ON QUASIIDENTITIES OF THE FINITE ALGEBRAIC SYSTEM

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All the quasiidentities (respectively, the identities) of a number of variables which do not exceed a finite number truly fixed on a finite algebraic system of finite signature, have finite base.

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1. INTRODUCTION

Even at the present time, one of the important problem in the quasivarieties theory area is the problem of existing the finite base of the real quasiidentities in a given algebra. It is well known that having no finite base, the quasiidentities of noncommutative nilpotent finite group [1], of a nonassociative or associative finite nilpotent ring with the no null produces [2,3] a noncommutative or nonassociative finite nilpotent Moufang loop [4, 5]. At this moment, it is necessary to investigate the finite base existence of the quasiidentities of a finite fixed number of variables of a finite algebra. In the present paper, this problem is solved for the finite algebraic system of finite signature.

2. MAIN RESULT

Let A be a finite algebraic system of finite signature $\sigma = \sigma^P \cup \sigma^P$ made of set σ^F of functional symbols and set σ^P of predicational symbols which do not intersect, $\sigma^P \cap \sigma^P = \emptyset$. To each $f \in \sigma^F$, $r \in \sigma^P$ a nonnegative number $\nu(f)$, $\nu(r)$, called arities of these symbols is attributed. At the same time, we mention that although the arities of functional symbols can be zero, the variety of predicational symbols is supposed to be greater than zero. We'll note with $QId(A)$ (respectively, $Id(A)$) the set of all the quasiidentities (respectively, the identities) of σ signature of the countable set variables $X_0 = \{x_1, x_2, \ldots\}$ real in the system A. It is said that the quasiidentities (respectively, the identities) of algebraic system A have finite base if there exists a finite subset $\Sigma \subseteq QId(A)$ (respectively, $\Sigma \subseteq Id(A)$); this means $\Sigma \models \Phi$ for any $\Phi \in QId(A)$ (respectively, $\Phi \in Id(A)$), *i.e.* the set Σ of formulate involves formula φ.

Theorem 2.1. *For any finite algebraic system A of finite signature σ and for any n natural number the sets $QId_n(A)$ and $Id_n(A)$ have finite bases.*

Proof. Let be $s = |A|^n$. Out of all elements $a_1 = (a_{11}, \ldots, a_{1n}),\ \ldots,\ a_s = (a_{s1},\ \ldots,\ a_{sn})$ of set A^s we make the matrix $[a_{ij}]_{s\times n}$. The lines of this matrix are all the elements $a_1,\ \ldots,\ a_s$ of set A^n, but the columns are elements $h_1 = (a_{11},\ \ldots,\ a_{s1}),\ \ldots,\ h_n = (a_{1n},\ \ldots,\ a_{sn})$ of set A^s. Both sets A^n and A^s are base sets Cartesian powers A^n and A^s of system A. The subsystem of algebraic system A^s generated by elements $h_1,\ \ldots,\ h_n \in A^s$ is noted with H.

Let $(\sigma(X_n), \sigma)$ be the algebraic system of terms of signature σ. The base set $\sigma(X_n)$ of this system is made up from all terms of signature σ of set X_n variables, but the operations and the predicates are defined like this:

$$f^{\sigma(X_n)}(t_1, \ldots, t_{\nu(f)}) = f(t_1, \ldots, t_{\nu(f)})$$

for any functional symbol $f \in \sigma^F$;

$$r = \begin{cases} \emptyset, & \text{if } \ r \neq \approx; \\ \{(t,t) \mid t \ - term\}, & \text{if } r = \approx, \end{cases}$$

where $\approx$ is binary predicational special symbol attributed to predicate $=$.

Let φ be the homomorphism from algebraic system $(\sigma(X_n), \sigma)$ on algebraic system H defined by the following equalities

$$\varphi(x_i) = h_i,\ i = 1,\ \ldots,\ n.$$

Let's note with θ the congruence (see [6]) on algebraic system $(\sigma(X_n), \sigma)$ equal nucleus $\ker\varphi$, with F– set $\sigma(X_n)/\theta$ and with F – factor-system $(\sigma(X_n)/\theta, \sigma)$. Further on we demonstrate the following two lemmas.

