LARGE DEVIATION ESTIMATES FOR AN OPERATOR OF ORDER FOUR WITH A POTENTIAL

RÉMI LÉANDRE

Communicated by Lucian Beznea

We give large deviation estimates when there is a potential, when the generated semi-group is not Markovian. This paper enters in the problematic of semi-classical asymptotics of Maslov and his school but with a different type of estimates.

AMS 2010 Subject Classification: 60F10, 81Q20.

Key words: large deviation theory, semi-classical expansion, non-Markovian semi-groups.

1. INTRODUCTION

Let us consider a symbol $a(x, \xi)$ on $R^d \times R^d$ which represents a pseudodifferential operator L. According to the terminology of Maslov-Fedoriuk [1], we consider the symbol $\frac{1}{\epsilon} a(x, \epsilon\xi)$ which represents another pseudodifferential operator L_ϵ. The object of semi-classical asymptotics is to get precise asymptotics of the Schroedinger operator (if it exists) $\exp[iL_\epsilon]$.

In the present context, we consider $X_i, i = 1, .., m$ m vector fields on $\mathbb{R}^d$ with bounded derivatives at each order and without divergence with respect to the Lebesgue measure on $\mathbb{R}^d$. Let V be a smooth bounded function on $\mathbb{R}^d$ with bounded derivatives at each order. The Hamiltonian is

$$(1) \qquad H(x, \xi) = \sum_1^m < X_i(x), \xi >^4 + V(x) = \sum_{\sum i_j = 4} A_{i_1,..,i_j}(x) \prod \xi_j^{i_j} + V(x),$$

We suppose that we are in an elliptic situation: there exists $C > 0$ such that

$$(2) \qquad \sum_1^m < X_i(x), \xi >^4 \geq C|\xi|^4$$

The Hamiltonian is the symbol of the operator:

$$(3) \qquad L = \sum_{\sum i_j = 4} A_{i_1,..,i_j}(x) \prod \frac{\partial^{i_j}}{\partial x_j^{i_j}} + V(x).$$

It generates a semi-group P_t solution of the parabolic equation:

$$\frac{\partial}{\partial t}P_t = -LP_t\,. \tag{4}$$

By elliptic theory [2–4],

$$P_t f(x) = \int_{\mathbb{R}^d} f(y)\mu_t(x, \mathrm{d}y), \tag{5}$$

where $\mu_t(x, \mathrm{d}y)$ is a bounded measure on $\mathbb{R}^d$. The symbol of the considered operator L_ϵ is

$$H_\epsilon(x,\xi) = \epsilon^{-1}\sum_1^m < X_i(x), \epsilon\xi >^4 + \frac{1}{\epsilon}V(x) = \epsilon^3\sum_1^m < X_i(x), \xi >^4 + \frac{1}{\epsilon}V(x). \tag{6}$$

L_ϵ generates a semi-group P_t^ϵ where

$$P_t^\epsilon f(x) = \int_{\mathbb{R}^d} f(y)\mu_t^\epsilon(x, \mathrm{d}y). \tag{7}$$

Instead of doing precise estimates of P_1^ϵ when $\epsilon \to 0$ as in [1], we perform in this paper rough logarithmic estimates of $|P_1^\epsilon|$, by adapting in this context the proof of Wentzel-Freidlin of such estimates, because we have an analog in this non-Markovian situation of the Girsanov formula and of the exponential martingales. See [5] for review. This paper enters in the general problematic to introduce stochastic analysis tools in the general framework of non-Markovian semi-groups [5].

In [6], we have translated the classical proof of Wentzel-Freidlin estimates, upper-bound, for Poisson processes, which was done originally for the whole process instead of the semi-group only as in [6]. Let us recall that the simplest proof for diffusions of Wentzel-Freidlin [7] estimates is done by the author in [8–10]. In the present paper, the estimates are of the type of those in [11], which are closely related to the classical Wentzel-Freidlin estimates. The proof follows closely the lines of [12]. See [13, 14] and [15] for related topics.

