Dedicated to Professor Dr. Andrei Duma on the occasion of his 70^{th} birthday

HITTING PROBABILITIES FOR RANDOM CONVEX BODIES AND LATTICES OF RECTANGLES WITH TRIANGULAR OBSTACLES

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In this paper, we calculate the probability that a small convex body hits a plane lattice that is a regular tiling with a finite number of in general different rectangles, each containing four triangular obstacles.

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1. INTRODUCTION

Let $\mathcal{R}_i$, $i \in \{1, 2, \ldots, n\}$, be the rectangle

$$\mathcal{R}_i = \left\{(x_i, y_i) \in \mathbb{R}^2 \,|\, 0 \le x_i \le a_i, 0 \le y_i \le b_i\right\}$$

with area $A_i = a_i b_i > 0$ (see Fig. 1). In every corner of $\mathcal{R}_i$ there is an *obstacle* $\mathcal{H}_{i,k}$ that is an isosceles right triangle with legs of length $h_i < \min(a_i, b_i)/2$. One easily finds that

$$\begin{aligned}
\mathcal{H}_{i,1} &:= \left\{(x_i, y_i) \in \mathbb{R}^2 \mid a_i - h_i \le x_i \le a_i,\ 0 \le y_i \le x_i - a_i + h_i\right\},\\
\mathcal{H}_{i,2} &:= \left\{(x_i, y_i) \in \mathbb{R}^2 \mid a_i - h_i \le x_i \le a_i,\ a_i + b_i - h_i - x_i \le y_i \le b_i\right\},\\
\mathcal{H}_{i,3} &:= \left\{(x_i, y_i) \in \mathbb{R}^2 \mid x_i \ge 0,\ x_i + b_i - h_i \le y_i \le b_i\right\},\\
\mathcal{H}_{i,4} &:= \left\{(x_i, y_i) \in \mathbb{R}^2 \mid x_i \ge 0,\ 0 \le y_i \le h_i - x_i\right\}.
\end{aligned}$$

We say that $\mathcal{K} = \bigcup_{i=1}^{n} \mathcal{R}_i$ is a *conglomerate* of $\mathcal{R}_1, \ldots, \mathcal{R}_n$ if:

– $\mathcal{R}_i \cap \mathcal{R}_j$, $i \ne j$, is either the empty set or a set of measure zero (point or line segment).
– $\mathcal{K}$ is simply connected.
– There exists at least one regular plane tiling with $\mathcal{K}$.

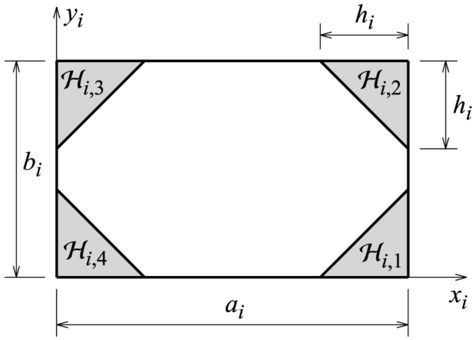

Fig. 1 – Rectangle $\mathcal{R}_i$ with obstacles.

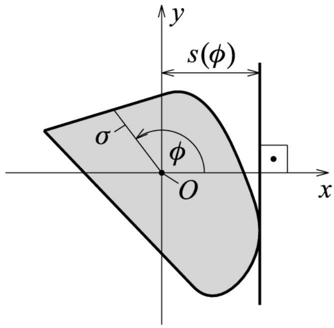

Fig. 2 – Convex body $\mathcal{C}$.

Note that $\mathcal{K}$ is a convex or non convex polygon with area $A = \sum_{i=1}^{n} A_i$. In the following, a regular tiling with $\mathcal{K}$ is called *lattice $\mathcal{R}$ of rectangles* (in short: *lattice $\mathcal{R}$*). We denote by $\partial\mathcal{R}_i$ the boundary of $\mathcal{R}_i$, and put

$$\mathcal{H}_i := \bigcup_{k=1}^{4} \mathcal{H}_{i,k}, \quad \mathcal{H} := \bigcup_{i=1}^{n} \mathcal{H}_i, \quad \widetilde{\mathcal{R}} := \bigcup_{i=1}^{n} \partial\mathcal{R}_i,$$

$$\mathcal{S}_i := \partial\mathcal{R}_i \cup \mathcal{H}_i, \quad \mathcal{S} := \widetilde{\mathcal{R}} \cup \mathcal{H} = \bigcup_{i=1}^{n} \mathcal{S}_i.$$

