In memory of my parents, Professors Iuliana and Ion Gh. Şabac

REMARKS ON THE SPECTRAL THEORY OF AN ARBITRARY OPERATOR

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The primary objects for the spectral theory of a linear bounded operator T on an arbitrary complex Banach space $\mathcal{X}$ derive from a function θ_T of complex variable λ given by the equality $\theta_T(\lambda) = \lambda - T$, $\lambda \in \mathbf{C}$. Some facts from spectral theory for T can be considered more generally, in a natural way, for an arbitrary analytic operator valued function θ of complex variable λ, $\theta(\lambda)$ linear bounded operator on $\mathcal{X}$ for $\lambda \in \mathbf{C}$. For instance, the spectrum of θ, $\sigma(\theta)$, consists of all $\lambda \in \mathbf{C}$ with $\theta(\lambda)$ not invertible, such that in particular $\sigma(\theta_T) = \sigma(T)$, the spectrum of T. In the following, Dunford's single valued extension property $(s.v.e.p)$, Bishop's spectral spaces and (β) property will be defined for θ in a similar way. Some natural localizations of these properties to the open subsets of complex numbers are defined and these localizations of $(s.v.e.p)$ or (β) property hold for θ in some open subset $G \subset \mathbf{C}$. These localizations of $(s.v.e.p)$ or Bishop's property (β) for θ_T (*i.e.* for T) for an arbitrary T hold for some open subsets of $\mathbf{C}$ although $(s.v.e.p)$ and (β) may be not true for T. First we use this in a particular case in [20] (Definition 2.1). The analysis of these localizations gives also an evaluation of $(s.v.e.p.)$ and (β) property for an arbitrary T. Finally, a class of spectral subspaces for an arbitrary analytic operator valued function θ of complex variable is defined. In particular, for θ_T this is a new class of spectral subspaces of an arbitrary bounded operator T, intermediate between the strong and weak Bishop's spectral spaces of T and having the restriction property to the spectrum of T.

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1. INTRODUCTION

Let $T \in \mathcal{B}(\mathcal{X})$ be a linear bounded operator on a complex Banach space $\mathcal{X}$ and let $\mathbf{C}$ be the field of complex numbers. Generally speaking, a spectral theory of T means a study of T in connection with a compact set of complex numbers, the spectrum of T usually denoted by $\sigma(T)$ and defined as $\sigma(T) = \mathbf{C} \setminus \rho(T)$, where

$$\rho(T) = \{\lambda \in \mathbf{C} \mid (\lambda I - T)^{-1} \in \mathcal{B}(\mathcal{X})\}$$

is the resolvent set of T.

The function $R(\lambda, T)$,the resolvent function of T is defined on $\rho(T)$ by

$$R(\lambda, T) = (\lambda I - T)^{-1},\ \lambda \in \rho(T).$$

It is well known that $R(., T)$ is an analytic operator-valued function on $\rho(T)$ and the boundedness of T implies that ∞(the compactification point of $\mathbf{C}$) is a regular point of it, actually $\rho(T) \cup \infty$ is a neighbourhood of ∞ containing $\{z \mid \mid z \mid > \| T \|\}$, $\lim_{\lambda\to\infty} R(\lambda, T) = 0$ and the spectrum of T is a compact subset of $\mathbf{C}$, $\sigma(T) \subset \{z \mid \mid z \mid \leq \| T \|\}$.

If $\mathcal{X}$ is of finite dimension, the objects defined above together with certain subspaces associated to closed subsets of $\sigma(T)$(actually finite subsets because $\sigma(T)$ is also finite in this case), named the spectral spaces associated to T, give a complete description of the operator T.

The general infinite dimensional case of $T \in \mathcal{B}(\mathcal{X})$, a linear bounded operator on a complex Banach space $\mathcal{X}$, is far from being so well understood. The important advances were made in the following two directions.

The first one concerns the study of certain operators having special properties while the second direction is primarily interested in spectral properties of a completely arbitrary bounded operator T, specifically looking for those spectral properties which hold for every T and are relevant for its structure. The important advances in this direction are the results concerning the following classes of operators. First, the Dunford's class of bounded operators having the single valued extension property($s.v.e.p$) and subclasses of Dunford scalar or spectral operators [9,10] with their extensions: the class of scalar generalized operators introduced by C. Foiaş [12] and the class of spectral generalized operators introduced by I. Colojoara [7]. Finally, observing that every spectral space $\mathcal{Y}$ from the Jordan model of a linear finite dimensional operator T contains all T-invariant subspaces $\mathcal{Z}$ having the property $\sigma(T \mid \mathcal{Z}) \subset \sigma(T \mid \mathcal{Y})$, C. Foiaş introduced the concept of a spectral maximal space and the corresponding infinite dimensional Jordan spectral model for a new class of operators called decomposable operators [13]. The class of decomposable operators is a subclass of the class of operators having ($s.v.e.p$) and contains all the others classes mentioned above, spectral and scalar generalized operators, spectral and scalar operators (see also [1, 2, 8, 15, 17, 22]. As E. Albrecht and J. Eschmeier proved [1], every bounded operator may be represented as a quotient of restriction of a decomposable operator. The restrictions and quotients of decomposable operators were becoming very important for the general spectral theory and not only for it if we remember that S. Brown's result concerning the existence of invariant subspaces for hyponormal operators with thick spectrum [6] was obtained after M. Putinar proved [18] that every hyponormal operator is a restriction of a decomposable operator.

The most relevant results for the second direction mentioned above are Bishop's results. E. Bishop attached [3] to each closed subset $F \subset \mathbf{C}$ two spectral subspaces for an arbitrary linear bounded operator $T \in \mathcal{B}(\mathcal{X})$. These spectral subspaces, called strong and weak respectively spectral manifolds of T corresponding to F, were denoted by $M(F,T)$ respectively $N(F,T)$. The generic element x of these manifolds is defined by the existence of a $\mathcal{X}$-valued analytic function f on $\mathbf{C} \setminus F$ which is an exact respectively approximative solution for the equation $(T-z)f(z) = x, z \in \mathbf{C} \setminus F$. Using invariant subspaces for T in particular these spectral manifolds, E. Bishop considers four types of spectral theory called "duality theories", establishing a certain duality of spectral point of view between T and its adjoint $T^* \in \mathcal{B}(\mathcal{X}^*)$. One of these dualities is valid for every T and the others, in some sense close to property of Foiaş-decomposability of T, have as consequences certain classical results in spectral theory such as the spectral theorem for Hermitian or unitary operators. For a reflexive Banach space $\mathcal{X}$ and T an arbitrary linear bounded operator on $\mathcal{X}$, E. Bishop's weakest duality theory (of type 4) holds for every T and means that the following inclusions hold:

$$M(F_1,T)^{\perp} \supset N(F_2,T^*), \ N(F_1,T)^{\perp} \supset M(F_2,T^*)$$

$$M(\overline{G}_1,T)^{\perp} \subset N(\overline{G}_2,T^*), \ N(\overline{G}_1,T)^{\perp} \subset M(\overline{G}_2,T^*)$$

for arbitrary $F_1, \ F_2$ disjoint compact subsets of the complex numbers $\mathbf{C}$ and arbitrary open sets $G_1, \ G_2$ which cover $\mathbf{C}$. The inclusion $M(F,T) \subset N(F,T)$ is always true. If $\mathcal{X}$ has finite dimension, then $M(F,T) = N(F,T)$ is a maximal spectral space derived from Jordan model (decomposability of T). Bishop's property (β) for an arbitrary bounded linear operator T is a sufficient condition for the equality $M(F,T) = N(F,T)$ for every closed subset $F \subset \mathbf{C}$ and the property (β) holds for every $T \in \mathcal{B}(\mathcal{X})$ when $\mathcal{X}$ has finite dimension.

Our aim in the paper is to enlarge the frame of spectral theory in those two directions described before. As mentioned above, the primary objects for the spectral theory of T are derived from an operator-valued function of complex variable given by $\theta_T(\lambda) = \lambda - T, \ \lambda \in \mathbf{C}$. It is then natural to try to develop a spectral theory starting from an arbitrary analytic operator valued function $\theta(\lambda), \ \lambda \in \mathbf{C}$. Some facts from spectral theory can be considered for an analytic operator valued function θ of complex variable λ, $\theta(\lambda)$ linear bonded operator on a complex Banach space $\mathcal{X}$, such that, for θ_T, they reduced to the facts from spectral theory of T and many of the results concerning the study of analytic operator valued functions can be found in the well known book of I.Gohberg and J.Leiterer [14]. As we mentioned above the spectrum of θ is defined such that in particular $\sigma(\theta_T) = \sigma(T)$, the spectrum of T (see [16, 21]).

Single valued extension property $(s.v.e.p.)$, Bishop's spectral spaces and (β) property have not been considered for an arbitrary analytic operator valued function θ. In the following, $(s.v.e.p.)$, spectral spaces and (β) property will all be generalized in the next Sections 2, 3, 4, 5, for an analytic operator valued function θ, such that,for θ_T, they reduce to the well known $(s.v.e.p.)$, Bishop's spectral spaces and (β) property of T, respectively.

In the Sections 3, 4, 5 we also define and analyse the localization of $(s.v.e.p.)$ and (β) property of θ to an arbitrary open subset of complex numbers $G \subset \mathbf{C}$. For instance, the localization to G of $(s.v.e.p.)$ for θ means the injectivity of the following map defined on $\mathcal{O}(G, \mathcal{X})$,

$$\mathcal{O}(G, \mathcal{X}) \ni f \to g \in \mathcal{O}(G, \mathcal{X}); g(\lambda) = \theta(\lambda) f(\lambda), \lambda \in G$$

and we say that G *is analytic spectral compatible with* θ or θ *has (s.v.e.p.) on* G. For θ_T, this means $f \in \mathcal{O}(G, \mathcal{X})$ and $(\lambda - T)f(\lambda) = 0$ for every $\lambda \in G$ implies $f \equiv 0$ ($f(\lambda) = 0$ for every $\lambda \in G$) such that G *is analytic spectral compatible with* T or T *has (s.v.e.p.) on* G [20, Def. 2.1]. For an arbitrary T these localizations of $(s.v.e.p.)$ or (β) can hold for some open subsets $G \subset \mathbf{C}$ although $(s.v.e.p.)$ or (β) property may be not true for T. For instance, T has $(s.v.e.p.)$on every ω an open subset of a set of analytic uniqueness of T (see Definition 1.1 in [22]). In particular, the above localization of $(s.v.e.p.)$ holds on every open set of analytic uniqueness of T. We observe also that the above open sets $G \subset \mathbf{C}$ spectral compatible with T (*i.e.* T has $(s.v.e.p.)$ on G) are not necessarily open sets of analytic uniqueness of T, they are more general sets. As T has $(s.v.e.p.)$ or (β) property if and only if each localization of this property holds for every open subset $G \subset \mathbf{C}$, the analysis of these localizations in Sections 3, 5, gives an evaluation of $(s.v.e.p.)$ or (β) property for an arbitrary T when these properties are not true for T. Finally, in Section 6 we obtain for θ_T, as a particular case of θ, a class of spectral spaces of T intermediate between Bishop's spectral manifolds $M(F, T)$ and $N(F, T)$. This new class is attached to a class of analytic functions on $\mathbf{C} \setminus F$, contains Bishop's spectral manifolds $M(F, T)$ and $N(F, T)$ (see Definition 6.3) and a new one denoted $L(F, T)$, intermediate between $M(F, T)$ and $N(F, T)$. $L(F, T)$ has the restriction property to the spectrum of T *i.e.* $L(F, T) = L(F \cap \sigma(T), T)$for every closed subset $F \subset \mathbf{C}$. We recall that $M(F, T)$ has the restriction property to the spectrum of T but $N(F, T)$ does not have this property.

2. OPERATOR VALUED FUNCTIONS OF COMPLEX VARIABLE

The basic concepts of spectral theory for a bounded linear operator $T \in \mathcal{B}(\mathcal{X})$ are derived from the invertibility of the operators $\lambda - T$ for $\lambda \in \mathbf{C}$. The following two operator-valued functions give the basic concepts of the spectral theory corresponding to $T \in \mathcal{B}(\mathcal{X})$:

$$\theta_T : \mathbf{C} \to \mathcal{B}(\mathcal{X}), \ \theta_T(\lambda) = \lambda - T$$

$$R(.,T) : \rho(T) \to \mathcal{B}(\mathcal{X}), \quad R(\lambda, T) = (\lambda - T)^{-1}$$

It is possible to define the concepts of resolvent and spectrum for an arbitrary operator-valued function θ defined on some set D of complex numbers, $\theta : D \to \mathcal{B}(\mathcal{X})$, $D \subset \mathbf{C}$, such that $\rho(\theta_T) = \rho(T)$ and $\sigma(\theta_T) = \sigma(T)$.

Definition 2.1. If $\theta : D \to \mathcal{B}(\mathcal{X})$, $D \subset \mathbf{C}$, is an operator valued function, we define

$$\rho(\theta) = \{\lambda \in D \mid \theta(\lambda)^{-1} \in \mathcal{B}(\mathcal{X})\} \ \textit{and} \ \sigma(\theta) = D \setminus \rho(\theta)$$

$\rho(\theta)$ is called the resolvent set of θ in D and $\sigma(\theta)$ is called the spectrum of θ in D. The function $R(\lambda, \theta) = \theta(\lambda)^{-1}$, $\lambda \in \rho(\theta)$, is called the resolvent function of θ. $R(.,\theta) : \rho(\theta) \to \mathcal{B}(\mathcal{X})$ is an operator valued function on $\rho(\theta)$, $\rho(R(.,\theta)) = \rho(\theta)$ and $R(.,R(.,\theta)) = \theta \mid \rho(\theta)$.

For $\theta : \mathbf{C} \to \mathcal{B}(\mathcal{X})$, $\rho(\theta)$ will be simply called the resolvent set of θ, $\mathbf{C} \setminus \rho(\theta) = \sigma(\theta)$ the spectrum of θ; $\rho(\theta \mid D)$, respectively $\sigma(\theta \mid D)$ denotes the resolvent set in D, respectively the spectrum in D of $\theta \mid D$. $R(.,\theta)$ is called the resolvent function of θ and $R(.,\theta \mid D) = R(.,\theta) \mid \rho(\theta \mid D)$.

Remark 2.2.

1. The resolvent set of θ_T is $\rho(\theta_T) = \rho(T)$, $\sigma(\theta_T) = \sigma(T)$ and $R(.,\theta_T) = R(.,T)$. The resolvent set of $R(.,T)$ in $\rho(T)$ is $\rho(T)$ and the spectrum of $R(.,T)$ in $\rho(T)$ is the empty set.
2. If $\mathcal{R}$ denotes the set of all invertible elements of $\mathcal{B}(\mathcal{X})$ and θ is as in 2.1, we have $\rho(\theta) = \theta^{-1}(\mathcal{R}) \subset D$, that is the inverse image of $\mathcal{R}$ through θ.

Obviously, the basic properties $\rho(\theta_T) = \rho(T) \neq \emptyset$ respectively $\sigma(\theta_T) = \sigma(T) \neq \mathbf{C}$ are not true for all operator functions, not even for all analytic ones. Then everywhere in the following we consider only operator-valued functions having a nonempty resolvent set and, for simplicity, defined on the complex field $\mathbf{C}$, $\theta : \mathbf{C} \to \mathcal{B}(\mathcal{X})$, $\rho(\theta) \neq \emptyset$. Everything can be easily restated for an operator valued function defined on an open subset of $\mathbf{C}$.

On the other hand for a continuous operator function, in particular for an analytic operator function θ with $\rho(\theta) \neq \emptyset$, $\rho(\theta)$ is like $\rho(\theta_T)$ an open set and $R(\lambda, \theta)$ is a continuous function.

Remark 2.3. It is well known that $\mathcal{R}$, the set of all invertible elements of $\mathcal{B}(\mathcal{X})$, is an open set in the Banach space $\mathcal{B}(\mathcal{X})$ and the function on $\mathcal{R}$ defined by $T \to T^{-1}$ is a continuous function.

This remark and 2 from Eemark 2.2, give the following.

PROPOSITION 2.4. *If $\theta : \boldsymbol{C} \to \mathcal{B}(\mathcal{X})$ is a continuous operator function with $\emptyset \neq \rho(\theta) = \theta^{-1}(\mathcal{R})$, then the resolvent set $\rho(\theta)$ is an open nonempty set of $\boldsymbol{C}$, and $R(.,\theta) : \rho(\theta) \to \mathcal{B}(\mathcal{X})$, $R(\lambda,\theta) = (\theta(\lambda))^{-1}$, the resolvent function of θ is a continuous function.*

The next proposition is also easy to prove.

PROPOSITION 2.5. *If $\theta : \boldsymbol{C} \to \mathcal{B}(\mathcal{X})$ is an analytic function (having a nonempty resolvent set), then the resolvent function $R(.,\theta)$ is an analytic function on the nonempty open set $\rho(\theta)$.*

Proof. Using the definition, we obtain the $\mathbf{C}-$*derivability* of $R(.,\theta)$ in a standard way as a consequence of Proposition 2.4:

$$\frac{d}{d\lambda}R(\lambda,\theta) = -\theta(\lambda)^{-1}[\frac{d}{d\lambda}\theta(\lambda)]\theta(\lambda)^{-1} \quad \square$$

In the following, we consider only analytic operator valued functions, called for simplicity analytic operator functions, having a nonempty resolvent set $\rho(\theta) \neq \emptyset(\sigma(\theta) \neq \mathbf{C}$, the spectrum $\sigma(\theta)$ a closed subset not necessarily compact).

3. SINGLE VALUED EXTENSION PROPERTY AND ANALYTIC OPERATOR-VALUED FUNCTIONS

In order to define in this context a single-valued extension property, $(s.v.e.p.)$, we use a general idea concerning operators acting in function spaces used also in [17] to describe properties in connection with $(s.v.e.p.)$ for $T \in \mathcal{B}(\mathcal{X})$.

Definition 3.1. If $\theta : \mathbf{C} \to \mathcal{B}(\mathcal{X})$ is an operator-valued function and the set of $\mathcal{X}$-valued functions on $E \subset \mathbf{C}$ is denoted $\{f \mid f : E \to \mathcal{X}\}$, we define:

$$\Phi_\theta : \{f \mid f : E \to \mathcal{X}\} \longrightarrow \{f \mid f : E \to \mathcal{X}\}$$

$$[\Phi_\theta f](z) = \theta(z)f(z) \text{ for every } z \in E.$$

Remark 3.2. For $\theta = \theta_T$, $\Phi_{\theta_T} = T_z$ (from [17]), $(T_z f)(\lambda) = (\lambda - T)f(\lambda)$ and for simplicity we denote $\Phi_T = \Phi_{\theta_T}$.

If we consider $\theta : \mathbf{C} \to \mathcal{B}(\mathcal{X})$ an analytic operator-valued function and $\mathcal{O}(G, \mathcal{X})$ the Fréchet space of analytic $\mathcal{X}$-valued functions on an arbitrary open subset $G \subset \mathbf{C}$, it is easy to observe the following.

Remark 3.3. The simple identity

$$\frac{\theta(z)f(z) - \theta(z_0)f(z_0)}{z - z_0} = \theta(z)\frac{f(z) - f(z_0)}{z - z_0} + \frac{\theta(z) - \theta(z_0)}{z - z_0}f(z_0)$$

implies that

$$\Phi_\theta : \mathcal{O}(G, \mathcal{X}) \longrightarrow \mathcal{O}(G, \mathcal{X}).$$

If for any $x \in \mathcal{X}$ we denote by x the constant function on G having value x, we write $x \in \mathcal{O}(G, \mathcal{X})$ for every $x \in \mathcal{X}$.

The following definition describes an adequate set of analytic function on G in order to introduce $(s.v.e.p.)$ for θ on the open subset G $\subset \mathbf{C}$. We mention that the same set will be used in the following Section 4 to define the strong Bishop's spectral spaces for θ.

Definition 3.4. The set $\Phi_\theta^{-1}(\{x\}) \cap \mathcal{O}(G, \mathcal{X})$, for every $x \in \mathcal{X}$ and an open subset G $\subset \mathbf{C}$, is called the (e.p.) index, the extention property index, attached to θ in x on the open subset G $\subset \mathbf{C}$.

