LARGE DEVIATION ESTIMATES FOR AN OPERATOR OF ORDER FOUR WITH A POTENTIAL

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We give large deviation estimates when there is a potential, when the generated semi-group is not Markovian. This paper enters in the problematic of semi-classical asymptotics of Maslov and his school but with a different type of estimates.

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1. INTRODUCTION

Let us consider a symbol \( a(x, \xi) \) on \( R^d \times R^d \) which represents a pseudodifferential operator \( L \). According to the terminology of Maslov-Fedoriuk [1], we consider the symbol \( \frac{1}{\epsilon} a(x, \epsilon \xi) \) which represents another pseudodifferential operator \( L_\epsilon \). The object of semi-classical asymptotics is to get precise asymptotics of the Schroedinger operator (if it exists) \( \exp[iL_\epsilon] \).

In the present context, we consider \( X_i, i = 1, \ldots, m \) \( m \) vector fields on \( R^d \) with bounded derivatives at each order and without divergence with respect to the Lebesgue measure on \( R^d \). Let \( V \) be a smooth bounded function on \( R^d \) with bounded derivatives at each order. The Hamiltonian is

\[
H(x, \xi) = \sum_{i=1}^{m} <X_i(x), \xi>^4 + V(x) = \sum \sum_{i_j=4} A_{i_1, \ldots, i_j}(x) \prod \xi^{i_j}_j + V(x),
\]

We suppose that we are in an elliptic situation: there exists \( C > 0 \) such that

\[
\sum_{i=1}^{m} <X_i(x), \xi>^4 \geq C|\xi|^4
\]

The Hamiltonian is the symbol of the operator:

\[
L = \sum \sum_{i_j=4} A_{i_1, \ldots, i_j}(x) \prod \frac{\partial^{i_j}}{\partial x^{i_j}_j} + V(x).
\]
It generates a semi-group $P_t$ solution of the parabolic equation:

$$\frac{\partial}{\partial t} P_t = -LP_t. \quad (4)$$

By elliptic theory [2–4],

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) \mu_t(x, dy), \quad (5)$$

where $\mu_t(x, dy)$ is a bounded measure on $\mathbb{R}^d$. The symbol of the considered operator $L_\epsilon$ is

$$H_\epsilon(x, \xi) = \epsilon^{-1} \sum_{i=1}^m <X_i(x), \xi>^4 + \frac{1}{\epsilon} V(x) = \epsilon^3 \sum_{i=1}^m <X_i(x), \xi>^4 + \frac{1}{\epsilon} V(x). \quad (6)$$

$L_\epsilon$ generates a semi-group $P_\epsilon^t$ where

$$P_\epsilon^t f(x) = \int_{\mathbb{R}^d} f(y) \mu_\epsilon^t(x, dy). \quad (7)$$

Instead of doing precise estimates of $P_\epsilon^t$ when $\epsilon \to 0$ as in [1], we perform in this paper rough logarithmic estimates of $|P_\epsilon^t|$, by adapting in this context the proof of Wentzel-Freidlin of such estimates, because we have an analog in this non-Markovian situation of the Girsanov formula and of the exponential martingales. See [5] for review. This paper enters in the general problematic to introduce stochastic analysis tools in the general framework of non-Markovian semi-groups [5].

In [6], we have translated the classical proof of Wentzel-Freidlin estimates, upper-bound, for Poisson processes, which was done originally for the whole process instead of the semi-group only as in [6]. Let us recall that the simplest proof for diffusions of Wentzel-Freidlin [7] estimates is done by the author in [8–10]. In the present paper, the estimates are of the type of those in [11], which are closely related to the classical Wentzel-Freidlin estimates. The proof follows closely the lines of [12]. See [13, 14] and [15] for related topics.

2. STATEMENT OF THE THEOREM

According to the general framework of the large deviation theory, we introduce the Legendre transform of the Hamiltonian $H(x, \xi)$:

$$L(x, p) = \sup_{\xi} (<p, \xi> - H(x, \xi>). \quad (8)$$

A simple computation shows that

$$C_1' + C_1|p|^{4/3} \geq L(x, p) \geq C_2|p|^{4/3} + C_2'. \quad (9)$$
If \([0, 1] \to \mathbb{R}^d\): \(t \to \phi_t\) is a piecewise continuous \(C^1\) curve, we introduce the action:

\[
S(\phi) = \int_0^1 L(\phi_t, d/dt\phi_t) dt.
\]

**Definition 1.** We put

\[
l(x, y) = \inf_{\phi_0 = x; \phi_1 = y} S(\phi).
\]

By using the Ascoli theorem, we deduce from (9) that:

**Proposition 2.** The control function \((x, y) \to l(x, y)\) is continuous. Moreover there exists at least one curve \(\phi\) joining \(x\) to \(y\) such that \(l(x, y) = S(\phi)\). Moreover \(l(x, y) \to \infty\) when \(y \to \infty\).

