SOME NON-NEWTONIAN EFFECTS IN HELE-SHAW DISPLACEMENTS

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We consider a non-Newtonian fluid displaced by air in a Hele-Shaw cell and study the modal linear stability. The particular flow geometry – a very thin Hele-Shaw cell – allows us to use some simplified flow equations. The novelty of the paper is an approximate formula for the growth constant (of perturbations), which can grow unbounded when the corresponding Weissenberg number is located in a certain range. We get a strong destabilization effect, compared with the Newtonian case studied by Saffman and Taylor (1958). This phenomenon was observed in experiments, during the flow of some complex fluids in thin Hele-Shaw cells. While we used the linear stability method, the (possible) very large values of the growth constant should be studied in the frame of the nonlinear stability.

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1. INTRODUCTION

A Hele-Shaw cell is a technical device consisting of two parallel plates at a small distance b and length l. The ratio $\epsilon = b/l$ is in general of order 10^{-3} or less. The main phenomenon studied by using this device is the displacement of two immiscible fluids – see Figure 3. We give below some details concerning this subject.

The flow of a viscous fluid in the small gap between two parallel plates was first considered by Hele-Shaw in [5]. An average procedure was used, starting with a Stokes fluid. Two main assumptions were used by Hele-Shaw: 1) the velocity component in the direction orthogonal on the plates is zero; 2) the partial derivatives in the directions contained in the plates plane are neglected in front of the derivative in the direction orthogonal on the plates. These two assumptions were confirmed in a large number of experiments. The main result obtained by Hele-Shaw was the following: the obtained averaged velocities (called in many papers as “filtration velocities”) verify an equation quite similar with the Darcy law for the flow in a porous medium, whose "permeability" is given by the Hele-Shaw gap b and the viscosity of the considered Stokes fluid. An important property of the "filtration fluid" is the following: even if we started with a Stokes (then viscous) fluid, the "filtration" velocities are not zero on the Hele-Shaw plates.

In [12] it was studied the linear stability of the interface between two immiscible fluids displacing in a Hele-Shaw cell. The main application of this study is related with the secondary oil recovery process – that of obtaining the oil from a porous media by displacing it with a second fluid (usually water or salted water). Saffman and Taylor obtained an exact formula for the growth constant, by using the modal linear stability method. This formula leads to a very important result: if the displacing fluid is less viscous, then the interface is unstable. It is interesting to see the section devoted to this problem in [1]: when the Hele-Shaw plates are transparent, we can can directly observe the evolution of the water-oil interface and the fluid lines between the injection and extraction wells.

As the water is less viscous compared with the oil, an interesting problem appears: how is it possible to minimize (or to suppress) the Saffman-Taylor instability? In some previous papers the use of an intermediate fluid layer between water and oil is suggested, where the viscosity is a parameter used to improve the interface stability. Some experiments were carried out by using polymer-solutes, with good results for an exponential growth of the viscosity in the intermediate layer. The polymers are non-Newtonian fluids, that means the constitutive relations between the stress and the strain-rate tensors are not linear. In this way, the problem of the stability of the interface(s) between non-Newtonian and Newtonian fluids displacing in a Hele-Shaw cell appears.

The non-Newtonian fluids are studied in a large number of papers – see [4, 11, 13, 14]. A particular type of non-Newtonian fluids, called second order fluids, are studied in [2, 3, 6]. Some numerical methods are used in [7, 8, 16], for studying the displacement of Oldroyd-B and Maxwell upper-convected fluids by air in a Hele-Shaw cell. In these papers, a strong destabilizing effect due to the non-Newtonian constitutive relations was obtained, compared with the case of a Newtonian fluid displaced by air. In [7, 8] it was obtained a blow-up of the growth constant of perturbations in the range of large and very large Weissenberg numbers W which appear in the constitutive relations. This possible singularity may be related to the fractures observed in the flows of some complex fluids in Hele-Shaw cells – see [9, 10, 17]. On the contrary, in [16] it was not reported a blow-up of the growth constant, but this result was (numerically) obtained for W near 1.

