Dedicated to Professor Nicu Boboc on the occasion of his 80^{th} birthday

BALAYAGE AND FINE CARRIER FOR EXCESSIVE FUNCTIONS

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We present minimal conditions for a proper sub-Markovian resolvent family of kernels, such that it is possible to develop a basic part of the potential theory, in the frame of the associated excessive structure. We characterize the regular excessive elements as being those excessive functions for which the associated pseudo-balayages are balayages, and we construct a fine carrier theory without using any kind of compactification.

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1. INTRODUCTION

Given a sub-Markovian resolvent family of kernels $\mathcal{V}$ on a measurable space $(X, \mathcal{B})$, we deal with the following two problems:
(1) describe the regular elements of the cone $\mathcal{E}_{\mathcal{V}}$ of all $\mathcal{V}$-excessive $\mathcal{B}$-measurable functions in terms of balayage theory on $\mathcal{E}_{\mathcal{V}}$;
(2) establish the link between the existence of fine carrier for the regular elements of $\mathcal{E}_{\mathcal{V}}$ and the property that any balayage operator B on $\mathcal{E}_{\mathcal{V}}$ may be represented on X under the form $B = R^A$, where R^A is the reduite in $\mathcal{E}_{\mathcal{V}}$ on the set A.

For this purpose we associate to any element $s \in \mathcal{E}_{\mathcal{V}}$, $s < \infty$, a pseudo-balayage B_s on $\mathcal{E}_{\mathcal{V}}$, defined by

$$B_s t = \sup\{u \in \mathcal{E}_{\mathcal{V}} | u \leqslant t, u \leqslant \alpha s \text{ for some } \alpha > 0\}.$$

This operator was considered in the frame of standard H-cones in [5] where s is universally continuous and, in this case, B_s is a balayage. In our paper we consider elements $s \in \mathcal{E}_{\mathcal{V}}$ such that B_s is a balayage and we show that s has this property if and only if it is regular, that is $\bigwedge_n R(s - s_n) = 0$ for any sequence $(s_n)_n$ increasing to s. This gives an answer to problem (1).

In the case when $\mathcal{V}$ is a resolvent having the properties from the last Remark 5 of this paper, then starting from a result in [1] and [2] which asserts that for any analytic, basic subset M of X there exists a regular excessive function whose fine carrier is contained in M, we show that the properties from (2) hold if and only if the balayage B_s is representable for any regular element s of $\mathcal{E}_{\mathcal{V}}$.

2. PRELIMINARIES AND FIRST RESULTS

Throughout, $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a proper sub-Markovian resolvent of kernels on a measurable space $(X, \mathcal{B})$. We denote by S the set of all $\mathcal{B}$-measurable numerical functions s which are supermedian, *i.e.* $s : X \longrightarrow [0, +\infty]$ and $\alpha V_\alpha s \leqslant s$ for all $\alpha > 0$. Let S^f be the set of all real-valued functions from S.

Let $\mathcal{E}$ be the set of all excessive, $\mathcal{B}$-measurable functions, which are finite $\mathcal{V}$-a.e., that is

$$\mathcal{E} = \{s \in S / \sup_\alpha \alpha V_\alpha s = s \text{ and } V_\alpha(1_{[s=\infty]}) = 0 \text{ for one (hence all) } \alpha \in \mathbb{R}_+\}.$$

For any $s \in S$ the family $(\alpha V_\alpha s)_{\alpha \in \mathbb{R}_+}$ is increasing and the function $\widehat{s}$ defined by

$$\widehat{s} = \lim_{\alpha\to\infty} \alpha V_\alpha s = \lim_{n\to\infty} nV_n s = \sup_n nV_n s,$$

called the regularized of s (with respect to $\mathcal{V}$) is dominated by s and the set $[\widehat{s} < s]$ is $\mathcal{V}$-negligible, *i.e.* $V_\alpha(1_{[\widehat{s}<s]}) = 0$ for one (hence all) $\alpha \in \mathbb{R}_+$.

We recall that for any $\mathcal{B}$-measurable function f on X the set

$$\{s \in S / s \geqslant f\}$$

possesses the smallest element denoted by $R_0 f$. If f is of the form $s_2 - s_1$ with $s_1, s_2 \in S$, then

$$R_0(s_2 - s_1) := \bigwedge\{s \in S / s_1 + s \geqslant s_2\} \underset{S}{\preccurlyeq} s_2,$$

where we have written $u \underset{S}{\preccurlyeq} v$ if there exists $s \in S$ such that $v = u + s$, u and v being positive functions on X. The relation $\underset{S}{\preccurlyeq}$ is the so called *specific order* induced by S.

If $A \in \mathcal{B}$ and $s \in S$ then the element $R_0(1_A \cdot s)$ is called *the reduite of s on the set A* and it will be denoted by $R_0^A s$. The following properties of the reduite operation are well known (see *e.g.* [4]):

If $s_1, s_2 \in \mathcal{E}$ then $R_0(s_2 - s_1) \in \mathcal{E}$ and $R_0(s_2 - s_1) \underset{\mathcal{E}}{\preccurlyeq} s_2$ where $\underset{\mathcal{E}}{\preccurlyeq}$ is the specific order given by $\mathcal{E}$.

