Dedicated to the 150th anniversary of the Romanian Academy

ON JORDAN ALGEBRAS AND UNIFICATION THEORIES

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Communicated by Vasile Brînzănescu

The current paper emerged after the 12th International Workshop on Differential Geometry and Its Applications, hosted by the Petroleum Gas University from Ploiesti, between September 23rd and September 26th, 2015. Jordan algebras and Lie algebras are the main non-associative structures. In the last years, several attempts to unify non-associative algebras and associative algebras led to UJLA structures. Another algebraic structure which we will use in order to unify non-associative algebras and associative algebras is the Yang-Baxter equation.

AMS 2010 Subject Classification: 16T10, 16T25, 17B01, 17B60, 17C05, 17C50, 17D99, 51C05, 65D20, 65L09, 68R99.

Key words: Jordan algebras, Lie algebras, associative algebras, Yang-Baxter equations.

1. INTRODUCTION

At the end of the 1950s, Professor Vranceanu began a systematic study of the spaces with constant affine connection. In 1966, he proposed the study of such spaces associated to finite-dimensional real Jordan algebras, study which was afterwards developed by Professors Iordanescu, Popovici and Turtoi (see [13]).

A systematic study of projective planes over large classes of associative rings was initiated by Dan Barbilian (see [13]). He proved that the rings which can be the underlying rings for projective geometries are (with a few exceptions) rings with a unit element in which any one-sided inverse is a two-sided inverse. Today, his name is listed in the AMS Classification.

The main non-associative structures are Lie algebras and Jordan algebras. Several Jordan structures have applications in quantum group theory and exceptional Jordan algebras play an important role in recent fundamental physical theories namely, in the theory of super-strings (see [11,13]). In the last years, attempts to unify non-associative structures and associative structures have led to interesting results (see, for example, [15–17, 30, 36]).

The discovery of the Yang-Baxter equation [40] in theoretical physics and statistical mechanics (see [2, 3, 43]) has led to many applications in these fields and in quantum groups, quantum computing, knot theory, braided categories, analysis of integrable systems, quantum mechanics, etc. (see [29]). The interest in this equation is growing, as new properties are found, and its solutions are not classified yet (see also [10, 37]). One of the newly discovered properties of this equation is its unifying feature for dual structures (see, for example, [21,22,33]).

The organization of our paper is the following. In Section 2, we review some properties of unifying structures and we present new properties and open problems. We define (weak) unifying structures and compatible structures. Compatible weak unifying structures give rise to another weak unifying structure, and this construction generalizes known results. In the next section, we discuss on the Yang-Baxter equation, and we will refer to the construction quantum gates and link invariants from solutions of it. The set-theoretical Yang-Baxter equation is presented. Section 4 deals with transcendental numbers and some applications. A conclusions section refers to other communications given at International Workshops on Differential Geometry and Its Applications, and it explains the connections between the sections of our paper.

2. NON-ASSOCIATIVE ALGEBRAS AND THEIR UNIFICATIONS

The main non-associative structures are Lie algebras and Jordan algebras. Arguable less known, Jordan algebras have applications in physics, differential geometry, ring geometries, quantum groups, analysis, biology, etc. (see [8, 11, 13, 14, 19, 20]).

We will define some structures which unify Jordan algebras, Lie algebras and (non-unital) associative algebras. The results for UJLA structures could be "decoded" in results for Jordan algebras, Lie algebras or (non-unital) associative algebras.

In this paper, tensor products are defined over the field k.

Definition 2.1. We define the unifying structure $(V,\ \eta)$, also called a "UJLA structure", in the following way. Let V be a vector space, and $\eta : V \otimes V \to V,\quad \eta(a \otimes b) = ab$, be a linear map, which satisfies the following axioms $\forall a, b, c \in V$:

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab), \tag{1}$$

$$(a^2b)a = a^2(ba), \tag{2}$$

$$(ab)a^2 = a(ba^2), \tag{3}$$

$$(ba^2)a = (ba)a^2, \tag{4}$$

$$a^2(ab) = a(a^2b). \tag{5}$$

If just the identity (1) holds, we call the structure $(V,\ \eta)$ a "weak unifying structure".