In this paper, we consider a non-Newtonian fluid displaced by air in a Hele-Shaw cell and study the modal linear stability of the interface, for small Weissenberg numbers – see the definition (19). We get *an approximate expression* of the amplitude f of the velocity perturbations, by using the equations of the pressure perturbations. We use an average procedure (across the Hele-Shaw gap) in the corresponding Laplace law and a method based on a continuity argument.

The novelty of our paper is an approximate formula of the growth constant of perturbations, obtained in the range of small Weissenberg numbers – see the formula (52). We get a strong non-Newtonian destabilizing effect, compared with the Newtonian case – see Figure 2. When $W = O(b/l)$ we obtain a blow-up of the growth rates in terms of the wave numbers of perturbations. Our main result is the following: the non-Newtonian fluid with constitutive relations (2) can give rise to the same destabilizing effect obtained in the previous papers [7, 8, 16]. The dispersion curves given in our Figure 2 are similar with those obtained (by using numerical methods) in [16], even if in this cited paper were considered Weissenberg numbers of order 1 – see Figure 1 of [16].

2. THE BASIC FLOW AND PERTURBATIONS

We consider a Hele-Shaw cell with plates parallel with the xOy plane. The Oz axis is orthogonal on the plates. The gap between plates is denoted by b and the length of the Hele-Shaw cell is l. Our problem is characterized by the small parameter $\epsilon = b/l << 1$.

We use the following notations: $\underline{\tau}, \underline{E}$ are the the extra-stress and strain-rate tensors; μ is the fluid viscosity; $x_1 = x, x_2 = y, x_3 = z$ are the spatial coordinates; $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ and $\underline{p}$ are the velocity components and pressure. $\underline{V}$ is the matrix of the velocity gradients and we have

$$(1) \qquad \underline{V}_{ij} = \partial \underline{u}_i / \partial x_j, \qquad (\underline{V}_{ij})^T = \underline{V}_{ji}, \quad \underline{E} = (\underline{V} + \underline{V}^T).$$

a) Our fluid is governed by the following *constitutive relations*:

$$(2) \qquad \underline{\tau} = \mu \underline{E} + \mu a (\underline{V}\,\underline{E} + \underline{E}\,\underline{V}^T), \quad a > 0,$$

where the dimension of a is *(time)*. Our constitutive relations are steady; it can be proved that (2) are frame-independent with respect to coordinate changes $x^+ = Qx$, where Q is an ortonormal matrix not depending on time. From now on we use the notation

$$(3) \qquad (\underline{u}_1, \underline{u}_2, \underline{u}_3) = (\underline{u}, \underline{v}, \underline{w}).$$

We consider an incompressible fluid, then we have

$$\underline{u}_x + \underline{v}_y + \underline{w}_z = 0 \tag{4}$$

where the lower indices denote the partial derivatives in terms of x, y, z. The no-slip conditions on the plates are imposed for the velocity, then

$$(\underline{u}, \underline{v}, \underline{w}) = 0 \quad \text{on} \quad z = 0, \quad z = b. \tag{5}$$

The flow equations of our fluid are given below:

$$\underline{p}_x = \underline{\tau}_{11,x} + \underline{\tau}_{12,y} + \underline{\tau}_{13,z}; \tag{6}$$

$$\underline{p}_y = \underline{\tau}_{21,x} + \underline{\tau}_{22,y} + \underline{\tau}_{23,z}; \tag{7}$$

$$\underline{p}_z = \underline{\tau}_{31,x} + \underline{\tau}_{32,y} + \underline{\tau}_{33,z}. \tag{8}$$

We consider below the particular basic flow described by the relations (9), (10). The characteristic velocity U of this basic flow will be used to define the Weissenberg dimensionless number, related with the parameter a - see the relations (18) and (19).