2. STATEMENT OF THE THEOREM

According to the general framework of the large deviation theory, we introduce the Legendre transform of the Hamiltonian $H(x, \xi)$:

$$L(x,p) = \sup_\xi(< p, \xi > -H(x, \xi >). \tag{8}$$

A simple computation shows that

$$C_1' + C_1|p|^{4/3} \geq L(x,p) \geq C_2|p|^{4/3} + C_2'. \tag{9}$$

If $[0,1] \to \mathbb{R}^d$: $t \to \phi_t$ is a piecewise continuous C^1 curve, we introduce the action:

$$S(\phi) = \int_0^1 L(\phi_t, \mathrm{d}/\mathrm{d}t\phi_t)\mathrm{d}t. \tag{10}$$

Definition 1. We put

$$l(x,y) = \inf_{\phi_0 = x; \phi_1 = y} S(\phi). \tag{11}$$

By using the Ascoli theorem, we deduce from (9) that:

PROPOSITION 2. *The control function $(x,y) \to l(x,y)$ is continuous. Moreover there exists at least one curve ϕ joining x to y such that $l(x,y) = S(\phi)$. Moreover $l(x,y) \to \infty$ when $y \to \infty$.*

The main result of this paper is the following and is a large deviation estimate of the type of [11].

THEOREM 3. *We have*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \,\mathrm{Log} |P_1^\epsilon|[1](x) \leq - \inf_{y \in \mathbb{R}^d} l(x,y). \tag{12}$$

Remark. It is nonsense in this non-Markovian situation to get a lower bound.

3. PROOF OF THE MAIN THEOREM

The proof follows slightly the lines of [10, 12], the main difference is that we consider the absolute values of semi-groups instead of the semi-group. This leads to two interpretations of the martingales exponentials.

LEMMA 4. *We consider the generator on $\mathbb{R}^d \times \mathbb{R}$*

$$L_\epsilon^{tot,\xi} = L_\epsilon + H(x,\xi)\frac{\partial}{\partial y}. \tag{13}$$

The parabolic equation issued of $f(x)\exp[\epsilon^{-1}(< x, \xi > - y)]$

$$\frac{\partial}{\partial t} P_{\epsilon,t}^{tot,\xi}[f(x')\exp[\epsilon^{-1}(< x' - x, \xi > - y')]](x,y) = - L_\epsilon^{tot,\xi} P_{\epsilon,t}^{tot,\xi}[f(x')[\exp[\epsilon^{-1}(< x', \xi > - y')]](x,y) \tag{14}$$

has a unique solution. Moreover, the map

$$f \to P_{\epsilon,t}^{tot,\xi}[f(x')\exp[\epsilon^{-1}(< x' - x, \xi > - y')]](x,0) \tag{15}$$

defines a bounded measures when $\epsilon \leq 1$ $t \leq 1$ and $|\xi| \leq C$ and all x.

Proof. Let P_t^ξ the semi-group on $\mathbb{R}^d$

$$f \to \exp[< x, \xi >] P_{\epsilon,t}[\exp[- < x', \xi >] f(x')](x). \tag{16}$$

Its generator is

$$L_\epsilon^\xi f(x) = \exp[< x, \xi >] L_\epsilon[exp[- < x, \xi >] f(x)] = L_\epsilon + \text{lower terms}. \tag{17}$$

By standard result, L_ϵ^ξ generates a semi group in bounded measures.

In order to solve the parabolic equation associated to (13), we use Volterra expansion.

$$\begin{gathered} P_{\epsilon,t}^{tot,\xi}[f(x')\exp[(< x' - x, \xi > -y')]](x, y) = \exp[-y] + \\ \sum_n (-1)^n \int_{I_n} \mathrm{d}s_1..\mathrm{d}s_n \\ P_{\epsilon,t-s_1}^\xi[H(x'_1, \xi)[P_{\epsilon,s_1-s_2}^\xi[H(x'_2, \xi)...P_{\epsilon,s_n}^\xi[H(x'_n, \xi) f(x'_n) \exp[-y]]](x), \end{gathered} \tag{18}$$

where I_n is the simplex of length n on $[0, t]$ ordered by decreasing order. By using the previous results, we can estimate the integral on the simplex by

$$\|P_{\epsilon,t}^\xi\|^n \|f\|_\infty \|H(., \xi)\|_\infty^n \frac{t^n}{n!} \tag{19}$$

Therefore the series converges. In order to prove the last statement of the lemma, we remark that

$$f \to P_{\epsilon,t}^{tot,\xi}[f(x') \exp[\epsilon^{-1}(< x' - x, \xi > -y')]](x, 0) \tag{20}$$

defines a semi-group because

$$\begin{gathered} P_{\epsilon,t}^{tot,\xi}[f(x') \exp[\epsilon^{-1}(< x' - x, \xi > -y')]](x, y) = \\ \exp[-y/\epsilon] P_{\epsilon,t}^{tot,\xi}[f(x') \exp[\epsilon^{-1}(< x' - x, \xi > -y')]](x, 0). \end{gathered} \tag{21}$$

We can compute easily the generator of this semi-group. We see that the diverging terms are cancelling, because they come when we apply L_ϵ only on $\exp[1/\epsilon < x, \xi >]$ when we apply chain rules and one derivatives of $\exp[-1/\epsilon y]$. Therefore the result. $\square$

The second lemma is an analog of exponential inequality which arises in stochastic analysis from the use of exponential martingales [16–18]. Here they follow from the standard Davies method [19].

