PERTURBED LAPLACE OPERATORS ON FINITE NETWORKS

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Communicated by Gabriela Marinoschi

On a finite network X, Δ denotes the Laplace operator and for any real-valued function $q(x)$ on X, the operator $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ represents a perturbation of Δ. Assuming that the conductance in X is not necessarily symmetric (non-reversible case) and that the function $q(x)$ is arbitrary (so that it is not anymore necessary the matrix associated to $-\Delta_q$ to be positive semi-definite), some results are proved using matrix methods which help solving the Poisson problem of finding a solution $u(x)$ to the equation $\Delta_q u(x) = f(x)$ on X for a given real-valued function $f(x)$. Consequently, Dirichlet-Poisson and Neumann-Poisson equations on proper subsets of X are solved.

AMS 2010 Subject Classification: 15A18, 35J05, 31C20, 60J10.

Key words: finite networks, discrete Laplace, Schrödinger, Helmholtz equations, Perron-Frobenius eigenvalue, Dirichlet-Poisson problem.

1. INTRODUCTION

Function theory on finite networks has a nice model in electrical networks. As an endomorphism of the family of real-valued functions on the finite network X, the Laplacian operator Δ plays a pivotal role in the development of the theory. In this model, when the operator Δ is identified with a matrix, the latter is a symmetric irreducible matrix with nonnegative off-diagonal entries. For a real-valued function $q(x)$ on X, the operator Δ_q defined by $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ is known as a perturbation of the Laplace operator Δ. When the matrix representing $-\Delta_q$ is positive definite or sometimes even positive semi-definite, the Poisson equation and discrete boundary-value problems like the Dirichlet problem, the Neumann problem and the Robin problem have been shown to have solutions, see Bendito *et al.* [3, 4, 7]; actually in [7] the symmetric case in a path has been studied even when the quadratic form associated to $-\Delta_q$ is not positive semi-definite.

It is relevant to remark here, from another perspective when the conductance in the network is symmetric, the local and global properties of the Laplacian eigenvectors and the study of nodal domains (Biyikoğlu *et al.* [6])

and certain spectral properties of $-\Delta_q$ like those of the second smallest eigenvalue of $-\Delta_q$ referred to as its algebraic connectivity and of the corresponding eigenvectors (Bapat *et al.* [3]) are of algebraic interest related to the function-theoretic content of this paper.

There is another model for function theory on finite networks, namely Markov chains in the finite state space X. In this model the transition probabilities $\{p(x,y)\}$ are generally not symmetric. However in this case usually an assumption (reversibility condition) that there exists a function $\phi(x) > 0$ on X such that $\phi(x)p(x,y) = \phi(y)p(y,x)$ for any two states x, y in X is introduced, which leads to a situation in X as in finite electrical networks with symmetric conductance.

In this note, without assuming the symmetry or the reversibility of the transition function on a finite network, we find matrix methods based on the Perron-Frobenius theorem to solve the Poisson equation $\Delta_q u(x) = f(x)$ and then the Δ_q-Dirichlet problem. In this general set-up the equation $\Delta_q u(x) = 0$ includes the discrete analogues of Laplace, Schrödinger and Helmholtz equations.

2. PERRON-FROBENIUS REPRESENTATION FOR FUNCTIONS ON A FINITE NETWORK

2.1. Perron-Frobenius theorem

Let us recall certain remarkable features of the Perron-Frobenius theorem (see Gantmacher [5]) which asserts the properties of the leading eigenvalue and of the associated eigenvectors for positive (or nonnegative) square matrices. Note that a matrix $A = (a_{ij})$ is called a positive matrix if all its entries are positive ($a_{ij} > 0$, for every i, j) and is called a nonnegative matrix if all its entries are nonnegative ($a_{ij} \geq 0$ for every i, j).

Let $A = (a_{ij})$ be an $n \times n$ positive matrix, that is $a_{ij} > 0$ for all entries. Then some of the important points of the Perron theorem are:

i) If $\rho = \rho(A)$ is the spectral radius of A, then ρ is positive and is a simple root of the characteristic polynomial of A whose associated eigenspace is of dimension 1.

ii) If λ is any other eigenvalue of A, then $|\lambda| < \rho$.

iii) There exists a positive $n \times 1$ right eigenvector u of A with associated eigenvalue ρ, that is there exists u such that $Au = \rho u.$ and $u_i > 0$ for every i.

iv) There exists a positive $1 \times n$ left eigenvector v^t such that $v^t A = \rho v^t$.

v) All other eigenvectors of A contain at least one entry that is nonpositive.

Frobenius extended the above results to the case when $A = (a_{ij})$ is a nonnegative matrix. To suit our purpose here, a part of this extension can be stated thus: Associate a directed graph G with the matrix A such that G has exactly n vertices and there is an edge from vertex i to vertex j if and only if $a_{ij} > 0$. Then the nonnegative matrix A is said to be irreducible if the graph G is strongly connected, that is given any two vertices i, j there exist directed paths from i to j and from j to i. Conversely to each strongly connected directed graph without self-loops, we associate a nonnegative irreducible matrix whose diagonal elements are 0. Let now A be an irreducible nonnegative square matrix with spectral radius ρ. Then A may have other eigenvalues whose absolute value is ρ. However, the other properties excluding ii) mentioned above for positive matrices are valid when A is only nonnegative but irreducible.

2.2. **Perron-Frobenius representation of functions**

Let X denote a set of a finite number of points called vertices. Let $t : X \times X \to \mathbb{R}^+$ be a function where $t(x, x) = 0$ for every $x \in X$; $t(x, y)$ and $t(y, x)$ do not have to be the same; and $t(x, y)$ is referred to as a transition function on X. If $t(x, y) = t(y, x)$ for all pairs of vertices x, y in X, then we call $\{t(x, y)\}$ a set of conductance in X and in this case, for the sake of clarity write $c(x, y)$ instead of $t(x, y)$. Say that a vertex y is a neighbour of x, denoted by $y \sim x$, if and only if $t(x, y) > 0$. Thus $\{X, t\}$ defines a finite directed graph. For any real-valued function $u(x)$ on X, define the Laplacian $\Delta u(x) = \sum\limits_{y \sim x} t(x, y)[u(y) - u(x)]$, $x \in X$. When the transition function t is actually a conductance, $\mathcal{L} = -\Delta$ is commonly referred to as the combinatorial Laplacian.