b) We consider the following *basic flow* (in the positive direction Ox), denoted by the super-index ^{0}:

$$\nabla p^0 = (p^0_x(x), 0, 0), \quad \mathbf{v^0} = (u^0(z), 0, 0), \tag{9}$$

$$\tau^0 = \mu\{E^0 + a(V^0E^0 + E^0V^{0T})\}, \quad a > 0, \tag{10}$$

where

$$V^0_{ij} = 0 \quad \forall \quad (i, j) \neq (1, 3); \quad V^0_{13} = u^0_z; \tag{11}$$

$$E^0_{ij} = 0 \quad \forall \quad (i, j) \neq (1, 3), (3, 1); \quad E^0_{13} = E^0_{31} = u^0_z.$$

Therefore from (10) we get the components of the basic extra-stress tensor:

$$\tau^0_{11} = 2a\mu(u^0_z)^2, \quad \tau^0_{12} = 0, \quad \tau^0_{13} = \mu u^0_z, \tag{12}$$

$$\tau^0_{22} = 0, \quad \tau^0_{23} = 0, \quad \tau^0_{33} = 0.$$

We have the following basic flow equations:

$$p^0_x = \tau^0_{11,x} + \tau^0_{12,y} + \tau^0_{13,z}, \tag{13}$$

$$p^0_y = \tau^0_{21,x} + \tau^0_{22,y} + \tau^0_{23,z}, \tag{14}$$

$$p^0_z = \tau^0_{31,x} + \tau^0_{32,y} + \tau^0_{33,z}. \tag{15}$$

The basic extra-stress tensor verifies the relations $\tau^0_{32} = \tau^0_{33} = 0$ and $\tau^0_{31} =$

$\tau^0_{31}(z)$, then the last above equation gives us

$$p^0_z = 0.$$

We can see also that τ^0_{11} is depending only on z and $\tau^0_{12} = \tau^0_{13} = 0$, then from the equation (13) it follows the important relation

$$p^0_x(x) = \tau^0_{13,z}(z) = \mu u^0_{zz} = G, \tag{16}$$

where G is a negative constant (the pressure is decreasing in terms of x, our flow is in the positive direction Ox). We point out that the fluid displacement is produced by the above *constant* pressure gradient G, which is considered the main parameter of the displacement process. The pressure gradient G is giving the basic flow with the velocity $(u^0(z), 0, 0)$ and the above relation (16) allows us to obtain the component u^0 of the basic velocity in terms of G:

$$u^0 = \frac{G}{\mu}(z^2 - bz)/2. \tag{17}$$

We introduce the average operator $< * >$ across the Hele-Shaw plates and use the above relation (17) for obtaining the *characteristic velocity* U of our basic flow:

$$U =< u^0 >:= \frac{1}{b}\int_0^b u^0(z)\mathrm{d}z = -(\frac{b^2}{12\mu})G. \tag{18}$$

The dimension of U is *(space)/(time)*. As the dimension of the parameter a in (2) is *(time)*, we introduce the *dimensionless* Weissenberg number W:

$$W = aU/l. \tag{19}$$

The main point of our paper is to study the modal linear stability of the above basic flow in the range of small W of order ϵ.

The relation (18) between $< u^0 >$ and G is quite similar with Darcy law for the flow in a porous media with "permeability" $= b^2/12\mu$. It is important to note that the averaged velocity $< u^0 >$ is not equal to zero on the Hele-Shaw plates, even if the "filtration" fluid described by (18) is a viscous one.

In [16] it is supposed that the pressure can depend on time – that means in the pressure expression we can add a constant not depending on x. As a consequence, the following dependence of the pressure in terms of the time t (which first appears here) is considered:

$$p^0 = G(x- < u^0 > t), \quad \text{for} \quad x >< u^0 > t. \tag{20}$$

We emphasize that our basic flow is steady. The basic moving interface between air and our fluid is

$$x =< u^0 > t. \tag{21}$$

c) *The perturbations* of the basic flow are denoted by $(u,v,w),p,\tau$. We use the dimensionless quantities (denoted by $'$)

