ON THE TOPOLOGY OF THE MILNOR FIBRATION
OF A HYPERPLANE ARRANGEMENT

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This note is mostly an expository survey, centered on the topology of complements of hyperplane arrangements, their Milnor fibrations, and their boundary structures. An important tool in this study is provided by the degree 1 resonance and characteristic varieties of the complement, and their tight relationship with orbifold fibrations and multinets on the underlying matroid. In favorable situations, this approach leads to a combinatorial formula for the first Betti number of the Milnor fiber and the algebraic monodromy. We also produce a pair of arrangements for which the respective Milnor fibers have the same Betti numbers, yet are not homotopy equivalent: the difference is picked up by isolated torsion points in the higher-depth characteristic varieties.

1. INTRODUCTION

1.1. The Milnor fibration

A construction due to J. Milnor [41] associates to each homogeneous polynomial $Q \in \mathbb{C}[z_0, \ldots, z_d]$ a fiber bundle, with base space $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$, total space the complement in $\mathbb{C}^{d+1}$ to the hypersurface V given by the vanishing of Q, and projection map $Q \colon \mathbb{C}^{d+1} \backslash V \to \mathbb{C}^*$.

The Milnor fiber $F = Q^{-1}(1)$ is a Stein manifold, and thus has the homotopy type of a finite, d-dimensional CW-complex. The monodromy of the fibration, $h \colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$, where n is the degree of Q. If the polynomial Q has an isolated singularity at the origin, then F is homotopy equivalent to a bouquet of d-spheres, whose number can be determined by algebraic means. In general, though, it is a rather hard problem to compute the homology groups of the Milnor fiber, even in the case when Q completely factors into distinct linear forms.

This situation is best described by a hyperplane arrangement, that is, a finite collection of codimension-1 linear subspaces in $\mathbb{C}^{d+1}$. Choosing a linear form f_H with kernel H for each hyperplane H in the arrangement $\mathscr{A}$, we obtain a homogeneous polynomial, $Q = \prod_{H \in \mathscr{A}} f_H$, which in turn defines the Milnor fibration of the arrangement, and the Milnor fiber, $F = F(\mathscr{A})$. A central question in the subject is to determine whether $\Delta_{\mathscr{A}}(t)$, the characteristic polynomial of the algebraic monodromy $h_* \colon H_1(F, \mathbb{C}) \to H_1(F, \mathbb{C})$, is determined by the intersection lattice of the arrangement, $L(\mathscr{A})$. We present here some recent progress on this and other related questions, mostly based on [52] and on joint work with G. Denham [14] and S. Papadima [47].

1.2. Complement and jump loci

Let U be the complement of the complexified arrangement, $\bar{\mathscr{A}} = \mathbb{P}(\mathscr{A})$. It turns out that the Milnor fiber F is a regular, cyclic n-fold cover of U, where $n = |\mathscr{A}|$. The classifying homomorphism for this cover, $\delta \colon \pi_1(U) \to \mathbb{Z}_n$, takes each meridian loop around a hyperplane to 1. Embedding $\mathbb{Z}_n$ into $\mathbb{C}^*$ by sending 1 to a primitive n-th root of unity, we may view δ as a character on $\pi_1(U)$, see [10,52]. The relative position of this character with respect to the characteristic varieties of U determines the Betti numbers of F, as well as the characteristic polynomial of the algebraic monodromy.

Since U is a smooth, quasi-projective variety, its characteristic varieties are finite unions of torsion-translates of algebraic subtori of the character group $\operatorname{Hom}(\pi_1(U), \mathbb{C}^*)$, cf. [1, 2, 7]. Since U is also a formal space, its resonance varieties (defined in terms of the Orlik–Solomon algebra of $\mathscr{A}$) coincide with the tangent cone at the origin to the corresponding characteristic varieties, cf. [11,21,22,34]. As shown by Falk and Yuzvinsky [28], the degree 1 resonance varieties may be described solely in terms of multinets on sub-arrangements of $\mathscr{A}$. In general, though, the degree 1 characteristic varieties of an arrangement may contain components which do not pass through the origin [14,50], and it is still an open problem whether such components are combinatorially determined.

Under simple combinatorial conditions, it is shown in [47] that the multiplicities of the factors of $\Delta_{\mathscr{A}}(t)$ corresponding to certain eigenvalues of order a power of a prime p are equal to the 'Aomoto–Betti numbers' $\beta_p(\mathscr{A})$, which in turn can be extracted from the intersection $L(\mathscr{A})$ by considering the resonance varieties of U over a field of characteristic p. When $\bar{\mathscr{A}}$ is an arrangement of projective lines with only double and triple points, this approach leads to a combinatorial formula for the algebraic monodromy.

1.3. **Boundary structures**

Both the projectivized complement U and the Milnor fiber F are boundaryless, non-compact manifolds. Removing a regular neighborhood of the arrangement yields a compact manifold with boundary, $\overline{U}$, onto which U deformretracts. Likewise, intersecting the Milnor fiber with a ball in $\mathbb{C}^{d+1}$ centered at the origin yields a compact manifold with boundary, $\overline{F}$, onto which F deformretracts.

We focus on the case $d = 2$, when both the boundary manifold of the arrangement, $\partial\overline{U}$, and the boundary of the Milnor fiber, $\partial\overline{F}$, are closed, orientable, 3-dimensional graph manifolds. Once again, there is a regular, cyclic n-fold cover $\partial\overline{F} \to \partial\overline{U}$, whose classifying map can be described in concrete terms. Various topological invariants of these manifolds, including the cohomology ring and the depth 1 characteristic variety of $\partial\overline{U}$ [12, 13], as well as the Betti numbers of $\partial\overline{F}$ [42], can be computed from the combinatorics of $\mathscr{A}$.

In [25], Falk produced a pair arrangements, $\mathscr{A}$ and $\mathscr{A}'$, for which the intersection lattices are non-isomorphic, but the projective complements, U and U', are homotopy equivalent. Nevertheless, the boundary manifolds are not homotopy equivalent, [13, 30], and thus the complements are not homeomorphic. We show here that the respective Milnor fibers, F and F', as well as their boundaries, $\partial\overline{F}$ and $\partial\overline{F}'$, have the same Betti numbers, but that $F \not\simeq F'$. The difference between the two Milnor fibers is detected by the depth 2 characteristic varieties: $\mathscr{V}_2(F) = \{\mathbf{1}\}$, whereas $\mathscr{V}_2(F') \cong \mathbb{Z}_3$.