If we represent X as $X = \{x_1, x_2, \ldots, x_n\}$, then any function $u(x)$ on X can also be considered as a (column) vector and in that case Δ is represented as a matrix $\{t_{ij}\}$ where $t_{ij} = t(x_i, x_j)$ if $i \neq j$ and $t_{ii} = -\sum\limits_{j \neq i} t_{ij}$. For any real-valued function $q(x)$ on X, the operator Δ_q defined as $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ is referred to as a perturbation of the Laplacian Δ. Note that Δ_q can be identified with the matrix $\{m_{ij}\}$ where $m_{ij} = t_{ij}$ if $i \neq j$ and $m_{ii} = t_{ii} - q_i$, $q_i = q(x_i)$. In the case of symmetric conductance, the symbol $\mathcal{L}_q$ is used in the place of $-\Delta_q$.

Suppose x, y are two vertices in $\{X, t\}$ such that there exists a sequence $\{x = a_0, a_1, \ldots, a_k = y\}$, $a_{i+1} \sim a_i$ for $0 \leq i \leq k-1$, so that $\prod\limits_{i=0}^{k-1} t(a_i, a_{i+1}) > 0$. Then y is said to be connected to x by a directed path. If any two vertices in X are connected by a directed path, $\{X, t\}$ is said to be strongly connected.

From now on, when we refer to $\{X, t\}$ as a finite network in this paper, we shall assume that it is strongly connected.

A real-valued function $s(x)$ on $\{X, t\}$ is said to be Δ-subharmonic on X if $\Delta u(x) \geq 0$ at every vertex $x \in X$; analogously u is Δ-harmonic if $\Delta u(x) = 0$ and Δ-superharmonic if $\Delta u(x) \leq 0$.

LEMMA 2.1. *Any Δ-subharmonic, -harmonic, -superharmonic function on a finite network $\{X, t\}$ is constant.*

Proof. Let us consider only the case where u is a Δ-subharmonic function on X, the other two remarks are proved analogously. The network X is a finite set, thus $\max\limits_{x \in X} u(x) = u(z)$ for some vertex $z \in X$. Let $u(z) = M$. Since X is strongly connected, for any $x \in X$, there exists a directed path $\{z = a_1, a_2, \dots, a_j = x\}$ from z to x. Now $\Delta u(z) \geq 0$ means $\sum\limits_{y \sim z} t(y, z)[u(y) - u(z)] \geq 0$. Since $u(y) - u(z) \leq 0$, we deduce that $u(y) - u(z) = 0$ for any $y \sim z$. In particular $u(a_2) = M$. Proceeding step by step, we show that $u(a_3) = M, \dots, u(a_j) = M$. Thus $u(x) = M$ for all $x \in X$. $\square$

Suppose $\{X, t\}$ is a finite network and $\xi(x) > 0$ is a function on X. Then $\{X, \hat{t}(x, y) = \xi(y)t(x, y)\}$ and $\left\{X, t(x, y) = \dfrac{\xi(y)}{\xi(x)}t(x, y)\right\}$ are also networks with their transition indices different from those of the initial network. We denote by $\widehat{\Delta}$ and $\widetilde{\Delta}$ the Laplacians associated with these two new networks respectively, so that

$$\widehat{\Delta}u(x) = \sum_{y \sim x} t(x, y)\xi(y)[u(y) - u(x)]$$

and

$$\widetilde{\Delta}u(x) = \sum_{y \sim x} t(x, y)\frac{\xi(y)}{\xi(x)}[u(y) - u(x)].$$

THEOREM 2.2. *Any real-valued function $q(x)$ on a finite network X is of the form $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, where c is a constant, $\xi(x) > 0$ and $\sum\limits_{x \in X} \xi(x) = 1$. In this representation $\xi(x)$ and c are uniquely determined.*

Proof. Let X have n vertices, $t_{ij} \geq 0$ represent the transition index from the vertex i to the vertex j. Then for any i, $t_{ii} = -\sum\limits_{j \neq i} t_{ij} < 0$. For $\lambda > 0$ large, let $A = (a_{ij})$ be the non-negative irreducible matrix where $a_{ij} = t_{ij}$ if $i \neq j$ and $a_{ii} = \lambda - q(x_i) + t_{ii} > 0$. Then by Perron-Frobenius theorem, there is an eigenvalue $\mu > 0$ which is simple and a right eigenvector v associated to μ chosen such that all its entries positive. That is, $Av = \mu v$ which can be written as $\Delta v - qv + \lambda v = \mu v$ where Δ is the classical Laplacian matrix.

Hence $q(x) = \dfrac{\Delta v(x)}{v(x)} + (\lambda - \mu)$. Take $\xi(x) = \dfrac{v(x)}{\sum\limits_x v(x)}$ and $c = \lambda - \mu$ to get the representation $q(x) = \dfrac{\Delta \xi(x)}{\xi(x)} + c$. Suppose $q(x) = \dfrac{\Delta \eta(x)}{\eta(x)} + c_1$ is another such representation, with $\sum\limits_{x \in X} \eta(x) = 1$. If $c \geq c_1$, then $\dfrac{\Delta \eta(x)}{\eta(x)} \geq \dfrac{\Delta \xi(x)}{\xi(x)}$ so that $\eta(x)$, $\xi(x)$ are proportional by the ensuing lemma, consequently $\eta(x) = \xi(x)$ for every $x \in X$. Then $c = c_1$ and hence the uniqueness of the representation. □

Definition 2.3. For a real-valued function $q(x)$ on X, we shall refer to the unique representation $q(x) = \dfrac{\Delta \xi(x)}{\xi(x)} + c$ as its Perron-Frobenius representation on X.