(22)
$$x'=x/l,\quad y'=y/l,\quad z'=z/b,\quad (u',v',w')=(u,v,w)/U,\quad \epsilon=b/l<<1$$

and from the free-divergence condition (4) it follows

$$\epsilon\{u'_{x'}+v'_{y'}\}+w'_{z'}=0.$$

Here is the point where we can use the particular flow geometry. The small parameter ϵ in front of the parenthesis $\{u'_{x'}+v'_{y'}\}$ could give us the following result

(23)
$$u'_{x'}+v'_{y'}=0,\quad w'_{z'}=0$$

which can be used to obtain

(24)
$$u_x+v_y=0,\quad w_z=0.$$

However, this result holds only if the quantity $\{u'_{x'}+v'_{y'}\}$ is bounded in terms of ϵ. We can have, for example

$$\{u'_{x'}+v'_{y'}\}\approx 1/\epsilon \quad \text{or} \quad 1/\sqrt{\epsilon}$$

and in this case we cannot obtain the relations (23). We will introduce later the Fourier expansion (30) for (u,v); this expansion must be bounded. If we avoid the blow-up of the growth constant σ in the relations (30), then $(u'_{x'}+v'_{y'})$ can be considered bounded and we have the relations (23). We use the no-slip boundary conditions for the vertical component w', then from $(23)_2$ we get $w'=0$ and $w=0$. Therefore, we can consider that *both relations* (24) *hold.*

As $w=0$, the perturbations V,E of V^0,E^0 are given by

(25)
$$V_{11}=u_x,\quad V_{12}=u_y,\quad V_{13}=u_z,$$
$$V_{21}=v_x,\quad V_{22}=v_y,\quad V_{23}=v_z,\quad V_{3j}=0\quad for\quad j=1,2,3;$$
$$E_{11}=2u_x,\quad E_{12}=(u_y+v_x),\quad E_{13}=u_z,$$
$$E_{22}=2v_y,\quad E_{23}=v_z,\quad E_{33}=0.$$

The equations of the linear perturbations are obtained by inserting the perturbations in (10). We neglect the second order terms involving the perturbations products and get

(26)
$$\tau=\mu\{E+a[V^0E+(V^0E)^T+VE^0+(VE^0)^T]\}.$$

The components of the extra-stress tensor in terms of the velocity perturbations are given in the relations

(27)
$$\tau_{11}=2\mu u_x+4a\mu u_z^0 u_z,\quad \tau_{12}=\mu(u_y+v_x)+2a\mu u_z^0 v_z,$$

$$\tau_{13} = \mu u_z + a\mu u_z^0 u_x,$$

(28) $$\tau_{22} = 2\mu v_y, \quad \tau_{23} = \mu v_z + a\mu u_z^0 v_x, \quad \tau_{33} = 0$$

and we get the corresponding pressure perturbations

(29) $$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z}, \ p_y = \tau_{21,x} + \tau_{22,y} + \tau_{23,z}, \ p_z = \tau_{31,x} + \tau_{32,y} + \tau_{33,z}.$$

We use the following Fourier decomposition for (u, v):

(30) $$u = f(z)\exp(\alpha x + \sigma t)\cos(ny), \quad v = f(z)\exp(\alpha x + \sigma t)\sin(ny),$$

where σ is the growth constant. Then from the free divergence relation $(24)_1$ it follows

(31) $$\alpha = -n; \quad u_x = (-n)u, \quad v_y = (n)u, \quad u_{xy} = (n^2)v, \quad v_{xy} = (-n^2)u,$$

(32) $$u_{xx} + u_{yy} = 0, \quad u_{zx} + v_{zy} = 0, \quad u_{xx} + v_{xy} = 0.$$

The equations (29)–(32) give us the pressure perturbations in terms of the velocity perturbations:

(33) $$p_x = a\mu \cdot (3u_z^0 u_{zx} + u_{zz}^0 u_x) + \mu u_{zz},$$

(34) $$p_x = (-n)a\mu \cdot (3u_z^0 u_z + u_{zz}^0 u) + \mu u_{zz}.$$

(35) $$p_z = \mu(u_z + au_x u_z^0)_x + \mu(v_z + av_x u_z^0)_y = 0,$$

(36) $$(p_x)_z = (-n)a\mu \cdot (3u_z^0 u_z + u_{zz}^0 u)_z + \mu u_{zzz} = 0.$$

We use the relations (33) and (27) to obtain the expression of $(p_x - \tau_{11,x})$:

(37) $$p_x - \tau_{11,x} = \mu(-2n^2 u) + a\mu n(u_z^0 u_z - u_{zz}^0 u) + \mu u_{zz}.$$

3. THE APPROXIMATE GROWTH RATE FORMULA

Our basic velocity is depending on z. However, as in [16], we consider the kinetic and dynamic boundary condition on the steady air-liquid interface given by the straight line $x =< u^0 > t$. Therefore, the perturbed interface is given by

(38) $$\psi = x - < u^0 > t.$$

This interface is a material one, then it follows

(39) $$\psi_t = u \Rightarrow \psi = u/\sigma.$$

The equality between the pressure jump and the surface tension multiplied with the interface curvature gives us the dynamic boundary condition on the air-liquid interface (that means *Laplace's law*):

$$(G\psi + p) - (\tau_{11}^0 + \tau_{11}) = \gamma \cdot \{\psi_{yy} + \psi_{zz}\} \Rightarrow \tag{40}$$

$$G\psi_x + (p_x - \tau_{11,x}) = \gamma \cdot (\psi_{yy} + \psi_{zz})_x, \tag{41}$$

where γ is the surface tension and $\{\psi_{yy} + \psi_{zz}\}$ is denoting the total curvature of the interface. From the equations $(39)_2$, (30) it follows

$$\psi_x = -\frac{n}{\sigma} f(z) \exp(\alpha x + \sigma t) \cos(ny),$$

$$\psi_{yy} = -\frac{n^2}{\sigma} f(z) \exp(\alpha x + \sigma t) \cos(ny),$$

$$\psi_{zz} = \frac{1}{\sigma} f_{zz} \exp(\alpha x + \sigma t) \cos(ny).$$

We insert the last three relations and the expression (37) in (41), then we get

$$\frac{G(-n)f}{\sigma} - 2n^2\mu f + a\mu n \cdot (u_z^0 f_z - u_{zz}^0 f) + \mu f_{zz} = \frac{\gamma}{\sigma}[-n^2 f + f_{zz}](-n). \tag{42}$$

We introduce the following notations:

$$R(a) = \int_0^b f^2, \quad S(a) = \int_0^b (f_z)^2, \quad J(a) = J(f,a) = R/S, \tag{43}$$

we use the conditions $f(0) = f(b) = 0$ and we obtain the relations

$$\int_0^b u_z^0 f_z f \mathrm{d}z = \int_0^b \frac{G}{\mu}(z - \frac{b}{2})(\frac{f^2}{2})_z = \frac{G}{2\mu} \int_0^b z(f^2)_z = -\frac{G}{2\mu} R(a), \tag{44}$$

$$\int_0^b u_{zz}^0 f^2 \mathrm{d}z = \frac{G}{\mu} R(a), \quad \int_0^b f_{zz} f = -S(a). \tag{45}$$

We multiply with f in the Laplace law (42), we integrate on $(0,b)$, we use the above values R, S, J and we get the *dimensional* formula of the growth constant $\sigma(a)$in terms of $J(a)$:

$$\sigma(a) = \frac{Un(12/b^2)J(a) - (\gamma/\mu)n^3 J(a) - (\gamma/\mu)n}{1 - aUn(18/b^2)J(a) + 2n^2 J(a)}. \tag{46}$$

Remark 1. To derive the growth constant (46), we need the value of $J(a)$. For this we use a method based on a "continuity" argument. We first compute $J(0)$. As in this paper we only consider very small values of W, we assume that $J(a) \approx J(0)$ for very small values of a. For obtaining this result, we use the relation (36) and see that (f/n) is continuous in terms of a. Finally, in the formula (46) for small a we use the value $J(0)$ instead of $J(a)$.