The main result of this paper is the following and is a large deviation estimate of the type of [11].

**Theorem 3.** We have

\[
\lim_{\epsilon \to 0} \epsilon \log |P_\epsilon^x[1](x)| \leq - \inf_{y \in \mathbb{R}^d} l(x, y).
\]

**Remark.** It is nonsense in this non-Markovian situation to get a lower bound.

## 3. PROOF OF THE MAIN THEOREM

The proof follows slightly the lines of [10, 12], the main difference is that we consider the absolute values of semi-groups instead of the semi-group. This leads to two interpretations of the martingales exponentials.

**Lemma 4.** We consider the generator on \(\mathbb{R}^d \times \mathbb{R}\)

\[
L^{\text{tot}, \xi}_\epsilon = L_\epsilon + H(x, \xi) \frac{\partial}{\partial y}.
\]

The parabolic equation issued of \(f(x) \exp[\epsilon^{-1}(<x, \xi > - y)]\)

\[
\frac{\partial}{\partial t} P^{\text{tot}, \xi}_{\epsilon, t}[f(x') \exp[\epsilon^{-1}(<x' - x, \xi > - y')]](x, y) = - L^{\text{tot}, \xi}_\epsilon P^{\text{tot}, \xi}_{\epsilon, t}[f(x') \exp[\epsilon^{-1}(<x', \xi > - y')]](x, y)
\]

has a unique solution. Moreover, the map

\[
f \to P^{\text{tot}, \xi}_{\epsilon, t}[f(x') \exp[\epsilon^{-1}(<x' - x, \xi > - y')]](x, 0)
\]

defines a bounded measures when \(\epsilon \leq 1\) \(t \leq 1\) and \(|\xi| \leq C\) and all \(x\).
Proof. Let $P_t^\xi$ the semi-group on $\mathbb{R}^d$

$$f \to \exp[<x, \xi>]P_{\epsilon, t}[\exp[-<x', \xi>]f(x')](x).$$

Its generator is

$$L^\xi_t f(x) = \exp[<x, \xi>]L_\epsilon[\exp[-<x, \xi>]f(x)] = L_\epsilon + \text{lower terms}.$$ 

By standard result, $L^\xi_t$ generates a semi group in bounded measures. In order to solve the parabolic equation associated to (13), we use Volterra expansion.

$$P_{\epsilon, t}^{\text{tot}, \xi}[f(x')\exp[(-<x' - x, \xi > - y')]](x, y) = \exp[-y] +$$

$$\sum_n (-1)^n \int_{I_n} ds_1 \ldots ds_n$$

$$P_{\epsilon, t-s_1}^\xi[H(x'_1, \xi)][P_{\epsilon, s_1-s_2}^\xi[H(x'_2, \xi)] \ldots P_{\epsilon, s_n}^\xi[H(x'_n, \xi)f(x'_n)\exp[-y]]](x),$$

where $I_n$ is the simplex of length $n$ on $[0, t]$ ordered by decreasing order. By using the previous results, we can estimate the integral on the simplex by

$$\|P_{\epsilon, t}^\xi\|^n\|f\|_\infty\|H(., \xi)\|_\infty\frac{t^n}{n!}$$

Therefore the series converges. In order to prove the last statement of the lemma, we remark that

$$f \to P_{\epsilon, t}^{\text{tot}, \xi}[f(x')\exp[\epsilon^{-1}(-<x' - x, \xi > - y')]](x, 0)$$

defines a semi-group because

$$P_{\epsilon, t}^{\text{tot}, \xi}[f(x')\exp[\epsilon^{-1}(-<x' - x, \xi > - y')]](x, y) =$$

$$\exp[-y/\epsilon]P_{\epsilon, t}^{\text{tot}, \xi}[f(x')\exp[\epsilon^{-1}(-<x' - x, \xi > - y')]](x, 0).$$

We can compute easily the generator of this semi-group. We see that the diverging terms are cancelling, because they come when we apply $L_\epsilon$ only on $\exp[1/\epsilon < x, \xi >]$ when we apply chain rules and one derivatives of $\exp[-1/\epsilon y]$. Therefore the result. \qed

The second lemma is an analog of exponential inequality which arises in stochastic analysis from the use of exponential martingales [16–18]. Here they follow from the standard Davies method [19].