For $\theta = \theta_T, T \in \mathcal{B}(\mathcal{X})$ the (e.p) index of θ_T is called (e.p.) index attached to T in $x \in \mathcal{X}$ on the open subset G $\subset \mathbf{C}$.

When (e.p.) index attached to θ on G in x is nonempty then there exists an analytic extension of $R(., \theta)x$ on G *i.e.* θ has "extension property", (e.p.), on G in x and it justifies the name.

Obviously, the (e.p.) index for θ in $x \in \mathcal{X}$ on G has at the most one element if and only if the (e.p.) index for θ in $0 \in \mathcal{X}$ on G is reduced to $0 \in \mathcal{O}(G, \mathcal{X})$. The $(s.v.e.p.)$ for θ on the open set G is given now by the following property of the (e.p.) index attached to θ.

Definition 3.5. Let $\theta : \mathbf{C} \to \mathcal{B}(\mathcal{X})$ be an analytic operator-valued function and let G be an arbitrary open subset $G \subset \mathbf{C}$. We say that θ has $(s.v.e.p.)$ on G(or G is spectral analytic compatible with θ) if one of the following equivalent conditions holds:

1. *the (e.p.) index attached to* θ *in* $0 \in \mathcal{X}$ *on* G *is reduced to* $0 \in \mathcal{O}(G, \mathcal{X})$,
2. Φ_θ *is injective on* $\mathcal{O}(G, \mathcal{X})$,
3. *the (e.p.) index attached to* θ *in* $x \in \mathcal{X}$ *on* G *has at the most one element* $f \in \mathcal{O}(G, \mathcal{X})$ for every $x \in \mathcal{X}$.

If θ_T has $(s.v.e.p.)$ on G we say that T has $(s.v.e.p.)$ on G (or G is spectral analytic compatible with T, see Definition 2.1 [20]).

We simply say θ has $(s.v.e.p.)$ if θ has $(s.v.e.p.)$ on every open subset $G \subset \mathbf{C}$. If θ_T has $(s.v.e.p.)$, this means that T has $(s.v.e.p.)$ on every open subset $G \subset \mathbf{C}$ *i.e.* T has $(s.v.e.p.)$ in Dunford's well known sense.

Remark 3.6. 1. By 3. of the above Definition, θ has $(s.v.e.p.)$ on G means that every nonempty (e.p.) index attached to θ in $x \in \mathcal{X}$ on G , $\emptyset \neq \Phi_\theta^{-1}(\{x\}) \cap \mathcal{O}(G, \mathcal{X})$ has a "single value", the unique analytic extension to G of the analytic function $R(., \theta)x$ ($R(., T)x$ for $\theta = \theta_T$) *i.e.* θ has a "single value" for its "extension property" in $x \in \mathcal{X}$ on G.

2. If $G_1 \subset G_2$ are two open subsets of $\mathbf{C}$ then the restriction to G_1 of the (e.p.) index attached to θ in $x \in \mathcal{X}$ on G_2 is contained in the (e.p.) index attached to θ in $x \in \mathcal{X}$ on G_1 and the $(s.v.e.p.)$ for θ can hold on G_2 but not on $G_1 \subset G_2$.

3. A set of analytic uniqueness for T (see [22]) is an open subset $\Omega \subset \mathbf{C}$ such that T has $(s.v.e.p.)$ on every open subset $\omega \subset \Omega$.

Let us consider now an analytic operator function $\theta : \mathbf{C} \to \mathcal{B}(\mathcal{X})$ ($\rho(\theta) \neq \emptyset$).

Examples 3.7. Using $\rho(\theta)$ the resolvent set of θ, a nonempty open subset of $\mathbf{C}$, we can describe open subsets $G \subset \mathbf{C}$ such that θ has $(s.v.e.p.)$ on G.

- e_1. θ has (s.v.e.p) on every open subset $G \subset \rho(\theta)$ because $\theta(z)$ is invertible for every $z \in \rho(\theta)$.
- e_2. θ has $(s.v.e.p.)$ on every open connected subset $D \subset \mathbf{C}$ having a nonempty intersection with $\rho(\theta)$ as a consequence of e_1 and the theorem of identity applied to the elements of the (e.p.) index attached to θ in $0 \in \mathcal{X}$ on D.
- e_3. If θ has $(s.v.e.p.)$ on an open subset V, $\emptyset \neq V \subset U$, U a connected open subset of $\mathbf{C}$, then θ has $(s.v.e.p.)$ on U. If θ has not $(s.v.e.p.)$ on the connected open subset U of $\mathbf{C}$, then θ has not $(s.v.e.p.)$ on every nonempty open subset V of U.
- e_4. If the sets $U_\alpha \subset \mathbf{C}$ are open for all $\alpha \in I$ and θ has $(s.v.e.p.)$ on U_α, then θ has $(s.v.e.p.)$ on $\bigcup\limits_{\alpha \in I} U_\alpha$.

LEMMA 3.8. *There exists a nonempty set D_θ^s, the largest open subset of $\boldsymbol{C}$ such that θ has (s.v.e.p.) on every open subset $G \subset D_\theta^s$.*

Always $D_\theta^s \supset \rho(\theta)$ and $D_\theta^s = \boldsymbol{C}$ means that θ has (s.v.e.p.).

This lemma extends to an arbitrary analytic operator function θ, a similar result of F.-H. Vasilescu [22] for $T \in \mathcal{B}(\mathcal{X})$.

Proof. Let us denote $\mathcal{G}_\theta^s = \{D \subset \mathbf{C}$ open $\mid \theta$ has $(s.v.e.p.)$ on every open $G \subset D\}$. If D_θ^s exists it would have to be the largest set in $\mathcal{G}_\theta^s$. We will prove that $D^s_\theta = \bigcup_{D\in \mathcal{G}^s_\theta} D$. Indeed, by e_1, $\rho(\theta) \in \mathcal{G}^s_\theta$ thus $\emptyset \neq \rho(\theta) \subset D^s_\theta$. Hence D^s_θ is a nonempty open set. If G is an open subset $G \subset D^s_\theta$, we have $G = \bigcup_{D\in \mathcal{G}^s_\theta} G\cap D$ and by e_4 θ has $(s.v.e.p.)$ on G because $G \cap D$ is an open subset of D for every $D \in \mathcal{G}^s_\theta$ and θ has $(s.v.e.p.)$ on $G \cap D$ by definition of $\mathcal{G}^s_\theta$. Hence $D^s_\theta \in \mathcal{G}^s_\theta$ and it is the largest set of $\mathcal{G}^s_\theta$ and the proof ends. □

Let $G \subset \mathbf{C}$ be now an arbitrary open subset and let $G = \bigcup_{\alpha\in\Lambda} G_\alpha$ be the decomposition of G given by its connected components (open subsets) G_α, $\alpha \in \Lambda$, of G. Using this decomposition we describe open subsets of G such that θ has $(s.v.e.p.)$ on it.

Definition 3.9. For the above decomposition of G we denote

$$\Lambda_\theta = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) \neq \emptyset\} \text{ and } \mathrm{G}_{\rho(\theta)} = \bigcup_{\alpha\in\Lambda_\theta} G_\alpha$$

and we call $G_{\rho(\theta)}$ the θ-spectral interior of G.

Remark 3.10. Obviously $G = \bigcup_{\alpha\in\Lambda_\theta} G_\alpha \cup \bigcup_{\alpha\in\Lambda\setminus\Lambda_\theta} G_\alpha = \mathrm{G}_{\rho(\theta)} \cup \bigcup_{\alpha: G_\alpha\subset\sigma(\theta)} G_\alpha$ and $(\mathrm{G}_{\rho(\theta)})_{\rho(\theta)} = \mathrm{G}_{\rho(\theta)}$

Observing that every analytic function h on G_α has an analytical extension f to G, $f \mid G_\beta = 0$ for every $\beta \neq \alpha$, it results from e_4 the following equivalence:

e_5. θ has $(s.v.e.p.)$ on G if and only if θ has $(s.v.e.p.)$ on every connected component of G, *i.e.* for every G_α, $\alpha \in \Lambda$.

From $\mathrm{e}_2, \mathrm{e}_4$ we deduce also the following property.

e_6. For every open subset $G \subset \mathbf{C}$, θ has $(s.v.e.p.)$ on $G_{\rho(\theta)}$ and $G_{\rho(\theta)}$ is a nonempty set if and only if $G \cap \rho(\theta) \neq \emptyset$.

The following property follows from e_5.

e_7. θ has $(s.v.e.p.)$ on G if and only if θ has $(s.v.e.p.)$ on every connected component of G contained in $\sigma(\theta)$. In other words, the connected components of G contained in $\sigma(\theta)$ determine whether θ has $(s.v.e.p.)$ on G. Thus, θ does not have $(s.v.e.p.)$ on G if and only if there exists a connected component G_α of G, $G_\alpha \subset \sigma(\theta)$ and θ has not $(s.v.e.p.)$ on every open subset of G_α (see e_3).

Now e_8 is an easy consequence of e_7.

e_8. θ has $(s.v.e.p.)$ (*i.e.* $(s.v.e.p.)$ on every open $G \subset \mathbf{C}$) if and only if θ has $(s.v.e.p.)$ on every connected open subset of $\sigma(\theta)$.

LEMMA 3.11. *For every open subset $G \subset \boldsymbol{C}$ there exists $S_\theta(G)$ the largest open subset of G such that θ has (s.v.e.p.) on it.*

Proof. If we denote

$$\mathcal{G}_\theta = \{\omega \mid \omega \text{ open subset in G}, \theta \text{ has } (s.v.e.p.) \text{ on } \omega\},$$

then using e_4, $S_\theta(G) = \bigcup\limits_{\omega \in \mathcal{G}_\theta} \omega \in \mathcal{G}_\theta$. □

Remark 3.12. If G has no nonempty open subsets on which θ has $(s.v.e.p.)$, then $S_\theta(G) = G_{\rho(\theta)} = \emptyset$ and $G \subset \sigma(\theta)$. Obviously $G_{\rho(\theta)} \subset S_\theta(G)$ and $S_\theta(G) \neq \emptyset$ if $G_{\rho(\theta)} \neq \emptyset$.

Let $G \subset \mathbf{C}$ be a fixed arbitrary open subset and let $G = \bigcup\limits_{\alpha \in \Lambda} G_\alpha$ be the decomposition given by the set of all connected components(open subsets) of G. If $S_\theta(G)$ is the set from Lemma 3.11 then its decomposition given by all its connected components can be given in terms of $\sigma(\theta)$ and connected components of $G_{\rho(\theta)}$(from Definition 3.9). For this we first make some useful remarks.

LEMMA 3.13. *If G, ω are two nonempty open subsets of $\boldsymbol{C}$, $G = \bigcup\limits_{\alpha \in \Lambda} G_\alpha$ and $\omega = \bigcup\limits_{\beta \in \Upsilon} \omega_\beta$ being the decompositions given by the connected components of G respectively ω, then the following implications hold:*

1. $\omega \subset G \Longrightarrow \omega_{\rho(\theta)} \subset \mathrm{G}_{\rho(\theta)}$
2. $G_{\rho(\theta)} \subset \omega \Longrightarrow G_{\rho(\theta)} \subset \omega_{\rho(\theta)}$
3. $G_{\rho(\theta)} \subset \omega \subset G \Longrightarrow \omega_{\rho(\theta)} = G_{\rho(\theta)}$ *and the connected components of G which have nonempty intersection with $\rho(\theta)$ are the connected components of ω which have nonempty intersection with $\rho(\theta)$.*

Proof. We recall that

$$\Lambda_\theta = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) \neq \emptyset\} \text{ and } \mathrm{G}_{\rho(\theta)} = \bigcup_{\alpha \in \Lambda_\theta} G_\alpha$$

and

$$\Upsilon_\theta = \{\beta \in \Upsilon \mid \omega_\beta \cap \rho(\theta) \neq \emptyset\} \text{ and } \omega_{\rho(\theta)} = \bigcup_{\beta \in \Upsilon_\theta} \omega_\beta$$

If $\omega \subset G$, then $\bigcup\limits_{\beta \in \Upsilon} \omega_\beta \subset \bigcup\limits_{\alpha \in \Lambda} G_\alpha$ and for $\beta \in \Upsilon$ there exists a unique $\alpha(\beta) \in \Lambda$ such that $\omega_\beta \subset G_{\alpha(\beta)}$ because G_α is a connected component of G, $\omega_\beta \subset \bigcup\limits_{\alpha \in \Lambda} G_\alpha$ and ω_β is a connected set as being a connected component of ω. If $\beta \in \Upsilon_\theta$ we have $\omega_\beta \cap \rho(\theta) \neq \emptyset$ and than obviously $G_{\alpha(\beta)} \cap \rho(\theta) \neq \emptyset$. Therefore, for every $\beta \in \Upsilon_\theta$ we have $\omega_\beta \subset G_{\alpha(\beta)} \subset \mathrm{G}_{\rho(\theta)}$ and 1. has been proved.

2. is an easy consequence of 1. because from $G_{\rho(\theta)} \subset \omega$ we deduce using 3.10 $\mathrm{G}_{\rho(\theta)} = (\mathrm{G}_{\rho(\theta)})_{\rho(\theta)} \subset \omega_{\rho(\theta)}$.

3. the equality $\omega_{\rho(\theta)} = G_{\rho(\theta)}$ is an easy consequence of 1. and 2. On the other hand, for $\alpha \in \Lambda_\theta, \beta \in \Upsilon_\theta$, G_α respectively ω_β are all the connected components of $\mathrm{G}_{\rho(\theta)}$ respectively $\omega_{\rho(\theta)}$ and the above equality shows that these are the same which concludes the proof of 3. and the lemma has been proved. □

Using 3.13 and 3.10 we can describe $(s.v.e.p.)$ for θ on ω when $G_{\rho(\theta)} \subset \omega \subset G$.

PROPOSITION 3.14. *Let $G_{\rho(\theta)} \subset \omega \subset G$ as in Lemma 3.13. If θ has (s.v.e.p.) on ω then we have*

$$\omega = \omega_{\rho(\theta)} \cup \bigcup_{\beta:\omega_\beta \subset \sigma(\theta)} \omega_\beta = \mathrm{G}_{\rho(\theta)} \cup \bigcup_{\beta:\omega_\beta \subset \sigma(\theta)} \omega_\beta$$

θ has (s.v.e.p.) on ω_β for every $\beta \in \Upsilon$ and by Examples 3.7(e_3) on every $G_{\alpha(\beta)}$ where $\alpha(\beta) \in \Lambda$ is unique with the property $\omega_\beta \subset G_{\alpha(\beta)}$.

For $\omega = S_\theta(G)$ a similar result as in the above proposition is more precise and describes its connected components.

PROPOSITION 3.15. *If $G \subset \boldsymbol{C}$ is an open subset of $\boldsymbol{C}$ and $G = \bigcup_{\alpha \in \Lambda} G_\alpha$ is the decomposition given by the set $\{G_\alpha\}_{\alpha \in \Lambda}$ of the connected components(open subsets) of G, then $S_\theta(G)$ the largest open subset of G on which θ has (s.v.e.p.) is given by*

$$S_\theta(G) = \bigcup_{\alpha \in \Lambda_S} G_\alpha,\ \Lambda_S = \{\alpha \in \Lambda \mid \theta \textit{ has (s.v.e.p.) on } G_\alpha\} \supset \Lambda_\theta$$

and $\{G_\alpha\}_{\alpha \in \Lambda_S}$ are the connected components of $S_\theta(G)$; in other words, the above equality is the decomposition of $S_\theta(G)$ given by its connected components.

Proof. By (e_3) θ has $(s.v.e.p.)$ on $\bigcup_{\alpha \in \Lambda_S} G_\alpha$ because θ has $(s.v.e.p.)$ on G_α for every $\alpha \in \Lambda_S$. Let ω be an arbitrary open subset of G and $\omega = \bigcup_{\beta \in \Upsilon} \omega_\beta$ the decomposition given by its connected components. If θ has $(s.v.e.p.)$ on ω then θ has $(s.v.e.p.)$ on ω_β for every $\beta \in \Upsilon$. But ω_β is a connected set and then there exists $\alpha(\beta) \in \Lambda$ such that $\omega_\beta \subset G_{\alpha(\beta)}$. So, for every $\beta \in \Upsilon$ we deduce from (e_3) that θ has $(s.v.e.p.)$ on $G_{\alpha(\beta)}$ and $\alpha(\beta) \in \Lambda_S$. Therefore we deduce $\omega \subset \bigcup_{\beta \in \Upsilon} G_{\alpha(\beta)} \subset \bigcup_{\alpha \in \Lambda_S} G_\alpha$. Thus we have proved that θ has $(s.v.e.p.)$ on $\bigcup_{\alpha \in \Lambda_S} G_\alpha$ and $\bigcup_{\alpha \in \Lambda_S} G_\alpha$ contains every open subset $\omega \subset G$ if θ has $(s.v.e.p.)$ on ω. This means $S_\theta(G) = \bigcup_{\alpha \in \Lambda_S} G_\alpha$(see Lemma 3.11) and obviously this equality is the decomposition of $S_\theta(G)$ given by its connected components. □

Corollary 3.16. *Let G be an open subset of $\boldsymbol{C}$, $G = \bigcup_{\alpha\in\Lambda} G_\alpha$ the decomposition given by its connected components and $G_{\rho(\theta)} = \bigcup_{\alpha\in\Lambda_\theta} G_\alpha$ the θ-spectral interior of G, $\Lambda_\theta = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) \neq \emptyset\}$(Definition 3.9). Let $S_\theta(G)$ be the largest open subset of G on which θ has(s.v.e.p.) (3.11), $S_\theta(G) = \bigcup_{\gamma\in\Gamma} S_\gamma$ the decomposition given by its connected components, $\Gamma_\theta = \{\gamma \in \Gamma \mid S_\gamma \cap \rho(\theta) \neq \emptyset\}$, $(S_\theta(G))_{\rho(\theta)} = \bigcup_{\gamma\in\Gamma_\theta} S_\gamma$ the θ-spectral interior of $S_\theta(G)$. Then we have,*

$$\{S_\gamma \mid \gamma \in \Gamma_\theta\} = \{G_\alpha \mid \alpha \in \Lambda_\theta\}, (S_\theta(G))_{\rho(\theta)} = G_{\rho(\theta)}$$

$$\{S_\gamma \mid \gamma \in \Gamma \setminus \Gamma_\theta\} = \{G_\alpha \mid \alpha \in \Lambda_S \setminus \Lambda_\theta\} =$$

$$= \{G_\alpha \mid G_\alpha \subset \sigma(\theta), \theta \text{ has } (s.v.e.p.) \text{ on } G_\alpha\}$$

and

$$S_\theta(G) = G_{\rho(\theta)} \cup \bigcup_{\alpha\in\Lambda_S\setminus\Lambda_\theta} G_\alpha = G_{\rho(\theta)} \cup s_{\sigma(\theta)}(G)$$

where we denoted $\bigcup_{\alpha\in\Lambda_S\setminus\Lambda_\theta} G_\alpha = s_{\sigma(\theta)}(G)$.

Proof. Indeed, by Proposition 3.15 we have

$$S_\theta(G) = \bigcup_{\alpha\in\Lambda_S} G_\alpha = \bigcup_{\alpha\in\Lambda_\theta} G_\alpha \cup \bigcup_{\alpha\in\Lambda_S\setminus\Lambda_\theta} G_\alpha,$$

where

$$\Lambda_S = \{\alpha \mid \theta \text{ has } (s.v.e.p.) \text{ on } G_\alpha\} \supset \Lambda_\theta.$$

So

$$\{S_\gamma \mid \gamma \in \Gamma_\theta\} = \{G_\alpha \mid \alpha \in \Lambda_\theta\},$$

$$G_{\rho(\theta)} = \bigcup_{\alpha\in\Lambda_\theta} G_\alpha = \bigcup_{\gamma\in\Gamma_\theta} S_\gamma = (S_\theta(G))_{\rho(\theta)}$$

and

$$S_\theta(G) = G_{\rho(\theta)} \cup s_{\sigma(\theta)}(G). \quad \square$$

Proposition 3.17. *If G^1, G^2 are two open subsets of $\boldsymbol{C}$, then*

$$G^1_{\rho(\theta)} \cup G^2_{\rho(\theta)} \subset (G^1 \cup G^2)_{\rho(\theta)}.$$

Proof. The following obvious property will be used.