The following Lemma 2.4 has a proof in [5, Lemma 2.1] in the symmetric conductance case:

LEMMA 2.4. *Let $\eta(x)$, $\xi(x)$ be two positive functions on X such that $\dfrac{\Delta \eta(x)}{\eta(x)} \geq \dfrac{\Delta \xi(x)}{\xi(x)}$. Then $\eta(x), \xi(x)$ must be proportional. Consequently, there is no pair of positive functions $\eta(x), \xi(x)$ on X such that $\dfrac{\Delta \eta(x)}{\eta(x)} > \dfrac{\Delta \xi(x)}{\xi(x)}$ for every $x \in X$.*

Proof. By hypothesis, $\xi(x)\Delta\eta(x) - \eta(x)\Delta\xi(x) \geq 0$ on X. That is, $\sum\limits_{y \sim x} t(x, y)$ $\xi(x)\eta(y) - \eta(x)\xi(y)] \geq 0$. Hence, $\xi(x) \sum\limits_{y \sim x} t(x, y)\xi(y) \left[\dfrac{\eta(y)}{\xi(y)} - \dfrac{\eta(x)}{\xi(x)}\right] \geq 0$, which shows that $\hat{\Delta}\left[\dfrac{\eta(x)}{\xi(x)}\right] \geq 0$ where $\widehat{\Delta}$ is the Laplacian associated to the finite network $\{X, t(x, y)\xi(y)\}$. Hence $\dfrac{\eta(x)}{\xi(x)}$ is a $\widehat{\Delta}$- subharmonic function on X, so that $\dfrac{\eta(x)}{\xi(x)}$ should be a constant (Lemma 2.1). This means $\dfrac{\Delta \eta(x)}{\eta(x)} = \dfrac{\Delta \xi(x)}{\xi(x)}$ for every $x \in X$, and hence the stated consequence. □

Note 1. Suppose $c(q)$ denotes the constant c in the unique representation of $q(x) = \dfrac{\Delta \xi(x)}{\xi(x)} + c$. Note that by Lemma 2.4, it follows that if $q_1 \geq q_2$, then $c(q_1) \geq c(q_2)$; if $q_1 \geq q_2$ and $c(q_1) = c(q_2)$, then $q_1 = q_2$.

As a consequence of Theorem 2.2, we have the following proposition using a Doob (Liouville) transform.

PROPOSITION 2.5. *Let* $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$ *on* X. *Then the operator* $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ *can be represented as* $\Delta_q u(x) = \xi(x)\left\{\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] - c\left[\dfrac{u(x)}{\xi(x)}\right]\right\}$ *where* $\widetilde{\Delta}$ *is the Laplacian associated to the finite network* $\left\{X, t(x,y)\dfrac{\xi(y)}{\xi(x)}\right\}$.

Proof.

$$\begin{aligned}
\Delta_q u(x) =& \Delta u(x) - \left[\frac{\Delta\xi(x)}{\xi(x)} + c\right] u(x) \\
=& \frac{1}{\xi(x)} \sum_{y\sim x} t(x,y)[\xi(x)u(y) - u(x)\xi(y)] - cu(x) \\
=& \sum_{y\sim x} t(x,y)\xi(y)\left[\frac{u(y)}{\xi(y)} - \frac{u(x)}{\xi(x)}\right] - cu(x) \\
=& \xi(x)\left\{\sum_{y\sim x} t(x,y)\frac{\xi(y)}{\xi(x)}\left[\frac{u(y)}{\xi(y)} - \frac{u(x)}{\xi(x)}\right] - c\left[\frac{u(x)}{\xi(x)}\right]\right\} \\
=& \xi(x)\left\{\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right] - c\left[\frac{u(x)}{\xi(x)}\right]\right\}. \quad \square
\end{aligned}$$

2.3. **Classification of perturbed Laplace operators**

Let $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$. Since $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, we have three different possibilities when we consider the equation $\Delta_q u(x) = 0$ on X, depending on $c = 0, c < 0, c > 0$. They resemble a generalised discrete version of the three fundamental differential equations on the real line $\mathbb{R}$, namely $y'' = 0$, $y'' + y = 0$, $y'' - y = 0$.

In the symmetric case where $t(x,y) = t(y,x)$, it is proved in [4, p. 782], that for a real-valued function $q(x)$ on X, $-\Delta_q$ is positive semidefinite if and only if there exists a positive function $\sigma(x)$ on X such that $q(x) \geq \dfrac{\Delta\sigma(x)}{\sigma(x)}$; moreover $-\Delta_q$ is a singular matrix if and only if $q = \dfrac{\Delta\sigma}{\sigma}$. Related to this result in the general case we are considering in this paper, we have the following two lemmas.

LEMMA 2.6. *Let* $q(x)$ *be a real-valued function on* X. *Take the representation* $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$. *Then for a positive function* $\sigma(x)$ *on* X, $q(x) \geq \dfrac{\Delta\sigma(x)}{\sigma(x)}$ *if and only if* $c \geq 0$.

Proof. Suppose $q(x) \geq \dfrac{\Delta\sigma(x)}{\sigma(x)}$. That is $\dfrac{\Delta\xi(x)}{\xi(x)} + c \geq \dfrac{\Delta\sigma(x)}{\sigma(x)}$. If $c < 0$, then $\dfrac{\Delta\xi(x)}{\xi(x)} > \dfrac{\Delta\sigma(x)}{\sigma(x)}$, not possible (Lemma 2.4). Hence $c \geq 0$; the converse is obvious. □

Lemma 2.7. *Let $q(x)$ be a real-valued function on X, with the representation $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$. Then the following are equivalent.*

i) $c > 0$.

ii) *There exists a positive function $\eta(x)$ such that $q(x) > \dfrac{\Delta\eta(x)}{\eta(x)}$ for every x in X.*

iii) *There exists a positive function $\sigma(x)$ in X such that $q(x) \geq \dfrac{\Delta\sigma(x)}{\sigma(x)}$ and at some vertex y, $q(y) > \dfrac{\Delta\sigma(y)}{\sigma(y)}$.*

Proof. Enough to check iii) $\Rightarrow$ i). Now if iii) holds, then by Lemma 2.6, $c \geq 0$. If $c = 0$, then by Lemma 2.4, $\xi(x)$ and $\sigma(x)$ are proportional so that $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} = \dfrac{\Delta\sigma(x)}{\sigma(x)}$ for all x in X. This is a contradiction, since by assumption $q(y) > \dfrac{\Delta\sigma(y)}{\sigma(y)}$. □

Remark 2.1. In line with Lemma 2.6, we remark the following:

1) Let $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$ be the canonical representation of a real-valued function $q(x)$. Then $c \leq 0$ if and only if $q(x) \leq \dfrac{\Delta\eta(x)}{\eta(x)}$ for some column vector $\eta(x)$ with all its entries positive.
2) In particular, if $q(x) \leq 0$, then $c \leq 0$. Similarly if $q(x) \geq 0$, then $c \geq 0$.
3) The representation $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$ can be expanded as $q(x) = \left[\sum\limits_{y\sim x} t(x,y)\dfrac{\xi(y)}{\xi(x)}\right] - t(x) + c$ where $t(x) = \sum\limits_{y\sim x} t(x,y)$. Hence, $[t(x)+q(x)] - c > 0$. Consequently, $c < \min_{x\in X}[t(x) + q(x)]$ so that if $t(a) + q(a) \leq 0$ for some $a \in X$, then $c < 0$.
4) Even if $c > 0$ in the representation $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, it does not mean that $q(x) \geq 0$. For example, take a positive nonconstant function $\xi(x)$ on X such that $\sum\limits_{x\in X} \xi(x) = 1$. Then $\dfrac{\Delta\xi(x)}{\xi(x)}$ takes both negative and positive

values since $\xi(x)$ is not constant. That is, if $\alpha = \min_{x\in X} \dfrac{\Delta\xi(x)}{\xi(x)}$ and $\beta = \max_{x\in X} \dfrac{\Delta\xi(x)}{\xi(x)}$, then $\alpha < 0$ and $\beta > 0$. Choose a constant c such that $0 < c < -\alpha$. With these choices if $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, then $q(x)$ takes both negative and positive values, while $c > 0$. Consequently even when $c > 0$, the matrix $-\Delta_q$ need not be even weakly diagonal dominant.

Proposition 2.8. *Let $q(x)$ be a real-valued function on X with representation $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$. Write $-\Delta_q u(x) = q(x)u(x) - \Delta u(x)$. Then,*

i) *$c = 0$ if and only if for some $\eta(x) > 0$, $-\Delta_q\eta(x) = 0$;*
ii) *$c > 0$ if and only if for some $\eta(x) > 0$, $-\Delta_q\eta(x) > 0$;*
iii) *$c < 0$ if and only if for some $\eta(x) > 0$, $-\Delta_q\eta(x) < 0$.*

Proof. i) If $c = 0$, then $-\Delta_q\xi(x) = 0$. Conversely, suppose $-\Delta_q\eta(x) = 0$ for some $\eta(x) > 0$. That is $\left[\dfrac{\Delta\xi(x)}{\xi(x)} + c\right]\eta(x) - \Delta\eta(x) = 0$. Hence, $\dfrac{\Delta\xi(x)}{\xi(x)} - \dfrac{\Delta\eta(x)}{\eta(x)} = -c$. In this case, whatever be the value of c, $\xi(x)$, $\eta(x)$ are proportional by Lemma 2.4. Consequently $c = 0$.

ii) If $c > 0$, then $-\Delta_q\xi(x) = c\xi(x) > 0$. Conversely, suppose $-\Delta_q\eta(x) > 0$ for some $\eta(x) > 0$. Then, $\dfrac{\Delta\xi(x)}{\xi(x)} - \dfrac{\Delta\eta(x)}{\eta(x)} > -c$. In this case if $c \le 0$ then by Lemma 2.4, $\xi(x), \eta(x)$ are proportional; hence $0 > -c$, a contradiction. Thus $c > 0$ is the only possibility given that $-\Delta_q\eta(x) > 0$.

iii) Proved as in ii). □

Lemma 2.9. *Let $\lambda = \alpha + i\beta$ be an eigenvalue of the Laplacian Δ (which maybe non-symmetric) on a finite network $\{X, t(x,y)\}$. Then $\alpha \le 0$.*

Proof. Suppose $\alpha > 0$. Then for each row j in Δ which is of the form (t_{ij}) where $-t_{jj} = \sum\limits_{k\ne j} t_{jk}$,

$$|t_{jj} - \lambda| \ge -t_{jj} + \alpha > -t_{jj} = \sum_{k\ne j} t_{jk}.$$

Hence the strictly diagonally dominant matrix $\Delta - \lambda I$ is non-singular (Minkowski), a contradiction with the fact that λ is an eigenvalue of Δ. Consequently, if $\lambda = \alpha + i\beta$ is an eigenvalue of Δ, then $\alpha \le 0$. □

Remark 2.2. This lemma can also be seen as a consequence of the Gerschgorin Theorem.

THEOREM 2.10. *Let $q(x)$ be a real-valued function on X. Let its unique representation be* $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$ *which can be written as* $\Delta_q\xi(x) = \Delta\xi(x) - q(x)\xi(x) = -c\xi(x)$.

i) *Thus, for the matrix $-\Delta_q$, the uniquely determined constant c is an eigenvalue.*

ii) *The eigenspace associated with c is of dimension one.*

iii) *The column vector $\xi(x)$ is the only eigenvector associated to c with all its entries positive and* $\sum_{x\in X} \xi(x) = 1$.

iv) *If β is any eigenvalue of $-\Delta_q$, then $Re\beta \geq c$.*

Proof. ii) To prove the eigenspace associated with c is of dimension 1, suppose $\Delta_q\eta(x) = -c\eta(x)$ for some column vector $\eta(x)$. Since $c = \dfrac{-\Delta_q\xi(x)}{\xi(x)}$, we have $\Delta_q\eta(x) = \left[\dfrac{\Delta_q\xi(x)}{\xi(x)}\right]\eta(x)$, which when simplified as earlier, can be written as

$$\sum_{y\sim x} t(x,y)\xi(x)\xi(y)\left[\frac{\eta(x)}{\xi(x)} - \frac{\eta(y)}{\xi(y)}\right] = 0.$$

This implies that $\dfrac{\eta(x)}{\xi(x)}$ is constant. The eigenspace associated with the eigenvalue c is of dimension 1.