Lemma 2.1. *Congruence θ has the following components*

$$\theta(r) = \{(t_1, \ldots, t_m) \in \sigma(X_n) \mid A \vDash r(t_1, \ldots, t_m)\},\ r \in \sigma^P,\ \nu(r) = m.$$

Proof. Indeed, if $r \in \sigma^P,\ \nu(r) = m$, and

$$(t_1(x_1,\ \ldots,\ x_n),\ \ldots,\ t_m(x_1,\ \ldots,\ x_n)) \in \theta(r),$$

then in algebraic system H takes place the following

$$r^H(t_1^H(h_1,\ \ldots,\ h_n),\ \ldots,\ t_m^H(h_1,\ \ldots,\ h_n)).$$

Moving onto components, we have in system A the following relations

$$r^A(t_1^A(a_{i1},\ \ldots,\ a_{in}),\ \ldots,\ t_m^A(a_{i1},\ \ldots,\ a_{in})),\ i = 1,\ \ldots,\ s.$$

In this way, in algebraic system A the identity $r(t_1(x_1,\ \ldots,\ x_n),\ \ldots,\ t_m(x_1,\ \ldots,\ x_n))$ is true. Reciprocally, if $A \vDash r(t_1(x_1,\ \ldots,\ x_n),\ \ldots,\ t_m(x_1,\ \ldots,\ x_n))$, then $A^s \vDash r(t_1(x_1,\ \ldots,\ x_n), \ldots,\ t_m(x_1, \ldots, x_n))$. From now on, since H is a subsystem of algebraic system A^s we have $H \vDash r(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$. Then it takes place $r^H(t_1^H(h_1, \ldots, h_n), \ldots, t_m^H(h_1, \ldots, h_m))$ and, because

$$t_i^H(h_1, \ldots, h_n) = h_i^H(\varphi(x_1), \ldots, \varphi(x_n)) = \varphi\left(h_i(x_1, \ldots, x_{n)}\right), \; i = 1, \ldots, m,$$

it results

$$r^H(\varphi(t_1(x_1, \ldots, x_n), \ldots, \varphi(t_m(x_1, \ldots, x_m)) .$$

From where it follows

$$(t_1(x_1, \ldots, x_n), \ldots, (t_m(x_1, \ldots, x_m)) \in \theta(r). \quad \square$$

LEMMA 2.2. *F is a free algebraic system of quasivarieties $Q(A)$.*

Proof. Firstly, we observe that algebraic system F, according to the theorem about homomorphisms, is isomorphic with algebraic system H. But H is a subsystem of Cartesian power A^s of algebraic system A. Since according to Mal'tsev's theorem (see [7], p. 271), the quasivariety is closed related to the Cartesian product and taking the subsystems we obtain that F belongs to quasivarieties generated by algebraic system A.

We consider $\overline{X} = \{x_1/\theta, \ldots, x_n/\theta\}$. If A isn't a system with only one element, then the elements $h_1, \ldots, h_n \in H$ taken by two are different. That's why set $\overline{X}$ with elements from F contains the same number of elements as set X. But if A is a system with one element, then $h_1 = \ldots = h_n$ and $x_1/\theta = \ldots = x_n/\theta$; that's why $\overline{X}$ contains one element, and in this case, we consider $n = 1$ and $\overline{X} = \{x_1/\theta\}$.

Let C be an algebraic system from quasivariety $Q(A)$. $\psi_0 : \overline{X} \to C$ a certain application of set $\overline{X}$ in base set C of algebraic system C. For every element $w \in F$, $w = t^F(x_1/\theta, \ldots, x_n/\theta)$ we consider $\psi(w) = t^C(\psi_0(x_1/\theta), \ldots, \psi_0(x_n))$. The image $\psi(w)$ doesn't depend on term t which represents element w. Indeed, if $p(x_1, \ldots, x_n)$ is a certain term from $\sigma(X)$, for which $w = p^F(x_1/\theta, \ldots, x_n/\theta)$, then $(t, p) \in \theta(\approx)$. From here, according to Lemma 2.1 $A \vDash t = p$. From this, it results $C \vDash t = p$, so $t^C(\psi_0(x_1/\theta), \ldots, \psi_0(x_n)) = p^C(\psi_0(x_1/\theta), \ldots, \psi_0(x_n))$.

In this way, we defined application $\psi : F \to C$, which is evidently an extension of application ψ_0. It remains to show that ψ is homomorphism. Indeed, let $r \in \sigma^P$, $\nu(r) = m$ be a certain predicate $r \in \sigma^P$ and $t_1^F(x_1/\theta, \ldots, x_n/\theta), \ldots, t_m^F(x_1/\theta, \ldots, x_n/\theta)$ elements from F so that

$$F \vDash r^F(t_1^F(x_1/\theta, \ldots, x_n/\theta), \ldots, t_m^F(x_1/\theta, \ldots, x_n/\theta)).$$