We derive the expression of $J(0)$ from the relation (36) with $a = 0$. This relation gives us $u_{zzz} = 0$ and we obtain

$$f(z) = Az(z - b), \tag{47}$$

where A is a constant. Therefore, from a simple calculus it follows

$$J(0) = \frac{\int_0^b [z(z-1)]^2 \mathrm{d}z}{\int_0^b (2z-1)^2 \mathrm{d}z} = \frac{b^2}{10}. \tag{48}$$

We use the above value $J(0)$ instead of $J(a)$ in (46) and obtain *the approximate dimensional growth constant for small* a, denoted by $\sigma(sa)$:

$$\sigma(sa) = \frac{(6/5)\{Un - (\gamma/\mu)(b^2/12)n^3\} - (\gamma/\mu)n}{1 - (9/5)aUn + (b^2/5)n^2}. \tag{49}$$

Saffman and Taylor (1959) obtained the following *dimensional* formula of the growth rate of a Newtonian liquid with viscosity μ displaced by air (air viscosity is considered equal to zero):

$$\sigma_{ST} = Un - (\gamma/\mu)(b^2/12)n^3. \tag{50}$$

Remark 2. In the case $a = 0$, the value (49) is quite similar with the Saffman-Taylor (50), but two new terms appear: 1) $(-\gamma/\mu)n$ in the numerator, given by the meniscus curvature; 2) $n^2(b^2/5)$ in the denominator, given by the x derivative of the velocity. Moreover, the coefficient of $\{Un - (\gamma/\mu)(b^2/12)n^3\}$ in the numerator of (49) are (6/5) instead of 1 in (50).

We introduce the following dimensionless quantities:

$$p' = p\frac{1}{Gl}, \quad t' = t\frac{U}{l}, \quad \sigma' = \sigma\frac{l}{U}; \quad f' = f/U; \quad \gamma' = \gamma\frac{1}{\mu U}; \quad n' = nl, \tag{51}$$

then from (49) we get the following *dimensionless approximate formula of the growth constant for small* W, denoted by σ'_{SW}:

$$\sigma'_{SW} = \frac{(6/5)\{n' - \gamma'(\epsilon^2/12)n'^3\} - \gamma' n'}{1 - (9/5)Wn' + (\epsilon^2/5)n'^2}. \tag{52}$$

The dimensionless quantities (51) and the formula (50) give us the *dimensionless* Saffman-Taylor growth constant

$$\sigma'_{ST} = n' - \gamma'(\epsilon^2/12)n'^3. \tag{53}$$

Remark 3. Consider $W = c\epsilon$ with $c > 0.496$. Then *the denominator of* (52) *has two real roots.* Indeed, we have

$$c^2\epsilon^2(9/10)^2 - \epsilon^2/5 > 0 \Leftrightarrow c^2 > 0.2469, \quad c > 0.496. \tag{54}$$

In Figure 1, we plot the *positive* dimensionless growth constant (53) in terms of the dimensionless wave number n', for the values $\epsilon = 0.006$ and $\gamma' = 0.1$.