LEMMA 5. *For all $\delta > 0$, all C there exists t_δ such that if $t < t_\delta$*

$$|P_{\epsilon,t}|[1_{B(x,\delta)^c}](x) \leq \exp[-C/\epsilon], \tag{22}$$

where $B(x, \delta)$ is the ball in $\mathbb{R}^d$ of center x and radius δ.

Proof. We consider the semi-group

$$\exp[- < x, \xi > /\epsilon] P_{\epsilon,t}[\exp[< x', \xi > /\epsilon] f(x')](x) \tag{23}$$

It has as generator

$$\overline{L}_\epsilon f(x) + H(x, \xi)/\epsilon f(x) \tag{24}$$

$\overline{L}_\epsilon$ generates a semi-group $\overline{P}_{\epsilon,t}$ in bounded measures such that $\|\overline{P}_{\epsilon,t}\|$ is bounded when $\epsilon \to 0$.

It is enough to show the estimate for $x = 0$. We use the Volterra expansion. We get

$$\begin{gathered} |P_{\epsilon,t}|[\exp[< x', \xi > /\epsilon] f(x')](0) \leq \|f\|_\infty + \\ \sum \int_{I_n} \mathrm{d}s_1..\mathrm{d}s_n \\ |\overline{P}_{\epsilon,t-s_1}|[H(x'_1, \xi)]|/\epsilon[|\overline{P}_{\epsilon,s_1-s_2}|[H(x'_2, \xi)]|/\epsilon..[|\overline{P}_{\epsilon,s_n}|[H(x'_n, \xi)]|/\epsilon|f(x'_n)|]..](x, 0) \\ \leq C \exp[tC|\xi|^{4/3}/\epsilon]\|f\|_\infty. \end{gathered} \tag{25}$$

Therefore

$$|P_{\epsilon,t}|[1_{B(x,\delta)^c}](0) \leq C \exp[-\delta|\xi|/\epsilon + Ct|\xi|^{4/3}/\epsilon]. \tag{26}$$

We extremize in ξ and choose t small in order to conclude. $\square$

Proof of Theorem 3. This follows closely the lines of [10] or [12]. We cut the time interval $[0, 1]$ is small intervals of length t_δ, $[t_i, t_{i+1}]$. We use by the semi-group property

$$\||P_{\epsilon,1}|[1](x) \leq |P_{\epsilon,t_1}|...|P_{\epsilon,1-t_n}|[[1]](x). \tag{27}$$

In $P_{\epsilon,t_i-t_{i+1}}$, we distinguish if $x_{t_{i+1}}$ and x_{t_i} are far or not. If they are, we use the previous lemma. If they are close, we deduce a **positive** measure $|W_\epsilon|$ on polygonal paths ϕ_t which join x_{t_i} to $x_{t_{i+1}}$ where the distance between two contiguous points is smaller than δ. By the previous lemma, it remains to estimate $|W_\epsilon|[1)]$. But $|W_\epsilon|$ is a **positive** measure. Therefore, we have the inequality

$$|W_\epsilon|[1] \leq |W_\epsilon|[\exp[S(\phi)/\epsilon]1] \exp[- \inf_{y \in \mathbb{R}^d} l(x, y)/\epsilon]. \tag{28}$$

Therefore, we have only to estimate $|W_\epsilon|[\exp[S(\phi)/\epsilon]1]$.

The sequel follows closely [14, p. 152] or [12, p. 188]. We remark that on the set of polygonal paths considered $\mathrm{d}/\mathrm{d}t\phi_t$ and ϕ_t remain bounded. We can choose some p_i in finite numbers such that if we put

$$L'(x, p) = \sup_i (L(x, p_i) + < \frac{\partial}{\partial p} L(x, p_i), p - p_i >) \tag{29}$$

we have for all polygonal paths considered

$$L(\phi_t, \mathrm{d}/\mathrm{d}t\phi_t) - L'(\phi_t, \mathrm{d}/\mathrm{d}t\phi_t) \leq \chi \tag{30}$$

for a small χ. Let us put

$$S'(\phi) = \int_0^1 L'(\phi_t, \mathrm{d}/\mathrm{d}t\phi_t)\mathrm{d}t. \tag{31}$$