But

$$r^F(t_1^F(x_1/\theta, \ldots, x_n/\theta), \ldots, t_m^F(x_1/\theta, \ldots, x_n/\theta))$$
$$= r^F(t_1^F(x_1, \ldots, x_n)/\theta, \ldots, t_m^F(x_1, \ldots, x_n)/\theta),$$

from where $(t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n)) \in \theta(r)$ and, according to Lemma 2.1, $A \models r(t_1, \dots, t_m)$. But then $C \models r(t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n))$ and in particular,

$$C \models r^C(t_1^C(\psi_0(x_1), \dots, \psi_0(x_n)), \dots, t_m(\psi_0(x_1), \dots, \psi_0(x_n))$$
$$= r^C(\psi(t_1^F(x_1, \dots, x_n)), \dots, \psi(t_m^F(x_1, \dots, x_n)). \quad \square$$

Remark 2.1. If A is a single element system, then the rank of the algebraic system F equals one; if not, the rank is a natural number $n \geq 1$.

Continuation of the proof of Theorem 2.1. So, according to Lemma 2.2, F is a free algebraic system of quasivariety $Q(A)$. According to the homomorphism theorem, system F is isomorphic with system H so it is finite. Consequently, algebraic system $(\sigma(X_n), \sigma)$ has a finite number adjacent classes adequate to congruence θ. We settle in each of these adjacent classes representatives $q_1, \dots, q_k$. We note with Σ_1 the set made up by the following identities:

$r(x_{i_1}, \dots, x_{i_m})$ if $(x_{i_1}, \dots, x_{i_m}) \in \theta(r)$, where $r \in \sigma^P$, $\nu(r) = m$, $1 \leq i_1 \leq \ldots \leq i_m \leq n$;

$r(q_{j_1}, \dots, q_{j_m})$ if $(q_{j_1}, \dots, q_{j_m}) \in \theta(r)$, where $r \in \sigma^P$, $\nu(r) = m$, $1 \leq j_1 \leq \ldots \leq j_m \leq k$.

With Σ_2 we denote the set made up by all the quasiidentities by the form

$$\&_{i=1}^{l} r_i(q_{i_1}, \dots, q_{i_{m_i}}) \Rightarrow r(q_{j_1}, \dots, q_{j_m}),$$

where $r_1, \dots, r_p$, $r \in \sigma^P$ is not repeated, $m_1 = \nu(r_1), \dots, m_p = \nu(r_p)$, $m = \nu(r)$; but the quasiidentity satisfies the condition: if for a certain $a_i \in A^I$ in A takes place the relations

$$r_k^A(q_{k_1}(a_{i1}, \dots, a_{in}), \dots, q_{k_{m_i}}(a_{i1}, \dots, a_{in})), \; k = 1, \dots, l,$$

then the following relation takes place

$$r^A(q_{j_1}(a_{i1}, \dots, a_{in}), \dots, q_{j_m}(a_{i1}, \dots, a_{in})).$$

Because $\sigma = \sigma^F \cup \sigma^P$ the signature is finite and the number of representatives $q_1, \dots, q_k$ is finite, sets Σ_1 and Σ_2 are finite. It is clear $Id_n(A) \models \Sigma_1$ and $QId_n(A) \models \Sigma_1 \cup \Sigma_2$, that's why it remains to show $\Sigma_1 \models Id_n(A)$ and $\Sigma_1 \cup \Sigma_2 \models QId_n(A)$.

Let be given a certain identity $r(t_1, \dots, t_{\nu(r)})$ from set $Id_n(A)$. We deduce through induction, upon the total number of functional symbols from formula $r(t_1, \dots, t_{\nu(r)})$, that $\Sigma_1 \models r(t_1, \dots, t_{\nu(r)})$. If this number equals zero, then the statement is true, because $t_1 = x_{i_1}, \dots, t_{\nu(r)} = x_{i_{\nu(r)}}$, where $1 \leq i_1 \leq \ldots \leq i_{\nu(r)} \leq n$, the identity has $r(x_{i_1}, \dots, x_{\nu(r)})$ form and belongs to set Σ_1. We suppose now that the statement is true for the identities that contain a less number of functional symbols as the given identity $r(t_1, \ldots, t_{\nu(r)})$, where we suppose to have a number of functional symbols greater than 1. Then for a certain $i \in \{1, \ldots, \nu(r)\}$ and a certain functional symbol $f \in \sigma^F$ we have $t_i = f(t_{ii}, \ldots, t_{i\nu(f)})$. We take terms

$$t_{i1} \equiv q_{i1}(\theta), \ldots, t_{i\nu(f)} \equiv q_{i\nu(f)}(\theta) \text{ and } f(q_{i1}, \ldots, q_{i\nu(f)}) \equiv q_j(\theta).$$