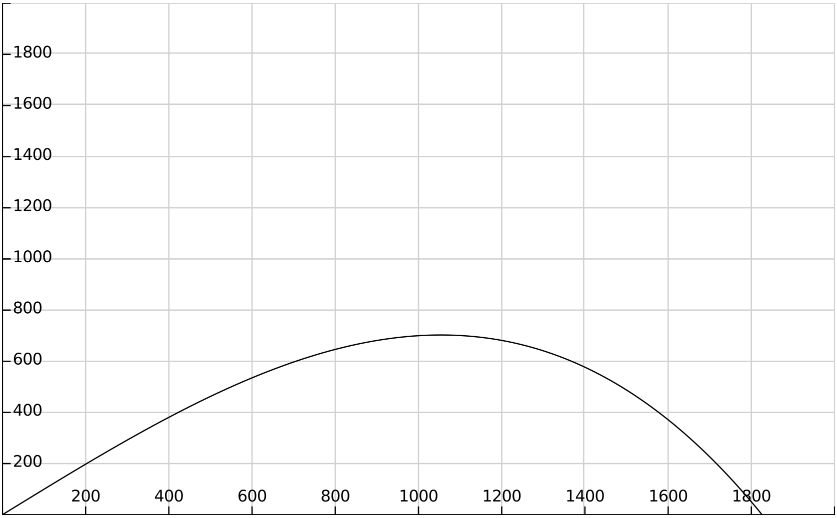

Fig. 1 – The positive Saffman-Taylor growth constant (53) (on vertical axis) *vs.* the wavenumber (on horizontal axis) for the values: $\epsilon = 0.006, \gamma' = 0.1$.

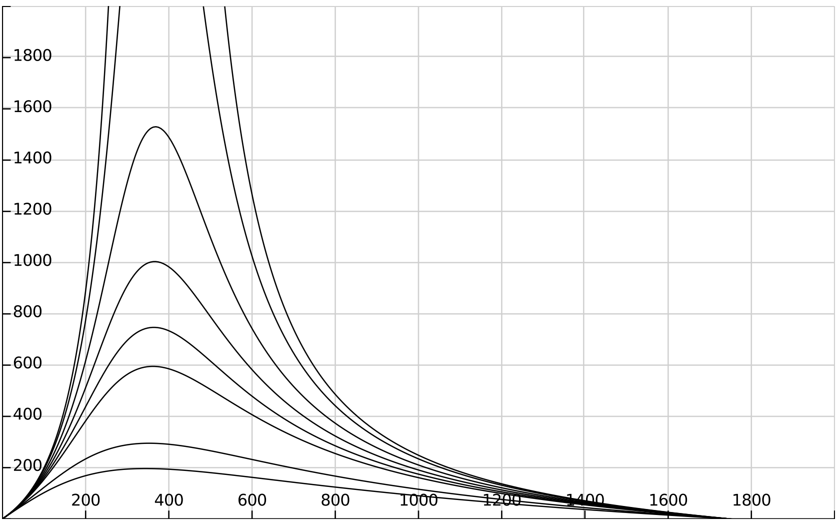

Fig. 2 – The positive growth constant (52) (on vertical axis) *vs.* the wavenumber (on horizontal axis) for the values: $\epsilon = 0.006, \gamma' = 0.1$. W=0 (lower curve), 0.001, 0.002, 0.0022, 0.0024, 0.0026, 0.0028, 0.0029 (upper curve).

In Figure 2, we plot the *positive* dimensionless growth constant (52) in terms of the dimensionless wave number n', for the same values $\epsilon = 0.006$ and $\gamma' = 0.1$. We can see the destabilizing effect of the non-Newtonian constitutive relations (2). The maximum value of σ'_{SW} is increasing in terms of W, from $W = 0$ (the lower curve) to 0.0028 (the upper curve). As we proved in the last *Remark*, the value $W \approx 0.002976$ is giving the blow-up of the growth constant σ'_{SW} .

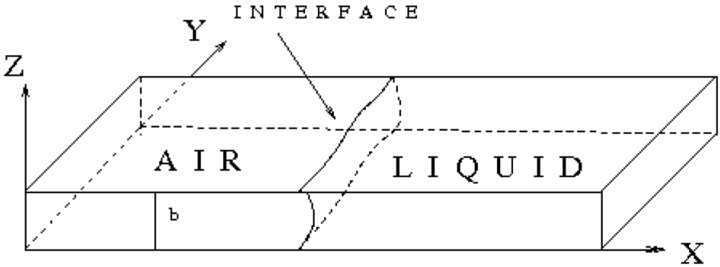

Fig. 3 – Air displacing a liquid in a Hele-Shaw cell.

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