Since $|W_\epsilon|$ is a positive measure, we have only to estimate

$$|W_\epsilon|[\exp[S'(\phi)/\epsilon]1]. \tag{32}$$

We remark that

$$\exp[\sup a_i] \leq \sum \exp[a_i]. \tag{33}$$

Moreover

$$L'(x,p) = \sup(< \xi_i, p > -H(x, \xi_i)), \tag{34}$$

where $\xi_i = \frac{\partial}{\partial p}L(x, p_i)$ Therefore it is enough to show that

$$\sup_{x, |\xi| < C} |P_{\epsilon, t_\delta}|[[\exp[\frac{(1+\chi)}{\epsilon}(< \xi, x' - x > -t_\delta H(x, \xi))]](x) \tag{35}$$

has a small exponential blowing up when $\epsilon \to 0$.

We consider $P^{tot,\xi}_{\epsilon,t_\delta}$ as in (14).

In order to estimate the previous quantity, we only have to estimate the quantity:

$$|P^{tot,\xi}_{\epsilon,t_\delta}|[\exp[\frac{(1+\chi)}{\epsilon}(< \xi, x' - x > -t_\delta H(x,\xi) + y - y)]](x, 0). \tag{36}$$

We distinguish in the previous quantity if $|t_\delta H(x,\xi) - y| > C_3 t_\delta \delta$ or not.

If $|t_\delta H(x,\xi) - y| < C_3 t_\delta \delta$, the result holds by (15) because $|P^{tot,\xi}_{\epsilon,t_\delta}|[.](x,0)$ is a **positive** measure.

If $|t_\delta H(x,\xi) - y| > C_3 t_\delta \delta$, we remark that in the definition of $|W_\epsilon|$, we keep values where $|x_{t_{i+1}} - x_{t_i}| < \delta$. Therefore, we have only to estimate

$$\exp[\frac{(1+\chi)\delta}{\epsilon}]|P^{tot,\xi}_{\epsilon,t_\delta}|[|y - t_\delta H(x,\xi)| > C_3 t_\delta \delta](x, 0). \tag{37}$$

Therefore, we have only to show that $|P^{tot,\xi}_{\epsilon,t_\delta}|[|y - t_\delta H(x,\xi)| > C_3 t_\delta \delta](x,0)$ has an enough big exponential decay.

We use the analog of (18). We get that if

$$|P^{tot,\xi}_{\epsilon,t_\delta}|[\exp[a|y - t_\delta H(x,\xi)|]](x,0) \leq \sum \int_{I_n} \mathrm{d}s_1..\mathrm{d}s_n \tag{38}$$

$$|P_{\epsilon,t-s_1}|[|a|\frac{|H(x_1',\xi)-H(x,\xi)|}{\epsilon}|P_{\epsilon,s_1-s_2}|[|a|\frac{|H(x_2',\xi)-H(x,\xi)|}{\epsilon}\ldots$$
$$|P_{\epsilon,s_n}|[|a|\frac{|H(x_n',\xi)-H(x,\xi)|}{\epsilon}]..](x).$$

The main remark is that ξ is bounded.

If one of the x_j' is such $|x_j-x|>C\delta$ for a big C, Lemma 5 shows that the series is smaller than

$$\exp[\frac{C_1|a|t_\delta}{\epsilon}]\exp[-\frac{C_0}{\epsilon}], \tag{39}$$

where

$$C_1=\|H(.,\xi)\|_\infty \sup_{s\leq t_\delta}\|P_{\epsilon,s}\|. \tag{40}$$

We can choose C_0 very big. If all the x_j' are such $|x_j-x|\leq C\delta$, we have an estimate of the series in $\exp[\frac{C_2|a|t_\delta\delta}{\epsilon}]$ where

$$C_2=\sup_{s\leq t_\delta}\|P_{\epsilon,s}\|\sup_{|x-x'|\leq C\delta}\frac{|H(x,\xi)-H(x',\xi)|}{C\delta}. \tag{41}$$

Therefore for $a>0$

$$|P^{tot,\xi}_{\epsilon,t_\delta}|[|y-t_\delta H(x,\xi)|>C_3t_\delta\delta](x,0)\leq$$
$$\exp[-\frac{aC_3t_\delta\delta}{\epsilon}](\exp[\frac{C_1at_\delta}{\epsilon}]\exp[-\frac{C_0}{\epsilon}]+\exp[\frac{C_2at_\delta\delta}{\epsilon}]). \tag{42}$$

We choose $a=t_\delta^{-1}$ and C_3 big in order to deduce that

$$|P^{tot,\xi}_{\epsilon,t_\delta}|[|y-t_\delta H(x,\xi)|>C_3t_\delta\delta](x,0)\leq\exp[-C_4\delta/\epsilon], \tag{43}$$

for a big C_4. The result holds. □

4. CONCLUSION

We have adapted the classical proof of Wentzel-Freidlin estimates for jump processes to the case of an operator of order four, where the semi-group is not Markovian and where there is a potential. Normalization are those classical of semi-classical asymptotics [1].