It is possible to cover the whole plane gaplessly with congruent rectangles $\overline{\mathcal{K}}$ so that every $\overline{\mathcal{K}} \cap \mathcal{R}$ (with inside line segments and obstacles) is a congruent copy of any other. Let $\bar{a}$ and $\bar{b}$ the side lengths of $\overline{\mathcal{K}}$. We define a Cartesian $\bar{x}, \bar{y}$-coordinate system for $\overline{\mathcal{K}}$ such that

$$\overline{\mathcal{K}} = \left\{ (\bar{x}, \bar{y}) \in \mathbb{R}^2 \,|\, 0 \le \bar{x} \le \bar{a}, 0 \le \bar{y} \le \bar{b} \right\}.$$

A planar convex body $\mathcal{C}$ is randomly thrown onto $\mathcal{R}$. We are asking for the probability that $\mathcal{C} \cap \mathcal{S} \neq \emptyset$. In this case, we say that *$\mathcal{C}$ hits $\mathcal{S}$*. Let $\mathcal{C}$ be provided with a fixed reference point O inside $\mathcal{C}$ and a fixed oriented line segment σ starting in O (see Fig. 2).

The *random throw of $\mathcal{C}$ onto $\mathcal{R}$* is defined as follows: The coordinates $\bar{x}$, $\bar{y}$ of the reference point O are random variables uniformly distributed in $[0, \bar{a}]$, $[0, \bar{b}]$, respectively; the angle ϕ between the $\bar{x}$-axis and the line segment σ is a random variable uniformly distributed in $[0, 2\pi]$. All three random variables are stochastically independent.

Let u denote the perimeter of $\mathcal{C}$, $A_{\mathcal{C}}$ the area of $\mathcal{C}$, and

$$s : \mathbb{R} \to \mathbb{R}\,, \quad \phi \mapsto s(\phi)$$

the support function of $\mathcal{C}$ with reference to the point O. We use s in the following sense (see Fig. 2): $\mathcal{C}$ rotates about the origin of a Cartesian x, y-coordinate system with rotation angle ϕ between the x-axis and segment σ, whereby O coincides with the origin. The support line, touching $\mathcal{C}$, is perpendicular to the x-axis. Now, the support function s is the distance between the origin and the support line. In [6, pp. 2–3], the support function, say $\tilde{s}$, is defined for fixed $\mathcal{C}$ and moving support line. Clearly, the relation between these two definitions is given by $s(\phi) = \tilde{s}(-\phi)$ if the fixed body $\mathcal{C}$ is in the position with $\phi = 0$, and O coinciding with the origin. The width of $\mathcal{C}$ in the direction ϕ is given by

$$w : \mathbb{R} \to \mathbb{R}, \quad \phi \mapsto w(\phi) = s(\phi) + s(\phi + \pi).$$

In the following, we assume $\mathcal{C}$ to be *small* with respect to $\mathcal{S}$. The criterion of the smallness of $\mathcal{C}$ (see Corollary 1) will be obtained later on.

Since Laplace calculated the hitting probability for a line segment (needle) and a lattice of rectangles (see [5, pp. 17–19]), problems of this kind are often called *Laplace type problems*. For recent results on Laplace type problems we refer to [1] and [2]. Results for triangular lattices with triangular obstacles are to be found in [3].

2. HITTING PROBABILITIES

2.1. One rectangle

In this subsection we consider only one rectangle $\mathcal{R}_i$, and assume that the coordinates x_i, y_i of O are random variables uniformly distributed in $[0, a_i]$, $[0, b_i]$, respectively. We calculate the probability of the event $\mathcal{C} \cap \mathcal{S}_i \neq \emptyset$ ("$\mathcal{C}$ hits $\mathcal{S}_i$"). Obviously,

$$\begin{aligned} P(\mathcal{C} \cap \mathcal{S}_i \neq \emptyset) &= P(\mathcal{C} \cap \partial\mathcal{R}_i \neq \emptyset) + P((\mathcal{C} \cap \mathcal{H}_i \neq \emptyset) \wedge (\mathcal{C} \cap \partial\mathcal{R}_i = \emptyset)) \\ &= P(\mathcal{C} \cap \partial\mathcal{R}_i \neq \emptyset) + \sum_{k=1}^{4} P((\mathcal{C} \cap \mathcal{H}_{i,k} \neq \emptyset) \wedge (\mathcal{C} \cap \partial\mathcal{R}_i = \emptyset)). \end{aligned}$$