2. COMPLEMENT, BOUNDARY MANIFOLD AND MILNOR FIBRATION

2.1. **The complement of a hyperplane arrangement**

An *arrangement of hyperplanes* is a finite set $\mathscr{A}$ of codimension-1 linear subspaces in a finite-dimensional, complex vector space $\mathbb{C}^{d+1}$. The combinatorics of the arrangement is encoded in its *intersection lattice*, $L(\mathscr{A})$, that is, the poset of all intersections of hyperplanes in $\mathscr{A}$ (also known as flats), ordered by reverse inclusion, and ranked by codimension. Given a flat X, we will denote by $\mathscr{A}_X$ the sub-arrangement $\{H \in \mathscr{A} \mid H \supset X\}$.

Without much loss of generality, we will assume throughout that the arrangement is *central*, that is, all the hyperplanes pass through the origin. For each hyperplane $H \in \mathscr{A}$, let $f_H \colon \mathbb{C}^{d+1} \to \mathbb{C}$ be a linear form with kernel H. The product

$$(1) \qquad Q(\mathscr{A}) = \prod_{H \in \mathscr{A}} f_H,$$

then, is a defining polynomial for the arrangement, unique up to a (non-zero) constant factor. Notice that $Q = Q(\mathscr{A})$ is a homogeneous polynomial of degree equal to $|\mathscr{A}|$, the cardinality of the set $\mathscr{A}$. The complement of the arrangement,

$$(2) \qquad M(\mathscr{A}) = \mathbb{C}^{d+1} \backslash \bigcup_{H \in \mathscr{A}} H,$$

is a connected, smooth complex quasi-projective variety. Moreover, $M = M(\mathscr{A})$ is a Stein manifold, and thus it has the homotopy type of a CW-complex of dimension at most $d + 1$. In fact, M splits off the linear subspace $\bigcap_{H \in \mathscr{A}} H$. Thus, we may safely assume that the arrangement $\mathscr{A}$ is *essential*, i.e., that this subspace is just 0.

The group $\mathbb{C}^*$ acts freely on $\mathbb{C}^{d+1} \backslash \{0\}$ via $\zeta \cdot (z_0, \dots, z_d) = (\zeta z_0, \dots, \zeta z_d)$. The orbit space is the complex projective space of dimension d, while the orbit map, $\pi \colon \mathbb{C}^{d+1} \backslash \{0\} \to \mathbb{CP}^d$, $z \mapsto [z]$, is the Hopf fibration. The set $\mathbb{P}(\mathscr{A}) = \{\pi(H) \colon H \in \mathscr{A}\}$ is an arrangement of codimension 1 projective subspaces in $\mathbb{CP}^d$. Its complement, $U = U(\mathscr{A})$, coincides with the quotient $\mathbb{P}(M) = M/\mathbb{C}^*$.

The Hopf map restricts to a bundle map, $\pi \colon M \to U$, with fiber $\mathbb{C}^*$. Fixing a hyperplane $H \in \mathscr{A}$, we see that π is also the restriction to M of the bundle map $\mathbb{C}^{d+1} \backslash H \to \mathbb{CP}^d \backslash \pi(H) \cong \mathbb{C}^d$. This latter bundle is trivial, and so we have a diffeomorphism $M \cong U \times \mathbb{C}^*$.

Fix now an order $H_1, \dots, H_n$ on the hyperplanes of $\mathscr{A}$, and denote the corresponding linear forms by $f_1, \dots, f_n$. We may then define a linear map $\iota \colon \mathbb{C}^{d+1} \to \mathbb{C}^n$ by $\iota(z) = (f_1(z), \dots, f_n(z))$. Since we assume $\mathscr{A}$ is essential, the map ι is injective. Its restriction to the complement yields an embedding $\iota \colon M \to (\mathbb{C}^*)^n$. As shown in [14, 52], this embedding is a classifying map for the universal abelian cover $M^{\mathrm{ab}} \to M$.

Clearly, the map $\iota \colon M \to (\mathbb{C}^*)^n$ is equivariant with respect to the diagonal action of $\mathbb{C}^*$ on both source and target. Thus, ι descends to a map $\bar{\iota} \colon M/\mathbb{C}^* \to (\mathbb{C}^*)^n/\mathbb{C}^*$. Since $\mathscr{A}$ is essential, this map defines an embedding $\bar{\iota} \colon U \hookrightarrow (\mathbb{C}^*)^{n-1}$, which is a classifying map for the universal abelian cover $U^{\mathrm{ab}} \to U$.

2.2. **The boundary manifold**

Let V be the union of the hyperplanes in $\mathscr{A}$, and let $W = \mathbb{P}(V)$. A regular neighborhood of the algebraic hypersurface $W \subset \mathbb{CP}^d$ may be constructed as follows. Let $\phi \colon \mathbb{CP}^d \to \mathbb{R}$ be the smooth function defined by $\phi([z]) = |Q(z)|^2 / \|z\|^{2n}$, where Q is a defining polynomial for the arrangement, and $n = |\mathscr{A}|$. Then, for sufficiently small $\delta > 0$, the space $\nu(W) = \phi^{-1}([0, \delta])$ is a closed, regular neighborhood of W. Alternatively, one may triangulate $\mathbb{CP}^d$

with W as a subcomplex, and take $\nu(W)$ to be the closed star of W in the second barycentric subdivision.

As shown by Durfee [24], these constructions yield isotopic neighborhoods, independent of the choices made. Plainly, $\nu(W)$ is a compact, orientable, smooth manifold with boundary, of dimension $2d$; moreover, $\nu(W)$ deform-retracts onto W. The *exterior* of the projectivized arrangement, denoted by $\overline{U}$, is the complement in $\mathbb{CP}^d$ of the open regular neighborhood $\text{int}(\nu(W))$. It is readily seen that $\overline{U}$ is a compact, connected, orientable, smooth $2d$-manifold with boundary, and that U deform-retracts onto $\overline{U}$.