If G is an open subset of $\mathbf{C}$ and G_α is a connected component of G, then for every connected subset $C \subset G$ we have either $C \cap G_\alpha = \emptyset$, or $C \subset G_\alpha$. So, if

$G = \bigcup_{j \in J} C_j$ and C_j are connected subsets for every $j \in J$, then every connected component G_α of G is given by

$$G_\alpha = \bigcup_{j \in J_\alpha} C_j, \quad J_\alpha = \{j \mid C_j \cap G_\alpha \neq \emptyset\}.$$

Let now G^1, G^2 be open subsets of $\mathbf{C}$ from the above proposition and $G^1 = \bigcup_{\alpha \in \Lambda_1} G^1_\alpha$, $G^2 = \bigcup_{\beta \in \Lambda_2} G^2_\beta$ the decompositions given by all the connected components of G^1, respectively G^2. Denote by $\{G^{1,2}_\nu, \nu \in \Lambda_{1.2}\}$ the set of all connected components of $G^1 \cup G^2$.

Obviously, every connected component of G^1 or G^2 is contained in some connected component of $G^1 \cup G^2$.

Every connected component of $G^1 \cup G^2$ is a union of some connected components of G^1 or G^2. The following equality explains this and follows by the above remark

$$G^{1,2}_\nu = \bigcup_{\alpha \in \Lambda_{1,\nu}} G^1_\alpha \cup \bigcup_{\beta \in \Lambda_{2,\nu}} G^2_\beta\ ,$$

where

$$\Lambda_{i,\nu} = \{\alpha \in \Lambda_i \mid G^i_\alpha \cap G^{1,2}_\nu \neq \emptyset\}, \quad i = 1, 2.$$

As mentioned above for every $\alpha \in \Lambda_1, \beta \in \Lambda_2$ there exist $\nu(\alpha), \nu(\beta)$ such that $G^1_\alpha \subset G^{1,2}_{\nu(\alpha)}$, $G^2_\beta \subset G^{1,2}_{\nu(\beta)}$. So $G^1_\alpha \subset G^1_\theta (i.e.\ G^1_\alpha \cap \rho(\theta) \neq \emptyset)$ implies $G^1_\alpha \subset G^{1,2}_{\nu(\alpha)} \subset (G^1 \cup G^2)_\theta$ and in a similar way $G^2_\beta \subset G^2_\theta$ implies $G^2_\beta \subset G^{1,2}_{\nu(\beta)} \subset (G^1 \cup G^2)_\theta$. Thus, by Definition 3.9 we obtain $G^1_{\rho(\theta)} \cup G^2_{\rho(\theta)} \subset (G^1 \cup G^2)_{\rho(\theta)}$. □

Remark 3.18. Some assertions from the above proof show that generally speaking the above inclusion given in Proposition 3.17 is not an equality. For example let G^1, G^2 be two nonempty connected open subsets of $\mathbf{C}$ such that $G_1 \cap \rho(\theta) = \emptyset$ and $G_2 \cap \rho(\theta) \neq \emptyset$ (in particular $G_1 \cap G_2 \cap \rho(\theta) = \emptyset$). If $G_1 \cap G_2 \neq \emptyset$, then $G_1 \cup G_2$ is a connected open set, $(G_1 \cup G_2)_{\rho(\theta)} = G_1 \cup G_2, G^1_{\rho(\theta)} = \emptyset, G^2_{\rho(\theta)} = G^2$, and $G_2 = G^1_{\rho(\theta)} \cup G^2_{\rho(\theta)} \underset{\neq}{\subset} (G_1 \cup G_2)_{\rho(\theta)} = G_1 \cup G_2$.

Corollary 3.19. *If G^1, G^2 are two open subsets of $\mathbf{C}$, $G^1 \cap G^2 = \emptyset$, then*

$$G^1_{\rho(\theta)} \cup G^2_{\rho(\theta)} = (G^1 \cup G^2)_{\rho(\theta)}.$$

Proof. Using the notations of Proposition 3.17, we have $G^1_\alpha \cap G^2_\beta = \emptyset$ for every $\alpha \in \Lambda_1, \beta \in \Lambda_2$. So, for every connected component $G^{1,2}_\nu$ of $G_1 \cup G_2$ there exists a unique $\alpha(\nu) \in \Lambda_1$ such that $G^{1,2}_\nu = G^1_{\alpha(\nu)}$, or there exists a unique $\beta(\nu) \in \Lambda_2$ such that $G^{1,2}_\nu = G^2_{\beta(\nu)}$. □

Remark 3.20. If $\theta = \theta_T$ for some $T \in \mathcal{B}(\mathcal{X})$ then $\rho(\theta) = \rho(T) \neq \emptyset$ and any of the above assertions concerning $(s.v.e.p.)$ for θ gives an assertion concerning $(s.v.e.p.)$ for T.

Let $T \in \mathcal{B}(\mathcal{X})$. We now translate for θ_T, hence for T, only $e_1, \ldots, e_8$, Lemma 3.8, Definition 3.9, Lemma 3.11, Proposition 3.15 and Corollary 3.16 but any other of the assertions concerning $(s.v.e.p.)$ for θ can also be rewritten in the same way for T.

f_1. T has $(s.v.e.p.)$ on every open subset $G \subset \rho(\theta)$.

f_2. T has $(s.v.e.p.)$ on every open connected subset $D \subset \mathbf{C}$ having a nonempty intersection with $\rho(T)$.

f_3. If T has $(s.v.e.p.)$ on an open subset V, $\emptyset \neq V \subset U$, U a connected open subset of $\mathbf{C}$, then T has $(s.v.e.p.)$ on U.

f_4. If $U_\alpha \subset \mathbf{C}$ are open for all $\alpha \in I$ and Thas $(s.v.e.p.)$ on $U_\alpha$for all $\alpha \in I$, then T has $(s.v.e.p.)$ on $\bigcup\limits_{\alpha \in I} U_\alpha$.

LEMMA 3.21. *There exists a nonempty set D_T^s, the largest open subset of $\boldsymbol{C}$ such that T has (s.v.e.p.) on every open subset $G \subset D_T^s$. (see Proposition 2.1 [22])*

Always $D_T^s \supset \rho(T)$ and $D_T^s = \boldsymbol{C}$ means that T has (s.v.e.p.).

Let $G \subset \mathbf{C}$ be now an arbitrary open subset and let $G = \bigcup_{\alpha \in \Lambda} G_\alpha$ be the decomposition of G given by G_α, $\alpha \in \Lambda$, the connected components (open subsets) of G.

Definition 3.22. For the above decomposition of G we denote

$$\Lambda_T = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(T) \neq \emptyset\} \text{ and } \mathrm{G}_{\rho(T)} = \bigcup_{\alpha \in \Lambda_T} G_\alpha$$

and we call $G_{\rho(T)}$ the T-spectral interior of G (a particular case of Definition 3.9).

f_5. T has $(s.v.e.p.)$ on G if and only if T has $(s.v.e.p.)$ on every connected component of G, *i.e.* for every G_α, $\alpha \in \Lambda$.

f_6. For every open subset $G \subset \mathbf{C}$, T has $(s.v.e.p.)$ on $G_{\rho(T)}$ and $G_{\rho(T)}$ is a nonempty set if and only if $G \cap \rho(T) \neq \emptyset$.

f_7. T has $(s.v.e.p.)$ on G if and only if T has $(s.v.e.p.)$ on every connected component of G contained in $\sigma(T)$. In other words, the connected components of G contained in $\sigma(T)$ determine whether T has $(s.v.e.p.)$ on G.

f_8. T has $(s.v.e.p.)$ (*i.e.* $(s.v.e.p.)$ on every open $G \subset \mathbf{C}$) if and only if T has $(s.v.e.p.)$ on every connected open subset of $\sigma(T)$.

LEMMA 3.23. *For every open subset $G \subset \boldsymbol{C}$ there exists $S_T(G)$ the largest open subset of G such that T has (s.v.e.p.) on it (see Lemma 3.11).*

PROPOSITION 3.24. *If $G \subset \boldsymbol{C}$ is an open subset of $\boldsymbol{C}$ and $G = \bigcup_{\alpha\in\Lambda} G_\alpha$ is the decomposition given by the set $\{G_\alpha\}_{\alpha\in\Lambda}$ of the connected components (open subsets) of G, then $S_T(G)$ the largest open subset of G on which T has (s.v.e.p.) is given by*

$$S_T(G) = \bigcup_{\alpha\in\Lambda_S} G_\alpha,\ \Lambda_S = \{\alpha \mid T \text{ has (s.v.e.p.) on } G_\alpha\} \supset \Lambda_T$$

and $\{G_\alpha\}_{\alpha\in\Lambda_S}$ are the connected components of $S_T(G)$; in other words, the above equality is the decomposition of $S_T(G)$ given by its connected components.

The decomposition of $S_T(G)$ given by its connected components can be detailed as in Proposition 3.15 and the structure of $S_T(G)$ can be described in the following corollary.

COROLLARY 3.25. *Let G be an open subset of $\boldsymbol{C}$, $G = \bigcup_{\alpha\in\Lambda} G_\alpha$ the decomposition given by its connected components and $\mathrm{G}_{\rho(T)} = \bigcup_{\alpha\in\Lambda_T} G_\alpha$ the T-spectral interior of G(Definition 3.22). Let $S_T(G)$ be the largest open subset of G on which T has(s.v.e.p.) (Lemma 3.23), $S_T(G) = \bigcup_{\gamma\in\Gamma} S_\gamma$ the decomposition given by its connected components, $\Gamma_T = \{\gamma \in \Gamma \mid S_\gamma \cap \rho(T) \neq \emptyset\}$ and $(S_T(G))_{\rho(T)} = \bigcup_{\gamma\in\Gamma_T} S_\gamma$ the T-spectral interior of $S_T(G)$. Then we have,*

$$\{S_\gamma \mid \gamma \in \Gamma_T\} = \{G_\alpha \mid \alpha \in \Lambda_T\}, (S_T(G))_{\rho(T)} = G_{\rho(T)},$$

$$\{S_\gamma \mid \gamma \in \Gamma \setminus \Gamma_T\} = \{G_\alpha \mid \alpha \in \Lambda_S \setminus \Lambda_T\} =$$

$$\{G_\alpha \mid G_\alpha \subset \sigma(T), T \text{ has (s.v.e.p.) on } \mathrm{G}_\alpha\}.$$

Denoting $\bigcup_{\alpha\in\Lambda_S\setminus\Lambda_T} G_\alpha = s_{\sigma(T)}(G)$ we obtain

$$S_T(G) = G_{\rho(T)} \cup \bigcup_{\alpha\in\Lambda_S\setminus\Lambda_T} G_\alpha = G_{\rho(T)} \cup s_{\sigma(T)}(G).$$

4. THE NONEMPTINESS OF THE EXTENSION PROPERTY INDEX AND SPECTRAL SPACES ATTACHED TO AN ANALYTIC OPERATOR VALUED FUNCTION ON A BANACH SPACE $\mathcal{X}$

Let us consider as before, a complex Banach space $\mathcal{X}$ and an analytic operator function $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$ with a nonempty resolvent $\rho(\theta) \neq \emptyset$. Then, for every open subset $G \subset \mathbf{C}$ we recall the map Φ_θ from Definition 3.1 defined on the Fréchet space $\mathcal{O}(G, \mathcal{X})$ of all analytic $\mathcal{X}$-valued functions on G:

$$\Phi_\theta : \mathcal{O}(G, \mathcal{X}) \longrightarrow \mathcal{O}(G, \mathcal{X}),\ [\Phi_\theta(f)](z) = \theta(z) f(z), \forall z \in G$$

Every $x \in \mathcal{X}$ is canonically identified with the constant analytical function on G with value x and $\Phi_\theta^{-1}(\{x\})$ was called the (e.p.) index of θ in x on the open subset G (Definition 3.4). Obviously Φ_θ is a linear continuous map in the Fréchet topology of $\mathcal{O}(G, \mathcal{X})$ and $\Phi_\theta^{-1}(\{x\})$ is a closed Fréchet subspace of $\mathcal{O}(G, \mathcal{X})$. The Banach space $\mathcal{X}$ is identified with the closed Fréchet subspace of $\mathcal{O}(G, \mathcal{X})$ consisting of all constant function on G. Thus we have $\bigcup\limits_{x \in \mathcal{X}} \Phi_\theta^{-1}(\{x\}) = \Phi_\theta^{-1}(\mathcal{X})$ also a Fréchet subspace of $\mathcal{O}(G, \mathcal{X})$.

We can define the strong spectral spaces of θ in the same way as Bhishop's strong spectral spaces of T (*i.e.* θ_T) is given by a property of (e.p.) index of θ_T in x on the open subset G(Definition 3.4). So, these strong spectral spaces of θ are the Banach subspaces of $\mathcal{X}$ given by the non-emptiness of the (e.p.) index of θ. Actually, such a subspace is attached to an arbitrary fixed open subset $G \subset \mathbf{C}$ and consists of the points $x \in \mathcal{X}$ for which the (e.p.) index of θ in x on $G \subset \mathbf{C}$ is a nonempty set.

Definition 4.1. For every closed subset $F \subset \mathbf{C}$ we call the strong spectral Bishop space of θ (or the strong θ-spectral Bishop space)corresponding to F, the following closed subspace of $\mathcal{X}$:

$$M(F, \theta) = \overline{M_0(F, \theta)}, \quad M_0(F, T) = \{x \in \mathcal{X} \mid \Phi_\theta^{-1}(\{x\}) \neq \emptyset\},$$

where $\Phi_\theta^{-1}(\{x\})$ is the extension property index attached to θ in the point $x \in \mathcal{X}$ on the open set $G = \mathbf{C} \setminus F$.

Remark 4.2.

1. Obviously, the map Φ_θ depends on the open set G. To keep the notation simple, we use the same notation Φ_θ for each G and we always specify the set G on which we consider the map Φ_θ and correspondingly the (e.p.) index of θ.

2. $x \in M_0(F, \theta)$ means the nonemptiness of the (e.p.) index of θ in x on the open subset $\mathbf{C} \setminus F$.

3. For $\theta = \theta_T, T \in \mathcal{B}(\mathcal{X}), M(F, \theta_T) = M(F, T)$ is the Bishop's strong spectral space of T corresponding to F (see [3]).

Definition 4.3. We call the *weak spectral Bishop space of θ, or the weak θ-spectral Bishop space* corresponding to a closed subset $F \subset \mathbf{C}$, the following closed subspace of $\mathcal{X}$:

$$N(F, \theta) =$$
$$\{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \| [\Phi_\theta(f_\epsilon)](z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\} =$$

$$\{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists g_\epsilon \in \Phi_\theta(\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})), \| g_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\}.$$

Remark 4.4.

1. $N(F, \theta)$ consists of all $x \in \mathcal{X}$ such that there exists a sequence $(f_n)_n$, $(f_n)_n \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ and

$$\theta(z)(f_n)(z) \to x, \text{ uniformly on } \mathbf{C} \setminus F, \text{ as } n \to \infty.$$

2. If we denote $\mathcal{M}(\mathbf{C} \setminus F, \mathcal{X})$ the Banach space of all bounded $\mathcal{X}$-valued functions on $\mathbf{C} \setminus F$ with $\| \ \|_\infty$, we have

$$N(F, \theta) = \overline{\Phi_\theta(\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})) \cap \mathcal{M}(\mathbf{C} \setminus F, \mathcal{X})}^{\infty} \cap \mathcal{X},$$

where $\overline{(\ )}^{\infty}$ means the closure in $\| \ \|_\infty$ and $\mathcal{X}$ is identified with the space of $\mathcal{X}$-valued constant functions, here on $\mathbf{C} \setminus F$.

3. If $\theta = \theta_T, T \in \mathcal{B}(\mathcal{X})$, then $N(F, \theta) = N(F, T)$ is Bishop's weak spectral space of T corresponding to F(see [3]) and all the results which follow can be rewritten for θ_T, and thus for T.

The following properties of the spectral spaces of θ follow directly from Definitions 4.1 and 4.3.

General Properties:

P1. $M(F, \theta) \subset N(F, \theta)$ for every closed subset $F \subset \mathbf{C}$.

P2. The maps $F \longrightarrow M(F, \theta)$, $F \longrightarrow N(F, \theta)$ are monotone.

P3. $\mathcal{X} = M(F, \theta) = N(F, \theta)$ for every closed subset $F \supset \sigma(\theta)$.

As we can see in 3.3, $\mathcal{X}$ can be identified with $\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})_{ct}$ the subspace of all $\mathcal{X}$-valued constant functions on $\mathbf{C} \setminus F$, a closed Fréchet subspace of $\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ and

$$\Phi_\theta : \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}) \longrightarrow \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$$

is a continuous linear map. So we denote $\mathcal{O}_\theta(\mathbf{C} \setminus F) = \Phi_\theta^{-1}(\mathcal{X})$ a closed(Fréchet) subspace of $\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ and obviously $M_0(F, \theta) = \Phi_\theta(\mathcal{O}_\theta(\mathbf{C} \setminus F))$. This proves the following property.

P4. The strong θ-spectral Bishop space attached to a closed subset $F \subset \mathbf{C}$ is the closure in $\mathcal{X}$ of the image of a linear continuous map Φ_θ restricted to a Fréchet space $\mathcal{O}_\theta(\mathbf{C} \setminus F)$.

$$M_0(F, \theta) = \Phi_\theta(\mathcal{O}_\theta(\mathbf{C} \setminus F)) \text{ and } M(F, \theta) = \overline{\Phi_\theta(\mathcal{O}_\theta(\mathbf{C} \setminus F))}$$

Properties in connection with $(s.v.e.p.)$

First, we prove some equivalent assertions concerning the consistency of (e.p.) index of an analytic operator valued function θ in $x \in \mathcal{X}$ on an open subset $D \subset \mathbf{C}$. As a consequence of these equivalences we can describe more precisely the general property P4 when θ has $(s.v.e.p.)$ on the open subset $\mathbf{C} \setminus F$. The next lemma follows directly from definitions.

LEMMA 4.5. *Let θ be an analytic operator valued function having (s.v.e.p.) on an open subset $D \subset \boldsymbol{C}$. The following assertions concerning the consistency of (e.p.) index of θ in $x \in \mathcal{X}$ on the open subset $D \subset \boldsymbol{C}$ are equivalent:*
(i) $\Phi_\theta^{-1}(\{x\}) \neq \emptyset$,
(ii) $\Phi_\theta^{-1}(\{x\})$ *consists of only one element.*

PROPOSITION 4.6. *Let D be a connected open subset $D \subset \boldsymbol{C}$ such that $D \cap \rho(\theta) \neq \emptyset$. The following assertions concerning the consistence of (e.p.) index of θ in $x \in \mathcal{X}$ on the open subset $D \subset \boldsymbol{C}$ are equivalent:*
(i) $\Phi_\theta^{-1}(\{x\}) \neq \emptyset$,
(ii) $\Phi_\theta^{-1}(\{x\})$ *consists of only one element,*
(iii) *The function $z \longrightarrow R(z,\theta)x = (\theta(z))^{-1}x$ has a unique analytic extension on D.*

Proof. We can apply the above lemma because by e_2 in 3.7, θ has $(s.v.e.p.)$ on D. The proof is completed by observing that a solution $f \in \mathcal{O}(D,\mathcal{X})$ of the equation $\Phi_\theta f = x$ satisfies the equality $f(z) = (\theta(z))^{-1}x = R(z,\theta)x$ for every $z \in D \cap \rho(\theta)$ and we can apply the identity theorem for analytic functions on the connected open set D. □

COROLLARY 4.7. *For an arbitrary open subset $G \subset \boldsymbol{C}$ let $G_{\rho(\theta)}$ be the θ-spectral interior of G (see Definition 3.9 and e_6), and $\Phi_\theta^{-1}(\{x\})$ the (e.p.) index of θ in a point $x \in \mathcal{X}$ on the open subset $G_{\rho(\theta)} \subset \boldsymbol{C}$. The following assertions are equivalent:*
(i) $\Phi_\theta^{-1}(\{x\}) \neq \emptyset$
(ii) $\Phi_\theta^{-1}(\{x\})$ *consists of only one element*
(iii) *The function $z \longrightarrow R(z,\theta)x$ has a unique analytic extension on $G_{\rho(\theta)}$.*

Proof. We apply the above proposition for every connected component of $G_{\rho(\theta)}$. □

Remark 4.8. If θ has $(s.v.e.p.)$ on $\mathbf{C} \setminus F$, then Φ_θ from P_4 of General properties is an isomorphism between $\mathcal{O}_\theta(\mathbf{C} \setminus F)$ and $M_0(F,\theta)$.