iii) Suppose a column vector $\sigma(x)$ with all its components positive is an eigenvector associated with some eigenvalue α of $-\Delta_q$. That is $-\Delta_q\sigma(x) = \alpha\sigma(x)$ which can be displayed as $-\Delta\sigma(x) + q(x)\sigma(x) = \alpha\sigma(x)$. Hence $q(x) = \dfrac{\Delta\sigma(x)}{\sigma(x)} + \alpha$ so that by the uniqueness of representation of $q(x)$, we have $\alpha = c$ and $\sigma(x)$ is proportional to $\xi(x)$.

iv) Let β be an eigenvalue of $-\Delta_q$, that is for some nonzero vector $\phi(x)$, $-\Delta_q\phi(x) = \beta\phi(x)$. Then, since $-\Delta_q\xi(x) = c\xi(x)$,

$$\begin{aligned} -\Delta_q\phi(x) =& c\phi(x) + (\beta - c)\phi(x) \\ =& \left[\frac{-\Delta_q\xi(x)}{\xi(x)}\right]\phi(x) + (\beta - c)\phi(x), \end{aligned}$$

which can be written as

$$\phi(x)\Delta\xi(x) - \xi(x)\Delta\phi(x) = \quad (\beta - c)\phi(x)\xi(x).$$

$$\begin{aligned}
\text{That is, } \sum_{y\sim x} t(x,y)[\xi(y)\phi(x)-\xi(x)\phi(y)] &= (\beta-c)\phi(x)\xi(x)\\
\sum_{y\sim x} t(x,y)\xi(x)\xi(y)\left[\frac{\phi(x)}{\xi(x)}-\frac{\phi(y)}{\xi(y)}\right] &= (\beta-c)\phi(x)\xi(x)\\
\sum_{y\sim x} t(x,y)\frac{\xi(y)}{\xi(x)}\left[\frac{\phi(x)}{\xi(x)}-\frac{\phi(y)}{\xi(y)}\right] &= (\beta-c)\left[\frac{\phi(x)}{\xi(x)}\right]\\
-\widetilde{\Delta}\left[\frac{\phi(x)}{\xi(x)}\right] &= -(c-\beta)\left[\frac{\phi(x)}{\xi(x)}\right]
\end{aligned}$$

where $\widetilde{\Delta}$ is the Laplacian operator associated with the network $\left\{X, t(x,y)\dfrac{\xi(y)}{\xi(x)}\right\}$. Hence $(c-\beta)$ is an eigenvalue of $\widetilde{\Delta}$, so that by Lemma 2.9, $Re(c-\beta)\leq 0$. That is, $Re\beta \geq c$. □

Example 2.3. For any $x\in X$, let $q(x)=\sum_{y\sim x}[t(y,x)-t(x,y)]$. Then there exists a unique positive function $\xi(x)$ on X such that $\sum_{x\in X}\xi(x)=1$ and the column matrix $q(x)=\dfrac{\Delta\xi(x)}{\xi(x)}$; and for any $x\in X$, $\sum_{y\sim x}[t(x,y)\xi(y)-t(y,x)\xi(x)]=0$.

Let $x_1,x_2,\ldots,x_n$ be the set of vertices of X. By Theorem 2.2, $q(x)=\dfrac{\Delta\xi(x)}{\xi(x)}+c$ so that, $-\Delta_q\xi(x)=q(x)\xi(x)-\Delta\xi(x)=c\xi(x)$, where $-\Delta_q$ is the $n\times n$ matrix having the j^{th} column with $(-\Delta_q)_{ij}=-t(x_i,x_j)$ if $i\neq j$ and $(-\Delta_q)_{jj}=\sum_{i\neq j}t(x_i,x_j)$. Since the sum of each column in $-\Delta_q$ is 0, there exists a column vector $\eta(x)$ with all positive entries such that $-\Delta_q\eta(x)=0$. Hence $\eta(x)$ is proportional to $\xi(x)$ and consequently $c=0$. That is $-\Delta_q\xi(x)=0$, which can be written in the form $\sum_{k\neq j}[t(x_j,x_k)\xi(x_k)-t(x_k,x_j)\xi(x_j)]=0$ for any j or reverting to the usual notation, we have $\sum_{y\sim x}[t(x,y)\xi(y)-t(y,x)\xi(x)]=0$ for all x (a useful result in the context of tournament matrices and a one-parameter system of bets, Moon and Pullman [6, p. 391]).

3. PERTURBATION IN THE DISCRETE LAPLACE EQUATION

As noted earlier, let us represent an arbitrary real-valued function $q(x)$ on X as $q(x)=\dfrac{\Delta\xi(x)}{\xi(x)}+c$. Let $\Delta_q u(x)=\Delta u(x)-q(x)u(x)=\xi(x)\left\{\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right]\right.$

$-c\left[\dfrac{u(x)}{\xi(x)}\right]\Big\}$. A function on X is treated as a column vector. Write as usual the matrix $A=(a_{ij})\geq 0$ if $a_{ij}\geq 0$, and $A>0$ if $a_{ij}>0$, for all pairs i,j. We say that a real-valued function $u(x)$ on X is said to be Δ_q-harmonic if and only if $u(x)$ is an eigenvector associated to 0, that is $\Delta_q u=0$; $u(x)$ is said to be Δ_q-superharmonic if $\Delta_q u\leq 0$.

Proposition 3.1. *If $\eta(x)$ is a non-negative function such that $\Delta_q\eta\leq 0$ in X, then $\eta(x)>0$ for all x or $\eta=0$.*

Proof. Suppose $\eta(z)=0$ for some z in X. Then

$$0=q(z)\eta(z)\geq\Delta\eta(z)=\sum_{y\sim x}t(z,y)[\eta(y)-\eta(z)]=\sum_{y\sim z}t(z,y)\eta(y)\geq 0.$$

This means, if $\eta(z)=0$ then $\eta(y)=0$ for all $y\sim z$. Since X is strongly connected, we conclude that $\eta(x)=0$ for all $x\in X$. Thus $\eta>0$ or $\eta=0$ on X. $\square$

Lemma 3.2. *Let $\{X,p(x,y)\}$ be a finite network with transition probabilities $p(x,y)$, that is $\sum\limits_{y\sim x}p(x,y)=1$ for all x. Let $\overline{\Delta}$ be its associated (non-symmetric) Laplacian matrix. Then there exists a unique vector v with all its entries positive such that $\overline{\Delta}^t v=0$ and $\sum\limits_{x\in X}v(x)=1$.*