The *boundary manifold* of the arrangement $\mathscr{A}$ is the common boundary $\partial\overline{U} = \partial\nu(W)$ of the exterior $\overline{U}$ and the regular neighborhood of W defined above. Clearly, $\partial\overline{U}$ is a compact, orientable, smooth manifold of dimension $2d - 1$. The inclusion map $\partial\overline{U} \to \overline{U}$ is a $(d-1)$-equivalence, see Dimca [16, Prop. 2.31]; in particular, $\pi_i(\partial\overline{U}) \cong \pi_i(U)$ for $i < d - 1$. Thus, $\partial\overline{U}$ is connected if $d \geqslant 2$, and $\pi_1(\partial\overline{U}) = \pi_1(U)$ if $d \geqslant 3$. For more information on the boundary manifolds of arrangements, we refer to [12, 13, 29, 32, 52].

2.3. The Milnor fibration

Once again, let $\mathscr{A}$ be a central arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. The polynomial map $Q = Q(\mathscr{A})\colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a map $Q\colon M(\mathscr{A}) \to \mathbb{C}^*$, where $M = M(\mathscr{A})$ is the complement of the arrangement. As shown by J. Milnor [41] in a more general context, this map is the projection map of a smooth, locally trivial bundle, known as the *Milnor fibration* of the arrangement. The typical fiber of this fibration,

$$(3) \qquad\qquad F(\mathscr{A}) = Q^{-1}(1)$$

is called the *Milnor fiber* of the arrangement. It is readily verified that $F = F(\mathscr{A})$ is a smooth, connected, orientable manifold of dimension $2d$. Moreover, F is a Stein domain of complex dimension d, and thus has the homotopy type of a finite CW-complex of dimension d.

For each $\theta \in [0, 1]$, let us denote by F_θ the fiber over the point $e^{2\pi i\theta} \in \mathbb{C}^*$, so that $F_0 = F_1 = F$. For each point $z \in M$, the path $\gamma_\theta\colon [0, 1] \to \mathbb{C}^*$, $t \mapsto e^{2\pi i t\theta}$ lifts to the path $\tilde{\gamma}_{\theta,z}\colon [0, 1] \to M$ given by $t \mapsto e^{2\pi i t\theta/n}z$. Clearly, $Q(\tilde{\gamma}_{\theta,z}(1)) = e^{2\pi i\theta}Q(z)$. Therefore, if $z \in F_0$, then $\tilde{\gamma}_{\theta,z}(1) \in F_\theta$; moreover, $\tilde{\gamma}_{\theta,z}(0) = z$.

By definition, the *monodromy* of the Milnor fibration is the diffeomorphism $h\colon F_0 \to F_1$ given by $h(z) = \tilde{\gamma}_{1,z}(1)$. In view of the preceding discussion, this diffeomorphism can be written as $h\colon F \to F$, $z \mapsto e^{2\pi i/n}z$. Clearly, h has
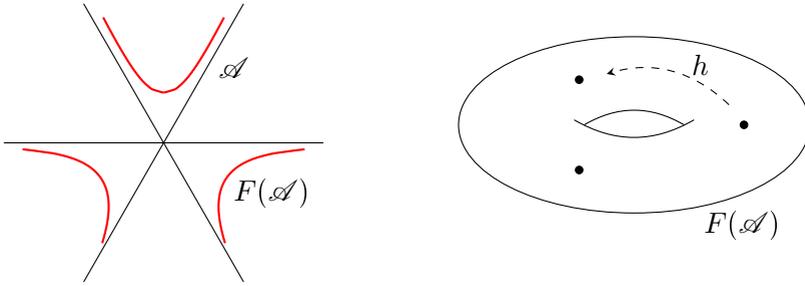
Fig. 1. Milnor fiber and monodromy for a pencil of 3 lines.

order n, and the complement M is homotopy equivalent to the mapping torus of h.

By homogeneity of the polynomial Q, we have that $Q(wz) = w^n Q(z)$, for every $z \in M$ and $w \in \mathbb{C}^*$. Thus, the restriction of Q to a fiber of the Hopf bundle map $\pi \colon M \to U$ may be identified with the covering projection $q \colon \mathbb{C}^* \to \mathbb{C}^*$, $q(w) = w^n$. Now, if both z and wz belong to F, then $Q(z) = Q(wz) = 1$, and so $w^n = 1$. Thus, the restriction

$$\pi \colon F(\mathscr{A}) \to U(\mathscr{A}) \tag{4}$$

is the orbit map of the free action of the geometric monodromy on $F(\mathscr{A})$. Hence, the Milnor fiber $F(\mathscr{A})$ may be viewed as a regular, cyclic n-fold cover of the projectivized complement $U(\mathscr{A})$, see for instance [10,43,52].

Example 2.1. Let $\mathscr{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$. Upon identifying the complement $M(\mathscr{B}_n)$ with the algebraic torus $(\mathbb{C}^*)^n$, we see that the map $Q(\mathscr{B}_n) \colon (\mathbb{C}^*)^n \to \mathbb{C}^*$, $z \mapsto z_1 \cdots z_n$ is a morphism of complex algebraic groups. Hence, the Milnor fiber $F(\mathscr{B}_n) = \ker(Q(\mathscr{B}_n))$ is an algebraic subtorus, which is isomorphic to $(\mathbb{C}^*)^{n-1}$. The monodromy automorphism h is isotopic to the identity, via the isotopy $h_t(z) = e^{2\pi \mathrm{i} t/n} z$. Thus, the Milnor fibration of the Boolean arrangement is trivial.

As noted in [52], the map $\iota \colon M(\mathscr{A}) \hookrightarrow M(\mathscr{B}_n)$ is compatible with the Milnor fibrations $Q(\mathscr{A}) \colon M(\mathscr{A}) \to \mathbb{C}^*$ and $Q(\mathscr{B}_n) \colon M(\mathscr{B}_n) \to \mathbb{C}^*$. It follows that the Milnor fiber $F(\mathscr{A})$ may be obtained by intersecting the complement $M(\mathscr{A})$, viewed as a subvariety of the algebraic torus $M(\mathscr{B}_n) = (\mathbb{C}^*)^n$ via the inclusion ι, with $F(\mathscr{B}_n) \cong (\mathbb{C}^*)^{n-1}$, viewed as an algebraic subgroup of $(\mathbb{C}^*)^n$; that is,

$$F(\mathscr{A}) = M(\mathscr{A}) \cap F(\mathscr{B}_n). \tag{5}$$

2.4. The closed Milnor fiber and its boundary

As before, let $\mathscr{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$. Intersecting the Milnor fiber $F(\mathscr{A})$ with a ball in $\mathbb{C}^{d+1}$ of large enough radius, we obtain a compact, smooth, orientable $2d$-dimensional manifold with boundary,