For abbreviation we denote the event $(\mathcal{C} \cap \mathcal{H}_{i,k} \neq \emptyset) \wedge (\mathcal{C} \cap \partial\mathcal{R}_i = \emptyset)$ by $H_{i,k}$. It is clear that $P(H_{i,1}) = P(H_{i,2}) = P(H_{i,3}) = P(H_{i,4})$. We already know [1, Theorem 1] that

$$P(\mathcal{C} \cap \partial\mathcal{R}_i \neq \emptyset) = \frac{u}{\pi a_i} + \frac{u}{\pi b_i} - \frac{1}{\pi a_i b_i} I\left(\frac{\pi}{2}\right) = \frac{1}{\pi A_i}\left[(a_i + b_i)u - I\left(\frac{\pi}{2}\right)\right]$$

with

$$I(x) := \int_0^\pi w(\phi) w(\phi + x)\, \mathrm{d}\phi \tag{1}$$

if

$$\max_{\phi \in [0,\pi)} w(\phi) \leq \min(a_i, b_i).$$

Using the law of total probability in the integral form (see [4, p. 282]), we have the Stieltjes integral

$$P(H_{i,k}) = \int_{-\infty}^{\infty} P(H_{i,k}\,|\,\phi)\,\mathrm{d}F(\phi)$$

for the probability of the event $H_{i,k}$, where F is the distribution function of the angle ϕ between the x_i-axis and the line segment σ,

$$F(\phi) = \begin{cases} 0\,, & -\infty < \phi < 0\,, \\ \phi/(2\pi)\,, & 0 \le \phi < 2\pi\,, \\ 1\,, & 2\pi \le \phi < \infty\,, \end{cases}$$

and $P(H_{i,k}\,|\,\phi)$ is the conditional probability of $H_{i,k}$ for fixed value of ϕ.

We calculate $P(H_{i,k})$ with the help of Fig. 3. For fixed angle ϕ, the event $H_{i,1}$ occurs if

$$O \in \mathcal{H}^*_{i,1}(\phi) := \Big\{(x_i, y_i) \in \mathbb{R}^2 \,\Big|\, a_i - s(\phi) - \delta_i(\phi) \le x_i < a_i - s(\phi),$$
$$s\Big(\phi + \frac{\pi}{2}\Big) < y_i \le x_i - a_i + h_i + \sqrt{2}\, s\Big(\phi + \frac{\pi}{4}\Big)\Big\}\,.$$

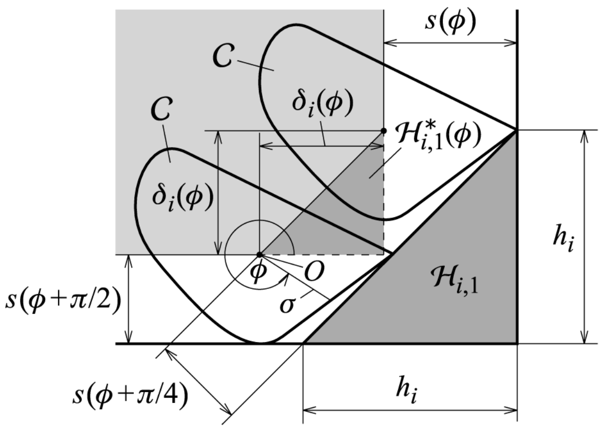

Fig. 3 – Set $\mathcal{H}^*_{i,1}(\phi)$.