The set $(\mathcal{E}, \underset{\mathcal{E}}{\preccurlyeq})$ is a conditionally σ-complete lattice, *i.e.* for any sequence $(s_n)_n \subset \mathcal{E}$ there exists the greatest lower bound noted by $\bigwedge_n s_n$ and we have:

$$s + \bigwedge_n s_n = \bigwedge_n (s + s_n) \text{ for all } s \in \mathcal{E}.$$

If $(s_n)_n \in \mathcal{E}$ is specifically dominated in $\mathcal{E}$ there exists the smallest upper bound denoted by $\bigvee_n s_n$ and we have:

$$s + \bigvee_n s_n = \bigvee_n (s + s_n) \text{ for all } s \in \mathcal{E}.$$

Moreover, if the sequence $(s_n)_n$ is specifically increasing (resp. decreasing) then we have:

$$\bigvee_n s_n = \sup_n s_n \text{ (resp. } \bigwedge_{\mathrm{n}} \mathrm{s_n} = \inf_{\mathrm{n}} \mathrm{s_n}),$$

where $\sup_n s_n$ (resp. $\inf_n s_n$) is the pointwise supremum (resp. infimum) of the sequence of functions $(s_n)_n$ on X. Particularly, the *Riesz decomposition property* holds in $\mathcal{E}$ and S, *i.e.* for any s, t_1, t_2 belonging to $\mathcal{E}$ (resp. S) with $s \preccurlyeq t_1 + t_2$ there exist s_1, s_2 in $\mathcal{E}$ (resp. S) such that $s_1 \preccurlyeq t_1,\ s_2 \preccurlyeq t_2,\ s = s_1 + s_2$. In fact, the *$\sigma$-Riesz decomposition property* may be immediately shown

$$s \preccurlyeq \sum_{i=1}^{\infty} t_i \Rightarrow s = \sum_{i=1}^{\infty} s_i,\ s_i \preccurlyeq t_i \quad \text{for all } i \in \mathbb{N}.$$

Other well known assertions from the vector lattice theory may be restated in the convex cones $\mathcal{E}$ and S. Among them the following one will be used: for any s_1, s_2 in $\mathcal{E}$ (resp. S) we have

$$s_1 \bigwedge s_2 + s_1 \bigvee s_2 = s_1 + s_2.$$

The Riesz decomposition property with respect to the pointwise order relation holds in S (respectively $\mathcal{E}$), *i.e.* for any s, t_1, t_2 in S (resp. $\mathcal{E}$) with $s \leqslant t_1 + t_2$ there exist s_1, s_2 in S (resp. $\mathcal{E}$) such that $s = s_1 + s_2,\ s_1 \leqslant t_1,\ s_2 \leqslant t_2$.

The following decomposition property is inspired by a similar one used by G. Mokobodzki in the study of subordinate resolvents (cf. [4] and [8]).

LEMMA 2.1. *For any $s \in S^f$, and any $A \in \mathcal{B}$ there exist s_A and s'_A such that*

$$s = s_A + s'_A \quad \text{and} \quad R_0^A s_A = s_A,\ R_0^{X \setminus A} s'_A = s'_A.$$

Proof. We define inductively two sequences $(s'_n)_n$ and $(s''_n)_n$ in S as follows:

$$s''_1 = R_0(s - R^A s), \quad s'_1 = s - R_0(s - R^A s)$$

$$s''_{n+1} = R(_0 s'_n - R^A s'_n), \quad s'_{n+1} = s'_n - R_0(s'_n - R^A s'_n).$$

Obviously, we have $s'_n = s'_{n+1} + s''_{n+1}$ and one can show that $s'_{n+1} = R^A s'_n \preccurlyeq s'_n$ and $s''_{n+1} = R^{X\setminus A} s''_{n+1}$. So, the sequence $(s'_n)_n$ is specifically decreasing in S and the sequence $(\sum\limits_{i=1}^{n} s''_i)_n$ is specifically increasing in S and we have

$$s = s'_n + \sum_{i=1}^{n} s''_i \quad \text{for all } \ n \in \mathbb{N}^*.$$

Therefore, $s = s_A + s'_A$ where we have denoted

$$s_A = \inf_n s'_n = \bigwedge_S s'_n, \ s'_A = \sum_{i=1}^{\infty} s''_i := \sup_n \sum_{i=1}^{n} s''_i = \bigvee_n \sum_{i=1}^{n} s''_i.$$

From the preceding considerations we deduce

$$R_0^A(\bigwedge_n s'_n) = \bigwedge_n R_0^A s'_n = \bigwedge_n s'_n, \ R_0^A s_A = s_A.$$

$$R_0^{X\setminus A}(\sum_{i=1}^{\infty} s''_i) = \sum_{i=1}^{\infty} R_0^{X\setminus A} s''_i = \sum_{i=1}^{\infty} s''_i, \ R_0^{X\setminus A} s'_A = s'_A. \quad \square$$