**Lemma 5.** For all $\delta > 0$, all $C$ there exists $t_\delta$ such that if $t < t_\delta$

$$|P_{\epsilon, t}[1_B(x, \delta)c](x) \leq \exp[-C/\epsilon],$$

where $B(x, \delta)$ is the ball in $\mathbb{R}^d$ of center $x$ and radius $\delta$. 

\[\]
Proof. We consider the semi-group
\begin{equation}
\exp[-<x, \xi>/\epsilon]P_{\epsilon,t}[\exp[-<x', \xi>/\epsilon]f(x')](x)
\end{equation}
It has as generator
\begin{equation}
L_\epsilon f(x) + H(x, \xi)/\epsilon f(x)
\end{equation}
\(L_\epsilon\) generates a semi-group \(P_{\epsilon,t}\) in bounded measures such that \(\|P_{\epsilon,t}\|\) is bounded when \(\epsilon \to 0\).

It is enough to show the estimate for \(x = 0\). We use the Volterra expansion. We get
\begin{equation}
|P_{\epsilon,t}||\exp[-<x', \xi>/\epsilon]f(x')|(0) \leq \|f\|_\infty + \sum \int_{I_n} ds_1...ds_n |P_{\epsilon,t-s_1}||H(x'_1, \xi)|/\epsilon |P_{\epsilon,s_1-s_2}||H(x'_2, \xi)|/\epsilon ... |P_{\epsilon,s_n}||H(x'_n, \xi)|/\epsilon f(x'_n)|.(x, 0)
\end{equation}
Therefore
\begin{equation}
|P_{\epsilon,t}|[1.B(x,\delta\epsilon)](0) \leq C \exp[-\delta|\xi|/\epsilon + Ct|\xi|^{4/3}/\epsilon].
\end{equation}
We extremize in \(\xi\) and choose \(t\) small in order to conclude. \(\square\)

Proof of Theorem 3. This follows closely the lines of [10] or [12]. We cut the time interval \([0, 1]\) is small intervals of length \(t_\delta, [t_i, t_i+1]\). We use by the semi-group property
\begin{equation}
\|P_{\epsilon,1}[[1](x) \leq |P_{\epsilon,t_1}|...|P_{\epsilon,1-t_n}||[1](x).
\end{equation}
In \(P_{\epsilon,t_i-t_i+1}\), we distinguish if \(x_{t_i+1}\) and \(x_{t_i}\) are far or not. If they are, we use the previous lemma. If they are close, we deduce a positive measure \(|W_\epsilon|\) on polygonal paths \(\phi_t\) which join \(x_{t_i}\) to \(x_{t_i+1}\) where the distance between two contiguous points is smaller than \(\delta\). By the previous lemma, it remains to estimate \(|W_\epsilon|[1]). But \(|W_\epsilon|\) is a positive measure. Therefore, we have the inequality
\begin{equation}
|W_\epsilon|[1] \leq |W_\epsilon|[\exp[S(\phi)/\epsilon]1]\exp[- \inf_{y \in \mathbb{R}^d} l(x, y)/\epsilon].
\end{equation}
Therefore, we have only to estimate \(|W_\epsilon|[\exp[S(\phi)/\epsilon]1].

The sequel follows closely [14, p. 152] or [12, p. 188]. We remark that on the set of polygonal paths considered \(d/dt\phi_t\) and \(\phi_t\) remain bounded. We can choose some \(p_i\) in finite numbers such that if we put
\begin{equation}
L'(x, p) = \sup_i (L(x, p_i) + <\partial/\partial p L(x, p_i), p - p_i>)
\end{equation}
we have for all polygonal paths considered
\[ L(\phi_t, d/dt\phi_t) - L'(\phi_t, d/dt\phi_t) \leq \chi \]
for a small \( \chi \). Let us put
\[ S'(\phi) = \int_0^1 L'(\phi_t, d/dt\phi_t) dt. \]

Since \( |W_\epsilon| \) is a positive measure, we have only to estimate
\[ |W_\epsilon| [\exp[S'(\phi)/\epsilon] 1]. \]

We remark that
\[ \exp[\sup a_i] \leq \sum \exp[a_i]. \]

Moreover
\[ L'(x, p) = \sup(<\xi_i, p> - H(x, \xi_i)), \]
where \( \xi_i = \frac{\partial}{\partial p} L(x, p_i) \)
Therefore it is enough to show that
\[ \sup_{x, |\xi| < C} |P_{t_0, t_\delta}[] [\exp[(1 + \chi)/\epsilon] (<\xi, x' - x > - t_\delta H(x, \xi))] |(x) \]
has a small exponential blowing up when \( \epsilon \to 0. \)

We consider \( P_{t_0, t_\delta}^\text{tot,}\xi \) as in (14).

