = S.$$

In this case, there is a closed subspace V_S of A such that

$$A = S \oplus_1 V_S,$$

the ℓ^1-direct sum of the spaces S and V_S.

Following [4], by an *admissible $*$-subalgebra* of A, we shall mean a topological complemented non-trivial $*$-subalgebra B of A for which both B^* and ${V_B}^*$ are W^*-algebras such that

$$A^* = B^* \oplus_\infty {V_B}^*,$$

the ℓ^∞-direct sum of the W^*-algebras B^* and ${V_B}^*$.

Note that for any admissible $*$-subalgebra B of A, the restriction of the identity element u_A of A^* to B is the identity of B^*, and hence B is a Lau $*$-algebra.

THEOREM 4.2. *Let A be a Lau $*$-algebra and let π be a $*$-representation of A on a Hilbert space H. Suppose that $(A_\gamma)_{\gamma\in\Gamma}$ is a net of admissible closed $*$-subalgebras of A such that*

(a) *$A_{\gamma_1} \subseteq A_{\gamma_2}$ whenever $\gamma_1 \le \gamma_2$;*

(b) *$\bigcup_{\gamma\in\Gamma} A_\gamma$ is dense in A.*

Then π is strongly amenable if and only if $\pi|_{A_\gamma}$ is strongly amenable for all $\gamma \in \Gamma$.

Proof. Suppose that $\pi|_{A_\gamma}$ is strongly amenable for all $\gamma \in \Gamma$, and let β_γ be a $\pi|_{A_\gamma}$-stable state. Then there is a subnet $(\beta_\delta)_{\delta\in\Delta}$ of $(\beta_\gamma)_{\gamma\in\Gamma}$ which is convergent to an element β of $S(B(H))$. For each $a \in A$ and any $\varepsilon > 0$, there exists $a_0 \in \bigcup_{\gamma\in\Gamma} A_\gamma$ such that

$$\|a - a_0\| < \varepsilon.$$

So, there exists $\gamma_0 \in \Gamma$ and $\delta_1 \in \Delta$ such that $a_0 \in A_{\gamma_0}$ and

$$|\beta_\delta(\pi(a_0)) - \beta(\pi(a_0))| < \varepsilon$$

for all $\delta \ge \delta_1$. Now, choose $\delta_2 \in \Delta$ with $\delta_2 \ge \delta_1$ and $\delta_2 \ge \gamma_0$. Then $a_0 \in A_{\delta_2}$, and hence

$$\beta_{\delta_2}(\pi(a_0)) = u_A(a_0).$$

If follows that

$$\begin{aligned}|\beta(\pi(a)) - u_A(a)| &= |\beta(\pi(a)) - \beta(\pi(a_0))| \\ &\quad + |\beta(\pi(a_0)) - \beta_{\delta_2}(\pi(a_0))| \\ &\quad + |u_A(a_0) - u_A(a)|\end{aligned}$$

which shows that

$$|\beta(\pi(a)) - u_A(a)| \le 3\,\varepsilon$$

for all $\varepsilon > 0$; that is, β is a π-stable state. So, π is strongly amenable. The converse in trivial. $\square$

Proposition 4.3. *Let A_1 and A_2 be Lau $*$-algebras. Suppose that π is a $*$-representation of A_2 on a Hilbert space H and $\vartheta : A_1 \longrightarrow A_2$ is a $*$-homomorphism with dense range such that $u_{A_2} \circ \vartheta = u_{A_1}$. Then π is strongly amenable if and only if $\pi \circ \vartheta$ is strongly amenable.*

Proof. Let β be a π-stable state. Then

$$\beta(\pi(\vartheta(a_1))) = u_{A_2}(\vartheta(a_1)) = u_{A_1}(a_1)$$

for all $a_1 \in A_1$, and therefore β is a $\pi \circ \vartheta$-stable state.

Conversely, if β is a $\pi \circ \vartheta$-stable state, then for $a_2 \in A_2$, there exists a sequence $(a_{1,n}) \subseteq A_1$ such that

$$\lim_n \vartheta(a_{1,n}) = a_2,$$

and so

$$\begin{aligned} \beta(\pi(a_2)) &= \beta(\pi(\lim_n \vartheta(a_{1,n}))) \\ &= \lim_n \beta(\pi(\vartheta(a_{1,n}))) \\ &= \lim_n u_{A_2}(\vartheta(a_{1,n})) \\ &= u_{A_2}(a_2). \end{aligned}$$

This shows that β is a π-stable state. $\square$

Before we give our next result, let us recall the following key lemma from [15], Lemma 2.3.

Lemma 4.4. *Let A be a Lau $*$-algebra and let I be an ideal of A which is not contained in $I_0(A)$. Suppose that p is a representable positive functional on A such that $p = u_A$ on I. Then $p = u_A$ on A.*

We end the work the following result which investigates the interaction between strong amenability of $*$-representations of a Lau $*$-algebra A and its restrictions to certain $*$-ideals of A.

Proposition 4.5. *Let A be a Lau $*$-algebra and let I be an admissible closed $*$-ideal of A with*

$$I \nsubseteq I_0(A).$$

Suppose that π is a $*$-representation of A on a Hilbert space H. Then π is strongly amenable if and only if $\pi|_I$ is strongly amenable.

Proof. The "only if" part follows from the definition. To prove the converse, suppose that β is a $\pi|_I$-stable state. Then $\beta \circ \pi$ is a representable positive functional on A such that

$$\beta \circ \pi|_I = u_A|_I.$$

So, it follows from Lemma 4.4 that

$$\beta \circ \pi = u_A;$$

that is, π is strongly amenable. $\square$

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