Proof. Let A be the row stochastic matrix with 0 in its diagonal and $p(x,y)$ in its xth row and yth column. Then by the Perron-Frobenius theorem, A has a left eigenvector v^t whose entries are all positive, $v^tA=v^t$, and $\sum\limits_{x\in X}v^t(x)=1$. Since $\overline{\Delta}=A-I$, we conclude that $\overline{\Delta}^t v=0$ with $\sum\limits_{x\in X}v(x)=1$. $\square$

Remark 3.1. Let $\{X,t(x,y)\}$ be a finite network, with Δ as its associated (non-symmetric) Laplacian. Let $t(x)=\sum\limits_{y\sim x}t(x,y)$. Write $p(x,y)=\dfrac{t(x,y)}{t(x)}$. Then the Laplacian $\overline{\Delta}$ associated to $\{X,p(x,y)\}$ has the property given in Lemma 3.2. Now if D is the diagonal matrix whose diagonal entries are given by $t(x)$, then $\Delta=D\overline{\Delta}$. Since D is nonsingular, there is a unique vector $u>0$ such that $Du=v$. Consequently, $\Delta^t u=\overline{\Delta}^t Du=\overline{\Delta}^t v=0$. If $\sum\limits_{x\in X}u(x)=\alpha$, then write $w(x)=\dfrac{u(x)}{\alpha}$ so that $\Delta^t w=0$ and $\sum\limits_{x\in X}w(x)=1$.

3.1. Δ_q-Poisson equation

Proposition 3.3. *In the unique representation of $q(x)$ as $\dfrac{\Delta\xi(x)}{\xi(x)}+c$, if $c=0$ then*

i) *the matrix Δ_q is singular and the eigenspace V associated to 0 is one-dimensional spanned by $\xi(x)$;*

ii) *if $\Delta_q u(x)$ is of the same sign, then $u \in V$;*

iii) *there exists a unique vector $\phi(x)$ with positive entries and $\sum\limits_{x\in X}\phi(x) = 1$ such that for a given real-valued function $f(x)$ in X, $\Delta_q u(x) = f(x)$ has a solution $u(x)$ if and only if $\sum\limits_{x\in X}\phi(x)f(x) = 0$.*

Proof. When $c=0$, $\Delta_q u(x) = \xi(x)\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right]$. Hence $\Delta_q u(x) = 0$ if and only if $\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] = 0$.

i) Since $\widetilde{\Delta}$ is the Laplacian operator in the finite network $\left\{X, t(x,y)\dfrac{\xi(y)}{\xi(x)}\right\}$, the matrix $\widetilde{\Delta}$ is singular and the eigenspace associated with 0 consists only of constants, we conclude that Δ_q is singular and the eigenspace V associated to 0 consists only of constant multiples of $\xi(x)$.

ii) Suppose $\Delta_q u(x) \le 0$. Then $\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] \le 0$ so that $\dfrac{u(x)}{\xi(x)}$ is constant [Lemma 2.1]. Hence $u \in V$.

iii) $\Delta_q u(x) = f(x)$ if and only if $\xi(x)\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] = f(x)$. Now, given the function $f(x)$, there exists a function $u(x)$ such that $\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] = \dfrac{f(x)}{\xi(x)}$ if and only if $\sum\limits_{x\in X} w(x)\left[\dfrac{f(x)}{\xi(x)}\right] = 0$ where $\widetilde{\Delta}^t w(x) = 0$ [2, Theorem 3.4]; such a function $w(x)$ with all its entries positive exists as shown in the Remark following Lemma 3.2. Set $\phi(x) = \dfrac{w(x)}{\xi(x)}$ which is a vector with all its entries positive. Then $\sum\limits_{x\in X}\phi(x)f(x) = 0$ if and only if there exists a function $u(x)$ such that $\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] = \dfrac{f(x)}{\xi(x)}$, that is $\Delta_q u(x) = f(x)$. $\square$

Remark 3.2. The problem of finding a solution $u(x)$ on X such that $\Delta_q u(x) = f(x)$ for a given function $f(x)$ is known as the Poisson problem for Δ_q. This is a basic problem in finite electrical networks where $q = 0$ and the Laplacian is symmetric so that ϕ is taken in this case as the unit vector [1, pp. 22–23]. Part iii) of the above Proposition 3.3 gives certain precisions for the vector $\phi(x)$ in the context of annihilators: if A is a matrix, the null space $N(A^t)$ is the same as $[R(A)]^\perp$ (which is a finite version of the Fredholm Alternative).

Recall that in Remark 4) following Lemma 2.7, we have mentioned that even if $c > 0$ the matrix $-\Delta_q$ need not be weakly diagonal dominant.

Proposition 3.4. *In the unique representation of $q(x)$ as $\dfrac{\Delta(x)}{\xi(x)} + c$, if $c > 0$ then*

i) *Δ_q is a non-singular matrix;*
ii) *given any $f(x)$ in X, there exists a unique function $u(x)$ such that $\Delta_q u(x) = f(x)$;*
iii) *given any vertex z in X, there exists a unique function $G_z(x) > 0$ (called the Green function with pole at z) such that $\Delta_q G_z(x) = -\delta_z(x)$ for x in X;*
iv) *if $p(x)$ is a real-valued function such that $\Delta_q p(x) \le 0$ on X, then $p(x)$ is non-negative and of the form $p(x) = \sum_{z\in X} [-\Delta_q p(z)] G_z(x)$.*

Proof. i) $\Delta_q u(x) = 0$ implies that $\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] = c\left[\dfrac{u(x)}{\xi(x)}\right]$. In this case if $u \neq 0$, then c is an eigenvalue of $\widetilde{\Delta}$; but this is not possible since $c > 0$ (Lemma 2.9). This contradiction shows that if $\Delta_q u(x) = 0$, then $u = 0$, hence the matrix Δ_q is invertible.

ii) Consequently if $\Delta_q u(x) = f(x)$, then $u(x) = \Delta_q^{-1}[f(x)]$ is uniquely determined.

iii) In particular if $f(x) = -\delta_z(x)$, we get the uniquely determined Green function $G_z(x)$ such that $\Delta_q G_z(x) = -\delta_z(x)$.