REFERENCES

[1] V.P. Maslov and M.V. Fedoriuk, *Semiclassical Approximation in Quantum Mechanics.* Reidel, Dordrecht, Netherland, 1981.

[2] J. Dieudonné, *Eléments d'analyse VII.* Gauthiers-Villars, Paris, France, 1977.

[3] L. Hoermander, *The Analysis of Linear Partial Operators III.* Springer, Berlin, Germany, 1984.

[4] L. Hoermander, *The Analysis of Linear Partial Operators IV.* Springer, Berlin, Germany, 1984.

[5] R. Léandre, *Stochastic analysis for a non-markovian generator: an introduction.* Russ. J. Math. Phys. **22** (2015), 39–52.

[6] R. Léandre, *Wentzel-Freidlin estimates for jump processes in semi-group theory: upper bound.* In: H. Arabnia and al (Eds.), Int. Conf. Scientific. Computing **10**, C.S.R.E.A. Press, Las-Vegas, U.S.A, 2010, pp. 187–189. Computer Technology and Application **2** (2011), 329–332.

[7] A.D. Wentzel and M.J. Freidlin *Random Perturbations of Dynamical Systems.* Springer, Berlin, Germany, 1984.

[8] R. Léandre, *A simple proof for a large deviation theorem.* In: D. Nualart (Ed.), *Barcelona Seminar on Stochastic Analysis*, Progr. Probab. **32**, Birkhauser. Basel, Schweiz, 1993, pp. 72–76.

[9] R. Léandre, *Wentzel-Freidlin estimates in semi-group theory.* In: Yeng Chai Soh and al (Eds.), *Control, Automation, Robotics and Vision*, C.D., I.E.E.E., 2008, pp. 2233–2236.

[10] R. Léandre, *Varadhan estimates without probability: upper bound.* In: M. Demiralp and al (Eds.), *Computational methods and applied computing.* W.S.E.A.S. Press, Athens, Greece, 2008, pp. 77–81. Transactions of Mathematics **7** (2008), 244–253.

[11] S.R.S. Varadhan, *Large Deviations and Applications.* S.I.A.M., Philadelphia, U.S.A., 1984.

[12] R. Léandre, *Wentzel-Freidlin estimates for an operator of order four.* In: B. Akhgar and al (Eds.), *2014 Int. Conf. Comp. Science and Comut. Intel.* IEEE Computer Society, Los-Alamitos, U.S.A., 2014, pp. 360–364.

[13] R. Léandre, *The Itô-Stratonovitch formula for an operator of order four.* In: G. Budzhan *et al.* (Eds.), *Festchrift in honour of A. Mukharjea, Ph. Feisilver and P. Feinsilver. Probability on algebraic and geometric structures.* Contemporary Mathematics **668** (2016), A.M.S. Providence, pp. 165–169.

[14] R. Léandre, *Large deviation estimates for a non-markovian Lévy generator of big order.* In: E. Vagenas *et al.* (Eds.), *4th Int. Conf. Math. Mod. Phys. Sciences.* J. Phys.: Conference Series **633** (2015), paper 012085.

[15] R. Léandre, *A class of non-markovian Peudodifferential operators of Lévy type.* To appear in: M.W. Wong (Ed.), *Pseudo-differential operators: groups, geometry and applications.*

[16] C. Dellacherie and P.A. Meyer, *Probabilités et potentiel. II.* Hermann, Paris, France, 1978.

[17] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edition. North-Holland, Amsterdam, Netherland, 1989.

[18] L.C.G. Rogers and D. Williams, *Stochastic Differential Equations and Martingales*, Vol. 2. Wiley, New-York, U.S.A., 1987.

[19] B. Davies, *Uniformly elliptic operators with measurable coefficients.* J. Funct. Anal. **132** (1995), 141–169.

Received 7 September 2015

Université de Franche-Comté,
Laboratoire de Mathématiques,
25030, Besançon,
France
remi.leandre@univ-fcomte.fr