Lemma 2.2 (A Choquet type lemma). *Let $(s_n)_n$ be a sequence in S and for any $n \in \mathbb{N}$ let $(s_{n,m})_{n,m}$ be a sequence in S which is specifically increasing to s_n.*

a. *We have*

$$\bigvee_S \{s_n / n \in \mathbb{N}\} = \bigvee_S \{t_n / n \in \mathbb{N}\},$$

where

$$t_n =: \bigvee_{i,j \leqslant n} s_{i,j}.$$

b. *If $s_n < \infty$, $n \in \mathbb{N}$, and for any sequence $\sigma = (m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ we denote*

$$s_\sigma = \bigwedge_S \{s_{n,m_n} / n \in \mathbb{N}\},$$

then we have

$$\bigwedge_S \{s_n / n \in \mathbb{N}\} = \sup\{s_\sigma / \sigma \in \mathbb{N}^{\mathbb{N}}\},$$

where sup *stands for the pointwise supremum and* $\mathbb{N}^{\mathbb{N}}$ *for the set of all sequences of natural numbers.*

Proof. a) Obviously we have

$$s_n = \bigvee_S \{s_{n,m} / m \in \mathbb{N}\} \preccurlyeq \bigvee_S \{t_k / k \in \mathbb{N}\} \preccurlyeq \bigvee_S \{s_k / k \in \mathbb{N}\}$$

and therefore

$$\bigvee_S \{s_n/n \in \mathbb{N}\} = \bigvee_S \{t_n/n \in \mathbb{N}\}.$$

b) Let $x \in X$ and let ε be a real number, $\varepsilon > 0$. Since the sequence $(s_{n,m})_m$ is specifically increasing (in S) to the element s_n of S we have

$$s_n(x) = \sup_m s_{n,m}(x) = \lim_{m\to\infty} s_{n,m}(x)$$

and therefore we may consider $m_n \in \mathbb{N}$ such that

$$s_n(x) \leqslant s_{n,m_n}(x) + \frac{\varepsilon}{2^n} \quad \text{or} \quad t_n(x) < \frac{\varepsilon}{2^n},$$

where $t_n \in S$ is such that $s_n = s_{n,m_n} + t_n$.

If we put $s_0 = \bigwedge_S \{s_n/n \in \mathbb{N}\}$, from the preceding consideration we have

$$s_0 \preccurlyeq s_{n,m_n} + s_0 \bigwedge (\sum_{i=1}^{n} t_i) \quad \text{for all} \quad n \in \mathbb{N},$$

$$s_0 \preccurlyeq s_{n,m_n} + \bigvee_S \{s_0 \bigwedge (\sum_{i=1}^{k} t_i)/k \in \mathbb{N}\},$$

$$s_0 \preccurlyeq \bigwedge_S \{s_{n,m_n}/n \in \mathbb{N}\} + \bigvee_S \{s_0 \bigwedge (\sum_{i=1}^{k} t_i)/k \in \mathbb{N}\}.$$

On the other hand at the point $x \in X$ the following inequality holds

$$\bigvee_S \{s_0 \bigwedge (\sum_{i=1}^{k} t_i)/k \in \mathbb{N}\}(x) = \lim_{k\to\infty} (s_0 \bigwedge (\sum_{i=1}^{k} t_i))(x) \leqslant$$

$$\leqslant \lim_{k\to\infty} \sum_{i=1}^{k} t_i(x) \leqslant \varepsilon$$

and therefore

$$s_0(x) \leqslant s_\sigma(x) + \varepsilon \quad \text{where} \quad \sigma = (m_n)_{n\in\mathbb{N}}.$$

The number ε being arbitrary we have

$$s_0(x) = \sup_{\sigma\in\mathbb{N}^{\mathbb{N}}} s_\sigma(x) \quad \text{for all} \quad x \in X. \quad \square$$

LEMMA 2.3. *Let $(s_n)_n$ be a sequence in $\mathcal{E}$ and for any $n \in \mathbb{N}$ let $(s_{n,m})_m$ be a sequence in $\mathcal{E}$ which is $\mathcal{E}$-specifically increasing to s_n.*

a. *If the sequence $(s_n)_n$ has a specific majorant in $\mathcal{E}$ then*

$$\bigvee_{\mathcal{E}} s_n = \bigvee_{\mathcal{E}} t_n$$

where

$$t_n =: \bigvee_{\mathcal{E}} \{s_{i,j}/i, j \leqslant n\}.$$

b. *If* $s_n < \infty$, $n \in \mathbb{N}$, *and for any sequence* $\sigma = (m_n)_{n\in\mathbb{N}}$ *in* $\mathbb{N}$ *we set*

$$s_\sigma = \bigwedge_{\mathcal{E}} \{s_{n,m_n}/n \in \mathbb{N}\},$$

then we have

$$\bigwedge_{\mathcal{E}} \{s_n/n \in \mathbb{N}\} = \sup\{s_\sigma/\sigma \in \mathbb{N}^{\mathbb{N}}\},$$

where sup *stands for the pointwise supremum and* $\mathbb{N}^{\mathbb{N}}$ *for the set of all sequences of natural numbers.*

Proof. We apply Lemma 2 and use the following properties of the specific order:

$$\bigvee_{\mathcal{E}} s_n = \bigvee_{S} s_n \ , \ \bigvee_{\mathcal{E}} t_n = \bigvee_{S} t_n \ , \ \bigwedge_{\mathcal{E}} \{s_{n,m_n}/n \in \mathbb{N}\} = \bigwedge_{S} \{s_{n,m_n}/n \in \mathbb{N}\}. \quad \square$$

3. PSEUDO-BALAYAGES ASSOCIATED WITH SUPERMEDIAN FUNCTIONS

A map $B : S \longrightarrow S$ is called *pseudo-balayage* on S if it is increasing (with respect to the pointwise order relation), additive, contractive ($Bs \leqslant s$) and idempotent ($B^2 s = B(Bs) = Bs$) for all $s \in S$.