In order to estimate the previous quantity, we only have to estimate the quantity:
\[ |P_{t_0, t_\delta}^\text{tot,}\xi [\exp[(1 + \chi)/\epsilon] (<\xi, x' - x > - t_\delta H(x, \xi)) + y - y)](x, 0). \]

We distinguish in the previous quantity if \( |t_\delta H(x, \xi) - y| > C_3 t_\delta \delta \) or not. If \( |t_\delta H(x, \xi) - y| < C_3 t_\delta \delta \), the result holds by (15) because \( |P_{t_0, t_\delta}^\text{tot,}\xi [\cdot](x, 0) \) is a positive measure.

If \( |t_\delta H(x, \xi) - y| > C_3 t_\delta \delta \), we remark that in the definition of \( |W_\epsilon| \), we keep values where \( |x_{t_{i+1}} - x_{ti}| < \delta \). Therefore, we have only to estimate
\[ \exp[(1 + \chi)/\epsilon] |P_{t_0, t_\delta}^\text{tot,}\xi [||y - t_\delta H(x, \xi)| > C_3 t_\delta \delta ](x, 0). \]

Therefore, we have only to show that \( |P_{t_0, t_\delta}^\text{tot,}\xi [||y - t_\delta H(x, \xi)| > C_3 t_\delta \delta ](x, 0) \) has an enough big exponential decay.

We use the analog of (18). We get that if
\[ |P_{t_0, t_\delta}^\text{tot,}\xi [\exp[a|y - t_\delta H(x, \xi)|]](x, 0) \leq \sum \int_{I_n} ds_1..ds_n \]

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\[
\left| P_{\epsilon,t-s_1} \right| \left| a \right| H(x_1', \xi) - H(x, \xi) \left/ \epsilon \right. \left| P_{\epsilon,s_1-s_2} \right| \left| a \right| H(x_2', \xi) - H(x, \xi) \left/ \epsilon \right. \cdots
\]

\[
\left| P_{\epsilon,s_n} \right| \left| a \right| H(x_n', \xi) - H(x, \xi) \left/ \epsilon \right. \right] \right)(x).
\]

The main remark is that \( \xi \) is bounded.

If one of the \( x_j' \) is such \( |x_j - x| > C\delta \) for a big \( C \), Lemma 5 shows that the series is smaller than

\[
\exp \left[ \frac{C_1|a|t_\delta}{\epsilon} \right] \exp \left[ - \frac{C_0}{\epsilon} \right],
\]

where

\[
C_1 = \| H(., \xi) \|_\infty \sup_{s \leq t_\delta} \| P_{\epsilon,s} \|.
\]

We can choose \( C_0 \) very big. If all the \( x_j' \) are such \( |x_j - x| \leq C\delta \), we have an estimate of the series in \( \exp \left[ \frac{C_2|a|t_\delta}{\epsilon} \right] \) where

\[
C_2 = \sup_{s \leq t_\delta} \| P_{\epsilon,s} \| \sup_{|x-x'| \leq C\delta} \frac{|H(x, \xi) - H(x', \xi)|}{C\delta}.
\]

Therefore for \( a > 0 \)

\[
P_{\epsilon,t_\delta}^\text{tot,} \left[ |y - t_\delta H(x, \xi)| > C_3 t_\delta \delta \right](x, 0) \leq \exp \left[ - \frac{aC_3 t_\delta \delta}{\epsilon} \right] \left( \exp \left[ \frac{C_1 a t_\delta}{\epsilon} \right] \exp \left[ - \frac{C_0}{\epsilon} \right] + \exp \left[ \frac{C_2 a t_\delta \delta}{\epsilon} \right] \right).
\]

We choose \( a = t_\delta^{-1} \) and \( C_3 \) big in order to deduce that

\[
P_{\epsilon,t_\delta}^\text{tot,} \left[ |y - t_\delta H(x, \xi)| > C_3 t_\delta \delta \right](x, 0) \leq \exp \left[ - C_4 \delta / \epsilon \right],
\]

for a big \( C_4 \). The result holds. \( \square \)

4. CONCLUSION

We have adapted the classical proof of Wentzel-Freidlin estimates for jump processes to the case of an operator of order four, where the semi-group is not Markovian and where there is a potential. Normalization are those classical of semi-classical asymptotics [1].

REFERENCES


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