By this remark we derive from P4 the following property.

P5. If F is a closed subset of $\mathbf{C}$ and θ an analytic operator valued function having $(s.v.e.p.)$ on $\mathbf{C} \setminus F$, then the strong θ-spectral Bishop space corresponding to F is the closure in $\mathcal{X}$ of the image of a linear continuous $\mathcal{X}$-valued injective map defined on a Fréchet space.

For every closed subset $F \subset \mathbf{C}$ with $(\mathbf{C} \setminus F) \cap \rho(\theta) \neq \emptyset$, θ has $(s.v.e.p.)$ on $(\mathbf{C} \setminus F)_{\rho(\theta)} \neq \emptyset$ the θ-spectral interior of $\mathbf{C} \setminus F$(see Definition 3.9 and e_6). So the complement of the θ-spectral interior of $(\mathbf{C} \setminus F)$ is a closed subset of $\mathbf{C}$

which exemplifies the above Remark 4.8. Some notations will be useful for describing this particular case given by $(\mathbf{C}\setminus F)_{\rho(\theta)}$. We recall that $G_{\rho(\theta)}$, the θ-spectral interior of G (see 3.9, e_6) was defined as the union of all the connected components of G having nonempty intersection with the resolvent set $\rho(\theta)$, $(G_{\rho(\theta)})_{\rho(\theta)} = G_{\rho(\theta)}$ and θ has $(s.v.e.p.)$ on $G_{\rho(\theta)}$.

Definition 4.9. For an arbitrary closed subset F of $\mathbf{C}$ and θ an analytic operator valued function, we denote $F^\theta = \mathbf{C}\setminus(\mathbf{C}\setminus F)_{\rho(\theta)}$, which is a closed subset of $\mathbf{C}$. We call F^θ the θ-spectral closure of F.

Obviously $F \subset F^\theta$, $F = (F^\theta)^\theta$, $\mathbf{C}\setminus F^\theta = (\mathbf{C}\setminus F)_{\rho(\theta)}$, θ has $(s.v.e.p.)$ on $\mathbf{C}\setminus F^\theta$, and $F = F^\theta$ if and only if every connected component of $\mathbf{C}\setminus F$ has a nonempty intersection with $\rho(\theta)$. Thus the consequence of Remark 4.8 can be easily written for F^θ as a particular case of property P5.

P6. If F is an arbitrary closed subset of $\mathbf{C}$ and θ an analytic operator valued function, then the strong θ-spectral Bishop space attached to F^θ is the closure in $\mathcal{X}$ of the image of a linear injective continuous map defined on a Fréchet space with values in $\mathcal{X}$.

Proof. Indeed $\Phi_\theta : \mathcal{O}_\theta(\mathbf{C}\setminus F^\theta) \longrightarrow \mathcal{X}$ is a continuous linear injective map and

$$\Phi_\theta(\mathcal{O}_\theta(\mathbf{C}\setminus F^\theta)) = M_0(F^\theta,\theta),\ \overline{\Phi_\theta(\mathcal{O}_\theta(\mathbf{C}\setminus F^\theta))} = M(F^\theta,\theta). \quad \square$$

Proposition 4.10. *Let us consider now F a closed subset of $\boldsymbol{C}$ and θ an analytic operator valued function having (s.v.e.p.) on $\boldsymbol{C}\setminus F$ (in particular F can be the θ-spectral closure of a closed arbitrary subset). Then the inverse operator of the restriction $\Phi_\theta \mid \mathcal{O}_\theta(\boldsymbol{C}\setminus F)$ given in P4 is a closed densely definite operator S in $M(F,\theta)$ (the corresponding to F strong θ-spectral Bishop space which is a Banach subspace of $\mathcal{X}$):*

$$S : M_0(F,\theta) \to \mathcal{O}(\boldsymbol{C}\setminus F), \quad \overline{M_0(F,\theta)} = M(F,\theta).$$

Proof. Indeed, by Definition 4.1 we can define for every $x \in M_0(F,\theta)$

$$Sx = \Phi_\theta^{-1}(\{x\}) \in \mathcal{O}_\theta(\mathbf{C}\setminus F)$$

because by Lemma 4.5, $x \in M_0(F,\theta)$ means $\Phi_\theta^{-1}(\{x\}) \neq \emptyset$ and contains only one element. Thus Sx is the unique solution $f_x \in \mathcal{O}_\theta(\mathbf{C}\setminus F)$ of the equation

$$\theta(z)f_x(z) = x, \text{ for every } z \in \mathbf{C}\setminus F.$$

Therefore,

$$S : M_0(F,\theta) \to \mathcal{O}(\mathbf{C}\setminus F), Sx = f_x, \quad \overline{M_0(F,\theta)} = M(F,\theta)$$

and it is easy to verify that S is a closed operator. Indeed, if $(x_n)_n \subset M_0(F,\theta)$, $x_n \to x \in \mathcal{X}, Sx_n = f_{x_n}$ and $Sx_n \to g$ in $\mathcal{O}(\mathbf{C}\setminus F)$, then $f_{x_n}(z) \to g(z)$ for every $z \in \mathbf{C} \setminus F$ and $\theta(z)g(z) = x$ for every $z \in \mathbf{C} \setminus F$ because $\theta(z) \in \mathcal{B}(\mathcal{X})$ and $x_n \to x \in \mathcal{X}$. So, $x \in M_0(F, \theta)$ and $g = f_x = Sx$ which proves that S is closed. □

Remark 4.11. All the results of Sections 3, 4, hold for θ_T, $T \in \mathcal{B}(\mathcal{X})$, and the reader can see that some of them (for instance Proposition 3.24, Corollary 3.25, Proposition 4.10) are new for θ_T hence for T. Note however, that an elementary property of Bishop's strong spectral spaces for θ_T, hence for T, which does not have an analogue for an arbitrary θ is $M(\emptyset, T) = 0$. This is a consequence of the fact that $\lim\limits_{\lambda \to \infty} R(\lambda, T) = 0$, but for an arbitrary θ this may not hold for $R(z, \theta)$.

5. BISHOP'S CONDITION β AND ANALYTIC OPERATOR VALUED FUNCTIONS

An analogue of condition β for $T \in \mathcal{B}(\mathcal{X})$ can be given for a general analytic operator valued function $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$. As for (*s.v.e.p.*), for every open subset $G \subset \mathbf{C}$ we consider a property of θ called *condition β on G*. This is in some way a *weak property β* relative to G because it could be possible that this property does not hold for all open subsets of $\mathbf{C}$.

Definition 5.1. An analytic operator valued function $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$ satisfies *condition β on the open subset G* of $\mathbf{C}$, if the following assertion holds:

(β') for every $(f_n)_n \subset \mathcal{O}(G, \mathcal{X})$ and $x \in \mathcal{X}$ such that $\theta(z)f_n(z) \to x$ uniformly on G as $n \to \infty$ it follows that the sequence $(f_n)_n$ is uniformly bounded on every compact subset of G.

Remark 5.2. For $\theta = \theta_T$, (β') is in some way the restriction to G of Bishop's condition β for T. It is natural to say in this case that T satisfies condition β on the open subset G.

In the same way as in [3], (β') can be reformulated as follows.

LEMMA 5.3. *For every θ an analytic operator valued function and G an open subset of $\mathbf{C}$, condition (β') is equivalent with*

(β'') for every $(f_n)_n \subset \mathcal{O}(G, \mathcal{X})$ and $x \in \mathcal{X}$, such that $\theta(z)f_n(z) \to 0$ uniformly on G as $n \to \infty$, it follows that $(f_n)_n$ is uniformly bounded on every compact subset of G.

Proof. The proof from [3] (see the remark after Definition 8) can be rewritten for this case. □

Remark 5.4. If one of the equivalent conditions (β'), (β'') is true for $\theta = \theta_T, T \in \mathcal{B}(\mathcal{X})$, we say that T *satisfies Bishop's condition* β *on the open subset* $G \subset \boldsymbol{C}$. If θ_T, hence T, satisfies condition β on every open subset $G \subset \mathbf{C}$, then T satisfies Bishop's condition β [3].

The above remark suggests the following definition.

Definition 5.5. If $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$, an analytic operator valued function, satisfies *condition* β *on every open subset* G of $\mathbf{C}$, we simply say that θ satisfies condition β.

One of the main consequences of the condition β for a bounded linear operator T can be reformulated for a general operator valued function θ. Let us recall first some useful objects and notations: an open set $G \subset \mathbf{C}$, $x \in \mathcal{X}$, $\Phi_\theta : \mathcal{O}(G, \mathcal{X}) \longrightarrow \mathcal{O}(G, \mathcal{X})$ and $f \in \mathcal{O}(G, \mathcal{X})$ such that $f \in \Phi_\theta^{-1}(\{x\})$ *i.e.*

(I) $\theta(z)f(z) = x$ for all $z \in G$.

A function $f \in \mathcal{O}(G, \mathcal{X})$ *which satisfies (I) is called an exact solution of (I) in* G *and a sequence* $(f_n)_n \subset \mathcal{O}(G, \mathcal{X})$ *such that* $\theta(z)f_n(z) \to x$ *uniformly on* G *as* $n \to \infty$ *is called an approximate solution of (I) in* G. It is not difficult to reformulate the proof of Theorem 4 from [3] and we obtain the following proposition.

PROPOSITION 5.6. *Every exact solution of (I) in* G *is an approximate solution of (I) in* G. *If* θ, *an analytic operator function, satisfies* β *on* G, *then any approximate solution of (I) in* G *gives an exact solution of (I) in* G, *in other words, (I) has exact solutions in* G *if and only if it has approximate solutions in* G.

Proof. If $(f_n)_n$ is an approximate solution of (I) on G and θ satisfies β on G, then $(f_n)_n$ is uniformly bounded on every compact subset $K \subset G$. So there exists a subsequence $(f_{n_k})_k$ converging pointwise in the weak topology of $\mathcal{X}$ to a function $f \in \mathcal{O}(G, \mathcal{X})$. $(f_{n_k})_k$ being also an approximate solution for (I) in G, we deduce that $\theta(z)f(z) = x$ for every $z \in G$, so f is an exact solution of (I) in G. □

$x \in M_0(F, \theta)$ respectively $x \in N(F, \theta)$ means (by Definition 4.1 respectively Remark 4.4) that equation $\theta(z)f(z) = x$ for all $z \in G$, has an exact solution $f \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ respectively an approximate solution $(f_n)_n \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$. In the other words, Proposition 5.6 is the following one.

COROLLARY 5.7. *If* θ, *an analytic operator function, satisfies condition* β *on* $\boldsymbol{C} \setminus F$ *(see Definition 5.1) for a closed subset* F *of* $\boldsymbol{C}$, *then we have the equality*

$$M_0(F, \theta) = N(F, \theta)$$

which implies that $M_0(F,\theta)$ *is a closed subspace of* $\mathcal{X}$. *If* θ *satisfies condition* β *(see Definition 5.5) then for every closed subset* $F \subset \boldsymbol{C}$ *we have the equality*

$$M_0(F,\theta) = N(F,\theta)$$

and $M_0(F,\theta)$ *is a closed subspace of* $\mathcal{X}$.

Remark 5.8. For $\theta = \theta_T$, $T \in \mathcal{B}(\mathcal{X})$, the above Corollary 5.7 is the following. If T, $T \in \mathcal{B}(\mathcal{X})$, satisfies condition β on $\mathbf{C} \setminus F$ (see Remark 5.4) for a closed subset F of $\mathbf{C}$, then we have the equality

$$M_0(F,T) = N(F,T)$$

which implies that $M_0(F,T)$ is a closed subspace of $\mathcal{X}$.
If T satisfies condition β (see Remark 5.4) then for every closed subset $F \subset \mathbf{C}$ we have the well-known equality

$$M_0(F,T) = N(F,T)$$

and $M_0(F,T)$ is a closed subspace of $\mathcal{X}$ for every closed subset F of $\mathbf{C}$ [3].

For an arbitrary analytic operator valued function θ and for an arbitrary $T \in \mathcal{B}(\mathcal{X})$ we would like to know how many open sets $G \subset \mathbf{C}$ there exists such that θ, respectively T satisfies condition β on G. Obviously $\rho(\theta)$ and $\rho(T)$ are open sets and θ, respectively T satisfies condition β on $\rho(\theta)$, respectively $\rho(T)$. We can also prove that for every analytic operator valued function θ(in particular for every $T \in \mathcal{B}(\mathcal{X})$ which generates θ_T)there exists the biggest open nonempty subset of $\mathbf{C}$ such that θ (respectively $T \in \mathcal{B}(\mathcal{X})$) satisfies condition β on every open subset of it.

Let us denote,

$$\mathcal{G}_\theta^\beta = \{G \text{ open set}, G \subset \mathbf{C} \mid \theta \text{ satisfies } \beta \text{ on } \Delta, \forall \text{ open set } \Delta \subset G\}$$

Obviously $\emptyset \neq \rho(\theta) \in \mathcal{G}_\theta^\beta$ and $\bigcup_{G \in \mathcal{G}_\theta^\beta} G$ is a nonempty open set. Indeed, if Δ is an open subset of $\rho(\theta)$ and $(f_n) \subset \mathcal{O}(\Delta,\mathcal{X}), \theta(z)f_n(z) \to x \in \mathcal{X}$ uniformly on Δ, as $n \to \infty$, we have for $z \in \Delta \subset \rho(\theta), f_n(z) \to \theta(z)^{-1}x$,as $n \to \infty$, uniformly on every compact $K \subset \Delta$. Thus, (f_n) is uniformly bounded on every compact $K \subset \Delta$ and θ satisfies condition β on Δ and $\rho(\theta) \in \mathcal{G}_\theta^\beta$.

Proposition 5.9. *Define* $D_\theta^\beta := \bigcup_{G \in \mathcal{G}_\theta^\beta} G \neq \emptyset$. *Then* D_θ^β *is the largest open subset in* $\boldsymbol{C}$ *such that* θ *satisfies condition* β *on every open subset* $\Delta \subset D_\theta^\beta$.

Proof. D_θ^β is a non-empty open set and $D_\theta^\beta \supset G \in \mathcal{G}_\theta^\beta$; hence D_θ^β contains every open subset $D \subset \mathbf{C}$ where θ satisfies condition β on every open subset of D. It remains to prove that θ satisfies condition β on every open subset $\Delta \subset D_\theta^\beta$. We have to prove that θ satisfies condition β on Δ a non-empty open subset of D_θ^β. Let us consider $(f_n)_n \subset \mathcal{O}(\Delta, \mathcal{X})$ and $x \in \mathcal{X}$ such that $\theta(z)f_n(z) \to x$ uniformly on Δ, as $n \to \infty$. We have to prove $(f_n)_n$ uniformly bounded on every compact $K \subset \Delta$. For a compact set $K \subset \Delta = \bigcup_{G \in \mathcal{G}_\theta^\beta} \Delta \cap G$, $\Delta \cap G$ being open subsets we deduce:

$$\forall z \in K,\ \exists G \in \mathcal{G}_\theta^\beta,\ \exists r_z^G > 0 \textit{ and } D(z, r_z^G) \subset \overline{D}(z, r_z^G) \subset G \cap \Delta.$$

K being a compact set, there exists $z_i \in K$, $G_i \in \mathcal{G}_\theta^\beta$ and $r_{z_i}^{G_i} > 0$,$i = 1, ..., m$ such that

$$K \subset \bigcup_{i=1}^{m} D(z_i, r_{z_i}^{G_i}),\ \ \overline{D}(z_i, r_{z_i}^{G_i}) \subset G_i \cap \Delta,\ \text{for every } i = 1, ..., m,$$

where $D(z, r) = \{\zeta \in \mathbf{C} \mid\mid \zeta - z \mid< r\}$, $\overline{D}(z, r) = \{\zeta \in \mathbf{C} \mid\mid \zeta - z \mid\leq r\}$. $G_i \cap \Delta$ are open subsets of $G_i \in \mathcal{G}_\theta^\beta$, hence θ satisfies condition β on $G_i \cap \Delta$ for $i = 1, ..., m$. But $\theta(z)f_n(z) \to x$ uniformly on $G_i \cap \Delta$ as $n \to \infty$ and $(f_n \mid_{G_i \cap \Delta})_n \subset \mathcal{O}(G_i \cap \Delta, \mathcal{X})$ for $i = 1, ...m$, because $G_i \cap \Delta \subset \Delta$. Hence, for every $i = 1, ..., m$

$$(f_n \mid \overline{D}(z_i, r_{z_i}^{G_i}))_n$$

is uniformly bounded because $\overline{D}(z_i, r_{z_i}^{G_i})$ is a compact subset of $G_i \cap \Delta$ and θ satisfies condition β on $G_i \cap \Delta$ for $i = 1, ...m$. We deduce $(f_n)_n$ uniformly bounded on K because $K \subset \bigcup_{i=1}^m \overline{D}(z_i, r_{z_i}^{G_i})$. Thus, we proved that θ satisfies condition β on every open subset $\Delta \subset D_\theta^\beta$, which concludes the proof. □

Corollary 5.10. *For every closed subset $F \supset \mathbf{C} \setminus D_\theta^\beta$ we have*

$$M_0(F, \theta) = N(F, \theta).$$

Proof. From $\mathbf{C} \setminus F \subset D_\theta^\beta$ it follows that θ satisfies condition β on $\mathbf{C} \setminus F$ and we can apply Corollary 5.7 of Proposition 5.6. □

Remark 5.11. The equality $D_\theta^\beta = \mathbf{C}$ means that θ satisfies condition β on every open subset of $\mathbf{C}$ or simply θ satisfies condition β.

For $\theta = \theta_T$, $T \in \mathcal{B}(\mathcal{X})$, denote $D_{\theta_T}^\beta = D_T^\beta$. Then Proposition 5.9 and Corollary 5.10 can be rewritten and this result is new for T.

Proposition 5.12. *For every $T \in \mathcal{B}(\mathcal{X})$, D_T^β is the largest open subset of $\mathbf{C}$ such that T satisfies condition β on every open subset of it. $D_T^\beta = \mathbf{C}$ means that T satisfies Bishop's condition β and for every closed subset $F \supset \mathbf{C} \setminus D_T^\beta$ we have $M_0(F, T) = N(F, T)$. For an arbitrary T, D_T^β gives an evaluation of Bishop's condition β for T.*

6. THE RESTRICTION PROPERTY FOR θ-SPECTRAL SPACES

We maintain the notation from the above sections. So, $\mathcal{B}(\mathcal{X})$ is the algebra of all bounded operators on a complex Banach space $\mathcal{X}$, $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$ is an analytic operator valued function with a nonempty resolvent $\rho(\theta) \neq \emptyset$ and $\sigma(\theta)$ is the spectrum of θ (Section 2). For F a closed subset of $\mathbf{C}$, we defined (Definitions 4.1, 4.3) the strong and weak Bishop θ-spectral spaces,

$$\overline{M_0(F,\theta)} = M(F,\theta) \subset N(F,\theta)$$

which for θ_T are the strong and weak Bishop spectral spaces of T (see [3]),

$$\overline{M_0(F,T)} = M(F,T) \subset N(F,T).$$

Recall that

$$M_0(F,\theta) = \{x \in \mathcal{X} \mid \exists f \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \theta(z)f(z) = x, \forall z \in \mathbf{C} \setminus F\},$$

$$M_0(F,T) = \{x \in \mathcal{X} \mid \exists f \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), (T-z)f(z) = x, \forall z \in \mathbf{C} \setminus F\},$$

$$\begin{aligned} &N(F,\theta) \\ &= \{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \| \theta(z)f_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\}, \end{aligned}$$

$$\begin{aligned} &N(F,T) \\ &= \{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \| (T-z)f_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\}. \end{aligned}$$

LEMMA 6.1. *For $x \in \mathcal{X}$ the following assertions are equivalent:*

1. *there exists $f \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \theta(z)f(z) = x$, every $z \in \mathbf{C} \setminus F$,*
2. *there exists $g \in \mathcal{O}(\mathbf{C} \setminus (F \cap \sigma(\theta)), \mathcal{X}), \theta(z)g(z) = x$ for all z, $z \in (\mathbf{C} \setminus F) \cup \rho(\theta)$.*

Proof. Obviously 2. $\Rightarrow$ 1.: f can be $g \mid \mathbf{C} \setminus F$.
1. $\Rightarrow$ 2.. by Definition 2.1 of $R(z,\theta), z \in \rho(\theta)$, we deduce from 1. that $f(z) = R(z,\theta)x$ for every $z \in (\mathbf{C} \setminus F) \cap \rho(\theta)$. Thus, the equalities

$$g(z) = \begin{cases} f(z) & \text{if } z \in \mathbf{C} \setminus F \\ R(z,\theta)x & \text{if } z \in \rho(\theta) \end{cases}$$

define $g \in \mathcal{O}((\mathbf{C} \setminus F) \cup \rho(\theta), \mathcal{X}) = \mathcal{O}((\mathbf{C} \setminus (F \cap \sigma(\theta)), \mathcal{X})$ and g verifies 2. □

Therefore $M_0(F,\theta) = M_0(F \cap \sigma(\theta), \theta)$ and the following property of the strong θ-spectral Bishop spaces holds.