A pseudo-balayage B is called *balayage* if it is σ-continuous in order from below, *i.e.* the sequence $(Bs_n)_n$ increases to Bs whenever the sequence $(s_n)_n$ increases to s.

A typical example of balayage on S is the map:

$$s \mapsto R_0^A s,$$

where $A \in \mathcal{B}$.

In the sequel, for any element $s \in S^f$ we associate a pseudo-balayage B_s such that $B_s s = s$. The procedure is inspired from a similar one developed in the frame of *standard H-cones.*

Proposition 3.1. *Let* $s \in S$ *be a finite element. Then for any* $t \in S$ *the set*

$$D_t := \{u \in S/u \leqslant t \text{ and } u \preccurlyeq \alpha s \text{ for some } \alpha > 0\}$$

has an upper bound in S *with respect to the pointwise order relation and the map*

$$t \mapsto \sup D_t := B_s t$$

is a pseudo-balayage with $B_s(s) = s$. *Moreover if* B *is a pseudo-balayage with* $B(s) = s$ *we have* $B_s \leqslant B$ *i.e.* $B_s t \leqslant Bt$ *for all* $t \in S$.

Proof. We consider the subset D_t^0 of D_t given by

$$D_t^0 = \{ns - R_0(ns - t)/n \in \mathbb{N}^*\}.$$

The set D_t^0 is countable and co-final in D_t *i.e.* for any $u \in D_t$ there exists $n \in \mathbb{N}$ such that

$$u \leqslant ns - R_0(ns - t).$$

Indeed, let $\alpha \in \mathbb{R}_+$ such that $u \preccurlyeq \alpha s$ and $u \leqslant t$. We have $u \preccurlyeq ns$ for $n \in \mathbb{N}$, $n \geqslant \alpha$ and we remark that

$$u = ns - R_0(ns - u).$$

On the other hand we notice that the sequence $(ns - R_0(ns - t))_n$ is increasing. Hence, the supremum of the set D_t^0 belongs to S and we have

$$B_s t = \sup D_t = \sup D_t^0 \leqslant t.$$

If $t = s$, obviously $s \in D_s$ and therefore $B_s s = s$.

The fact that the map B_s is increasing follows from the definition of B_s because if $t_1 \leqslant t_2$ then $D_{t_1} \subset D_{t_2}$.

Using the definition of the sets D_{t_1}, D_{t_2} and $D_{t_1+t_2}$ for $t_1, t_2 \in S$ we deduce, using Riesz decomposition property (with respect to the pointwise order relation) that

$$D_{t_1} + D_{t_2} = D_{t_1+t_2}.$$

So, we have

$$B_s(t_1 + t_2) = \sup D_{t_1+t_2} = \sup D_{t_1} + \sup D_{t_2} = B_s(t_1) + B_s(t_2).$$

For any $t \in S$ and any $u \in D_t$, we have $u \leqslant B_s t$, and by the definition of $D_{B_s t}$ we have $u \in D_{B_s t}$. Hence

$$u \leqslant B_s(B_s t), \quad B_s(t) \leqslant B_s(B_s(t)), \quad B_s(t) = B_s^2 t.$$

If B is a pseudo-balayage on S such that $Bs = s$, then for any $u \in S$, $u \preccurlyeq \alpha s$ for some $\alpha > 0$ we have

$$B(\alpha s) = \alpha B s = \alpha s,$$
$$B(u) + B(\alpha s - u) = B(\alpha s) = \alpha s = u + (\alpha s - u),$$
$$Bu \leqslant u, \quad B(\alpha s - u) \leqslant \alpha s - u$$

and therefore $Bu = u$, $B(\alpha s - u) = \alpha s - u$.

Let now $t \in S$ and $u \in D_t$. From the preceding consideration we deduce

$$Bu = u \quad \text{for all } u \in D_t, \quad B_s t = \sup_{u \in D_t} u = \sup_{u \in D_t} Bu \leqslant Bt. \quad \square$$

Remark. For the convex cone $\mathcal{E}$ we have similar definition of the pseudo-balayage or balayage operator $B : \mathcal{E} \to \mathcal{E}$.

Proposition 3.2. Proposition 2. *For any element $s \in \mathcal{E}^f$, the restriction to $\mathcal{E}$ of the map B_s is a pseudo-balayage on $\mathcal{E}$.*

Proof. We remark that for any t in S which is finite $\mathcal{V}$-a.e. we have $B_s t \in \mathcal{E}$. Indeed, we have $D_t \subset \mathcal{E}$ and therefore the supremum of the increasing and dominated sequence $(ns - R(ns-t))_n$ is an element of $\mathcal{E}$. □

4. FINE CARRIER FOR EXCESSIVE FUNCTIONS

In the sequel, we shall denote by $\mathcal{E}^0$ the set of all finite excessive functions s on X such that for any specific minorant $u \in \mathcal{E}$ $(u \preccurlyeq s)$ the associated pseudo-balayage B_u is a balayage on $\mathcal{E}$.