($\mathcal{H}^*_{i,1}(\phi)$ is an isosceles right triangle with legs of length $\delta_i(\phi)$.) We assume that $\delta_i(\phi) \ge 0$ for every $\phi \in [0, 2\pi)$. The area of $\mathcal{H}^*_{i,1}(\phi)$ is equal to $\delta_i^2(\phi)/2$, hence

$$P(H_{i,1}\,|\,\phi) = \frac{\delta_i^2(\phi)}{2A_i}\,.$$

Therefore

$$P(H_{i,1}) = \frac{1}{2\pi}\int_0^{2\pi} P(H_{i,1}\,|\,\phi)\,\mathrm{d}\phi = \frac{1}{4\pi A_i}\int_0^{2\pi} \delta_i^2(\phi)\,\mathrm{d}\phi$$

and

$$P(\mathcal{C} \cap \mathcal{S}_i \neq \emptyset) = P(\mathcal{C} \cap \partial\mathcal{R}_i \neq \emptyset) + 4P(H_{i,1})$$
$$= \frac{1}{\pi A_i}\Big[(a_i+b_i)u - I\Big(\frac{\pi}{2}\Big)\Big] + 4\,\frac{1}{4\pi A_i}\int_0^{2\pi} \delta_i^2(\phi)\,\mathrm{d}\phi\,,$$

hence

$$P(\mathcal{C} \cap \mathcal{S}_i \neq \emptyset) = \frac{1}{\pi A_i}\Big[(a_i+b_i)u - I\Big(\frac{\pi}{2}\Big) + \int_0^{2\pi} \delta_i^2(\phi)\,\mathrm{d}\phi\Big]. \tag{2}$$

Now, we determine the function δ_i. The line containing the hypotenuse of $\mathcal{H}^*_{i,1}(\phi)$ intersects the line $y_i = s(\phi+\pi/2)$ in the point whose x_i-coordinate is

$$s\Big(\phi+\frac{\pi}{2}\Big) + a_i - h_i - \sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big).$$

Hence

$$\delta_i(\phi) = a_i - s(\phi) - \Big[s\Big(\phi+\frac{\pi}{2}\Big) + a_i - h_i - \sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big)\Big]$$
$$= h_i - \Big[s(\phi) - \sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big) + s\Big(\phi+\frac{\pi}{2}\Big)\Big].$$

From the above-mentioned assumption $\delta_i(\phi) \geq 0$, it follows that

$$\sqrt{2}\max_{\phi\in[0,2\pi)}\Big[s(\phi) - \sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big) + s\Big(\phi+\frac{\pi}{2}\Big)\Big] \leq \sqrt{2}\,h_i\,,$$

where $\sqrt{2}\,h_i$ is the length of the hypotenuse of $\mathcal{H}_{i,k}$. This gives rise to the definition that $\mathcal{C}$ is *small with respect to* $\mathcal{S}_i$ if

$$\sqrt{2}\max_{\phi\in[0,2\pi)}\Big[s(\phi) - \sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big) + s\Big(\phi+\frac{\pi}{2}\Big)\Big] \leq \min(\sqrt{2}\,h_i,\, a_i - 2h_i,\, b_i - 2h_i)\,. \tag{3}$$

We have

$$\delta_i^2(\phi) = s^2(\phi) + 2s^2\Big(\phi+\frac{\pi}{4}\Big) + s^2\Big(\phi+\frac{\pi}{2}\Big) - 2\sqrt{2}\,s(\phi)\,s\Big(\phi+\frac{\pi}{4}\Big)$$
$$+ 2s(\phi)\,s\Big(\phi+\frac{\pi}{2}\Big) - 2\sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big)\,s\Big(\phi+\frac{\pi}{2}\Big)$$
$$- 2h_i\Big[s(\phi) - \sqrt{2}\,s\Big(\phi+\frac{\pi}{4}\Big) + s\Big(\phi+\frac{\pi}{2}\Big)\Big] + h_i^2\,.$$