iv) Let $p(x)$ be such that $\Delta_q p(x) \le 0$ for all x in X. Write $s(x) = \sum_{z\in X} (-\Delta_q p(z)) G_z(x)$ which is non-negative. If $h(x) = s(x) - p(x)$, then $(-\Delta_q)h(x) = 0$ for all x in X; hence $h = 0$ since Δ_q is invertible. Consequently, $p(x) = s(x) = \sum_{z\in X} (-\Delta_q p(z)) G_z(x)$ for all x in X. □

Proposition 3.5. *In the unique representation of $q(x)$ as $\dfrac{\Delta\xi(x)}{\xi(x)} + c$, if $c < 0$ then*

i) *if $u \ge 0$ and $\Delta_q u(x) \le 0$, then $u = 0$;*

ii) *if* $u(x)$ *is a non-zero function such that* $\Delta_q u(x) = 0$, *then* $u(x)$ *takes both positive and negative values;*

iii) *the vector subspace* $V = \{u : \Delta_q u = 0\}$ *consisting of all the* Δ_q-*harmonic functions in* X *is of dimension* $d > 0$ *if and only if* c *is an eigenvalue of multiplicity* d *associated to the Laplacian* $\widetilde{\Delta}$.

Proof. i) Suppose $u \geq 0$ is such that $\Delta_q u(x) \leq 0$. Then $\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right] \leq c\left[\frac{u(x)}{\xi(x)}\right] \leq 0$ for all $x \in X$ and $\widetilde{\Delta}\left[\frac{u(z)}{\xi(z)}\right] < 0$ if $u(z) > 0$. Now $\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right] \leq 0$ for all x indicates that $\left[\frac{u(x)}{\xi(x)}\right]$ is constant so that $\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right] = 0$. Hence there is no vertex z where $u(z) > 0$, that is $u = 0$.

ii) Since $\Delta_q u(x) = 0$, then $\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right] = c\left[\frac{u(x)}{\xi(x)}\right]$. If $u(x)$ takes only non-negative or non-positive values, then $\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right]$ is of the same sign in X, hence $\left[\frac{u(x)}{\xi(x)}\right]$ is constant. That means $c = 0$, a contradiction. Hence if $u \neq 0$, then $u(x)$ takes both positive and negative values on X.

iii) The vector subspace V which consists of all the Δ_q-harmonic functions is of dimension $d > 0$ if and only if for some non-zero function $u(x)$, $\widetilde{\Delta}\left[\frac{u(x)}{\xi(x)}\right] = c\left[\frac{u(x)}{\xi(x)}\right]$, that is if and only if c is an eigenvalue of multiplicity d associated to the Laplacian $\widetilde{\Delta}$. □

Corollary 3.6. *Let* $q(x)$ *be a real-valued function on* X *with the unique representation* $q(x) = \frac{\Delta\xi(x)}{\xi(x)} + c$ *where* $c < 0$. *Let* $f(x)$ *be a given function on* X.

i) *If* c *is not an eigenvalue of* $\widetilde{\Delta}$, *then there exists a unique solution* $u(x)$ *such that* $\Delta_q u(x) = f(x)$.

ii) *If* c *is an eigenvalue of* $\widetilde{\Delta}$ *and if the subspace* $V = \{v : (\widetilde{\Delta} - cI)^t v = 0\}$ *is of dimension* m, *then there exist* m *linearly independent vectors* $\{\phi_j(x)\}$ *in* V, $1 \leq j \leq m$, *such that* $\Delta_q u = f$ *has a solution* $u(x)$ *if and only if* $\sum_{x \in X} \phi_j(x) f(x) = 0$ *for each* j.

Proof. i) If c is not an eigenvalue of $\widetilde{\Delta}$, then Δ_q is invertible since $\Delta_q u(x) = \xi(x)\left[(\widetilde{\Delta} - cI)\left(\frac{u(x)}{\xi(x)}\right)\right]$. Hence $\Delta_q u(x) = f(x)$ has a unique solution $u(x)$.

ii) This follows as in the Remark following Proposition 3.3 since $N(\Delta_q^t) = [R(\Delta_q)]^\perp$. $\square$

3.2. $\boldsymbol{\Delta_q}$-Dirichlet-Poisson equation when $\boldsymbol{c \geq 0}$

We have already remarked that when $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, the matrix Δ_q is non-singular when $c > 0$ and singular when $c = 0$. However some special submatrices of Δ_q are non-singular when $c \geq 0$. Precisely, we have the following lemma [1, Theorems 2.2.4 and 2.4.4]:

LEMMA 3.7. *Let* $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, $c \geq 0$, *on a network* X *with* n *vertices. For any* $k, 1 \leq k \leq n-1$, *let* Δ_q^k *be a proper submatrix of* Δ_q *by selecting the rows and the columns corresponding to the vertices of a proper subset of* X *with* k *vertices. Then* Δ_q^k *is non-singular.*

THEOREM 3.8 (Δ_q-Dirichlet-Poisson equation). *Let* F *be a proper subset of a finite network. Let* $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, $c \geq 0$. *Let* f, g *be two real-valued functions,* f *defined on* F *and* g *defined on* $X \setminus F$. *Then there exists a unique function* u *on* X *such that* $\Delta_q u = f$ *on* F *and* $u = g$ *on* $X \setminus F$.