Then, according to the hypothesis of induction,

$$\Sigma_1 \vDash t_{i1} = q_{i1}, \ \ldots, \Sigma_1 \vDash t_{i\nu(f)} = q_{i\nu(f)},$$

from where

$$\Sigma_1 \vDash f(t_{i1}, \ldots, t_{i\nu(f)}) = f(q_{i1}, \ldots, q_{i\nu(f)})$$

and

$$\Sigma_1 \vDash t_i = q_j.$$

From here

$$A \vDash t_i = q_j,$$

which together with

$$A \vDash r(t_1, \ldots, t_{\nu(r)}),$$

implies

$$A \vDash r(t_1, \ldots, t_{i-1}, q_j, t_{i+1}, \ldots, t_{\nu(r)}).$$

But $r(t_1, \ldots, t_{i-1}, q_j, t_{i+1}, \ldots, t_{\nu(r)})$ contains less functional symbols as $r(t_1, \ldots, t_{\nu(r)})$, so according to the hypothesis

$$\Sigma_1 \vDash r(t_1, \ldots, t_{i-1}, q_j, t_{i+1}, \ldots, t_{\nu(r)}).$$

Therefore,

$$\Sigma_1 \vDash t_i = q_j$$

and

$$\Sigma_1 \vDash r(t_1, \ldots, t_{i-1}, q_j, t_{i+1}, \ldots, t_{\nu(r)}),$$

from where we obtain

$$\Sigma_1 \vDash (t_1, \ldots, t_{\nu(r)}).$$

Finally, let

$$\Phi = (\&_{i=1}^{l} r_i(t_{i_1}, \ldots, t_{i_{\nu(r_i)}}) \Rightarrow r(t_{j_1}, \ldots, t_{j_{\nu(r)}}))$$

be an arbitrary quasiidentity from set $QId_n(A)$. We take terms

$$q_{i_1}, \ldots, q_{i_{\nu(r_i)}} \, (i = 1, \ldots, I), \ q_{j_1}, \ldots, q_{j_{\nu(r_)}}$$

so as

$$t_{i_1} \equiv q_{i_1}(\theta), \ldots, t_{i_{\nu(r_i)}} \equiv q_{i_{\nu(r_i)}}(\theta) \ \ (i = 1, \ldots, l),$$

$$t_{j_1} \equiv q_{j_1}(\theta), \ldots, t_{j_{\nu(r)}} \equiv q_{j_{\nu(r)}}(\theta),$$

then we substitute the respective terms from formula Φ with these terms. If the left side of the quasiidentity contains an expression which coincides with the right side expression, then this quasiidentity is obvious and it is equivalent with the identity $x_1 = x_1$ contained in Σ_1. But if the left side of the obtained quasiidentity contains repeated expressions, then, except one, we exclude the others from the formula. As a result we obtain the quasiidentity

$$\Phi' = (\&_{i=1}^{l'} r_i(q_{i_1}, \ldots, q_{i_{\nu(r_i)}}) \Rightarrow r(\, q_{j_1}, \ldots, q_{j_{\nu(r)}})).$$

Now we easily realize that $\Phi' \in \Sigma_2$. Consequently, $\Sigma_1 \cup \Sigma_2 \models \Phi$. The theorem is proved. □

By Theorem 2.1 easily follows

COROLLARY 2.1. *Any algebraic system of finite signature, which contains functional symbols only of arities at most one, has finite base of identities.*

Indeed, if σ is a finite signature, which contains only functional symbols of arities one then every term of this signature contains at most a variable. Therefore any identity $r(t_1, ..., t_{\nu(r)})$ contains no more than $\nu(r)$ variables. Because signature σ is finite we obtain that the number of variables contained in the identities of signature σ is bounded. Then, according to the theorem, any finite algebraic system of signatures σ has finite base of identities.

From Corollary 2.2, we obtain the following result of G. Birkhoff [8]:

COROLLARY 2.2. *A finite algebra finished with a finite number of unary operations are based finite of identities.*

Remark 2.2. In [11], V.K. Kartashov showed that the statement analogous to Corollary 2.1 is true only if finite algebra has one unary operation, and in [10], V.A. Gorbunov proved that any algebra of finite signature with two basic elements is finite based of quasiidentities and also built an algebra with three elements and two unary operations which is not finite based of quasiidentities.

Remark 2.3. The algebraic systems of signature σ with a single element are also called σ-points. In [9] is mentioned that σ-points are successful models for the quasiidentity investigation, because here the terms have a single variable and the situation becomes more transparent. It is clear that in any σ-point the identity $x_1 \approx x_2$ is true. But then, the set of all the true quasiidentities in any σ-point is equivalent with a set of quasiidentities with a single variable. From here, according to the theorem, we obtain the following: *the algebraic system of finite signature with a single element has finite base of quasiidentities.*

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