Using the the relations

$$\int_0^{2\pi} s^2\Big(\phi+\frac{\pi}{4}\Big)\,\mathrm{d}\phi = \int_0^{2\pi} s^2\Big(\phi+\frac{\pi}{2}\Big)\,\mathrm{d}\phi = \int_0^{2\pi} s^2(\phi)\,\mathrm{d}\phi\,,$$

$$\int_0^{2\pi} s\Big(\phi+\frac{\pi}{4}\Big)\,s\Big(\phi+\frac{\pi}{2}\Big)\,\mathrm{d}\phi = \int_0^{2\pi} s(\phi)\,s\Big(\phi+\frac{\pi}{4}\Big)\,\mathrm{d}\phi\,,$$

$$\int_0^{2\pi} s\Big(\phi+\frac{\pi}{4}\Big)\,\mathrm{d}\phi = \int_0^{2\pi} s\Big(\phi+\frac{\pi}{2}\Big)\,\mathrm{d}\phi = \int_0^{2\pi} s(\phi)\,\mathrm{d}\phi\,,$$

that result from the 2π-periodicity of s, and

$$u = \int_0^{2\pi} s(\phi)\,\mathrm{d}\phi$$

(see [6, p. 3]), it follows that

$$\begin{aligned}\int_0^{2\pi} \delta_i^2(\phi)\,\mathrm{d}\phi &= 4\int_0^{2\pi} s^2(\phi)\,\mathrm{d}\phi - 4\sqrt{2}\int_0^{2\pi} s(\phi)\,s\Big(\phi+\frac{\pi}{4}\Big)\,\mathrm{d}\phi \\ &\quad + 2\int_0^{2\pi} s(\phi)\,s\Big(\phi+\frac{\pi}{2}\Big)\,\mathrm{d}\phi - 2\left(2-\sqrt{2}\right)h_i\int_0^{2\pi} s(\phi)\,\mathrm{d}\phi \\ &\quad + h_i^2\int_0^{2\pi}\mathrm{d}\phi \\ &= 4J(0) - 4\sqrt{2}J\Big(\frac{\pi}{4}\Big) + 2J\Big(\frac{\pi}{2}\Big) - 2\left(2-\sqrt{2}\right)h_i u + 2\pi h_i^2\end{aligned}$$

with

$$J(x) := \int_0^{2\pi} s(\phi)\,s(\phi+x)\,\mathrm{d}\phi\,. \tag{4}$$

With (2), we have proved the following theorem.

THEOREM 1. *If O is chosen uniformly at random from $\mathcal{R}_i$, and $\mathcal{C}$ is small with respect to $\mathcal{S}_i$ according to* (3)*, then, with* (1) *and* (4)*,*

$$\begin{aligned}P(\mathcal{C}\cap\mathcal{S}_i\neq\emptyset) = \frac{1}{\pi A_i}\Big[&(a_i+b_i)u - I\Big(\frac{\pi}{2}\Big) + 4J(0) - 4\sqrt{2}J\Big(\frac{\pi}{4}\Big) \\ &+ 2J\Big(\frac{\pi}{2}\Big) - 2\left(2-\sqrt{2}\right)h_i u + 2\pi h_i^2\Big]\,.\end{aligned} \tag{5}$$

Remark 1. For all convex bodies $\mathcal{C}$ where O may be chosen such that $w(\phi) = 2s(\phi)$ for every $\phi\in[0,2\pi)$, we have $I(x) = 2J(x)$, and therefore

$$P(\mathcal{C}\cap\mathcal{S}_i\neq\emptyset) = \frac{1}{\pi A_i}\Big[(a_i+b_i)u + 2I(0) - 2\sqrt{2}\,I\Big(\frac{\pi}{4}\Big) - 2\left(2-\sqrt{2}\right)h_i u + 2\pi h_i^2\Big].$$

2.2. Lattice $\mathcal{R}$

Now we consider the random throw of $\mathcal{C}$ onto a lattice $\mathcal{R}$ that is a regular tiling with a conglomerate $\mathcal{K} = \bigcup_{i=1}^n \mathcal{R}_i$ of rectangles $\mathcal{R}_1,\ldots,\mathcal{R}_n$ as defined in Section 1, and calculate the probability of the event $\mathcal{C}\cap\mathcal{S}\neq\emptyset$ ("$\mathcal{C}$ hits $\mathcal{S}$"). Using the law of total probability, we have

$$P(\mathcal{C}\cap\mathcal{S}\neq\emptyset) = \sum_{i=1}^n P(\mathcal{C}\cap\mathcal{S}\neq\emptyset \mid O\in\mathcal{R}_i)\,P(O\in\mathcal{R}_i)\,.$$