Property (r). *For every closed subset $F \subset \mathbf{C}$ and for an analytic operator valued function $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$ we have*

$$M(F,\theta) = M(F \cap \sigma(\theta), \theta).$$

We say that the strong θ-spectral Bishop spaces have the restriction property to the spectrum of θ, or simply have the Property (r).

Remark 6.2. We can prove in a similar way a similar property(r) for the weak θ-spectral spaces $N(F,\theta)$ only in the case when $(\mathbf{C}\setminus F)\cap\rho(\theta)=\emptyset$; generally speaking there is a natural obstruction to use the above proof for $N(F,\theta)$ for every closed subset $F\subset\mathbf{C}$.

The inclusion $N(F,\theta)\supset N(F\cap\sigma(\theta),\theta)$ is obviously true as above. But for proving the opposite one, in the same way as above, we have to obtain a sequence(see Remark 4.4)$(g_n)\subset\mathcal{O}((\mathbf{C}\setminus F)\cup\rho(\theta),\mathcal{X})$ with the property

$$\theta(z)g_n(z)\to x \text{ uniformly on}(\mathbf{C}\setminus F)\cup\rho(\theta),\ \text{as } n\to\infty,$$

when we have $(f_n)\subset\mathcal{O}(\mathbf{C}\setminus F,\mathcal{X})$ with a weak similar property

$$\theta(z)f_n(z)\to x \text{ uniformly on } \mathbf{C}\setminus F,\ \text{as } n\to\infty,$$

which gives only

$$\theta(z)f_n(z)\to x \text{ for } z\in(\mathbf{C}\setminus F)\cap\rho(\theta)$$

This last assertion is, generally speaking, not enough for obtaining $(g_n)_n$; excepting the case when $(\mathbf{C}\setminus F)\cap\rho(\theta)=\emptyset$, the equalities

$$g_n(z)=\begin{cases} f_n(z) & \text{if } z\in\mathbf{C}\setminus F\\ R(z,\theta)x & \text{if } z\in\rho(\theta)\end{cases}$$

cannot always define an analytic functions on $(\mathbf{C}\setminus F)\cup\rho(\theta)$. For instance, if $D_1\subset D$ are open subsets in $\mathbf{C}$ and $\varphi\in\mathcal{O}(D_1)$ does not have an analytic extension to D, then $f_n=1/n\ \varphi$ gives a sequence $(f_n)_n\subset\mathcal{O}(D_1), f_n\to 0$ as $n\to\infty$ and for every n, f_n does not have an analytic extension to D.

The obstruction described above does not exist for a new class of θ-spectral subspaces which are intermediate between the strong and weak θ-spectral spaces. We will describe this new class in what follows. First we describe a large class of such a θ-spectral spaces attached to the closed subsets $F\subset\mathbf{C}$ and being between $M(F,\theta)$ and $N(F,\theta)$ the strong, respectively weak θ-spectral spaces. In order to describe a class of such spaces, a natural way is to look for common elements of the definitions of $M(F,\theta)$ and $N(F,\theta)$, F being as above a closed set of complex numbers. So, as above, $\mathcal{X}$ being a complex Banach space and $\mathcal{B}(\mathcal{X})$ the algebra of all linear bounded operators on $\mathcal{X}$ we can enumerate the following elements:

1. the space of the $\mathcal{X}$-valued analytic functions $\mathcal{O}(\mathbf{C}\setminus F,\mathcal{X})$ defined on the open subset $G=\mathbf{C}\setminus F$,

2. the map

$$\Phi_\theta:\mathcal{O}(G,\mathcal{X})\longrightarrow\mathcal{O}(G,\mathcal{X})$$

associated to an analytic operator function $\theta:\mathbf{C}\to\mathcal{B}(\mathcal{X})$ having nonempty resolvent $\rho(\theta)\neq\emptyset$ (Section 3),

$$[\Phi_\theta f](z)=\theta(z)f(z) \text{ for every } f\in\mathcal{O}(\mathbf{C}\setminus F,\mathcal{X}) \text{ and every } z\in G=\mathbf{C}\setminus F,$$

3. the subspace of $\mathcal{X}$-valued constant function on $\mathbf{C} \setminus F$,

$$\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$$

$\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}) = \widetilde{\mathcal{X}}$ can be identified with $\mathcal{X}$ as usual.

Using 1., 2., 3., from above and Definitions 4.1, 4.3 we can write $N(F, \theta) =$

$$\{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists h_\epsilon \in Range\Phi_\theta, \| h_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\},$$

$M(F, \theta) =$

$$\{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists h_\epsilon \in Range\Phi_\theta \cap \mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}), \| h_\epsilon(z) - x \| < \epsilon, \\ \forall z \in \mathbf{C} \setminus F\}.$$

Indeed, the first equality is obvious and for the second we observe that $h_\epsilon \in Range\Phi_\theta \cap \mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X})$ is equivalent with the assertion " $\exists f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \exists x_\epsilon \in \mathcal{X}, h_\epsilon(z) = \Phi_\theta(f_\epsilon)(z) = x_\epsilon \ \forall z \in \mathbf{C} \setminus F$ " *i.e.* $x_\epsilon \in M_0(F, \theta)$ and in this case the property of x can be obviously rewritten

$$\forall \epsilon > 0 \ \ \exists x_\epsilon \in M_0(F, \theta) \text{ such that } \ \| x_\epsilon - x \| < \epsilon,$$

which means that $x \in \overline{M_0(F, \theta)} = M(F, \theta)$. This description of $M(F, \theta)$ and $N(F, \theta)$ given by the above two equalities proves the existence of a similar description for both of this subspaces and their inclusion in a large class of subspaces between $M(F, \theta)$ and $N(F, \theta)$. Indeed, if we consider a subspace $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$,

$$\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$$

where the inclusions are not strict, we associate to this functional subspace $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$, a θ-spectral subspace in the same way as $N(F, \theta)$ corresponds to $\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ and $M(F, \theta)$ corresponds to $\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X})$. So we define:

Definition 6.3. Let $\theta : \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{X})$ be an analytic operator valued function with a nonempty resolvent set $\rho(\theta) \neq \emptyset$, $F = \overline{F} \subset \mathbf{C}$ an arbitrary closed subset of $\mathbf{C}$ and $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$ a subspace of $\mathcal{X}$-valued functions on $\mathbf{C} \setminus F$,

$$\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}).$$

The θ-spectral space $Y(F, \theta)$ associated to F and defined by $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$ is described by the following equality:

$$Y(F, \theta) = \{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists h_\epsilon \in Range\Phi_\theta \cap \mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}), \\ \| h_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\}.$$

and will be called θ-spectral space "associated to $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$" or simply "associated to F" when no misunderstanding is possible.

The property (r), the restriction property to the spectrum $\sigma(\theta)$, of the class of θ-spectral spaces $Y(F, \theta)$, is the set of equalities $Y(F, \theta) = Y(F \cap \sigma(\theta), \theta)$ for every closed subset $F \subset \mathbf{C}$.

Remark 6.4. Obviously we have:

1. $M(F,\theta) \subset Y(F,\theta) \subset N(F,\theta)$ for every $F = \overline{F} \subset \mathbf{C}$,
2. $Y(F,\theta) = N(F,\theta)$ if $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$,
   $Y(F,\theta) = M(F,\theta)$ if $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X})$.

Definition 6.3 can be rewritten using sequences of analytic functions obtained replacing "$\forall \epsilon > 0$" by the values of some positive numerical sequence converging to 0, for example $(1/n)_n$. Then for every natural number n for $\epsilon = 1/n$ we denote h_ϵ by h_n. Thus we obtain an equivalent statement of Definition 6.3.

LEMMA 6.5. *$x \in Y(F,\theta)$ if and only if there exists a sequence $(h_n)_n \subset Range\Phi_\theta \cap \mathcal{Y}(\boldsymbol{C} \setminus F, \mathcal{X})$ such that $h_n(z) \longrightarrow x$ uniformly for $z \in \boldsymbol{C} \setminus F$, or in detail $x \in Y(F,\theta)$ if and only if there exists a sequence $(f_n)_n \subset \mathcal{O}(\boldsymbol{C} \setminus F, \mathcal{X})$ such that $\Phi_\theta(f_n) \in \mathcal{Y}(\boldsymbol{C} \setminus F, \mathcal{X})$ and for $n \longrightarrow \infty$ $\Phi_\theta(f_n)(z) = \theta(z) f_n(z) \longrightarrow x$ uniformly for $z \in \boldsymbol{C} \setminus F$.*

In the following, we introduce a class of functional spaces $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$, a particular case of $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$ considered above, such that

$$\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$$

and the class of θ-spectral spaces $L(F,\theta)$ associated to $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ by Definition 6.3. The class of θ-spectral spaces $L(F,\theta)$ contains only the weak θ-spectral spaces $N(F,\theta)$ corresponding to the closed subsets $F \subset \mathbf{C}$, $(\mathbf{C} \setminus F) \cap \rho(\theta) = \emptyset$ and the above mentioned obstruction (see Remark 6.2) concerning a property(r) for the weak θ-spectral spaces $N(F,\theta)$ does not exist for these θ-spectral spaces $L(F,\theta)$.

As we will see, first we discuss the relation between the decompositions given by the connected components of the open subsets $\mathbf{C} \setminus F$, $\rho(\theta)$ and $(\mathbf{C} \setminus F) \cup \rho(\theta)$. This relation is described in general for the arbitrary open subsets $G, \rho, G \cup \rho$ of $\mathbf{C}$ and used in the particular case of the sets $\mathbf{C} \setminus F$, $\rho(\theta)$ and $(\mathbf{C} \setminus F) \cup \rho(\theta)$ for F a closed subset of $\mathbf{C}$.

So, let G, ρ, be now two open arbitrary open subsets of $\mathbf{C}$ and

$$G = \bigcup_{\alpha \in \Lambda} G_\alpha, \quad \rho = \bigcup_{i \in I} \rho_i$$

the decomposition given by the connected components $\{G_\alpha\}_{\alpha \in \Lambda}$ of G and $\{\rho_i\}_{i \in I}$ of ρ. In the following, we obtain from these decompositions the decomposition of $G \cup \rho$ given by its connected components. We will denote for $\alpha \in \Lambda$,

$$I_\alpha = \{j \in I \mid \rho_j \cap G_\alpha \neq \emptyset\}$$

and

$$I_0 = \{k \in I \mid \rho_k \cap G = \emptyset\} = \{k \in I \mid \rho_k \cap G_\alpha = \emptyset, \forall \alpha \in \Lambda\} = I \setminus \bigcup_{\alpha \in \Lambda} I_\alpha.$$

Then we have,

$$(*) \quad G \cup \rho = \bigcup_{\alpha\in\Lambda} G_\alpha \cup \bigcup_{i\in I} \rho_i = \bigcup_{\alpha\in\Lambda} (G_\alpha \cup \bigcup_{j\in I_\alpha} \rho_j) \cup \bigcup_{k\in I_0} \rho_k = \bigcup_{\alpha\in\Lambda} \widehat{G}_\alpha \cup \bigcup_{k\in I_0} \rho_k$$

where

$$\widehat{G}_\alpha := G_\alpha \cup \bigcup_{j\in I_\alpha} \rho_j \text{ and } \widehat{G}_\alpha \cap \bigcup_{k\in I_0} \rho_k = \emptyset, \forall \alpha \in \Lambda.$$

It is easy to prove that $\widehat{G}_\alpha$ is a connected set. Indeed for every $j \in I_\alpha$ we have $G_\alpha \cap \rho_j \neq \emptyset$, $G_\alpha \cup \rho_j$ is a connected set, $G_\alpha \cup \rho_j \supset G_\alpha$ and $\widehat{G}_\alpha = \bigcup_{j\in I_\alpha} (G_\alpha \cup \rho_j)$.

It is also easy to verify that for $\alpha \neq \beta$ $\widehat{G}_\alpha \cap \widehat{G}_\beta \neq \emptyset$ if and only if $I_\alpha \cap I_\beta \neq \emptyset$ *i.e.* there exists $j \in I, \rho_j \cap G_\alpha \neq \emptyset, \rho_j \cap G_\beta \neq \emptyset$.

Let us consider now the following equivalence relation in the set $\{\widehat{G}_\alpha \mid \alpha \in \Lambda\}$.

Definition 6.6. We say that $\widehat{G}_\alpha$ and $\widehat{G}_\beta$ are equivalent and denote $\widehat{G}_\alpha \sim \widehat{G}_\beta$, if there exists $\{\alpha_1, ..., \alpha_n\} \subset \Lambda$ such that,

$$\widehat{G}_\alpha \cap \widehat{G}_{\alpha_1} \neq \emptyset, ..., \widehat{G}_{\alpha_k} \cap \widehat{G}_{\alpha_{k+1}} \neq \emptyset, ..., \widehat{G}_{\alpha_n} \cap \widehat{G}_\beta \neq \emptyset$$

We can consider $\mathcal{C}_\alpha = \{\widehat{G}_\beta \mid \widehat{G}_\beta \sim \widehat{G}_\alpha\}$ for $\alpha \in \Lambda$, the equivalence class of $\widehat{G}_\alpha$ given by the above defined equivalence $\sim$ and

$$\{\mathcal{C}_\alpha \mid \alpha \in \Lambda_1\}, \quad \Lambda_1 \subset \Lambda$$

the set of all distinct equivalence classes (two arbitrary equivalent classes $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ with $\alpha, \beta \in \Lambda$ are disjoint or coincide).

Remark 6.7. 1. If $\alpha \in \Lambda$ and $I_\alpha = \emptyset$, *i.e.* $G_\alpha \cap \rho = \emptyset$, we have $\widehat{G}_\alpha = G_\alpha, \mathcal{C}_\alpha = \{G_\alpha\}$ and $\alpha \in \Lambda_1$.

2. If $\widehat{G}_\alpha \sim \widehat{G}_\beta$ as in Definition 6.6 we have $\widehat{G}_\beta \in \mathcal{C}_\alpha$, $\widehat{G}_{\alpha_k} \in \mathcal{C}_\alpha$ for every $k = 1, ..., n$ and $\widehat{G}_\alpha \cup \widehat{G}_{\alpha_1} \cup ... \cup \widehat{G}_{\alpha_k} \cup ... \cup \widehat{G}_{\alpha_n} \cup \widehat{G}_\beta$ is a connected set. [Indeed as we have already remarked, $\widehat{G}_\alpha$ is connected for every α and $\widehat{G}_\alpha \cap \widehat{G}_{\alpha_1} \neq \emptyset$ gives $\widehat{G}_\alpha \cup \widehat{G}_{\alpha_1}$ a connected set, $\widehat{G}_{\alpha_1} \cap \widehat{G}_{\alpha_2} \neq \emptyset$ gives $(\widehat{G}_\alpha \cup \widehat{G}_{\alpha_1}) \cap \widehat{G}_{\alpha_2} \neq \emptyset$ hence $(\widehat{G}_\alpha \cup \widehat{G}_{\alpha_1}) \cup \widehat{G}_{\alpha_2} \neq \emptyset$ is a connected set, etc.]

Lemma 6.8. *If we denote* $\mathcal{G}_\alpha = \bigcup\limits_{\widehat{G}_\beta \in \mathcal{C}_\alpha} \widehat{G}_\beta = \bigcup\limits_{\widehat{G}_\beta \sim \widehat{G}_\alpha} \widehat{G}_\beta, \alpha \in \Lambda_1,$ *then* $\mathcal{G}_\alpha$ *is an open connected subset of* $\boldsymbol{C}$ *for every* α, $\mathcal{G}_{\alpha_1} \cap \mathcal{G}_{\alpha_2} = \emptyset$ *for every* $\alpha_1, \alpha_2 \in \Lambda_1, \alpha_1 \neq \alpha_2$ *and* $\bigcup\limits_{\alpha \in \Lambda_1} \mathcal{G}_\alpha = \bigcup\limits_{\alpha \in \Lambda} \widehat{G}_\alpha$.

Proof. $\mathcal{G}_\alpha$ are open subsets because $\widehat{G}_\beta$ are open sets for every $\beta \in \Lambda$. We have $\mathcal{G}_\alpha = \bigcup\limits_{\widehat{G}_\beta \sim \widehat{G}_\alpha} \widehat{G}_\alpha \cup \widehat{G}_{\alpha_1} \cup ... \cup \widehat{G}_{\alpha_k} \cup ... \cup \widehat{G}_{\alpha_n} \cup \widehat{G}_\beta$, with $\widehat{G}_{\alpha_k}$ given by the equivalence $\widehat{G}_\alpha \sim \widehat{G}_\beta$, because $\widehat{G}_\alpha \sim \widehat{G}_{\alpha_k}$ for every $k = 1, ..., n$ by definition. On the other hand, for every β with $\widehat{G}_\beta \sim \widehat{G}_\alpha$, the sets $\widehat{G}_\alpha \cup \widehat{G}_{\alpha_1} \cup ... \cup \widehat{G}_{\alpha_k} \cup ... \cup \widehat{G}_{\alpha_n} \cup \widehat{G}_\beta$ contain $\widehat{G}_\alpha$ and are connected sets by 2. of the above Remark 6.7. Hence their union *i.e.* $\mathcal{G}_\alpha$ is a connected set. We recall that for $\alpha_1, \alpha_2 \in \Lambda_1, \alpha_1 \neq \alpha_2$ we have $\mathcal{C}_{\alpha_1} \neq \mathcal{C}_{\alpha_2}$. If there exists $\alpha_1, \alpha_2 \in \Lambda_1, \alpha_1 \neq \alpha_2$ such that $\mathcal{G}_{\alpha_1} \cap \mathcal{G}_{\alpha_2} \neq \emptyset$ then(by Definition of $\mathcal{G}_\alpha$) there exists $\widehat{G}_\beta \in \mathcal{C}_{\alpha_1}$, $\widehat{G}_\gamma \in \mathcal{C}_{\alpha_2}$ and $\widehat{G}_\beta \cap \widehat{G}_\gamma \neq \emptyset$ which means $\widehat{G}_\beta \sim \widehat{G}_\gamma$ and $\widehat{G}_\beta \in \mathcal{C}_{\alpha_1}$, $\widehat{G}_\gamma \in \mathcal{C}_{\alpha_2}$ give $\mathcal{C}_{\alpha_1} = \mathcal{C}_{\alpha_2}$. But $\mathcal{C}_{\alpha_1} \neq \mathcal{C}_{\alpha_2}$. Hence $\mathcal{G}_{\alpha_1} \cap \mathcal{G}_{\alpha_2} = \emptyset$ for every $\alpha_1, \alpha_2 \in \Lambda_1, \alpha_1 \neq \alpha_2$. The equality $\bigcup\limits_{\alpha \in \Lambda_1} \mathcal{G}_\alpha = \bigcup\limits_{\alpha \in \Lambda} \widehat{G}_\alpha$ results from the definition of Λ_1 which gives $\bigcup\limits_{\alpha \in \Lambda_1} \mathcal{C}_\alpha = \{\widehat{G}_\alpha \mid \alpha \in \Lambda\}$ and the lemma has been proved. □