Remark 2.2. If $(A,\ \theta)$, where $\theta : A \otimes A \to A,\quad \theta(a \otimes b) = ab$, is a (non-unital) associative algebra, then we define a UJLA structure $(A,\ \theta')$, where $\theta'(a \otimes b) = \alpha ab\ +\ \beta ba$, for some $\alpha,\ \beta \in k$. For $\alpha = \beta = \frac{1}{2}$, $(A,\ \theta')$ is a Jordan algebra, and for $\alpha = 1 = -\beta$, $(A,\ \theta')$ is a Lie algebra.

THEOREM 2.3. *For V a k-space, $f : V \to k$ a k-map, $\alpha,\ \beta \in k$, and $e \in V$ such that $f(e) = 1$, the following structures can be associated.*

(i) *(V, M), a non-unital associative algebra, where $M(v \otimes w) = f(v)w$;*

(ii) *(V, M, e), a unital associative algebra, where $M(v \otimes w) = f(v)w + vf(w) - f(v)f(w)e$;*

(iii) *$(V,\ [,])$, a Lie algebra, where $[v,\ w] = f(v)w - vf(w)$;*

(iv) *$(V,\ \mu)$, a Jordan algebra, where $\mu(v \otimes w) = f(v)w + vf(w)$;*

(v) *$(V, M_{\alpha,\beta})$, a UJLA structure, where $M_{\alpha,\beta}(v \otimes w) = \alpha f(v)w + \beta vf(w)$.*

Proof. Let us observe that (iii) and (iv) follow from (i).

We now prove (ii), which is more general than (i). We denote by $x \cdot y = M(x \otimes y)$, we observe that " $\cdot$ " is commutative, and e is the unity of our algebra:

$x \cdot e = f(x)e + f(e)x - f(e)f(x)e = x = e \cdot x$.

We prove the associativity of " $\cdot$ ":

$$(x \cdot y) \cdot z = f(x)f(y)z + f(x)f(z)y - f(x)f(y)f(z)e + f(x)f(y)z + xf(y)f(z) \\ - f(x)f(y)f(z)e - f(x)f(y)z;$$

$$x \cdot (y \cdot z) = f(x)f(y)z + f(x)yf(z) - f(x)f(y)f(z)e + xf(y)f(z) - f(x)f(y)f(z)e.$$

It follows that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

We leave the proof of (v) as an exercise for the reader. $\square$

Remark 2.4. In the above theorem, $(V, M_{\alpha,\beta})$, where $M_{\alpha,\beta}(v \otimes w) = \alpha f(v)w + \beta vf(w)$, is also an alternative algebra (*i.e.* $x(yx) = (xy)x)$). Alternative algebras are not necessarily UJLA structures, and, obviously, UJLA structures might not be alternative algebras.

THEOREM 2.5 (Nichita [36]). *Let (V, η) be a UJLA structure, and $\alpha,\ \beta \in k$. Then, (V, η'), $\eta'(a \otimes b) = \alpha ab + \beta ba$ is a UJLA structure.*

Remark 2.6. Let (V, η) be a UJLA structure. Then, $\delta_a : V \to V$, $\delta_a(x) = xa - ax$, is a derivation for the following UJLA structure: (V, η'), $\eta'(a \otimes b) = ab - ba$.

OPEN PROBLEM. A UJLA structure is power associative. We know that a UJLA structure is power associative for dimensions less or equal then 5.

Remark 2.7. The classification of UJLA structures is also an open problem.