As in the introduction, for any subset A of X and any element $t \in \mathcal{E}$ we denote

$$R^A t := \inf\{t' \in \mathcal{E}/t' \geqslant t \text{ on } A\}.$$

We denote also by $\mathcal{E}^0$ the set of all elements $s \in \mathcal{E}^f$ for which the pseudo-balayage B_u on $\mathcal{E}$ (see Proposition 2) is a balayage for all $u \in \mathcal{E}$, $u \lll s$.

Generally, the function $R^A t$ is not $\mathcal{B}$-measurable but if it is then this function belongs to S and the function

$$x \mapsto \sup \alpha V_\alpha (R^A t)(x)$$

is denoted by $B^A t$. Obviously, $B^A t \in \mathcal{E}$.

Definition 4.1. The set A is called *subbasic* if the function $B^A s$ is defined for all $s \in \mathcal{E}$ and we have $B^A s = s$ on A.

A subbasic set M is called *a basic set* if we have

$$M = \{x \in X/B^M s(x) = s(x) \ \text{ for all } \ s \in \mathcal{E}\}.$$

Remark 4.1. Arguing as in [4], Proposition 1.7.1, one can show that a subset M of X is subbasic if and only if the function $R^A s$ belongs to $\mathcal{E}$ and therefore $R^A t = B^A s$ for all $s \in \mathcal{E}$.

Remark 4.2. If M is subbasic then the map on $\mathcal{E}$

$$s \mapsto B^M s$$

is a balayage on $\mathcal{E}$.

Remark 4.3. If M is a subbasic set and $b(M)$ is given by

$$b(M) = \{x \in X/B^M s(x) = s(x) \ \text{ for all } \ s \in \mathcal{E}\}$$

then $B^{b(M)} s = s$ for all $s \in \mathcal{E}$ and $b(M) \in \mathcal{B}$.

The last assertion follows immediately from the fact that

$$b(M) = [B^M V f_0 = V f_0],$$

where f_0 is a $\mathcal{B}$-measurable, $0 < f_0 < 1$ and $V f_0 < \infty$.

On the space X, we consider the *fine topology i.e.*, the coarsest topology τ on X making continuous all functions of the vector lattice $\mathcal{E}_b - \mathcal{E}_b$ of bounded functions on X. We suppose here that $\mathcal{E}$ is min-stable and $1 \in \mathcal{E}$.

Recall that all elements $s \in \mathcal{E}$ are continuous with respect to τ and any point $x_o \in X$ has a base of neighbourhoods of the form $x_0 \in [s - t > 0]$ with $s, t \in \mathcal{E}$, $t \leqslant s \leqslant 1$. Obviously, the elements of this base belong to $\mathcal{B}$.

Definition 4.2. DEFINITION. We say that a balayage B on $\mathcal{E}$ is *representable* if there exists a basic set in X denoted by $b(B)$ such that

$$Bs = B^{b(B)}s$$

for all $s \in \mathcal{E}$.

The space X is called *nearly saturated* if all balayages on $\mathcal{E}$ are representable. By Theorem 5.3.8 from [5] one can see that this definition agrees with that from [3] and [4].

From now on, we suppose that X is nearly saturated and the convex cone $\mathcal{E}$ is min-stable and contains the positive constant functions.

Definition 4.3. For any element $s \in \mathcal{E}^0$ we associate the subset $b(B_s)$ the base of the balayage B_s. We shall denote it by *carr* s and we shall call it the *fine carrier* of s (with respect to $\mathcal{E}$).

From Remark 4.3 we deduce that the set *carr* s is finely closed and we have

$$carr\ s = \varnothing \Leftrightarrow s = 0.$$

PROPOSITION 4.1. *The following assertions hold.*

1. *$\mathcal{E}^0$ is a solid convex sub-cone of $\mathcal{E}$ with respect to the specific order.*
2. *$carr(s_1 + s_2) = carr\ s_1 \cup carr\ s_2$ for all $s_1, s_2 \in \mathcal{E}^0$.*
3. *If $(s_n)_n$ is a sequence in $\mathcal{E}^0$ then the function $\sum\limits_{n=1}^{\infty} s_n$ belongs to $\mathcal{E}^0$ provided that the sum is finite, and the set $carr(\sum\limits_{n=1}^{\infty} s_n)$ is the closure (with respect to τ) of the set $\bigcup\limits_{n=1}^{\infty} carr\ s_n$.*