Clearly,

$$P(\mathcal{C} \cap \mathcal{S} \neq \emptyset \mid O \in \mathcal{R}_i) = P(\mathcal{C} \cap \mathcal{S}_i \neq \emptyset \mid O \in \mathcal{R}_i)\,, \quad i \in \{1, \ldots, n\}\,.$$

Due to the assumption $O \in \mathcal{R}_i$ in Theorem 1, the conditional probabilities $P(\mathcal{C} \cap \mathcal{S}_i \neq \emptyset \mid O \in \mathcal{R}_i)$ are given by formula (5). Obviously,

$$P(O \in \mathcal{R}_i) = \frac{A_i}{A}\,, \quad i \in \{1, \ldots, n\}\,.$$

It follows that

$$\begin{aligned} P(\mathcal{C} \cap \mathcal{S} \neq \emptyset) &= P\big(\mathcal{C} \cap (\widetilde{\mathcal{R}} \cup \mathcal{H}) \neq \emptyset\big) = P\big((\mathcal{C} \cap \widetilde{\mathcal{R}} \neq \emptyset) \vee (\mathcal{C} \cap \mathcal{H} \neq \emptyset)\big) \\ &= \sum_{i=1}^{n} \frac{1}{\pi A_i} \Big[(a_i + b_i)u - 2\left(2 - \sqrt{2}\right) h_i u + 2\pi h_i^2 \\ &\quad - I\Big(\frac{\pi}{2}\Big) + 4J(0) - 4\sqrt{2}J\Big(\frac{\pi}{4}\Big) + 2J\Big(\frac{\pi}{2}\Big)\Big] \frac{A_i}{A}\,, \end{aligned}$$

hence

$$\begin{aligned} (6) \quad P(\mathcal{C} \cap \mathcal{S} \neq \emptyset) = \frac{1}{\pi A} \Bigg\{ u \left[\sum_{i=1}^{n} (a_i + b_i) - 2\left(2 - \sqrt{2}\right) \sum_{i=1}^{n} h_i\right] + 2\pi \sum_{i=1}^{n} h_i^2 \\ - n\left[I\Big(\frac{\pi}{2}\Big) - 4J(0) + 4\sqrt{2}J\Big(\frac{\pi}{4}\Big) - 2J\Big(\frac{\pi}{2}\Big)\right] \Bigg\}. \end{aligned}$$

Taking into account that every $\mathcal{S}_i$, $i \in \{1, \ldots, n\}$, has to be considered checking the smallness of $\mathcal{C}$ according to inequality (3), we have found the following corollary.

Corollary 1. *$\mathcal{C}$ is randomly thrown onto a lattice $\mathcal{R}$ with conglomerate $\mathcal{K} = \bigcup_{i=1}^{n} \mathcal{R}_i$. If inequality*

$$\begin{aligned} \sqrt{2} \max_{\phi \in [0, 2\pi)} \Big[s(\phi) - \sqrt{2}\, s\Big(\phi + \frac{\pi}{4}\Big) + s\Big(\phi + \frac{\pi}{2}\Big)\Big] \\ \leq \min_{i \in \{1, \ldots, n\}} \left(\sqrt{2}\, h_i,\, a_i - 2h_i,\, b_i - 2h_i\right) \end{aligned}$$

holds, then the probability that $\mathcal{C}$ hits $\mathcal{S}$ is given by (6) *with* (1) *and* (4).

Remark 2. *If $I(x) = 2J(x)$, then*

$$\begin{aligned} P(\mathcal{C} \cap \mathcal{S} \neq \emptyset) = \frac{1}{\pi A} \Bigg\{ u \left[\sum_{i=1}^{n} (a_i + b_i) - 2\left(2 - \sqrt{2}\right) \sum_{i=1}^{n} h_i\right] \\ + 2\pi \sum_{i=1}^{n} h_i^2 + 2n\left[I(0) - \sqrt{2}\, I\Big(\frac{\pi}{4}\Big)\right] \Bigg\}. \end{aligned}$$

3. SPECIAL CASES

3.1. $\mathcal{C}$ is a rectangle

Let $\mathcal{C}$ be a rectangle with side lengths $\widetilde{m}$ and $\tilde{n}$. Clearly, $\mathcal{C}$ is small with respect to $\mathcal{S}$ if

$$\max(\widetilde{m},\, \tilde{n}) \leq \min_{i\in\{1,\ldots,n\}} \left(\sqrt{2}\, h_i,\, a_i - 2h_i,\, b_i - 2h_i\right).$$