THEOREM 2.8. *Let (V, η) be a UJLA structure. Then, (V, η'), $\eta'(a \otimes b) = ab - ba$ is a Lie algebra.*

Proof. The proof follows from formula (1). □

THEOREM 2.9. *Let (V, η) be a UJLA structure. Then, (V, η'), $\eta'(a \otimes b) = \frac{1}{2}(ab + ba)$ is a Jordan algebra.*

Proof. The proof follows from formulas (2), (3), (4) and (5). □

Definition 2.10. Two weak unifying structures defined over the same vector space, $(V, \cdot)$ and $(V, *)$, are called "compatible" if

$$\{a \cdot b, c\} + \{b \cdot c, a\} + \{c \cdot a, b\} = a * [b, c] + b * [c, a] + c * [a, b], \tag{6}$$

where $[,]$ and $\{,\}$ are the brackets associated to the two weak unifying structures: $[a, b] = a \cdot b - b \cdot a$, $\{a, b\} = a * b - b * a$.

Remark 2.11. A Poisson structure (see [30]) gives rise to compatible weak unifying structures if and only if the algebra structure is commutative.

Remark 2.12. Let (V, η) be a weak unifying structure. Then, (V, η) and (V, η'), where $\eta'(a \otimes b) = ab - ba$, are compatible.

THEOREM 2.13. *If two weak unifying structures are compatible, then we can define a new weak unifying structure on the same vector space: $a \circ b = ab + a * b$.*

Proof. The proof follows from formula (1). □

Remark 2.14. A Lie-Jordan structure (see [9, 20]) gives rise to compatible weak unifying structures in a natural way. The above theorem can be seen as a generalization for one of the main properties of Lie-Jordan structures.

3. YANG-BAXTER EQUATIONS

The Yang-Baxter equation first appeared in theoretical physics, and it has applications in many areas of physics, informatics and mathematics. The classification of its solutions is an open problem (see [7, 17, 29, 31, 32, 35–38]).

For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map, $\tau(v \otimes w) = w \otimes v$, and by $I : V \to V$ the identity map; for $R : V \otimes V \to V \otimes V$ a k-linear map, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, $R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 3.1. A *Yang-Baxter operator* is an invertible k-linear map $R : V \otimes V \to V \otimes V$, which satisfies the braid condition (the *Yang-Baxter equation*):

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{7}$$

Both $R \circ \tau$ and $\tau \circ R$ satisfy then the *quantum Yang-Baxter equation* (QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{8}$$

For A be a (unitary) associative k-algebra, and $\alpha, \beta, \gamma \in k$, the authors of [7] defined the k-linear map $R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A$,

$$a \otimes b \mapsto \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b \tag{9}$$

which is a Yang-Baxter operator in some special cases.

Remark 3.2. An interesting property of (9), appears in knot theory, where the link invariant associated to $R^A_{\alpha,\beta,\gamma}$ is the Alexander polynomial (cf. [25,42]). The reciprocal approach was also studied by other authors (see [6]).

Remark 3.3. In dimension two, $R^A_{\alpha,\beta,\alpha}$ leads to a universal quantum gate (see [17]), which, according to [18], is related to the CNOT gate. It is an open problem to relate the operator (9) to the abstract controlled-not (obtained by [44]).

For $(L, [,])$ a Lie super-algebra over k, $z \in Z(L) = \{z \in L : [z, x] = 0 \;\; \forall\, x \in L\}$, $|z| = 0$ and $\alpha \in k$, the authors of the papers [23] and [38] defined the following Yang-Baxter operator: $\phi^L_\alpha : L \otimes L \longrightarrow L \otimes L$,

$$x \otimes y \mapsto \alpha[x, y] \otimes z + (-1)^{|x||y|} y \otimes x \, . \tag{10}$$

Formulas (9) and (10) could be seen as a unification for the associative algebras and Lie algebras.

Remark 3.4. For a connection of the QYBE with Jordan triple systems see S. Okubo (University of Rochester Report UR-1334, 1993) and also [13], pp. 114–115.