Proof. 1. and 2. First, we remark that if M_1, M_2 are basic sets then so is $M_1 \cup M_2$ and for any element $t \in \mathcal{E}$ we have

$$B^{M_1 \cup M_2} t = B^{M_1} t \vee B^{M_2} t.$$

Hence, if we take $M_1 = carr\ s_1$, $M_2 = carr\ s_2$ then

$$s_1 + s_2 = B^{M_1} s_1 + B^{M_2} s_2 \leqslant B^{M_1 \cup M_2} s_1 + B^{M_1 \cup M_2} s_2 = B^{M_1 \cup M_2}(s_1 + s_2) \leqslant s_1 + s_2,$$

$$B^{M_1 \cup M_2}(s_1 + s_2) = s_1 + s_2$$

and therefore for any $u \in \mathcal{E}$, $u \lll \alpha(s_1 + s_2)$ we have

$$B^{M_1 \cup M_2} u = u.$$

Hence for any $t \in \mathcal{E}$ and any $u \in \mathcal{E}$, $u \leqslant t$, $u \preccurlyeq \alpha(s_1 + s_2)$ for some $\alpha > 0$ we have

$$u = B^{M_1 \cup M_2} u \leqslant B^{M_1 \cup M_2} t, \quad B_{s_1+s_2} t \leqslant B^{M_1 \cup M_2} t.$$

We also have $B^{M_1 \cup M_2} t \leqslant B_{s_1+s_2}$ because $B_{s_i} \leqslant B_{s_1+s_2}$ for $i = 1, 2$. Hence the map on $\mathcal{E}$

$$t \mapsto B_{s_1+s_2} t = B^{M_1 \cup M_2} t$$

is a balayage on $\mathcal{E}$. The preceding considerations show that $s_1 + s_2 \in \mathcal{E}^0$ for all $s_1, s_2 \in \mathcal{E}^0$ and

$$carr(s_1 + s_2) = b(B_{s_1+s_2}) = b(B_{s_1}) \cup b(B_{s_2}) = carr\ s_1 \cup carr\ s_2.$$

The last assertion follows by the fact that a countable union of basic sets is a subbasic set. $\square$

A map $\mu : \mathcal{E} \longrightarrow \overline{\mathbb{R}}_+$ is called *σ-H-integral* if it is additive, increasing, σ-continuous in order from below, and for each $s \in \mathcal{E}$ there exists a sequence $(s_n)_n$ in $\mathcal{E}$, increasing to s such that $\mu(s_n) < \infty$ for all $n \in \mathbb{N}$. We would like to mark that the space of all σ-H-integrals is in one to one correspondence with the space of all σ-finite excessive measures on $(X, \mathcal{B})$, via the *energy functional* associated with $\mathcal{V}$; see [4], Theorem 1.4.6.

Proposition 4.2. *The following assertions hold for $u \in \mathcal{E}^0$.*

a) *If $s \geqslant u$ on $carr\ u$ then $s \geqslant u$ on X.*

b) *The set $carr\ u$ is fine closed and $\mathcal{B}$-measurable subset of X.*

c) *If F is a fine closed subset of X such that $s \geqslant u$ on X whenever $s \in \mathcal{E}$ and $s \geqslant u$ on F, then $carr\ u \subset F$.*

d) *$carr\ u = \{x \in X / \mu\ \sigma$-$H$-integral, $\mu \leqslant \varepsilon_x$ on $\mathcal{E}$, $\mu(u) = u(x) \Rightarrow \mu = \varepsilon_x\}$.*

Proof. a) We have

$$u = B_u u = B^{carr\ u} u \leqslant B^{carr\ u} s \leqslant s \text{ if } s \in \mathcal{E},\ s \geqslant u \text{ on } carr\ u.$$

Assertion b) follows from the fact that

$$carr\ u = b(B_u) = \{x \in X / B_u s(x) = s(x) \text{ for all } s \in \mathcal{E}\} =$$
$$= \{x \in X / B_u V f(x) = V f(x)\}$$

where f is $\mathcal{B}$-measurable, $0 < f < 1$, and $Vf < \infty$.

c) Using the hypothesis, we have

$$u \geqslant R^F u \geqslant u,\ R^F u = u,\ R^F(\alpha u) = \alpha u \quad \text{for all } \ \alpha \in \mathbb{R}_+.$$

Since $R^F s \leqslant s$ we deduce that

$$R^F v = v \quad \text{for all } \ v \in \mathcal{E},\ v \preccurlyeq u,$$

and therefore, for any $v \in \mathcal{E},\ v \leqslant s, v \preccurlyeq \alpha u$ for some $\alpha > 0$ we have

$$R^F s \geqslant R^F v = v.$$

The element v being arbitrary we get $B_u s \leqslant R^F s$ for any $s \in \mathcal{E}$.

Let now $x_0 \in carr\ u \backslash F$ and let $s_1, s_2 \in \mathcal{E},\ s_1 \leqslant s_2$ be such that

$$s_1 \leqslant s_2,\ s_1(x_0) < s_2(x_0),\ s_1 = s_2 \text{ on } F.$$

From the preceding considerations, we get the contradictory relations

$$R^F s_1 = R^F s_2 \text{ on } X,$$
$$0 < s_2(x_0) - s_1(x_0) = B^{carr\ u} s_2(x_0) - B^{carr\ u} s_1(x_0),$$

$$s_i \geqslant R^F s_i \geqslant B_u s_i\ i = 1, 2,\ s_i(x_0) \geqslant R^F s_i(x_0) \geqslant B_u s_i(x_0) = s_i(x_0),\ i = 1, 2.$$

Hence $carr\ u \backslash F = \varnothing$.

d) Let $x_0 \in X$ such that if μ is a σ-H-integral on $\mathcal{E}$ with $\mu \leqslant \varepsilon_{x_0}$ on $\mathcal{E}$ and $\mu(u) = u(x_0)$ then $\mu = \varepsilon_{x_0}$ on $\mathcal{E}$.