$$(6) \qquad \overline{F}(\mathscr{A}) = F(\mathscr{A}) \cap D^{2(d+1)},$$

which we call the *closed Milnor fiber* of the arrangement. The *boundary of the Milnor fiber* of the arrangement $\mathscr{A}$ is the compact, smooth, orientable, $(2d-1)$-dimensional manifold

$$(7) \qquad \partial \overline{F}(\mathscr{A}) = F(\mathscr{A}) \cap S^{2d+1}.$$
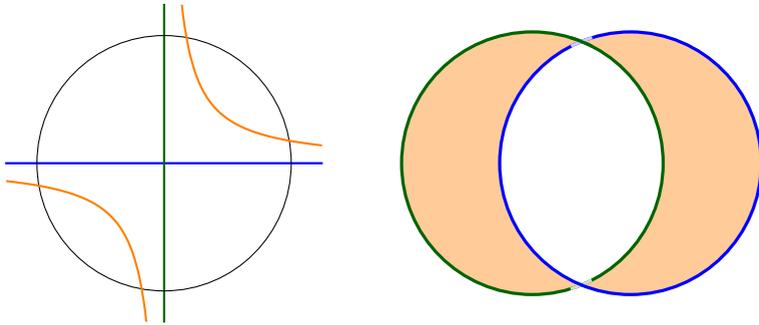

Fig. 2. Local Milnor fibration and closed Milnor fiber for 2 lines.

We will drop $\mathscr{A}$ from the notation when the arrangement is understood. As noted in [16, Proposition 2.4], the pair $(\overline{F}, \partial \overline{F})$ is $(d-1)$-connected. In particular, if $d \geqslant 2$, the boundary of the Milnor fiber is connected, and the inclusion-induced homomorphism $\pi_1(\partial \overline{F}) \to \pi_1(\overline{F})$ is surjective. Furthermore, as shown in [52, Lemma 7.5], the map $\pi \colon \mathbb{C}^{d+1}\backslash\{0\} \to \mathbb{CP}^d$ restricts to regular, cyclic n-fold covers, $\pi \colon \overline{F} \to \overline{U}$ and $\pi \colon \partial \overline{F} \to \partial \overline{U}$. In summary, we have a commuting ladder

$$(8)$$

where the horizontal arrows are inclusions, and the maps denoted by π are principal bundles with fiber either $\mathbb{Z}_n$ or $\mathbb{C}^*$, as indicated.

2.5. Classifying homomorphisms for cyclic covers

As before, let $\mathscr{A}$ be a central arrangement in $\mathbb{C}^{d+1}$, and set $n = |\mathscr{A}|$. Fix a basepoint for the complement $M = M(\mathscr{A})$. For each $H \in \mathscr{A}$, let x_H denote the based homotopy class of a compatibly oriented meridian curve about the hyperplane H. A standard application of the van Kampen theorem shows that the fundamental group $\pi_1(M)$ is generated by these elements. To simplify notation, we will denote the image of x_H in $H_1(M, \mathbb{Z})$ by the same symbol. Similarly, we will denote by $\overline{x}_H$ the image of x_H in both $\pi_1(U)$ and its abelianization. We then have that $H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^n$, with basis $\{x_H \colon H \in \mathscr{A}\}$, and $H_1(U, \mathbb{Z})$ is isomorphic to the quotient of $\mathbb{Z}^n$ by the cyclic subgroup generated by $\sum_{H \in \mathscr{A}} x_H$.

Let Q be a defining polynomial for $\mathscr{A}$, and let $Q \colon M \to \mathbb{C}^*$ be the Milnor fibration. By [52, Prop. 4.6], the induced homomorphism $Q_\sharp \colon \pi_1(M) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$ sends each generator x_H to 1. Recall that the Hopf fibration restricts to a regular, cyclic n-fold cover $\pi \colon F \to U$. As shown for instance in [8, 10, 51, 52], this cover is classified by the homomorphism $\delta \colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ given by $\overline{x}_H \mapsto 1$. If $d \geqslant 3$, we know that $\pi_1(\partial \overline{U}) = \pi_1(U)$, and so the n-fold cover $\pi \colon \partial \overline{F} \to \partial \overline{U}$ is classified by the same epimorphism δ.

In the critical case $d = 2$, the 3-dimensional manifold $\partial \overline{U}$ is a graph manifold, with underlying graph Γ the bipartite graph whose vertices correspond to the lines and the intersection points of the projectivized line arrangement in $\mathbb{CP}^2$, and with edges (ℓ, P) joining a line vertex ℓ to a point vertex P if $P \in \ell$; see Figure 2.5 for an illustration. Furthermore, each vertex manifold is the product of S^1 with a sphere S^2 with a number of open 2-disks removed. For more details on this construction we refer to [13, 29–32].
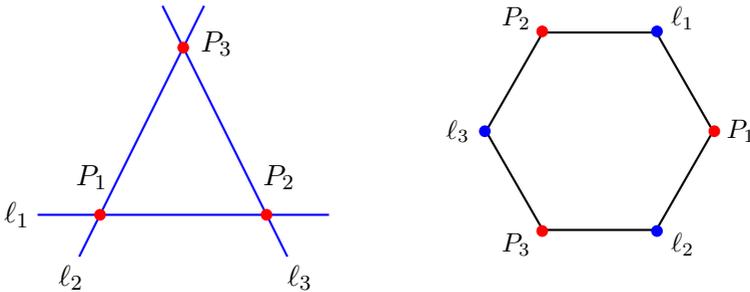
Fig. 3. An arrangement of lines and its associated graph.

The group $\pi_1(\partial \overline{U})$, then, has generators $\overline{x}_H$ for $H \in \mathscr{A}$ and generators y_c corresponding to the cycles in Γ. As shown in [52, Prop. 7.6], the regular $\mathbb{Z}_n$-cover $\pi \colon \partial \overline{F} \to \partial \overline{U}$ is classified by the homomorphism $\overline{\delta} \colon \pi_1(\partial \overline{U}) \twoheadrightarrow \mathbb{Z}_n$ given by $\overline{x}_H \mapsto 1$ and $y_c \mapsto 0$.