In 1994, S. Okubo [39] has obtained some solutions of a triple product equation.

Definition 3.5. For an arbitrary set X, the map $S : X \times X \to X \times X$, is a solution for the *set-theoretical Yang-Baxter equation* if

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}. \tag{11}$$

(Here $S^{12} = S \times I$, $S^{23} = I \times S$.)

The set-theoretical Yang-Baxter equation represents some kind of compatibility condition in logic: for three logical sentences p, q and r, let us suppose that if all of them are true, then the conclusion A could be drawn, and if p, q, r are all false then the conclusion C can be drawn. Modeling this situation by the map $(p, q) \mapsto (p' = p \vee q,\ q' = p \wedge q)$, helps us to comprise our analysis. The Yang-Baxter equation guarantees that the order in which we start this analysis is not important. Thus, the map $(p, q) \mapsto (p \vee q,\ p \wedge q)$ is a solution for (11).

The sorting of numbers (see, for example, [17]) is an important problem in informatics, and the set-theoretical Yang-Baxter equation is related to it. Ordering three numbers is related to a common solution for the set-theoretical QYBE and the set-theoretical braid condition: $R(a, b) = (min(a, b), max(a, b))$. In a similar manner, one can find the greatest common divisor and the least common multiple of three numbers, using another common solution for the set-theoretical QYBE and the set-theoretical braid condition: $R'(a, b) = (gcd(a, b), lcm(a, b))$. Since R and R' can be extended to braidings in certain monoidal categories, we obtain interpretations for the cases when we deal with more numbers.

THEOREM 3.6 (Nichita [36]). *The following is a two-parameter family of solutions for the set-theoretical QYBE:*
$S : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \quad (x, y) \mapsto (x^{\beta} y^{1-\alpha\beta}, y^{\alpha}) \quad \forall \alpha, \beta \in \mathbb{N}^*.$

THEOREM 3.7 (Nichita [36]). *The following is a two-parameter family of solutions for the set-theoretical QYBE:*
$R : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}, \quad (z, w) \mapsto (\ \beta z + (1 - \alpha\beta) w, \alpha w) \quad \forall \alpha, \beta \in \mathbb{C}.$

THEOREM 3.8. *The following is a solution for the set-theoretical QYBE:*
$S' : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^* \times \mathbb{R}^*, \quad (x, y) \mapsto (x^2, \frac{y}{x}).$

Proof. The direct proof is the shortest. Notice the relationship of S' with S from Theorem 3.6. $\square$

We consider a three dimensional Euclidean space, and a point $P(a, b, c)$ of it.

The symmetry of the point $P(a, b, c)$ about the origin, $S_O(a, b, c) = (-a, -b, -c)$, the symmetries of the point $P(a, b, c)$ about the axes OX, OY, OZ, $S_{OX}(a, b, c) = (a, -b, -c)$, $S_{OY}(a, b, c) = (-a, b, -c)$ and $S_{OZ}(a, b, c) = (-a, -b, c)$, the symmetries of the point $P(a, b, c)$ about the planes XOY, XOZ, YOZ, $S_{XOY}(a, b, c) = (a, b, -c)$, $S_{XOZ}(a, b, c) = (a, -b, c)$ and $S_{YOZ}(a, b, c) = (-a, b, c)$, with the identity map form a group: $\{I, S_{OX}, S_{OY}, S_{OZ}, S_{XOY}, S_{XOZ}, S_{YOZ}, S_O\}$, which is isomorphic to the group of the following matrices

$$(12)\qquad \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

The following instances of the QYBE hold for the above symmetries:
$S_{XOY} \circ S_{XOZ} \circ S_{YOZ} = S_{YOZ} \circ S_{XOZ} \circ S_{XOY}$,
$S_{OX} \circ S_{OY} \circ S_{OZ} = S_{OZ} \circ S_{OY} \circ S_{OX}$.

Moreover, Theorem 3.7 and the gluing procedure from [4] are related to the above solutions of the set-theoretical QYBE.