We already know, see [1], that

$$I(\alpha) = \frac{\left[(\pi - 2\alpha)\left(\widetilde{m}^2 + \tilde{n}^2\right) + 4\widetilde{m}\tilde{n}\right]\cos\alpha + 2\left(\widetilde{m}^2 + \tilde{n}^2 + 2\alpha\widetilde{m}\tilde{n}\right)\sin\alpha}{2}.$$

For $\alpha = \pi/4$, one finds

$$I\Big(\frac{\pi}{4}\Big) = \frac{(\pi+4)\,(\widetilde{m}+\tilde{n})^2}{4\sqrt{2}}.$$

The formula for $I(\alpha)$ is valid for $\alpha = 0$, too, and gives

$$I(0) = \frac{1}{2}\left[\pi\left(\widetilde{m}^2 + \tilde{n}^2\right) + 4\widetilde{m}\tilde{n}\right].$$

Therefore, we have

$$\begin{aligned} I(0) - \sqrt{2}\, I\Big(\frac{\pi}{4}\Big) &= \frac{1}{4}\left[(\pi - 4)\left(\widetilde{m}^2 + \tilde{n}^2\right) - 2\pi\widetilde{m}\tilde{n}\right] \\ &= \frac{\pi}{4}\,(\widetilde{m} - \tilde{n})^2 - \left(\widetilde{m}^2 + \tilde{n}^2\right), \end{aligned}$$

and

$$\begin{aligned} P(\mathcal{C}\cap\mathcal{S}\neq\emptyset) = \frac{1}{\pi A}\Bigg\{ & 2(\widetilde{m}+\tilde{n})\left[\sum_{i=1}^{n}(a_i+b_i) - 2\left(2-\sqrt{2}\right)\sum_{i=1}^{n} h_i\right] \\ & + 2\pi\sum_{i=1}^{n} h_i^2 + n\left[\frac{\pi}{2}\,(\widetilde{m}-\tilde{n})^2 - 2\left(\widetilde{m}^2+\tilde{n}^2\right)\right]\Bigg\}. \end{aligned}$$

3.2. A special lattice $\mathcal{R}$

Now, we consider the special lattice shown in Fig. 4. Four obstacles $\mathcal{H}_{1,1}$, $\mathcal{H}_{1,2}$, $\mathcal{H}_{1,3}$, $\mathcal{H}_{1,4}$ form a square $\widetilde{\mathcal{H}}_1$ with side length $\sqrt{2}\, h_1$. At first, we calculate the probability that the convex body $\mathcal{C}$ hits $\mathcal{H}$ under the condition

$$\max_{\phi\in[0,\pi)} w(\phi) \leq \min(a_1 - 2h_1, b_1 - 2h_1)\,. \tag{7}$$

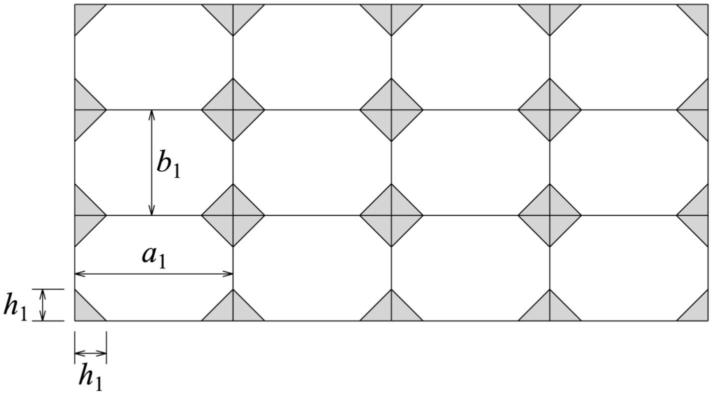

Fig. 4 – Lattice $\mathcal{R}$ consisting of rectangles $\mathcal{R}_1$.