3. MULTINETS AND PENCILS

3.1. Multinets

For our purposes here, it will be enough to assume that the arrangement $\mathscr{A}$ lives in $\mathbb{C}^3$, in which case $\bar{\mathscr{A}} = \mathbb{P}(\mathscr{A})$ is an arrangement of (projective) lines in $\mathbb{CP}^2$. This is clear when the rank of $\mathscr{A}$ is at most 2, and may be achieved otherwise by taking a generic 3-slice. This operation does not change the poset $L_{\leqslant 2}(\mathscr{A})$, nor does it change the monodromy action on $H_1(F(\mathscr{A}), \mathbb{C})$.

For a rank-3 arrangement, the set $L_1(\mathscr{A})$ is in 1-to-1 correspondence with the lines of $\bar{\mathscr{A}}$, while $L_2(\mathscr{A})$ is in 1-to-1 correspondence with the intersection points of $\bar{\mathscr{A}}$. Moreover, the poset structure of $L_{\leqslant 2}(\mathscr{A})$ mirrors the incidence structure of the point-line configuration $\bar{\mathscr{A}}$. We will say that a rank-2 flat X has multiplicity q if $|\mathscr{A}_X| = q$, or, equivalently, if the point $\mathbb{P}(X)$ has exactly q lines from $\bar{\mathscr{A}}$ passing through it. The following notion, due to Falk and Yuzvinsky [28], will play an important role in the sequel.

Definition 3.1 ([28]). A *multinet* $\mathcal{N}$ on an arrangement $\mathscr{A}$ consists of the following data:
  (i) An integer $k \geqslant 3$, and a partition of $\mathscr{A}$ into k subsets, say, $\mathscr{A}_1, \ldots, \mathscr{A}_k$.
  (ii) An assignment of multiplicities on the hyperplanes, $m \colon \mathscr{A} \to \mathbb{N}$.
  (iii) A subset $\mathcal{X} \subseteq L_2(\mathscr{A})$, called the base locus.
Moreover, the following conditions must be satisfied:
  1. There is an integer d such that $\sum_{H \in \mathscr{A}_i} m_H = d$, for all $i \in [k]$.
  2. For any two hyperplanes H and K in different classes, $H \cap K \in \mathcal{X}$.
  3. For each $X \in \mathcal{X}$, the sum $n_X := \sum_{H \in \mathscr{A}_i \colon H \supset X} m_H$ is independent of i.
  4. For each $i \in [k]$, the space $\left(\bigcup_{H \in \mathscr{A}_i} H \right) \backslash \mathcal{X}$ is connected.

We say that a multinet as above has k classes and weight d, and refer to it as a (k, d)-multinet, or simply as a k-multinet. Without essential loss of generality, we may assume that $\gcd\{m_H\}_{H \in \mathscr{A}} = 1$. If all the multiplicities are equal to 1, the multinet is said to be *reduced*. If, furthermore, every flat in $\mathcal{X}$ is contained in precisely one hyperplane from each class, the multinet is called a (k, d)-*net*.

The various possibilities are illustrated in Fig. 4. The first picture shows a $(3, 2)$-net on a planar slice of the reflection arrangement of type A_3. The second picture shows a non-reduced $(3, 4)$-multinet on a planar slice of the reflection arrangement of type B_3. Finally, the third picture shows a simplicial arrangement of 12 lines in $\mathbb{CP}^2$ supporting a reduced $(3, 4)$-multinet which is not a 3-net.
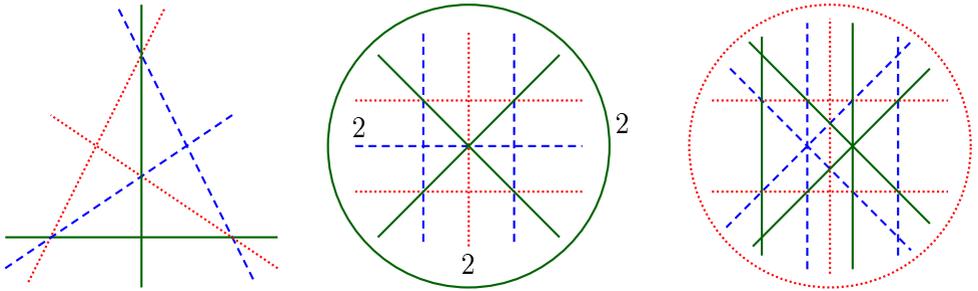
Fig. 4. A $(3,2)$-net; a $(3,4)$-multinet; and a reduced $(3,4)$-multinet which is not a 3-net.

Work of Yuzvinsky [56,57] and Pereira–Yuzvinsky [48] shows that, if $\mathcal{N}$ is a k-multinet on an arrangement $\mathcal{A}$, with base locus of size greater than 1, then $k = 3$ or 4; moreover, if the multinet $\mathcal{N}$ is not reduced, then $k = 3$. Although several infinite families of multinets with $k = 3$ are known, only one multinet with $k = 4$ is known to exist: the $(4,3)$-net on the Hessian arrangement. For more examples and further discussion, we refer to [6,28,54,55].

As noted in [47, Lemma 2.1], if $\mathcal{A}$ has no 2-flats of multiplicity kr, for any $r > 1$, then every reduced k-multinet on $\mathcal{A}$ is a k-net. The next lemma provides an alternative definition of nets.

LEMMA 3.2 ([47]). *A k-net on an arrangement $\mathcal{A}$ is a partition with non-empty blocks, $\mathcal{A} = \coprod_{\alpha \in [k]} \mathcal{A}_\alpha$, with the property that, for every $H \in \mathcal{A}_\alpha$ and $K \in \mathcal{A}_\beta$ with $\alpha \neq \beta$ we have that $|H \cap K \cap \mathcal{A}_\gamma| = 1$, for every $\gamma \in [k]$.*

In particular, a 3-net on $\mathcal{A}$ is a partition into 3 non-empty subsets with the property that, for each pair of hyperplanes $H, K \in \mathcal{A}$ in different classes, we have $H \cap K = H \cap K \cap L$, for some hyperplane L in the third class. Nets of type $(3,d)$ are intimately related to Latin squares of size d, i.e., $d \times d$ matrices with each row and column a permutation of the set $[d]$. Indeed, if $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are the parts of such a 3-net, then the multi-colored 2-flats define a Latin square: if we label the hyperplanes of $\mathcal{A}_\alpha$ as $H_1^\alpha, \dots, H_d^\alpha$, then the (p,q)-entry of this matrix is the integer r given by the requirement that $H_p^1 \cap H_q^2 \cap H_r^3 \in L_2(\mathcal{A})$. A similar procedure shows that a k-net is encoded by a $(k-2)$-tuple of orthogonal Latin squares.