If $x_0 \notin carr\ u$ then using b) we may consider two functions

$$s_1, s_2 \in \mathcal{E},\ s_1 \leqslant s_2 \text{ on } X,\ s_1(x_0) < s_2(x_0) \text{ and } s_1 = s_2 \text{ on } carr\ u.$$

We take as σ-H-integral μ on $\mathcal{E}$ the map

$$s \mapsto B_u s(x_0) = \mu(s).$$

Clearly we have $\mu(s) \leqslant s(x_0)$ for all $s \in \mathcal{E}$ and $\mu(u) = u(x_0)$ and therefore, using the hypothesis $\mu(s) = s(x_0)$ for all $s \in \mathcal{E}$.

The last assertion gives us

$$B_u s_1 = B_u s_2 \text{ on } X,\ B_u s_i = \mu(s_i) = s_i(x_0),\ i = 1, 2,$$
$$s_1(x_0) = s_2(x_0).$$

This contradicts the choice of s_1 and s_2.

Let now $x_o \in carr\ u$ and let μ be a σ-H-integral such that $\mu \leqslant \varepsilon_{x_0}$ on $\mathcal{E}$, $\mu(u) = u(x_0)$. We get the relation

$$\mu(v) = v(x_0) \text{ for all } v \in \mathcal{E},\ v \preccurlyeq \alpha u \text{ for some } \alpha \in \mathbb{R}_+.$$

Hence taking $s \in \mathcal{E}$, $v \in \mathcal{E}$, $v \preccurlyeq \alpha u$ for some $\alpha \in \mathbb{R}_+$ and $v \leqslant s$, we have

$$\mu(s) \geqslant \mu(v) = v(x_0).$$

The element v being arbitrary we have

$$\mu(s) \geqslant B_u s(x_0) = s(x_0),\ \mu = \varepsilon_{x_0} \text{ on } \mathcal{E}.$$

Definition 4.4. An element $s \in \mathcal{E}^f$ is called *regular* if for any increasing sequence $(s_n)_n$ with $\sup s_n = s$ we have

$$\bigwedge_n R(s - s_n) = 0.$$

Note that the potentials are regular elements.

The following result is well known in standard H-cones (see [4, 5]).

PROPOSITION 4.3. *If s is a regular element of $\mathcal{E}$ then the associated pseudo-balayage B_s is a balayage.*

Proof. Let $(s_n)_n$ be a sequence in $\mathcal{E}$ increasing to s and for any $n \in \mathbb{N}$ let $u_n \in \mathcal{E}$ be such that

$$R(s - s_n) + u_n = s.$$

The sequence $(R(s-s_n))_n$ is decreasing and the sequence $(u_n)_n$ is increasing with respect to the pointwise order relation. Therefore, we have

$$u := \sup_n u_n \in \mathcal{E}, \text{ and } \inf_n R(s - s_n) \in S.$$

But since

$$u + \inf_n R(s - s_n) = s$$

we deduce that $\inf\limits_n R(s - s_n) \in \mathcal{E}$. Hence, using the regularity of s we have

$$\inf_n R(s - s_n) = \bigwedge_n R(s - s_n) = 0, \; u = s.$$

With the above notations we have

$$u_n \leqslant s_n \leqslant s, \; u_n = B_s u_n \leqslant B_s s_n \text{ for all } n \in \mathbb{N}.$$

Therefore $\sup B_s s_n = s$. Obviously, for any $\alpha \in \mathbb{R}_+$ and any sequence (s_n) in $\mathcal{E}$ increasing to αs we have $\sup B_s s_n = \alpha s$.

Now if $u \in \mathcal{E}$, $u \preccurlyeq s$ and $(u_n)_n$ is a sequence in $\mathcal{E}$ increasing to u we claim that

$$\sup B_s u_n = u.$$

Indeed, if we denote $v = s - u$ then the sequence $(u_n + v)_n$ increasing to s and therefore

$$\sup_n B_s(u_n + v) = s, \; \sup_n B_s u_n + v = s, \text{ and } \sup_n B_s u_n = u.$$

To finish the proof we consider an arbitrary element t of $\mathcal{E}$ and a sequence $(t_n)_n$ in $\mathcal{E}$ increasing to t. Let $u \in D_t$ and recall that

$$D_t = \{u \in S / u \preccurlyeq t \text{ and } u \preccurlyeq \alpha s \text{ for some } \alpha > 0\}.$$

Since $u \leqslant t$ then the sequence $(\inf(u, t_n))_n$ is in $\mathcal{E}$ and increases to $\inf(u, t) = u$. But $u \preccurlyeq \alpha s$ for some $\alpha \in \mathbb{R}_+$. By the above considerations we have

$$\sup_n B_s(\inf(u, t_n)) = B_s u = u.$$

Hence

$$\sup_n B_s t_n \geqslant \sup_n B_s(\inf(u, t_n)) = B_s u = u.$$

But u being arbitrary we get

$$\sup_n B_s t_n \geqslant B_s t, \ \sup_n B_s t_n = B_s t. \quad \square$$

THEOREM 4.1. *The element $s \in \mathcal{E}^f$ is regular if and only if for any $u \in \mathcal{E}$, $u \preccurlyeq s$ the pseudo-balayage B_u is a balayage on $\mathcal{E}$ i.e. $s \in \mathcal{E}^0$.*