4. TRANSCENDENTAL NUMBERS AND APPLICATIONS

The well-known identity containing e and π (see [24, 34]), $e^{i\pi} + 1 = 0$, can be interpreted using the matrix J (for $\alpha \in \mathbb{R}^*$), where

$$(13)\qquad J = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\alpha} i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ \alpha i & 0 & 0 & 0 \end{pmatrix},$$

as follows: $e^{\pi\ J} + I_4 = 0_4$ (here $J,\ I_4, 0_4 \in \mathcal{M}_4(\mathbb{C})$).

Now, $R(x) = e^{xJ} : V^{\otimes 2} \to V^{\otimes 2}$ satisfies the colored Yang-Baxter equation:

$$(14)\qquad R^{12}(x) \circ R^{23}(x+y) \circ R^{12}(y) \ = \ R^{23}(y) \circ R^{12}(x+y) \circ R^{23}(x)\ .$$

Also, $R(x)$ is a solution for the following differential matrix equation:

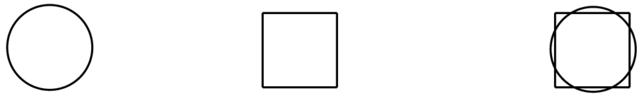
$$(15)\qquad Y' = JY\ .$$

The combination of the properties (14) and (15) leads to computations of Hamiltonians of many body systems in physics.

Remark 4.1. The above solution to the equation (14) can be used to find solutions for Okubo's triple product equation (see Remark 3.4).

Other recent problems with e and π are the following:
$\mid e^{1-z} + e^{\bar{z}} \mid > \pi \ \forall z \in \mathbb{C}$, $\qquad x^2 + e > \pi x \quad \forall x \in \mathbb{R}$.
The last inequality holds because $\Delta = \pi^2 - 4e = -1,003522913... < 0$, and we conjecture that $4e - \pi^2 = 1,003522913...$ is a transcendental number.

The geometrical interpretation of the formula $\pi^2 < 4e$ could be stated as: "The length of the circle with diameter π is almost equal (and less) to the perimeter of a square with edges of length e".



The area of the above circle is greater than the area of the square, because $\pi^3 > 4e^2$.

OPEN PROBLEMS. For an arbitrary convex closed curve, we consider the largest diameter (D). (It can be found by considering the center of mass of a body which corresponds to the domain inside the given curve.) In a similar manner, one can define the smallest diameter (d). Alternatively, d and D can be defined using the concept of "cut locus".

i) If L is the length of the given curve and the domain inside the given curve is a convex set, then we conjecture that:

$$\frac{L}{D} \leq \pi \leq \frac{L}{d} \quad .$$

(ii) If A is the area inside the given curve, the equation

$$x^2 - \frac{L}{2}x + A = 0 \tag{16}$$

and its implications are not completely understood. For example, if the given curve is an ellipse, solving this equation in terms of the semi-axes of the ellipse is an unsolved problem.

(iii) The following system of equations is some kind of an inverse of (16). We consider two real functions with second order derivatives, such that

$f : [0, D] \to \mathbb{R},\ f \geq 0,\ f'' \leq 0,\quad g : [0, D] \to \mathbb{R},\ g \leq 0,\ g'' \geq 0,$
$\int_0^D \sqrt{1 + (f'(x))^2} + \sqrt{1 + (g(x)')^2}\ dx\ = L\ ,\quad \int_0^D f(x) - g(x)\ dx\ = A\ .$

5. FURTHER COMMENTS AND CONCLUSIONS

At the 12th International Workshop on Differential Geometry and Its Applications, hosted by the Petroleum Gas University from Ploieşti (23–26 September, 2015) we commemorated 115 years from the birth of Prof. Vranceanu and 120 from the birth of Prof. Barbilian.