Proposition 6.9. *The decomposition of* $G \cup \rho$ *given by its connected components is*

$$G \cup \rho = \bigcup_{\alpha \in \Lambda_1} \mathcal{G}_\alpha \cup \bigcup_{k \in I_0} \rho_k$$

i.e. $\{\mathcal{G}_\alpha, \rho_k \mid \alpha \in \Lambda_1, k \in I_0\}$ *is the set of all connected components of* $G \cup \rho$. *If we consider the subset*

$$\Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho = \emptyset \quad i.e. \quad I_\alpha = \emptyset\}$$

similar to $I_0 \subset I$, *by 1. of the above Remark 6.7 we have* $\Lambda_0 \subset \Lambda_1$ *and the decomposition of* $G \cup \rho$ *given by its connected components can be rewritten in a symmetrical form*

$$G \cup \rho = \bigcup_{\alpha \in \Lambda_0} G_\alpha \cup \bigcup_{\alpha \in \Lambda_1 \setminus \Lambda_0} \mathcal{G}_\alpha \cup \bigcup_{k \in I_0} \rho_k.$$

Proof. The first equality derives from $(*)$ equality(before Definition 6.6)

$$G \cup \rho = \bigcup_{\alpha \in \Lambda} \widehat{G}_\alpha \cup \bigcup_{k \in I_0} \rho_k,$$

and the last equality from Lemma 6.8. Then, $\{\mathcal{G}_\alpha, \rho_k \mid \alpha \in \Lambda_1, k \in I_0\}$ are the connected components of $G \cup \rho$ because by Lemma 6.8 and definition of I_0 they are open, connected and mutually disjoint sets. The second equality is an easy consequence of 1. Remark 6.7 because $\alpha \in \Lambda_0$ means $I_\alpha = \emptyset$ *i.e.* $\widehat{G}_\alpha = G_\alpha$ so $\mathcal{C}_\alpha = \{G_\alpha\}, \alpha \in \Lambda_1$ and $\mathcal{G}_\alpha = G_\alpha$. □

LEMMA 6.10. *For every $\beta \in \Lambda \setminus \Lambda_0$ there exists a unique $\alpha(\beta) \in \Lambda_1 \setminus \Lambda_0$ such that $G_\beta \subset \mathcal{G}_{\alpha(\beta)}$.*

Proof. For every $\beta \in \Lambda \setminus \Lambda_0$, G_β is a connected component of $G \subset G \cup \rho$ and $G_\beta \cap \rho_k = \emptyset$ for every $k \in I_0$. Obviously, $G_\beta \cap G_\alpha = \emptyset$ for every $\alpha \neq \beta$ in particular for every $\beta \in \Lambda \setminus \Lambda_0$ and $\alpha \in \Lambda_0$. On the other hand, as a connected subset of $G \cup \rho$,G_β is contained in only one connected component of $G \cup \rho$ and this connected component cannot be one of G_α with $\alpha \in \Lambda_0$ or ρ_k with $k \in I_0$ because, as we proved before, G_β has a void intersection with these. So by Proposition 6.9 this unique connected component of $G \cup \rho$ which contains G_β can be denoted $\mathcal{G}_{\alpha(\beta)}$ with $\alpha(\beta) \in \Lambda_1 \setminus \Lambda_0$ and the lemma is proved. □

A similar proof gives the following lemma.

LEMMA 6.11. *For every $k \in I \setminus I_0$ there exists a unique $\alpha(k) \in \Lambda_1 \setminus \Lambda_0$ such that $\rho_k \subset \mathcal{G}_{\alpha(k)}$.*

LEMMA 6.12. *Let $\mathcal{G}_\alpha$ be a connected component of $G \cup \rho$ with $\alpha \in \Lambda_1 \setminus \Lambda_0$ (Proposition 6.9). Then*

$$\alpha \in \Lambda_\alpha = \{\beta \in \Lambda \mid \mathrm{G}_\beta \subset \mathcal{G}_\alpha\}$$

and

$$G_\gamma \cap \rho \neq \emptyset \ \text{ for every } \ G_\gamma \subset \mathcal{G}_\alpha, \quad \gamma \in \Lambda$$

i.e. $\Lambda_\alpha \subset \Lambda \setminus \Lambda_0$.

Proof. $\alpha \in \Lambda_\alpha$ because for $\alpha \in \Lambda_1 \setminus \Lambda_0$ we have $\widehat{G}_\alpha = G_\alpha \cup \bigcup_{j \in I_\alpha} \rho_j \in \mathcal{C}_\alpha$ and by definition we have $\mathcal{G}_\alpha = \bigcup_{\widehat{G}_\beta \in \mathcal{C}_\alpha} \widehat{G}_\beta$. For proving the second part of the lemma we start by assuming the opposite: that for some $\alpha \in \Lambda_1 \setminus \Lambda_0$ and $G_\gamma \subset \mathcal{G}_\alpha, \gamma \in \Lambda$, we have $G_\gamma \cap \rho = \emptyset$. Now we prove that this is impossible, which conclude the proof. Obviously, $G_\gamma \cap \rho = \emptyset$ implies $\gamma \in \Lambda_0$, such that $G_\gamma = \widehat{G}_\gamma$, $\mathcal{C}_\gamma = \{G_\gamma\}$, $\gamma \in \Lambda_1$ (1. Remark 6.7). Then $\mathcal{G}_\gamma = G_\gamma \subset \mathcal{G}_\alpha$ which is impossible because for $\gamma \in \Lambda_0$ and $\alpha \in \Lambda_1 \setminus \Lambda_0$ we have $\alpha \neq \gamma$ hence $\mathcal{G}_\gamma \cap \mathcal{G}_\alpha = \emptyset$, and the lemma has been proved. □

COROLLARY 6.13. *For every $\alpha \in \Lambda_1 \setminus \Lambda_0$ and $\beta \in \Lambda$ the following assertions are equivalent:*

1. $\mathrm{G}_\beta \subset \mathcal{G}_\alpha$
2. $\widehat{G}_\beta = G_\beta \cup \bigcup_{i\in I_\beta} \rho_i \subset \mathcal{G}_\alpha$ *and* $I_\beta \neq \emptyset$*(i.e.* $\mathrm{G}_\beta \cap \rho \neq \emptyset$*).*

We have also $\bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \Lambda_\alpha = \Lambda \setminus \Lambda_0$ *and for every* $\alpha \in \Lambda_1 \setminus \Lambda_0$*, we have*

$$\mathcal{G}_\alpha = \bigcup_{\widehat{G}_\beta\in\mathcal{C}_\alpha} \widehat{G}_\beta = \bigcup_{\widehat{G}_\beta\sim\widehat{G}_\alpha} \widehat{G}_\beta = \bigcup_{\beta:G_\beta\subset\mathcal{G}_\alpha} \widehat{G}_\beta = \bigcup_{\beta:G_\beta\subset\mathcal{G}_\alpha} G_\beta \cup \bigcup_{i\in I_\beta} \rho_i.$$

Proof. If $G_\beta \subset \mathcal{G}_\alpha$,$\alpha \in \Lambda_1\setminus\Lambda_0$, by Lemma 6.12 we deduce $G_\beta\cap\rho \neq \emptyset$. Then $\beta \in \Lambda\setminus\Lambda_0, I_\beta \neq \emptyset, \widehat{G}_\beta = G_\beta \cup \bigcup_{i\in I_\beta} \rho_i \subset \mathcal{C}_\beta = \mathcal{C}_{\delta(\beta)}$ for some $\delta(\beta) \in \Lambda_1\setminus\Lambda_0$. Thus $\mathrm{G}_\beta \subset \widehat{G}_\beta \subset \mathcal{G}_{\delta(\beta)}$ by definition of $\mathcal{G}_{\delta(\beta)}$, $\delta(\beta) \in \Lambda_1$. Then $\mathcal{G}_{\delta(\beta)} = \mathcal{G}_\alpha$ because $G_\beta \subset \mathcal{G}_{\delta(\beta)} \cap \mathcal{G}_\alpha$. So $\widehat{G}_\beta \subset \mathcal{G}_{\delta(\beta)} = \mathcal{G}_\alpha$, 1. implies 2. and the assertions 1. and 2. are equivalent because obviously 2. implies 1.. We prove now the equality $\bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \Lambda_\alpha = \Lambda \setminus \Lambda_0$. The inclusion $\bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \Lambda_\alpha \subset \Lambda \setminus \Lambda_0$ is a consequence of the Lemma 6.12. For proving the second inclusion $\Lambda \setminus \Lambda_0 \subset \bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \Lambda_\alpha$ we can proceed as in Lemma 6.10. By definitions, we first note that $\gamma \in \Lambda \setminus \Lambda_0$ gives $I\gamma \neq \emptyset$ and $\widehat{G}_\gamma \in \mathcal{C}_\gamma = \mathcal{C}_{\alpha(\gamma)}$ for some $\alpha(\gamma) \in \Lambda_1$. Observing that $\widehat{G}_\gamma \in \mathcal{C}_{\alpha(\gamma)}$ gives $\widehat{G}_\gamma \sim \widehat{G}_{\alpha(\gamma)}$, we deduce $\alpha(\gamma) \notin \Lambda_0$ by the definition of equivalence $\sim$. So $\mathrm{G}_\gamma \subset \mathcal{G}_{\alpha(\gamma)}$ with $\alpha(\gamma) \in \Lambda_1 \setminus \Lambda_0$ and $\gamma \in \Lambda_{\alpha(\gamma)}$. Hence $\gamma \in \bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \Lambda_\alpha$ for every $\gamma \in \Lambda\setminus\Lambda_0$, which concludes the proof of the second inclusion and the corollary, the last equalities are obvious. □

Remark 6.14. We mention now a particular case of the above Proposition 6.9 namely the case of G, ρ open subsets of $\mathbf{C}$, $G \cup \rho = G \supset \rho$ and $\{G_\alpha\}_{\alpha\in\Lambda}$, $\{\rho_i\}_{i\in I}$, the sets of all connected components of G respectively ρ as in Proposition 6.9. In this case, we have $I_0 = \emptyset$ and for every $\alpha \in \Lambda$ and $j \in I$, $G_\alpha \cap \rho_j \neq \emptyset$ means $G_\alpha \supset \rho_j$, $\widehat{G}_\alpha = G_\alpha$, $\mathcal{C}_\alpha = \{G_\alpha\}$, $\mathcal{G}_\alpha = G_\alpha$ and $\Lambda_1 = \Lambda$.

The decomposition of $G\cup\rho = G$ given by its connected components from Proposition 6.9 corresponds to a setting of terms $G_\alpha, \alpha \in \Lambda$ as in the following:

$$G \cup \rho = G = \bigcup_{\alpha\in\Lambda_0} G_\alpha \cup \bigcup_{\alpha\in\Lambda\setminus\Lambda_0} G_\alpha, \rho \subset \bigcup_{\alpha\in\Lambda\setminus\Lambda_0} G_\alpha$$

where $\Lambda_0 = \{\alpha \mid G_\alpha \cap \rho = \emptyset\}$, $\Lambda \setminus \Lambda_0 = \{\alpha \mid G_\alpha \cap \rho \neq \emptyset\}$.

The decomposition given by Proposition 6.9 will be written now for $G = \mathbf{C} \setminus F$, F a closed subset $F \subset \mathbf{C}$ and $\rho = \rho(\theta)$ the resolvent set of an operator analytic function θ with $\rho(\theta) \neq \emptyset$. We use the same notation as before for the decompositions of $G = \mathbf{C}\backslash F$ and $\rho = \rho(\theta)$ given by their connected components,

$$\mathbf{C} \setminus F = \bigcup_{\alpha\in\Lambda} G_\alpha, \quad \rho(\theta) = \bigcup_{i\in I} \rho_i$$

So we have the following proposition.

Proposition 6.15. *If $\{G_\alpha \mid \alpha \in \Lambda\}$ are the connected components of $\mathbf{C} \setminus F$, $F = \overline{F} \subset \mathbf{C}$, and $\{\rho_i \mid i \in I\}$ are the connected components of $\rho(\theta)$ for an operator analytic function θ with $\rho(\theta) \neq \emptyset$, then the decomposition of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ given by its connected components is*

$$(\mathbf{C} \setminus F) \cup \rho(\theta) = \bigcup_{\alpha\in\Lambda_0} G_\alpha \cup \bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \mathcal{G}_\alpha \cup \bigcup_{k\in I_0} \rho_k$$

where $\Lambda_0, \Lambda_1, \mathcal{G}_\alpha, I_0$ are as in Proposition 6.9.

Remark 6.16. Obviously all the properties concerning this decomposition in the general case (Proposition 6.9) can be used in the case $F = \overline{F} \subset \mathbf{C}$, $G = \mathbf{C} \setminus F$ and $\rho = \rho(\theta)$. In this case, we can also recognize some previously introduce objects as $\Lambda_0 = \Lambda \setminus \Lambda_\theta$(see Definition 3.9), $\Lambda_1 \setminus \Lambda_0 = \Lambda_1 \cap \Lambda_\theta$ and $\Lambda \setminus \Lambda_0 = \Lambda_\theta$, $\bigcup\limits_{\alpha\in\Lambda\setminus\Lambda_0} G_\alpha = \bigcup\limits_{\alpha\in\Lambda_\theta} G_\alpha = (\mathbf{C} \setminus F)_{\rho(\theta)}$ the θ-interior of $\mathbf{C} \setminus F$.

Now we introduce the class of functional spaces $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$, a particular case of functional spaces $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X})$ from Definition 6.3.

Definition 6.17. Let F and the decomposition of $\mathbf{C} \setminus F$ from Proposition 6.15. We denoted $\Lambda_\alpha = \{\beta \in \Lambda \mid \mathrm{G}_\beta \subset \mathcal{G}_\alpha\}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$, (see Lemma 6.12). Then the functional space $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ attached to an arbitrary closed subset $F = \overline{F} \subset \mathbf{C}$ is

$$\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}) = \{g \mid g \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}), \forall\alpha \in \Lambda_1 \setminus \Lambda_0 \;\; \exists x_\alpha \in \mathcal{X},$$

$$g|_{\bigcup\limits_{\beta\in\Lambda_\alpha} G_\beta} = x_\alpha\}$$

x_α denoting as usual the constant $\mathcal{X}$-valued function with value x_α.

Remark 6.18. 1. Explicitly(see Remark 6.16), $g \in \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ means that the following properties hold:
(i) $g \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$
(ii) for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ there exists $x_\alpha \in \mathcal{X}$, such that $g\,|_{G_\beta} = x_\alpha$ for every connected component G_β of G, $G_\beta \subset \mathcal{G}_\alpha$.

2. If K is a closed subset of $\mathbf{C}$, $K \subset \sigma(\theta)$, then $\mathbf{C} \setminus K \supset \rho(\theta)$ and by Remark 6.14 the above condition (ii) is simpler because in this case $G_\alpha = \mathcal{G}_\alpha$ for every $\alpha \in \Lambda, \Lambda = \Lambda_1$ and $\Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) = \emptyset\}$. So $h \in \mathcal{L}(\mathbf{C} \setminus K, \mathcal{X})$ signifies that the following assertions hold:

(i) $h \in \mathcal{O}(\mathbf{C} \setminus K, \mathcal{X})$
(ii) for every $\alpha \in \Lambda \setminus \Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) \neq \emptyset\}$ there exists $y_\alpha \in \mathcal{X}$, such that $g|_{G_\alpha} = y_\alpha$ *i.e.* g is constant on every connected component G_α of $G = \mathbf{C} \setminus K \supset \rho(\theta)$, $G_\alpha \cap \rho(\theta) \neq \emptyset$.
In other words, if K is a closed subset of $\mathbf{C}$, $K \subset \sigma(\theta)$, then $\mathcal{L}(\mathbf{C} \setminus K, \mathcal{X})$ consists of all analytic functions $h : \mathbf{C} \longrightarrow \mathcal{X}$, h a constant function on every connected component of $\mathbf{C} \setminus K$ having a nonempty intersection with $\rho(\theta)$.

It is easy to see that we have

$$\mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}).$$

If θ is an analytic operator function on $\mathbf{C}$, F a closed subset $F \subset \mathbf{C}$ and $\Phi_\theta : \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}) \longrightarrow \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ as in the beginning of Section 6, the θ-spectral space corresponding to $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ is the subspace of $\mathcal{X}$ denoted by $L(F, \theta)$ and given by Definition 6.3 for $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ as follows.

Definition 6.19. We define the θ-spectral space $L(F, \theta)$ associated to a closed subset $F \subset \mathbf{C}$ by the following equality:

$$L(F, \theta) = \{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists g_\epsilon \in Range\Phi_\theta \cap \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}),$$
$$\| g_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\}.$$

A particular case of the spaces $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ respectively$L(F, \theta)$ are $\mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X})$ respectively $L_s(F, \theta)$, defined by the equalities:

$$\mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}) \cap \mathcal{O}_{ct}(\bigcup_{\beta \in \Lambda \setminus \Lambda_0} G_\beta, \mathcal{X})$$

$$L_s(F, \theta) = \{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists g_\epsilon \in Range\Phi_\theta \cap \mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X}),$$
$$\| g_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F\}.$$

By Definition 3.9 we have $\bigcup_{\beta \in \Lambda \setminus \Lambda_0} G_\beta = (\mathbf{C} \setminus F)_{\rho(\theta)}$ the θ-interior of $\mathbf{C} \setminus F$ and the above equalities can be rewritten,

$$\mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}) \cap \mathcal{O}_{ct}((\mathbf{C} \setminus F)_{\rho(\theta)}, \mathcal{X})$$

$$L_s(F, \theta) = \{x \in \mathcal{X} \mid \forall \epsilon > 0, \exists g_\epsilon \in Range\Phi_\theta \cap \mathcal{O}_{ct}((\mathbf{C} \setminus F)_{\rho(\theta)}, \mathcal{X})$$
$$\| g_\epsilon(z) - x \| < \epsilon \ \forall z \in \mathbf{C} \setminus F\} =$$

$\{x \in \mathcal{X} \mid \forall \epsilon > 0, \ \exists g_\epsilon \in Range\Phi_\theta, \ \exists x_\epsilon \in \mathcal{X}, \| g_\epsilon(z) - x \| < \epsilon \ \forall z \in \mathbf{C} \setminus F$ and $g_\epsilon|_{(\mathbf{C} \setminus F)_{\rho(\theta)}} = x_\epsilon\}$.

Remark 6.20. 1. If $\rho(\theta) \cap (\mathbf{C} \setminus F) = \emptyset i.e. (\mathbf{C} \setminus F)_{\rho(\theta)} = \emptyset$, then $L_s(F, \theta) = L(F, \theta) = N(F, \theta)$ and has the property(r) (Remark 6.2).

2. If $\Lambda_0 = \emptyset$ *i.e.* $(\mathbf{C} \setminus F)_{\rho(\theta)} = \mathbf{C} \setminus F$, then $L_s(F, \theta) = M(F, \theta)$
3. $M_0(F, \theta) \subset L_s(F, \theta) \subset L(F, \theta) \subset N(F, \theta)$ for every closed subset F of $\mathbf{C}$.

Now the properties of these new θ -spectral spaces will be proved.

Proposition 6.21 (Monotonicity property). *The maps given by* $F \longmapsto L(F, \theta)$, $F \longmapsto L_s(F, \theta)$ *for* F *closed subset of* $\mathbf{C}$ *are monotone i.e.* $L(F, \theta) \subset L(F_1, \theta)$ *and* $L_s(F, \theta) \subset L_s(F_1, \theta)$ *for every* F, F_1 *closed subsets,* $F \subset F_1 \subset \mathbf{C}$.

Proof. Let F, F_1 be two closed subsets of $\mathbf{C}$, $F \subset F_1$ as above. We first observe that the inclusions $L(F, \theta) \subset L(F_1, \theta)$ and $L_s(F, \theta) \subset L_s(F_1, \theta)$ are(by Definition 6.19) easy consequences of the inclusions $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$, $\mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}_s(\mathbf{C} \setminus F_1, \mathcal{X})$.

First we shall prove the inclusion $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$ for every F, F_1 closed subsets of $\mathbf{C}$, $F \subset F_1$, $\mathbf{C} \setminus F_1 \subset \mathbf{C} \setminus F$.