3.2. Pencils

Let $\mathcal{A}$ be a (central) arrangement in $\mathbb{C}^3$, with defining polynomial $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H$. Suppose we have a (k,d)-multinet $\mathcal{N}$ on $\mathcal{A}$, with parts $\mathcal{A}_\alpha$ and multiplicity vector m. Write $Q_\alpha = \prod_{H \in \mathcal{A}_\alpha} f_H^{m_H}$, and define a rational map $f \colon \mathbb{C}^3 \to \mathbb{CP}^1$ by $f(x) = (Q_1(x) : Q_2(x))$. There is then a set $D = \{(a_1 : b_1), \dots, (a_k : b_k)\}$

of k distinct points in $\mathbb{CP}^1$ such that each of the degree d polynomials $Q_1, \ldots, Q_k$ can be written as $Q_\alpha = a_\alpha Q_2 - b_\alpha Q_1$, and, furthermore, the image of $f\colon M(\mathscr{A}) \to \mathbb{CP}^1$ misses D. The *pencil* associated to the multinet $\mathcal{N}$, then, is the restriction of f to the complement of the arrangement,

$$(9) \qquad\qquad f = f_{\mathcal{N}}\colon M(\mathscr{A}) \to \mathbb{CP}^1 \backslash D.$$

The map f can also be viewed as an 'orbifold fibration', or, in the terminology of Arapura [1], an 'admissible map'. To compute the homomorphism induced in first homology by this map, let $\gamma_1, \ldots, \gamma_k$ be compatibly oriented simple closed curves on $S = \mathbb{CP}^1 \backslash D$, going around the points of D, so that $H_1(S, \mathbb{Z})$ is generated by the homology classes $c_\alpha = [\gamma_\alpha]$, subject to the single relation $\sum_{\alpha=1}^k c_\alpha = 0$. The following lemma was proved in [47] using an approach based on de Rham cohomology. We give here another proof.

LEMMA 3.3 ([47]). *Let* $f\colon M(\mathscr{A}) \to S$ *be the pencil associated to a multinet* $\mathcal{N}$ *on an arrangement* $\mathscr{A}$. *The induced homomorphism* $f_*\colon H_1(M(\mathscr{A}), \mathbb{Z}) \to H_1(S, \mathbb{Z})$ *is then given by*

$$f_*(x_H) = m_H c_\alpha, \quad \text{for } H \in \mathscr{A}_\alpha.$$

Proof. Each polynomial Q_α defines a map $Q_\alpha\colon M(\mathscr{A}_\alpha) \to \mathbb{C}^*$. If we let $\theta_\alpha\colon S \to \mathbb{C}^*$ be the map given by $\theta_\alpha(z_1 : z_2) = a_\alpha z_2 - b_\alpha z_1$, and let $\iota_\alpha\colon M(\mathscr{A}) \to M(\mathscr{A}_\alpha)$ be the inclusion map, we obtain a commuting diagram,

$$(10) \qquad\qquad
\begin{array}{ccc}
M(\mathscr{A}) & \xrightarrow{\ f\ } & S \\
\downarrow{\scriptstyle \iota_\alpha} & & \downarrow{\scriptstyle \theta_\alpha} \\
M(\mathscr{A}_\alpha) & \xrightarrow{\ Q_\alpha\ } & \mathbb{C}^*
\end{array}$$

Apply now the $H_1(-, \mathbb{Z})$ functor to this diagram, and identify $H_1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}$. Clearly, if $H \in \mathscr{A}_\beta$, then $(\iota_\alpha)_*$ takes x_H to $\delta_{\alpha\beta} a_H$, whereas $(\theta_\alpha)_*$ takes c_β to $\delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta. On the other hand, $(Q_\alpha)_*$ is given by $x_H \mapsto m_H$, see [52, Prop. 4.6]. This completes the proof. $\qquad\square$

4. COHOMOLOGY JUMP LOCI

4.1. Resonance varieties of a graded algebra

Let A be a graded, graded-commutative algebra over a field $\Bbbk$. We will assume that each graded piece A^i is free and finitely generated over $\Bbbk$, and $A^0 = \Bbbk$. We will also assume that $a^2 = 0$, for all $a \in A^1$, a condition which is automatically satisfied if $\operatorname{char}(\Bbbk) \neq 2$, by graded-commutativity of multiplication in A. For each element $a \in A^1$, we turn the algebra A into a cochain

complex,

$$(11) \qquad (A, \delta_a): \ A^0 \xrightarrow{\ \delta_a\ } A^1 \xrightarrow{\ \delta_a\ } A^2 \longrightarrow \cdots,$$

with differentials the maps $\delta_a(b) = ab$. The (degree i, depth s) *resonance varieties* of A are then defined as the jump loci for the cohomology of this 'Aomoto' complex,

$$(12) \qquad \mathscr{R}^i_s(A) = \{a \in A^1 \mid \operatorname{rank}_{\Bbbk} H^i(A, \delta_a) \geqslant s\}.$$

These sets are Zariski-closed, homogeneous subsets of the affine space A^1. Furthermore, these varieties respect field extensions: if $\Bbbk \subseteq \mathbb{K}$, then $\mathscr{R}^i_s(A) = \mathscr{R}^i_s(A \otimes \mathbb{K}) \cap A^1$. As shown in [46,53], the resonance varieties obey the following 'product formulas':

$$(13) \qquad \begin{aligned} \mathscr{R}^1_s(A \otimes B) &= \mathscr{R}^1_s(A) \times \{0\} \cup \{0\} \times \mathscr{R}^1_s(B), \\ \mathscr{R}^i_1(A \otimes B) &= \bigcup_{j+k=i} \mathscr{R}^j_1(A) \times \mathscr{R}^k_1(B). \end{aligned}$$

For our purposes here, we will mainly consider the degree one resonance varieties, $\mathscr{R}_s(A) = \mathscr{R}^1_s(A)$. Clearly, these varieties depend only on the degree 2 truncation of A. More explicitly, $\mathscr{R}_s(A)$ consists of 0, together with all elements $a \in A^1$ for which there exist $b_1, \ldots, b_s \in A^1$ such that the span of $\{a, b_1, \ldots, b_s\}$ has dimension $s + 1$ and $ab_1 = \cdots = ab_s = 0$ in A^2.