Due to this condition, the probability that $\mathcal{C}$ hits more than one square $\widetilde{\mathcal{H}}_1$ is equal to zero. Using a formula from [6, p. 140], we easily find

$$P(\mathcal{C} \cap \mathcal{H} \neq \emptyset) = P(\mathcal{C} \cap \widetilde{\mathcal{H}}_1 \neq \emptyset) = \frac{\pi\left(2h_1^2 + A_{\mathcal{C}}\right) + 2\sqrt{2}h_1 u}{\pi a_1 b_1}.$$

Next, if (7) and

$$\sqrt{2} \max_{\phi \in [0,2\pi)} \left[s(\phi) - \sqrt{2}\, s\left(\phi + \frac{\pi}{4}\right) + s\left(\phi + \frac{\pi}{2}\right)\right] \leq \sqrt{2}\, h_1$$

hold, we get the probability that $\mathcal{C}$ hits $\widetilde{\mathcal{R}}$ and $\mathcal{H}$ at the same time:

$$\begin{aligned}
&P\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \cap \mathcal{H} \neq \emptyset\big) = P\big(\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \neq \emptyset\big) \wedge \big(\mathcal{C} \cap \mathcal{H} \neq \emptyset\big)\big) \\
&= P\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \neq \emptyset\big) + P\big(\mathcal{C} \cap \mathcal{H} \neq \emptyset\big) - P\big(\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \neq \emptyset\big) \vee \big(\mathcal{C} \cap \mathcal{H} \neq \emptyset\big)\big) \\
&= P\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \neq \emptyset\big) + P\big(\mathcal{C} \cap \mathcal{H} \neq \emptyset\big) - P\big(\mathcal{C} \cap \big(\widetilde{\mathcal{R}} \cup \mathcal{H}\big) \neq \emptyset\big) \\
&= P\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \neq \emptyset\big) + P\big(\mathcal{C} \cap \mathcal{H} \neq \emptyset\big) - P\big(\mathcal{C} \cap \mathcal{S} \neq \emptyset\big) \\
&= \frac{1}{\pi a_1 b_1}\left\{(a_1 + b_1)u - I\left(\frac{\pi}{2}\right) + 2\pi h_1^2 + \pi A_{\mathcal{C}} + 2\sqrt{2}h_1 u\right. \\
&\quad - \left[(a_1 + b_1)u - I\left(\frac{\pi}{2}\right) + 4J(0) - 4\sqrt{2}J\left(\frac{\pi}{4}\right) + 2J\left(\frac{\pi}{2}\right)\right. \\
&\quad \left.\left. - 4h_1 u + 2\sqrt{2}h_1 u + 2\pi h_1^2\right]\right\} \\
&= \frac{1}{\pi a_1 b_1}\left[\pi A_{\mathcal{C}} + 4h_1 u - 4J(0) + 4\sqrt{2}J\left(\frac{\pi}{4}\right) - 2J\left(\frac{\pi}{2}\right)\right].
\end{aligned}$$

If $\mathcal{C}$ is a rectangle whose side lengths $\widetilde{m}$ and $\tilde{n}$ satisfy the inequalities

$$\max(\widetilde{m}, \tilde{n}) \leq \sqrt{2}\, h_1 \quad \text{and} \quad \sqrt{\widetilde{m}^2 + \tilde{n}^2} \leq \min(a_1 - 2h_1, b_1 - 2h_1),$$

then, with

$$2J(x) = I(x)\,, \quad I(0) = 2\widetilde{m}\tilde{n} + \frac{\pi}{2}\left(\widetilde{m}^2 + \tilde{n}^2\right), \quad I\Big(\frac{\pi}{2}\Big) = \widetilde{m}^2 + \tilde{n}^2 + \pi\widetilde{m}\tilde{n}\,,$$

$$2\sqrt{2}\, I\Big(\frac{\pi}{4}\Big) - I\Big(\frac{\pi}{2}\Big) = 4\widetilde{m}\tilde{n} + \frac{1}{2}\,(\pi+2)\left(\widetilde{m}^2 + \tilde{n}^2\right),$$

one finds

$$P\big(\mathcal{C} \cap \widetilde{\mathcal{R}} \cap \mathcal{H} \neq \emptyset\big) = \frac{2\pi\widetilde{m}\tilde{n} + 16h_1(\widetilde{m} + \tilde{n}) - (\pi - 2)\left(\widetilde{m}^2 + \tilde{n}^2\right)}{2\pi a_1 b_1}\,.$$

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