In order to describe $g \in \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$(Definition 6.17) we need the decomposition of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ given by its connected components (Proposition 6.15):

$$(\mathbf{C} \setminus F) \cup \rho(\theta) = \bigcup_{\alpha \in \Lambda_0} G_\alpha \cup \bigcup_{\alpha \in \Lambda_1 \setminus \Lambda_0} \mathcal{G}_\alpha \cup \bigcup_{k \in I_0} \rho_k$$

where $\{G_\alpha\}_{\alpha \in \Lambda}$, respectively $\{\rho_i\}_{i \in I}$ are the connected components of $\mathbf{C} \setminus F$, respectively $\rho(\theta)$, $\Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) = \emptyset\}$, $I_0 = \{k \in I \mid \rho_k \cap (\mathbf{C} \setminus F) = \emptyset\}$, $\Lambda_\alpha = \{\beta \in \Lambda \mid \mathrm{G}_\beta \subset \mathcal{G}_\alpha\} = \{\beta \in \Lambda \setminus \Lambda_0 \mid \widehat{G_\beta} \sim \widehat{G_\alpha}\}$ if $\alpha \in \Lambda_1 \setminus \Lambda_0$ (from Corollary 6.13).

Similarly, for $\mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$, we consider the decomposition of $(\mathbf{C} \setminus F_1) \cup \rho(\theta)$ given by its connected components (Proposition 6.15):

$$(\mathbf{C} \setminus F_1) \cup \rho(\theta) = \bigcup_{\gamma \in \Delta_0} D_\gamma \cup \bigcup_{\gamma \in \Delta_1 \setminus \Delta_0} \mathcal{D}_\gamma \cup \bigcup_{j \in J_0} \rho_j$$

$\{D_\gamma\}_{\gamma \in \Delta}$,respectively $\{\rho_i | i \in I\}$ are the connected components of $\mathbf{C} \setminus F_1$, respectively $\rho(\theta)$, $\Delta_0 = \{\gamma \in \Delta \mid D_\gamma \cap \rho(\theta) = \emptyset\}$, $J_0 = \{j \in I \mid \rho_j \cap \mathbf{C} \setminus F_1 = \emptyset\} \supset I_0$ and $\Delta_\gamma = \{\delta \in \Delta \mid D_\delta \subset \mathcal{D}_\gamma\} = \{\delta \in \Delta \setminus \Delta_0 \mid \widehat{D_\delta} \sim \widehat{D_\gamma}\}$ for $\gamma \in \Delta_1 \setminus \Delta_0$ (Corollary 6.13).

Let be now $g \in \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$. We will prove $g \in \mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$.

First we observe by Definition 6.17 that $g \in \mathcal{O}(\mathbf{C} \setminus F, \theta)$ and $g|_{\bigcup_{\beta \in \Lambda_\alpha} G_\beta}$ is a constant function for every $\alpha \in \Lambda_1 \setminus \Lambda_0$. But $F \subset F_1$ gives $\mathbf{C} \setminus F_1 \subset \mathbf{C} \setminus F$ and $\mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F_1, \mathcal{X})$.

Then $g \in \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ implies $g \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{O}(\mathbf{C} \setminus F_1, \mathcal{X})$.

Thus for proving $g \in \mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$ it remains to prove that $g|_{\bigcup_{\delta \in \Delta_\gamma} D_\delta}$ is a constant function for every $\gamma \in \Delta_1 \setminus \Delta_0$.

So, let us consider $\gamma \in \Delta_1 \setminus \Delta_0$, $\mathcal{D}_\gamma$ the corresponding connected component of $(\mathbf{C} \setminus F_1) \cup \rho(\theta)$ and $\bigcup_{\delta \in \Delta_\gamma} D_\delta = \bigcup_{\delta : D_\delta \subset \mathcal{D}_\gamma} D_\delta$.

We have to prove that $g|_{D_\delta}$ is the same constant function for every $D_\delta \subset \mathcal{D}_\gamma$ if $\gamma \in \Delta_1 \setminus \Delta_0$.

We recall from the above written decomposition of $(\mathbf{C} \setminus F_1) \cup \rho(\theta)$ given by its connected components, that $\mathcal{D}_\gamma \cap \rho_j = \emptyset$ for every $\gamma \in \Delta_1 \setminus \Delta_0$, $j \in J_0$. But $I_0 \subset J_0$ because $\mathbf{C} \setminus F_1 \subset \mathbf{C} \setminus F$ and $\rho_k \cap \mathbf{C} \setminus F_1 = \emptyset$ if $\rho_k \cap \mathbf{C} \setminus F = \emptyset$.

So we have in particular $\mathcal{D}_\gamma \cap \rho_k = \emptyset$ for every $k \in I_0 \subset J_0$.

Recall also that $(\mathbf{C} \setminus F_1) \cup \rho(\theta) \subset (\mathbf{C} \setminus F) \cup \rho(\theta)$ and $\mathcal{D}_\gamma$ is a connected subset of $(\mathbf{C} \setminus F_1) \cup \rho(\theta)$. Hence there exists a connected component of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ containing $\mathcal{D}_\gamma$. This connected component is one of $\mathcal{G}_\alpha$, $\alpha \in \Lambda_1$, because as we have observed before $\mathcal{D}_\gamma \cap \rho_k = \emptyset$ for every $k \in I_0$.

So there exists $\mu(\gamma) \in \Lambda_1$ such that $\mathcal{D}_\gamma \subset \mathcal{G}_{\mu(\gamma)}$.

On the other hand, $\gamma \in \Delta_1 \setminus \Delta_0$ gives $D_\gamma \cap \rho(\theta) \neq \emptyset$ and $D_\gamma \subset \mathcal{D}_\gamma$ for $\gamma \in \Delta_1 \setminus \Delta_0$(Lemma 6.12). So, $D_\gamma \subset \mathcal{D}_\gamma \subset \mathcal{G}_{\mu(\gamma)}$, $D_\gamma \cap \rho(\theta) \neq \emptyset$ gives $\mathcal{G}_{\mu(\gamma)} \cap \rho(\theta) \neq \emptyset$ and by the decomposition of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ given by its connected components we deduce that $\mu(\gamma) \in \Lambda_1 \setminus \Lambda_0$.

Therefore, we have proved that for $\gamma \in \Delta_1 \setminus \Delta_0$ there exists $\mu(\gamma) \in \Lambda_1 \setminus \Lambda_0$, $\mathcal{D}_\gamma \subset \mathcal{G}_{\mu(\gamma)}$.

Then for every $D_\delta \subset \mathcal{D}_\gamma$ we have $D_\delta \subset \mathcal{G}_{\mu(\gamma)}$. D_δ being a connected subset of $\mathbf{C} \setminus F_1 \subset \mathbf{C} \setminus F$, for every $D_\delta \subset \mathcal{D}_\gamma$ there exists a unique connected component of $\mathbf{C} \setminus F$ denoted $G_{\alpha(\delta)}$, $\alpha(\delta) \in \Lambda$, such that $D_\delta \subset G_{\alpha(\delta)}$. In the same way, $G_{\alpha(\delta)}$ being in particular a connected subset of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ there exists $\mathcal{G}_{\nu(\delta)}$, for some $\nu(\delta) \in \Lambda_1$, a unique connected component of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ such that $G_{\alpha(\delta)} \subset \mathcal{G}_{\nu(\delta)}$.

Then, for every $D_\delta \subset \mathcal{D}_\gamma$ there exists a unique connected component of $\mathbf{C} \setminus F$ denoted $G_{\alpha(\delta)}$, $\alpha(\delta) \in \Lambda$, and a unique connected component of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ denoted $\mathcal{G}_{\nu(\delta)}$, $\nu(\delta) \in \Lambda_1$, such that $D_\delta \subset G_{\alpha(\delta)} \subset \mathcal{G}_{\nu(\delta)}$.

Now we notice that the above determined sets $\mathcal{G}_{\mu(\gamma)}$, $\mathcal{G}_{\nu(\delta)}$ as connected components of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ are disjoint or coincide.

Because $D_\delta \subset \mathcal{G}_{\mu(\gamma)} \cap \mathcal{G}_{\nu(\delta)}$ it results that $\mathcal{G}_{\mu(\gamma)} = \mathcal{G}_{\nu(\delta)}$ and $\mu(\gamma) = \nu(\delta)$ because $\mu(\gamma), \nu(\delta) \in \Lambda_1$ (Definition 6.6).

Thus for every $\gamma \in \Delta_1 \setminus \Delta_0$ there exists $\mu(\gamma)(= \nu(\delta)) \in \Lambda_1 \setminus \Lambda_0$ and for every $D_\delta \subset \mathcal{D}_\gamma$, $\delta \in \Delta$, there exists $\alpha(\delta) \in \Lambda$ with $D_\delta \subset G_{\alpha(\delta)} \subset (\mathcal{G}_{\nu(\delta)} =)\mathcal{G}_{\mu(\gamma)}$.

Then, if $g \in \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ we have $g|_{\bigcup_{l : G_l \subset \mathcal{G}_\mu} G_l}$ is a constant function for every $\mu \in \Lambda_1 \setminus \Lambda_0$. In particular for $\mu = \mu(\gamma) \in \Lambda_1 \setminus \Lambda_0$, for every $\gamma \in \Delta_1 \setminus \Delta_0$ there exists $x_{\mu(\gamma)} \in \mathcal{X}$ such that $g|_{\bigcup_{l : G_l \subset \mathcal{G}_{\mu(\gamma)}} G_l} = x_{\mu(\gamma)}$.

As we have proved, for every $\gamma \in \Delta_1 \setminus \Delta_0$, every $D_\delta \subset \mathcal{D}_\gamma$ verifies $D_\delta \subset$

$G_{\alpha(\delta)} \subset \mathcal{G}_{\mu(\gamma)}$ for some $\alpha(\delta) \in \Lambda$. Then we have

$$g|D_\delta = g|G_{\alpha(\delta)} = x_{\mu(\gamma)}.$$

Therefore for every $\gamma \in \Lambda_1 \setminus \Lambda_0$, $g|\bigcup_{\delta: D_\delta \subset \mathcal{D}_\gamma} D_\delta$ is a constant function$(x_{\mu(\gamma)})$.

Thus for every $g \in \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$ we proved $g \in \mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$ and the inclusions $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}(\mathbf{C} \setminus F_1, \mathcal{X})$ for every closed subsets F, F_1, $F \subset F_1 \subset \mathbf{C}$. The inclusions $\mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X}) \subset \mathcal{L}_s(\mathbf{C} \setminus F_1, \mathcal{X})$ for every closed subsets $F \subset F_1 \subset \mathbf{C}$ can be proved in the same way and conclude the proof. $\square$

PROPOSITION 6.22. *$L(F, \theta)$ and $L_s(F, \theta)$ are closed subspaces of $\mathcal{X}$ for every closed subset $F \subset \boldsymbol{C}$.*

Proof. Obviously $L(F, \theta)$ and $L_s(F, \theta)$ are subspaces of $\mathcal{X}$. We will prove that $L(F, \theta)$ is closed; the closure property for $L_s(F, \theta)$ can be proved in the same way.

First recall that $x \in L(F, \theta)$ means that

$$\forall \epsilon > 0, \exists g_\epsilon \in Range\Phi_\theta \cap \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}), \| g_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F.$$

Let $\overline{L(F, \theta)}$ be the closure in $\mathcal{X}$ of $L(F, \theta)$ and $y \in \overline{L(F, \theta)}$. Then for every $\epsilon > 0$ there exists $x \in L(F, \theta)$ such that $\| y - x \| < \epsilon$. If g_ϵ is attached to $\epsilon > 0$ and $x \in L(F, \theta)$ as above we have

$$\| g_\epsilon(z) - y \| < \| g_\epsilon(z) - x \| + \| x - y \| < 2\epsilon \text{ for every } z \in \mathbf{C} \setminus F$$

Thus $y \in L(F, \theta)$ and $\overline{L(F, \theta)} \subset L(F, \theta)$ which conclude the proof. $\square$

COROLLARY 6.23. *For every closed subset $F \subset \boldsymbol{C}$ we have,*

$$M(F, \theta) = \overline{M_0(F, \theta)} \subset L_s(F, \theta) \subset L(F, \theta) \subset N(F, \theta)$$

Proof. It results by Remark 6.20, 3. and the closure property of $L_s(F, \theta)$. $\square$

Now we prove that the θ-spectral spaces $L(F, \theta)$, $L_s(F, \theta)$ (Definition 6.19) have the property(r) (see after Lemma 6.1), the restriction property to the spectrum $\sigma(\theta)$.

THEOREM 6.24. *For every analytic operator valued function θ with nonempty resolvent set and every closed subset $F \subset \boldsymbol{C}$, the θ-spectral spaces $L(F, \theta)$, $L_s(F, \theta)$ have the property (r), the restriction property to the spectrum $\sigma(\theta)$, i.e.*

$$L(F, \theta) = L(F \cap \sigma(\theta), \theta), L_s(F, \theta) = L_s(F \cap \sigma(\theta), \theta),$$

for every closed subset $F \subset \boldsymbol{C}$.

Proof. We shall give only the proof for $L(F,\theta) = L(F \cap \sigma(\theta), \theta)$ for every closed subset $F \subset \mathbf{C}$, the similar equalities for $L_s(F,\theta)$ can be proved in a similar way. Let us consider an arbitrary closed subset $F \subset \mathbf{C}$. First we observe that $L(F \cap \sigma(\theta), \theta) \subset L(F,\theta)$ because the map $F \longmapsto L(F,\theta)$ is monotone 6.21 (monotonicity property).

So it remains to prove $L(F,\theta) \subset L(F \cap \sigma(\theta), \theta)$ for every closed subset $F \subset \mathbf{C}$.

For proving this, we consider an arbitrary $x \in L(F,\theta)$. Then by Definition 6.19 we have

$$\forall \epsilon > 0, \exists g_\epsilon \in Range\Phi_\theta \cap \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}), \| g_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F.$$

Explicitly this means:

(i_1) for every $\epsilon > 0$ there exists $f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$

(i_2) $g_\epsilon = \Phi_\theta f_\epsilon$, $g_\epsilon(z) = \theta(z) f_\epsilon(z)$ for every $z \in \mathbf{C} \setminus F$ verifies the following properties a), b):

a) $g_\epsilon|_{\bigcup\limits_{\beta : G_\beta \subset \mathcal{G}_\alpha} G_\beta} = x_{\alpha,\epsilon} \in \mathcal{X}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$

b) $\| g_\epsilon(z) - x \| < \epsilon$, for every $z \in \mathbf{C} \setminus F$.

Recall that $\Lambda, \Lambda_0, \Lambda_1, G_\alpha, \mathcal{G}_\alpha$ are from Proposition 6.9, 6.15 and used before in the proof of Proposition 6.21. So we have:

$\{G_\alpha\}_{\alpha \in \Lambda}$ are all the connected components of $\mathbf{C} \setminus F$,

$\Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) = \emptyset\}$,

$\mathcal{G}_\alpha$ for $\alpha \in \Lambda_1$ are connected components of $(\mathbf{C} \setminus F) \cup \rho(\theta)$,

$\mathcal{G}_\alpha$ for $\alpha \in \Lambda_1 \setminus \Lambda_0$ are all $\mathcal{G}_\alpha$ having nonempty intersection with $\rho(\theta)$. Recall also, from Corollary 6.13, $G_\beta \cap \rho(\theta) \neq \emptyset$ for every $G_\beta \subset \mathcal{G}_\alpha$ with $\alpha \in \Lambda_1 \setminus \Lambda_0$ and for every $G_\beta, \mathrm{G}_\beta \cap \rho(\theta) \neq \emptyset (\beta \in \Lambda_1 \setminus \Lambda_0)$ there exists $\alpha \in \Lambda_1 \setminus \Lambda_0$ such that $G_\beta \subset \mathcal{G}_\alpha$.

We will prove that $x \in L(F,\theta)$ implies $x \in L(F \cap \sigma(\theta), \theta)$, proving that x satisfying the above assertions (i_1), (i_2) verifies also the Definition 6.19 with $F \cap \sigma(\theta)$ instead of F.

So, for x verifying the above assertions (i_1), (i_2) we have to prove the following assertion:

$$\forall \epsilon > 0, \exists h_\epsilon \in Range\Phi_\theta \cap \mathcal{L}(\mathbf{C} \setminus (F \cap \sigma(\theta)), \mathcal{X}), \| h_\epsilon(z) - x \| < \epsilon,$$

$$\forall z \in \mathbf{C} \setminus (F \cap \sigma(\theta)).$$

Obviously $\mathbf{C} \setminus (F \cap \sigma(\theta)) = (\mathbf{C} \setminus F) \cup \rho(\theta) \supset \rho(\theta)$ and we used 2. from Remark 6.18 for $K = F \cap \sigma(\theta) \subset \sigma(\theta)$. We deduce in this case that the space $\mathcal{L}(\mathbf{C} \setminus (F \cap \sigma(\theta)), \mathcal{X})$ is the space of all $\mathcal{X}$-valued analytic functions on $\mathbf{C} \setminus (F \cap \sigma(\theta)) = (\mathbf{C} \setminus F) \cup \rho(\theta)$ which are constant on every connected component of $\mathbf{C} \setminus (F \cap \sigma(\theta)) = (\mathbf{C} \setminus F) \cup \rho(\theta)$ having nonempty intersection with $\rho(\theta)$.

The decomposition of $(\mathbf{C} \setminus F) \cup \rho(\theta)$ given by its connected components is described in Proposition 6.9, Remark 6.14 and 6.18. Using the notations of this decomposition from 6.9, the connected components of $\mathbf{C} \setminus (F \cap \sigma(\theta)) = (\mathbf{C} \setminus F) \cup \rho(\theta)$ having a nonempty intersection with $\rho(\theta)$ are $\mathcal{G}_\alpha$ for $\alpha \in \Lambda_1 \setminus \Lambda_0$ and ρ_k for $k \in I_0$. Thus $h \in \mathcal{L}(\mathbf{C}\setminus(F \cap \sigma(\theta)), \mathcal{X})$ means $h \in \mathcal{O}((\mathbf{C}\setminus F) \cup \rho(\theta), \mathcal{X})$ and h locally constant on $\bigcup_{\alpha \in \Lambda_1 \setminus \Lambda_0} \mathcal{G}_\alpha \cup \bigcup_{k \in I_0} \rho_k$ *i.e.* h constant on $\mathcal{G}_\alpha$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ and h constant on ρ_k for every $k \in I_0$.

Thus, what we need to prove can be rewritten as

(r_1) for every $\epsilon > 0$ there exists $\varphi_\epsilon \in \mathcal{O}((\mathbf{C} \setminus F) \cup \rho(\theta), \mathcal{X})$

(r_2) $h_\epsilon = \Phi_\theta \varphi_\epsilon$, $h_\epsilon(z) = \theta(z)\varphi_\epsilon(z)$ for every $z \in (\mathbf{C} \setminus F) \cup \rho(\theta)$, verifies the following properties a), b):

a) $h_\epsilon|_{\mathcal{G}_\alpha} = x_{\alpha,\epsilon} \in \mathcal{X}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ and

$h_\epsilon|_{\rho_k} = y_{k,\epsilon} \in \mathcal{X}$ for every $k \in I_0$

b) $\| h_\epsilon(z) - x \| < \epsilon$, for every $z \in (\mathbf{C} \setminus F) \cup \rho(\theta)$.