This article is a survey paper based on a talk given at that workshop. At that time, we discussed about the applications of Jordan algebras in Differential Geometry (see [19]) and we presented some parts of some previous articles [27, 28] and talks. The current paper also contains some new results and open problems.

In the last years, several attempts to unify non-associative structures led to interesting results. The UJLA structures are not the only structures which realize such a unification. The formulas (9) and (10) lead to the unification of associative algebras and Lie (super)algebras in the framework of Yang-Baxter structures (see [12,30]). For the invertible elements of a Jordan algebra, one can associate a symmetric space, and, after that, a Yang-Baxter operator. Thus, the Yang-Baxter equation can be thought as a unifying equation. From some solutions of the Yang-Baxter equation, one could construct abstract universal gates from quantum computing or knot invariants.

Professor Osman Gursoy's talk at the 12th International Workshop on Differential Geometry and Its Applications was related to our open problems about convex closed curves (presented in Section 4).

It is worth mentioning here another paper on Jordan algebras [26], which was based on a talk at the 11th International Workshop on Differential Geometry and Its Applications. The following question arises in regard to that paper: "What kind of cones could be associated to UJLA structures, if we think the UJLA structures as generalizations for Jordan algebras?".

In his talk at the 12th International Workshop on Differential Geometry and Its Applications, Florin Caragiu explained that there exists a special mathematical discourse, called "proofs without words", which uses pictures or diagrams in order to boost the intuition of the reader (see [5]). We thus have pictorial/diagrammatic style of mathematical language which is much appreciated by both educators and researchers in mathematics. Examples of this type of language appear in knot theory (where the Yang-Baxter equation plays an important role), category theory, (differential) geometry (see for example [1]), differential topology, quantum field theory, etc.

Related to the equation (14) there is a long standing open problem. The following system of equations, obtained in [37] and extended in [10], is not completely classified:

$$\begin{aligned}&(\beta(v,w)-\gamma(v,w))(\alpha(u,v)\beta(u,w)-\alpha(u,w)\beta(u,v))\\ &\qquad+(\alpha(u,v)-\gamma(u,v))(\alpha(v,w)\beta(u,w)-\alpha(u,w)\beta(v,w))=0\end{aligned} \tag{17}$$

$$\begin{aligned}&\beta(v,w)(\beta(u,v)-\gamma(u,v))(\alpha(u,w)-\gamma(u,w))\\ &\qquad+(\alpha(v,w)-\gamma(v,w))(\beta(u,w)\gamma(u,v)-\beta(u,v)\gamma(u,w))=0\end{aligned} \tag{18}$$

$$\begin{aligned}&\alpha(u,v)\beta(v,w)(\alpha(u,w)-\gamma(u,w))+\alpha(v,w)\gamma(u,w)(\gamma(u,v)-\alpha(u,v))\\ &\qquad+\gamma(v,w)(\alpha(u,v)\gamma(u,w)-\alpha(u,w)\gamma(u,v))=0\end{aligned} \tag{19}$$

$$\begin{aligned}&\alpha(u,v)\beta(v,w)(\beta(u,w)-\gamma(u,w))+\beta(v,w)\gamma(u,w)(\gamma(u,v)-\beta(u,v))\\ &\qquad+\gamma(v,w)(\beta(u,v)\gamma(u,w)-\beta(u,w)\gamma(u,v))=0\end{aligned} \tag{20}$$

$$\alpha(u,v)(\alpha(v,w)-\gamma(v,w))(\beta(u,w)-\gamma(u,w))$$

$$+(\beta(u,v)-\gamma(u,v))(\alpha(u,w)\gamma(v,w)-\alpha(v,w)\gamma(u,w))=0 \tag{21}$$

We think that some techniques from [41] might help in solving the above system.

Acknowledgments. The author would like to thank the referee. Also, special thanks are addressed to the Simion Stoilow Institute of Mathematics of the Romanian Academy and to the Petroleum Gas University from Ploieşti.

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Received 3 May 2016

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