The degree 1 resonance varieties enjoy the following naturality property: if $\varphi: A \to B$ is a morphism of commutative graded algebras, and φ is injective in degree 1, then the $\Bbbk$-linear map $\varphi^1: A^1 \to B^1$ embeds $\mathscr{R}_s(A)$ into $\mathscr{R}_s(B)$, for each $s \geqslant 1$.

Finally, suppose X is a connected, finite-type CW-complex. We define then the resonance varieties of X to be the sets $\mathscr{R}^i_s(X, \Bbbk) := \mathscr{R}^i_s(H^*(X, \Bbbk))$, viewed as homogeneous subsets of the affine space $H^1(X, \Bbbk)$.

4.2. The resonance varieties of the Orlik–Solomon algebra

The cohomology ring of a hyperplane arrangement complement was computed by E. Brieskorn in the early 1970s, building on pioneering work of V.I. Arnol'd on the cohomology ring of the pure braid group. In [44], Orlik and Solomon gave a simple description of this ring, solely in terms of the intersection lattice of the arrangement.

Once again, let $\mathscr{A}$ be a central arrangement, with complement $M = M(\mathscr{A})$. Fix a linear order on $\mathscr{A}$, and let E be the exterior algebra over a field $\Bbbk$ with generators $\{e_H \mid H \in \mathscr{A}\}$ in degree 1. Next, define a differential

$\partial\colon E \to E$ of degree -1, starting from $\partial(1) = 0$ and $\partial(e_H) = 1$, and extending ∂ to a linear operator on E, using the graded Leibniz rule. Finally, let I be the ideal of E generated by all elements of the form $\partial\big(\prod_{H\in\mathscr{B}} e_H\big)$, where $\mathscr{B} \subset \mathscr{A}$ and $\operatorname{codim}\bigcap_{H\in\mathscr{B}} H < |\mathscr{B}|$. Then $H^*(M, \Bbbk)$ is isomorphic, as a graded $\Bbbk$-algebra, to the quotient ring $A = E/I$.

Under this isomorphism, the basis $\{e_H\}$ of A^1 is dual to the basis of $H_1(M, \Bbbk) = H_1(M, \mathbb{Z}) \otimes \Bbbk$ given by the meridians $\{x_H\}$ around the hyperplanes, oriented compatibly with the complex orientations on $\mathbb{C}^{d+1}$ and the hyperplanes. Since A is a quotient of an exterior algebra, we have that $a^2 = 0$ for all $a \in A^1$. Thus, we may define the resonance varieties $\mathscr{R}^i_s(\mathscr{A}, \Bbbk)$ of our arrangement $\mathscr{A}$ (over the field $\Bbbk$) as the corresponding resonance varieties of the Orlik–Solomon algebra $H^*(M(\mathscr{A}), \Bbbk)$.

As usual, let $U = U(\mathscr{A})$ be the projectivized complement. The diffeomorphism $M \cong U \times \mathbb{C}^*$, together with the Künneth formula and the product formulas for resonance from (13) yields identifications

(14)
$$\mathscr{R}^1_s(\mathscr{A}, \Bbbk) \cong \mathscr{R}^1_s(U(\mathscr{A}), \Bbbk),$$
$$\mathscr{R}^i_1(\mathscr{A}, \Bbbk) \cong \mathscr{R}^i_1(U(\mathscr{A}), \Bbbk) \cup \mathscr{R}^{i-1}_1(U(\mathscr{A}), \Bbbk).$$

If $\mathscr{B} \subset \mathscr{A}$ is a proper sub-arrangement, the inclusion $M(\mathscr{A}) \hookrightarrow M(\mathscr{B})$ induces a morphism $H^*(M(\mathscr{B}), \Bbbk) \to H^*(M(\mathscr{B}), \Bbbk)$ which is injective in degree 1. We thus obtain an embedding $\mathscr{R}^1_s(\mathscr{B}, \Bbbk) \hookrightarrow \mathscr{R}^1_s(\mathscr{A}, \Bbbk)$. The irreducible components of $\mathscr{R}^1_s(\mathscr{A}, \Bbbk)$ that lie in the image of such an embedding are called *non-essential*; the remaining components are called *essential*.

The description of the Orlik–Solomon algebra given above makes it clear that the resonance varieties $\mathscr{R}^i_s(\mathscr{A}, \Bbbk)$ depend only on the intersection lattice, $L(\mathscr{A})$, and on the characteristic of the field $\Bbbk$.

The complex resonance varieties $\mathscr{R}^i_1(\mathscr{A}, \mathbb{C})$ were first defined and studied by Falk in [26]. Soon after, Cohen–Suciu [11], Libgober [34], and Libgober–Yuzvinsky [37] showed that the varieties $\mathscr{R}_s(\mathscr{A}) = \mathscr{R}^1_s(\mathscr{A}, \mathbb{C})$ consist of linear subspaces of the vector space $\mathbb{C}^{\mathscr{A}}$, intersecting transversely at 0. Moreover, all such subspaces have dimension at least two, and the cup-product map vanishes identically on each one of them. Finally, $\mathscr{R}_s(\mathscr{A})$ is the union of all components of $\mathscr{R}_1(\mathscr{A})$ of dimension greater than s.

The resonance varieties $\mathscr{R}^1_s(\mathscr{A}, \Bbbk)$ for $\Bbbk$ a field of positive characteristic were first defined and studied by Matei and Suciu in [40]. The nature of these varieties is much less predictable; for instance, their irreducible components need not be linear, and, even when they are linear, they may intersect non-transversely. We refer to [14, 27, 47, 49] for more on this subject.

4.3. **Multinets and complex resonance**

The work of Falk and Yuzvinsky [28] greatly clarified the nature of the (degree 1) resonance varieties of arrangements. Let us briefly review their construction.

Recall that every k-multinet $\mathcal{N}$ on an arrangement $\mathscr{A}$ with parts $\mathscr{A}_1, \ldots,$ $\mathscr{A}_k$ and multiplicities m_H for each $H \in \mathscr{A}$ gives rise to an orbifold fibration (or, for short, a pencil) $f \colon M(\mathscr{A}) \to S$, where $S = \mathbb{CP}^1 \backslash \{k \text{ points}\}$. In view of Lemma 3.3, the induced map in cohomology, $f^* \colon H^*(S, \mathbb{Z}) \to H^*(M, \mathbb{Z})$, is given in degree 1 by $f^*(c_\alpha^*) = u_\alpha$, where $u_\alpha = \sum_{H \in \mathscr{A}_\alpha} m_H e_H$. Consequently, the homomorphism $f^* \colon H^1(S, \mathbb{C}) \to H^1(M, \mathbb{C})$ is injective, and thus sends $\mathscr{R}_1(S, \mathbb{C})$ to $\mathscr{R}_1(M, \mathbb{C})$.