In order to define such a φ_ϵ for every $\epsilon > 0$, we use the above functions f_ϵ attached to $x \in L(F, \theta)$, verifying $(i_1), (i_2)$ by definition of $L(F, \theta)$. Then the following property of f_ϵ derived from $(i_2)a)$:
for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ there exists $x_{\alpha,\epsilon} \in \mathcal{X}$ such that,

$$f_\epsilon|_{\bigcup_{\beta : G_\beta \subset \mathcal{G}_\alpha} G_\beta \cap \rho(\theta)} = R(.,\theta)x_{\alpha,\epsilon}$$

i.e. for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ there exists $x_{\alpha,\epsilon} \in \mathcal{X}$ such that

$$f_\epsilon(z) = R(z,\theta)x_{\alpha,\epsilon} \text{ for every } z \in \bigcup_{\beta : G_\beta \subset \mathcal{G}_\alpha} G_\beta \cap \rho(\theta).$$

Indeed, g_ϵ, f_ϵ, given before by $x \in L(F, \theta)$, verify (i_2) a). By Corollary 6.13 $G_\beta \subset \mathcal{G}_\alpha$ if and only if $\widehat{G}_\beta = G_\beta \cup \bigcup_{i \in I_\beta} \rho_i \subset \mathcal{G}_\alpha$ and $I_\beta \neq \emptyset$. Thus we obtain by (i_2) a):

$$\forall \alpha \in \Lambda_1 \setminus \Lambda_0, \exists x_{\alpha,\epsilon} \in \mathcal{X}, g_\epsilon(z) = \theta(z)f_\epsilon(z) = x_{\alpha,\epsilon}, \forall z \in G_\beta \subset \mathcal{G}_\alpha.$$

We also have for every $G_\beta \subset \mathcal{G}_\alpha$, $G_\beta \cap \rho(\theta) = G_\beta \cap \bigcup_{i \in I_\beta} \rho_i \neq \emptyset$ because $I_\beta = \{i \in I \mid \rho_i \cap G_\beta \neq \emptyset\}$, $\{\rho_i\}_{i \in I}$ being the all connected components of $\rho(\theta)$. For $z \in G_\beta \cap \rho(\theta) \subset G_\beta \subset \mathcal{G}_\alpha$ obviously we have $g_\epsilon(z) = \theta(z)f_\epsilon(z) = x_{\alpha,\epsilon}$ and the above mentioned f_ϵ, has the announced property,

$$f_\epsilon|_{\bigcup_{\beta : G_\beta \subset \mathcal{G}_\alpha} G_\beta \cap \rho(\theta)} = R(.,\theta)x_{\alpha,\epsilon}.$$

Resuming, we have for every $x \in L(F, \theta)$:

for every $\epsilon > 0$ there exists $f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$,
for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ there exists $x_{\alpha,\epsilon} \in \mathcal{X}$ such that

$$f_\epsilon(z) = R(z,\theta)x_{\alpha,\epsilon}, \text{for every } z \in \bigcup_{\beta: G_\beta \subset \mathcal{G}_\alpha} G_\beta \cap \rho(\theta) = \bigcup_{\beta: G_\beta \subset \mathcal{G}_\alpha} G_\beta \cap \bigcup_{i \in I_\beta} \rho_i,$$

where $G_\beta \cap \rho(\theta) \neq \emptyset$, for every $G_\beta \subset \mathcal{G}_\alpha$,
and

$$\| \theta(z) f_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F.$$

For easier reading we recall also (see Proposition 6.9, 6.15):
$\{G_\alpha\}_{\alpha \in \Lambda}$ and $\{\rho_i\}_{i \in I}$ are all the connected components of $\mathbf{C} \setminus F$ respectively $\rho(\theta)$, $(\mathbf{C} \setminus F) \cup \rho(\theta) = \bigcup_{\alpha \in \Lambda_0} G_\alpha \cup \bigcup_{\alpha \in \Lambda_1 \setminus \Lambda_0} \mathcal{G}_\alpha \cup \bigcup_{k \in I_0} \rho_k$ is the decomposition given by the connected components of $(\mathbf{C} \setminus F) \cup \rho(\theta)$,
$\Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(\theta) = \emptyset\}$, $G_\alpha = \mathcal{G}_\alpha$ for every $\alpha \in \Lambda_0$,
$I_0 = \{i \in I \mid \rho_i \cap \rho(\theta) = \emptyset\}$, $I_\beta = \{i \in I \mid G_\beta \cap \rho_i \neq \emptyset\}$,
$G_\beta \cap \rho(\theta) = G_\beta \cap \bigcup_{i \in I_\beta} \rho_i$.

Using the function f_ϵ attached to $x \in L(F,\theta)$ and its properties derived before from $(i_1), (i_2)$ of definition of $L(F,\theta)$, we can prove now that every $x \in L(F,\theta)$ is verifying $(r_1), (r_2)$ *i.e.* $x \in L(F \cap \sigma(\theta), \theta)$.

More precisely, having f_ϵ as above given by $x \in L(F,\theta)$ we will define for every $\epsilon > 0$ a function $\varphi_\epsilon \in \mathcal{O}((\mathbf{C} \setminus F) \cup \rho(\theta), \mathcal{X})$ verifying (r_2). First we define $\psi_{\alpha,\epsilon} \in \mathcal{O}(\mathcal{G}_\alpha, \mathcal{X})$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$,

$$\psi_{\alpha,\epsilon}(z) = \begin{cases} f_\epsilon(z) & \text{if } z \in G_\beta \text{ for every } G_\beta \subset \mathcal{G}_\alpha \\ R(z,\theta)x_{\alpha,\epsilon} & \text{if } z \in \rho_i \quad \text{for every } \rho_i \subset \mathcal{G}_\alpha \end{cases}$$

For every $\alpha \in \Lambda_1 \setminus \Lambda_0$, the function $\psi_{\alpha,\epsilon}$ is well defined and analytic on $\mathcal{G}_\alpha$ because $\mathcal{G}_\alpha = \bigcup_{\beta: G_\beta \subset \mathcal{G}_\alpha} G_\beta \cup \bigcup_{i \in I_\beta} \rho_i$(Corollary 6.13), f_ϵ and $R(.,\theta)$ are analytic functions on $\mathbf{C} \setminus F$ respectively $\rho(\theta)$ and by the above proved property of f_ϵ, $f_\epsilon(z) = R(z,\theta)x_{\alpha,\epsilon}$ if $z \in G_\beta \cap \rho_i, i \in I_\beta$.

Because $\mathcal{G}_\alpha \cap \mathcal{G}_\beta = \emptyset$ for every $\alpha, \beta \in \Lambda_1, \alpha \neq \beta$, the following equalities:

$$\psi_\epsilon(z) = \psi_{\alpha,\epsilon}(z) \quad \text{for every} \quad z \in \mathcal{G}_\alpha \quad \text{and} \quad \alpha \in \Lambda_1 \setminus \Lambda_0,$$

define ψ_ϵ as analytic function on $\bigcup_{\alpha \in \Lambda_1 \setminus \Lambda_0} \mathcal{G}_\alpha$.

Then, the following equalities define $\varphi_\epsilon \in \mathcal{O}((\mathbf{C} \setminus F) \cup \rho(\theta), \mathcal{X})$,

$$\varphi_\epsilon(z) = \begin{cases} f_\epsilon(z) & \text{if } z \in \bigcup_{\beta \in \Lambda_0} G_\beta \\ \psi_\epsilon(z) & \text{if } z \in \bigcup_{\alpha \in \Lambda_1 \setminus \Lambda_0} \mathcal{G}_\alpha \\ R(z,\theta)x & \text{if } z \in \bigcup_{k \in I_0} \rho_k \end{cases}$$

because $\bigcup_{\beta\in\Lambda_0} G_\beta$, $\bigcup_{\alpha\in\Lambda_1\setminus\Lambda_0} \mathcal{G}_\alpha$, $\bigcup_{k\in I_0} \rho_k$ are open disjoint subsets of $\mathbf{C}$.

Thus (r_1) holds for every $x \in L(F,\theta)$.

We will prove that φ_ϵ verifies (r_2). Indeed we have,

$$h_\epsilon(z) = \theta(z)\varphi_\epsilon(z) = \begin{cases} \theta(z)f_\epsilon(z) & \text{if } z \in \bigcup\limits_{\beta\in\Lambda_0} G_\beta \\ \theta(z)\psi_\epsilon(z) & \text{if } z \in \bigcup\limits_{\alpha\in\Lambda_1\setminus\Lambda_0} \mathcal{G}_\alpha \\ x & \text{if } z \in \bigcup\limits_{k\in I_0} \rho_k \end{cases}$$

For $\alpha \in \Lambda_1 \setminus \Lambda_0$ and $z \in \mathcal{G}_\alpha$, from definition of $\psi_{\alpha,\epsilon}$, we have the equality $f_\epsilon(z) = R(z,\theta)x_{\alpha,\epsilon}$ for every $z \in G_\beta \cap \rho(\theta) \neq \emptyset$ and the identity theorem for analytic functions gives:

$$\theta(z)\psi_\epsilon(z) = \begin{cases} \theta(z)f_\epsilon(z) = x_{\alpha,\epsilon} & \text{if } z \in G_\beta \text{ for every } G_\beta \subset \mathcal{G}_\alpha \\ \theta(z)R(z,\theta)x_{\alpha,\epsilon} = x_{\alpha,\epsilon} & \text{if } z \in \rho_i \quad \text{for every } \rho_i \subset \mathcal{G}_\alpha \end{cases}$$

i.e. $\theta(z)\psi_\epsilon(z) = x_{\alpha,\epsilon}$ for every $z \in \mathcal{G}_\alpha$ and $\alpha \in \Lambda_1 \setminus \Lambda_0$.

Then we obtain,

$$h_\epsilon(z) = \theta(z)\varphi_\epsilon(z) = \begin{cases} \theta(z)f_\epsilon(z) = g_\epsilon(z) & \text{if } z \in \bigcup\limits_{\beta\in\Lambda_0} G_\beta \\ x_{\alpha,\epsilon} & \text{if } z \in \mathcal{G}_\alpha, \alpha \in \Lambda_1 \setminus \Lambda_0 \\ x & \text{if } z \in \bigcup\limits_{k\in I_0} \rho_k \end{cases}$$

Therefore $(r_2)a)$ holds for $h_\epsilon = \Phi_\theta\varphi_\epsilon$ *i.e.* φ_ϵ verifies $(r_2)a)$ and we have

$$h_\epsilon(z) - x = \begin{cases} g_\epsilon(z) - x & \text{if } z \in \bigcup\limits_{\beta\in\Lambda_0} G_\beta \\ x_{\alpha,\epsilon} - x & \text{if } z \in \mathcal{G}_\alpha, \alpha \in \Lambda_1 \setminus \Lambda_0 \\ 0 & \text{if } z \in \bigcup\limits_{k\in I_0} \rho_k \end{cases}$$

Then we deduce

$$\| h_\epsilon(z) - x \| < \epsilon, \text{ for every } z \in (\mathbf{C} \setminus F) \cup \rho(\theta),$$

because by $(i_2)b)$ we have $\| g_\epsilon(z) - x \| < \epsilon$ for every $z \in \mathbf{C} \setminus F$, and by $(i_2)a)$ $\| x_{\alpha,\epsilon} - x \| < \epsilon$ for $z \in G_\beta$, for every $G_\beta \subset \mathcal{G}_\alpha$ and $\alpha \in \Lambda_1 \setminus \Lambda_0$.

Thus h_ϵ verifies also $(r_2)b)$ *i.e.* φ_ϵ verifies $(r_2)b)$.

Therefore, φ_ϵ verifies $(r_2)a), r_2)b)$ hence (r_2) and x verifies $(r_1), (r_2)$ if $x \in L(F,\theta)$.

Hence $x \in L(F \cap \sigma(\theta), \theta)$ if $x \in L(F,\theta)$, the inclusion $L(F,\theta) \subset L(F \cap \sigma(\theta))$ has been proved for every closed subset $F \subset \mathbf{C}$ and this concludes the proof. $\square$

Remark 6.25. All the above mentioned θ-spectral spaces can be of interest for a closed subset $F \subset \mathbf{C}$ only for analytic operator valued functions θ which do not satisfy condition β on$\mathbf{C} \setminus F$; for θ satisfying condition β on $\mathbf{C} \setminus F$ (see Definition 5.1), by Corollary 5.7 we have $M_0(F,\theta) = L_s(F,\theta) = L(F,\theta) = N(F,\theta)$.

As usual, all the results proved above for θ are true for $\theta = \theta_T$, $T \in \mathcal{B}(\mathcal{X})$, that is for spectral spaces of a general bounded operator T, attached to a closed subset of complex numbers. Thus we can consider spectral spaces for $T \in \mathcal{B}(\mathcal{X})$ given by $Y(F,T) = Y(F,\theta_T)$ which are defined using $\mathcal{X}$-valued function spaces $\mathcal{Y}(\mathbf{C}\setminus F, \mathcal{X})$ (Definition 6.3). As in Remark 6.4, 6.25, all this spectral spaces are between the strong and weak Bishop's spectral spaces and are of interest only for T which does not satisfy Bishop's condition β on the complement $\mathbf{C} \setminus F$ of a closed subset $F \subset \mathbf{C}$.

1. $M(F,T) \subset Y(F,T) \subset N(F,T)$ for every $F = \overline{F} \subset \mathbf{C}$.
2. $Y(F,T) = N(F,T)$ if $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$,
   $Y(F,T) = M(F,T)$ if $\mathcal{Y}(\mathbf{C} \setminus F, \mathcal{X}) = \mathcal{O}_{ct}(\mathbf{C} \setminus F, \mathcal{X})$.

If T satisfies Bishop's condition β, then it is well known (see [3]) that $M_0(F,T) = N(F,T)$, hence $M_0(F,T) = Y(F,T) = N(F,T)$ for every $F = \overline{F} \subset \mathbf{C}$.

Now for an arbitrary $T \in \mathcal{B}(\mathcal{X})$ we can consider spectral spaces $L(F,\theta_T)$, $L_s(F,\theta_T)$, defined by $\theta_T(z) = z - T, z \in \mathbf{C}$ and $\mathcal{X}$-valued function spaces $\mathcal{L}(\mathbf{C} \setminus F, \mathcal{X})$, respectively $\mathcal{L}_s(\mathbf{C} \setminus F, \mathcal{X})$ (definitions 6.3, 6.17, 6.19).

These new spectral spaces of T are closed subspaces between Bishop's spectral spaces $M(F,T), N(F,T)$ and have the restriction property to the spectrum $\sigma(T)$ of T. We close with a short description of these.

As an open subset of $\mathbf{C}$, $\rho(T)$ the resolvent set of T has a decomposition given by its connected components $\{\rho_i\}_{i\in I}$, $\rho(T) = \bigcup\limits_{i\in I} \rho_i$. For every closed subset $F \subset \mathbf{C}$ we denote $L(F,T) = L(F,\theta_T)$, $L_s(F,T) = L_s(F,\theta_T)$ (a particular case of Definition 6.19). Explicitly, $x \in L(F,T) = L(F,\theta_T)$ can be written as follows.

$x \in L(F,T)$ if and only if the following condition is fulfilled:

$$\forall \epsilon > 0, \exists g_\epsilon \in Range\Phi_{\theta_T} \cap \mathcal{L}(\mathbf{C} \setminus F, \mathcal{X}), \| g_\epsilon(z) - x \| < \epsilon, \forall z \in \mathbf{C} \setminus F.$$

In order to explain in detail we recall some notations from θ in the particular case of θ_T:

$\{G_\alpha\}_{\alpha\in\Lambda}$ respectively $\{\rho_i\}_{i\in I}$ all the connected components of $\mathbf{C}\setminus F$ respectively $\rho(T)$,

$(\mathbf{C} \setminus F) \cup \rho(T) = \bigcup\limits_{\alpha\in\Lambda_0} G_\alpha \cup \bigcup\limits_{\alpha\in\Lambda_1\setminus\Lambda_0} \mathcal{G}_\alpha \cup \bigcup\limits_{k\in I_0} \rho_k$ the decomposition given by the connected components of $(\mathbf{C} \setminus F) \cup \rho(T)$,

$\Lambda_0 = \{\alpha \in \Lambda \mid G_\alpha \cap \rho(T) = \emptyset\}$,
$I_0 = \{i \in I \mid \rho_i \cap \rho(T) = \emptyset\}$,
$\Lambda_1 \subset \Lambda$ and $\{\mathcal{G}_\alpha\}_{\alpha\in\Lambda_1}$ (connected components of $(\mathbf{C} \setminus F) \cup \rho(T)$) were defined in Lemma 6.8,
$\{\mathcal{G}_\alpha\}_{\alpha\in\Lambda_1\setminus\Lambda_0}$ are $\mathcal{G}_\alpha$, $\alpha \in \Lambda_1$, having a nonempty intersection with $\rho(T)$.
Then, $x \in L(F,T)$ is defined by the following assertions:

(i_1) for every $\epsilon > 0$ there exists $f_\epsilon \in \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$

(i_2) $g_\epsilon(z) = (z-T)f_\epsilon(z)$ for every $z \in \mathbf{C}\setminus F$ verifies the following properties a), b):

a) $g_\epsilon|_{\bigcup\limits_{\beta:G_\beta\subset\mathcal{G}_\alpha} G_\beta} = x_{\alpha,\epsilon} \in \mathcal{X}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$

b) $\| (z-T)f_\epsilon(z) - x \| < \epsilon$, for every $z \in \mathbf{C} \setminus F$.

Replacing $\epsilon > 0$ by $1/n$, we have as in Lemma 6.5 the following equivalent assertion for $x \in L(F,T)$:

(i_1) there exists $(f_n)_{n\in N} \subset \mathcal{O}(\mathbf{C} \setminus F, \mathcal{X})$ a sequence of $\mathcal{X}$-valued analytic functions on $\mathbf{C} \setminus F$,

(i_2) $g_n(z) = (z-T)f_n(z)$ for every $z \in \mathbf{C} \setminus F$ and verifies the following properties a), b):

a) for every $\alpha \in \Lambda_1 \setminus \Lambda_0, n \in N$, $g_n|_{\bigcup\limits_{\beta:G_\beta\subset\mathcal{G}_\alpha} G_\beta} = x_{\alpha,n} \in \mathcal{X}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$,

b) $(z-T)f_n(z) \longrightarrow x$ for $n \longrightarrow \infty$, uniformly for every $z \in \mathbf{C} \setminus F$.

Obviously $x \in L_s(F,T)$ means $x \in L(F,T)$ and $x_{\alpha,\epsilon} = x_\epsilon \in \mathcal{X}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ in first formulation, $x_{\alpha,n} = x_n \in \mathcal{X}$ for every $\alpha \in \Lambda_1 \setminus \Lambda_0$ correspondingly in the second formulation. The properties of $L(F,T)$ and $L_s(F,T)$ derived from the properties of $L(F,\theta)$ and $L_s(F,\theta)$ when $\theta = \theta_T$, are contained in the following proposition.

Proposition 6.26. *For every closed subset $F \subset \mathbf{C}$ and $T \in \mathcal{B}(\mathcal{X})$, the spectral subspaces $L(F,T)$ and $L_s(F,T)$ are closed subspaces of $\mathcal{X}$ and the following inclusions hold:*

$$M(F,T) = \overline{M_0(F,T)} \subset L_s(F,T) \subset L(F,T) \subset N(F,T)$$

The maps $F \longmapsto L(F,T), F \longmapsto L_s(F,T)$ are monotone and the spectral subspaces $L(F,T)$ and $L_s(F,T)$ have the property(r), the restriction property to the spectrum $\sigma(T)$:

$$(r) L_s(F \cap \sigma(T),\, T) = L_s(F,T) \text{ and } L(F \cap \sigma(T),\, T) = L(F,T),$$

for every closed subset $F \subset \mathbf{C}$.

The new spectral spaces $L_s(F,T), L(F,T)$, intermediate between $M(F,T)$ and $N(F,T)$, are particular cases of the more general spectral spaces $Y(F,\theta_T)$

described in this paper. These spaces answer the first part of Bishop's [3] question concerning "the existence of a third type of spectral manifold intermediate between $M(F,T)$ and $N(F,T)$, which is self-dual". The second part, the duality of these spaces will be considered in a future paper.

In closing, as in the general case of θ, it is necessary to make the following remark.

Remark 6.27. All the above mentioned spectral spaces M_0, M, L_s, L, N can be of interest for a closed subset $F \subset \mathbf{C}$ only for an operator $T \in \mathcal{B}(\mathcal{X})$ which does not satisfy Bishop's condition β on the open set $\mathbf{C} \setminus F$; when T satisfies Bishop's condition β on $\mathbf{C} \setminus F$(see Remark 5.4), then $M_0(F,T) = N(F,T)$ and by the above proposition obviously we have:

$$M(F,T) = M_0(F,T) = L_s(F,T) = L(F,T) = N(F,T).$$

If T satisfies Bishop's condition β, then $M_0(F,T) = N(F,T)$ for every closed subset $F \subset \mathbf{C}$ and the above equalities hold for every closed subset $F \subset \mathbf{C}$.

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