Proof. Let $s \in \mathcal{E}^0$ and let $(s_n)_n$ be a sequence in $\mathcal{E}$ increasing to s. For $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and $n \in \mathbb{N}$, $n > 0$ we denote by A_n the subset of X given by

$$A_n = \left[s < s_n + \left(1 - \frac{1}{n}\right)\varepsilon\right] = \{x \in X / s(x) < s_n(x) + \left(1 - \frac{1}{n}\right)\varepsilon\}.$$

Clearly we have $\overline{A}_n \subset A_n$ and A_n is fine open for every $n \in \mathbb{N}$, $n > 0$. Moreover $\bigcup_{n=1}^{\infty} A_n = X$.

Let $u_n = R\left(s - s_n - \left(1 - \frac{1}{n}\right)\varepsilon\right)$ and $v_n = s - u_n$. Then

$$u_n = R^{X \backslash A_n} u_n =^{\mathcal{E}} R^{X \backslash A_n} u_n \text{ since } \left[s > s_n + \left(1 - \frac{1}{n}\right)\varepsilon\right] \subset X \backslash A_n$$

and therefore

$$u_{n+m} = R^{X \backslash A_{n+m}} u_{n+m} \leqslant R^{X \backslash A_n} u_{n+m} \quad \text{for all } n, m \in \mathbb{N}^*,$$
$$u_{n+m} = R^{X \backslash A_n} u_{n+m} \quad \text{for all } n, m \in \mathbb{N}^*.$$

Since $u_n + v_n = s$, and the sequence $(u_n)_n$ is decreasing it follows that the sequence $(v_n)_n$ is increasing to an element $v \in \mathcal{E}$, and if we denote $u = \inf_n u_n$, we have

$$u \in S, \ u + v = s, \ \widehat{u} + \widehat{v} = \widehat{s}, \ \widehat{u} + v = s, \ u = \widehat{u}.$$

Hence $u \in \mathcal{E}^0$ and from the preceding consideration, it follows

$$R^{X \backslash A_n}(u_{n+m} + v_{n+m}) = R^{X \backslash A_n} s \quad \text{for all } n, m \in \mathbb{N}^*,$$
$$u_{n+m} + R^{X \backslash A_n}(v_{n+m}) = R^{X \backslash A_n} s \quad \text{for all } n, m \in \mathbb{N}^*.$$

Leting $m \to \infty$ we obtain

$$u + R^{X \backslash A_n} v = R^{X \backslash A_n} s.$$

On the other hand
$$R^{X\backslash A_n}u + R^{X\backslash A_n}v = R^{X\backslash A_n}s$$
and therefore $R^{X\backslash A_n}u = u$. The set $X\backslash A_n$ being finely closed we deduce *carr* $u \subset X\backslash A_n$ for any $n \in \mathbb{N}$. But $\bigcap\limits_{n=1}^{\infty} X\backslash A_n = \varnothing$ and therefore $u = 0$, $\inf_n R\left(s - s_n - \left(1 - \frac{1}{n}\right)\varepsilon\right) = 0$.

The relations
$$^{\mathcal{E}}R(s-s_n) \leqslant {}^{\mathcal{E}}R\left(s - s_n - \left(1 - \frac{1}{n}\right)\varepsilon\right) + \left(1 - \frac{1}{n}\right)\varepsilon \quad \text{for all } n \geqslant 1 \text{ and } \varepsilon > 0$$
lead to
$$\inf_n {}^{\mathcal{E}}R(s - s_n) \leqslant \varepsilon, \quad \inf_n {}^{\mathcal{E}}R(s - s_n) = 0,$$
that is s is a regular element of $\mathcal{E}$.

Conversely, if s is regular then any element $u \in \mathcal{E}$, $u \preccurlyeq s$ is also regular and by Proposition 3 the pseudo-balayage B_u is a balayage, *i.e.* $s \in \mathcal{E}^0$. $\square$

Remark 4.4. In their papers concerning the semi-polar sets and regular excessive functions respectively balayages on excessive measures L. Beznea and N. Boboc (see [1] and [2]) show that for any basic set M which is analytic there exists a bounded regular excessive function q such that its fine carrier is contained in M.

Remark 4.5. We may prove the following assertion:

Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a resolvent family of kernels on a measurable space $(X, \mathcal{B})$ *i.e.*,

a) $\mathcal{V}$ is a proper sub-Markovian resolvent of kernels.

b) The convex cone $\mathcal{E}$ of all excessive functions with respect to v is min-stable and contains the positive constant functions.

c) There exists a distance d on X such that the associated topology τ_d is smaller than the fine topology on X.

d) The Borel structure associated with the distance d coincides with $\mathcal{B}$. In this case if the space $(X, \mathcal{B})$ is such that for any regular and bounded excessive function p with respect to the resolvent $\mathcal{V}$, the balayage associated as above to p is representable, then all balayages on $\mathcal{E}$ are representable.

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