Proof. Let $u = (u_1, u_2, \ldots, u_n)^t$ and $v = (v_1, v_2, \ldots, v_n)^t$ be the column vectors such that $u(x) = g(x)$ if $x \in X \setminus F$ and $v(x) = f(x)$ if $x \in F$. Now if we write $\Delta_q u = v$, then the value of $u(x)$ for $x \in F$ can be calculated, since the submartix of Δ_q determined by the vertices of F is non-singular as given by the above lemma. Consequently, we have a function u in X such that $\Delta_q u(x) = f(x)$ if $x \in F$ and $u(x) = g(x)$ if $x \in X \setminus F$. The uniqueness of the solution $u(x)$ follows from the Minimum Principle for Δ_q [1, Corollary 2.2.2 and Lemma 2.4.3]. $\square$

3.3. $\boldsymbol{\Delta_q}$-Poisson-Neumann equation

Let A be a proper non-empty strongly connected finite subset of the network X. A vertex a is said to be in the interior of A, if a and all its neighbours in X are in A. Let $\mathring{A}$ denote the set of all interior vertices of A; write $\partial A = A \setminus \mathring{A}$. Let $\aleph$ be the restriction of the Laplacian operator Δ on A, that is $\aleph u(x) = \Delta u(x)$ if $x \in \mathring{A}$, and $\aleph u(\zeta) = \dfrac{\partial u(\zeta)}{\partial n^-} = \sum\limits_{y \sim \zeta, y \in A} t(\zeta, y)[u(y) - u(\zeta)]$ if $\zeta \in \partial A$. Let f be defined on $\mathring{A}$, g be defined on ∂A and q be defined on

A. Then $q(x)$ has the unique representation $q(x) = \dfrac{\aleph\xi(x)}{\xi(x)} + c$ where $\xi(x) > 0$ and $\sum\limits_{x\in A} \xi(x) = 1$. Write $\aleph_q u(x) = \aleph u(x) - q(x)u(x)$. Then, using the results of Section 3.1 for the network A with its Laplacian $\aleph$, we obtain the following solution to the Δ_q-Poisson-Neumann equation for $\{A, \aleph_q, f, g\}$.

i) Let $c = 0$. Then there exists a function $u(z)$ on A such that $\aleph_q u(x) = \Delta u(x) - q(x)u(x) = f(x)$ when $x \in \mathring{A}$ and $\aleph_q u(\zeta) = \dfrac{\partial u(\zeta)}{\partial n^-} - q(\zeta)u(\zeta) = g(\zeta)$ when $\zeta \in \partial A$, if and only if $\sum\limits_{x\in\mathring{A}} \eta(x)f(x) = -\sum\limits_{\zeta\in\partial A} \eta(\zeta)g(\zeta)$ where $\eta(z)$ is the unique function on A such that $\eta(z)$ is positive, $\sum\limits_{z\in A} \eta(z) = 1$ and $\aleph_q^t \eta(z) = 0$. If there is another such solution $v(z)$ on A, then $v(z) = u(z) + \lambda\xi(z)$ for some constant λ.

ii) Let $c > 0$. Then there exists a unique function $u(z)$ on A such that $\Delta u(x) - q(x)u(x) = f(x)$ when $x \in \mathring{A}$ and $\dfrac{\partial u(\zeta)}{\partial n^-} - q(\zeta)u(\zeta) = g(\zeta)$ when $\zeta \in \partial A$.

iii) Let $c < 0$. In this case, we distinguish two cases when 0 is or is not an eigenvalue of $\aleph_q$:

  (a) When 0 is not an eigenvalue of $\aleph_q$, then the unique solution exists as in ii).

  (b) When 0 is an eigenvalue of $\aleph_q$, then the solution exists as in i) if and only if $\sum\limits_{x\in\mathring{A}} \eta(x)f(x) = -\sum\limits_{\zeta\in\partial A} \eta(\zeta)g(\zeta)$ for every $\eta(z)$ of a linearly independent base which generates the null space $N(\aleph_q^t)$.

4. CONCLUDING REMARKS

Let X be a set consisting of n elements called vertices. Let $\mathcal{M}$ be the family of all $n \times n$ matrices such that $M \in \mathcal{M}$ if and only if M has the following two properties:

(1) All its non-diagonal entries are nonnegative. That is, if $t(x, y)$ is the entry in M corresponding to the i^{th} row and the j^{th} column, then $t(x, y) \geq 0$ if $x \neq y$.

(2) Let D denote the diagonal matrix whose entries are $t(x, x)$. If we write $M = D + A$, then A is irreducible.

Note that X can be made into a directed graph by constructing an edge from x to y if and only if $t(x, y) > 0$. Then A can be thought of as a weighted adjacency matrix on the directed graph X without self-loops. Thus, if $t(x,x) = 0$ for all x in X, then M is the adjacency matrix of X; and if $t(x,x) = -\sum_{y\sim x} t(x,y)$, then M is the Laplacian matrix Δ of X. For a general $M \in \mathcal{M}$, the question of finding the existence and the uniqueness of the solution of $Mu(x) = f(x)$ where $f(x)$ is a given real-valued function (column vector) on X is known as the Poisson problem for M. Similarly the Dirichlet-Poisson problem and the Poisson-Neumann problem for M are posed.

To solve the above problems, note that the matrix $M \in \mathcal{M}$ can be represented as $Mu(x) = \Delta u(x) - q(x)u(x) = \Delta_q u(x)$ where $-q(x) = t(x) + \sum_{y\sim x} t(x,y)$ for $x \in X$. Thus actually M is a perturbed Laplacian matrix according to the usage in the above text. In this form the spectral properties of M are studied conveniently using the Perron-Frobenius results, as shown above. Using the unique representation $q(x) = \dfrac{\Delta\xi(x)}{\xi(x)} + c$, we see that $Mu(x) = \xi(x)\left\{\widetilde{\Delta}\left[\dfrac{u(x)}{\xi(x)}\right] - c\left[\dfrac{u(x)}{\xi(x)}\right]\right\}$ where $\widetilde{\Delta}$ is the Laplacian operator associated with the network $\left\{X, t(x,y)\dfrac{\xi(y)}{\xi(x)}\right\}$. Consequently, $\eta(x)$ is an eigenvector associated to an eigenvalue λ for M if and only if $\dfrac{\eta(x)}{\xi(x)}$ is an eigenvector associated to the eigenvalue $\lambda + c$ for $\widetilde{\Delta}$. Thus the spectral properties of M can be related to the spectral properties of $\widetilde{\Delta}$. For example, c is the smallest eigenvalue of $-M$ in the sense that if α is any eigenvalue of $-M$, then $c < Re\alpha$. Note that the graph structure of X is the same whether we consider the matrix M or the Laplacian $\widetilde{\Delta}$, in the sense that for any two vertices x, y in X, if $M = D + A$, then A determines a directed path from x to y if and only if $\widetilde{\Delta}$ determines a directed path from x to y in the graph X.

Acknowledgments. We are grateful for the referee's comments. The first author thanks the DST India for the research grant SR/S4/MS:718/10.

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Received 14 November 2014

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