Let us identify $\mathscr{R}^1(S, \mathbb{C})$ with $H^1(S, \mathbb{C}) = \mathbb{C}^{k-1}$, and view $P_\mathcal{N} = f^*(H^1(S, \mathbb{C}))$ as lying inside $\mathscr{R}_1(\mathscr{A})$. It follows from the preceding discussion that $P_\mathcal{N}$ is the $(k-1)$-dimensional linear subspace spanned by the vectors $u_2 - u_1, \ldots, u_k - u_1$. In fact, as shown in [28, Thms. 2.4–2.5], this subspace is an essential component of $\mathscr{R}_1(\mathscr{A})$.

More generally, suppose there is a sub-arrangement $\mathscr{B} \subseteq \mathscr{A}$ supporting a multinet $\mathcal{N}$. In this case, the inclusion $M(\mathscr{A}) \hookrightarrow M(\mathscr{B})$ induces a monomorphism $H^1(M(\mathscr{B}), \mathbb{C}) \hookrightarrow H^1(M(\mathscr{A}), \mathbb{C})$, which restricts to an embedding $\mathscr{R}_1(\mathscr{B}) \hookrightarrow \mathscr{R}_1(\mathscr{A})$. The linear space $P_\mathcal{N}$, then, lies inside $\mathscr{R}_1(\mathscr{B})$, and thus, inside $\mathscr{R}_1(\mathscr{A})$. Conversely, as shown in [28, Thm. 2.5] all (positive-dimensional) irreducible components of $\mathscr{R}_1(\mathscr{A})$ arise in this fashion. Consequently,

$$(15) \qquad \mathscr{R}_s(\mathscr{A}) = \bigcup_{\mathscr{B} \subseteq \mathscr{A}} \bigcup_{\substack{\mathcal{N} \text{ a multinet on } \mathscr{B} \\ \text{with at least } s+2 \text{ parts}}} P_\mathcal{N}.$$

4.4. **Characteristic varieties and finite abelian covers**

We switch now to a different type of jump loci, involving this time homology with twisted coefficients. Let X be a connected, finite-type CW-complex, let $\pi = \pi_1(X, x_0)$, and let $\mathrm{Hom}(\pi, \mathbb{C}^*)$ be the affine algebraic group of $\mathbb{C}$-valued, multiplicative characters on π, which we will identify with $H^1(X, \mathbb{C}^*)$. The (degree i, depth s) *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$(16) \qquad \mathscr{V}_s^i(X) = \{\xi \in \mathrm{Hom}(\pi, \mathbb{C}^*) \mid \dim_\mathbb{C} H_i(X, \mathbb{C}_\xi) \geqslant s\}.$$

By construction, these loci are Zariski-closed subsets of the character group.

To a large degree, the characteristic varieties control the Betti numbers of regular, finite abelian covers $Y \to X$. For instance, suppose that the deck-transformation group is cyclic of order n, and fix an inclusion $\iota \colon \mathbb{Z}_n \hookrightarrow \mathbb{C}^*$, by

sending $1 \mapsto e^{2\pi i/n}$. With this choice, the epimorphism $\nu \colon \pi \twoheadrightarrow \mathbb{Z}_n$ defining the n-fold cyclic cover $Y \to X$ yields a torsion character, $\rho = \iota \circ \nu \colon \pi \to \mathbb{C}^*$. A standard argument using Maschke's theorem yields an isomorphism of $\mathbb{C}[\mathbb{Z}_n]$-modules,

$$(17) \qquad H_i(Y, \mathbb{C}) \cong H_i(X, \mathbb{C}) \oplus \bigoplus_{1 < r | n} (\mathbb{C}[t]/\Phi_r(t))^{\mathrm{depth}(\rho^{n/r})},$$

where $\Phi_r(t)$ is the r-th cyclotomic polynomial, and the depth of a character $\xi \colon \pi \to \mathbb{C}^*$, defined as the dimension of $H_i(X, \mathbb{C}_\xi)$, is given by $\mathrm{depth}(\xi) = \max\{s \mid \xi \in \mathscr{V}_s^i(X)\}$.

The exponents in formula (17) arising from prime-power divisors can be estimated in terms of the corresponding Aomoto–Betti numbers. More precisely, suppose n is divisible by $r = p^s$, for some prime p. Composing the canonical projection $\mathbb{Z}_n \twoheadrightarrow \mathbb{Z}_p$ with ν defines a cohomology class $\bar{\nu} \in H^1(X, \mathbb{F}_p)$. Assuming that $H_*(X, \mathbb{Z})$ is torsion-free, it was shown in [45, Thm. 11.3] that

$$(18) \qquad \dim_{\mathbb{C}} H_i(X, \mathbb{C}_{\rho^{n/r}}) \leqslant \dim_{\mathbb{F}_p} H^i(H^\bullet(X, \mathbb{F}_p), \delta_{\bar{\nu}}).$$

4.5. Characteristic varieties of arrangements

Let us consider again a hyperplane arrangement $\mathscr{A}$, with complement $M = M(\mathscr{A})$. The varieties $\mathscr{V}_s(\mathscr{A}) := \mathscr{V}_s^1(M(\mathscr{A}))$ are closed algebraic subsets of the character torus $\mathrm{Hom}(\pi_1(M), \mathbb{C}^*) = (\mathbb{C}^*)^n$, where $n = |\mathscr{A}|$. Since M is diffeomorphic to $U \times \mathbb{C}^*$, where $U = U(\mathscr{A})$, the character torus $H^1(M, \mathbb{C}^*)$ splits as $H^1(U, \mathbb{C}^*) \times \mathbb{C}^*$. Under this splitting, the characteristic varieties $\mathscr{V}_s(\mathscr{A})$ get identified with the varieties $\mathscr{V}_s^1(U)$ lying in the first factor.

Since M is a smooth, quasi-projective variety, a general result of Arapura [1], as refined by Artal Bartolo–Cogolludo–Matei [2] and Budur–Wang [7], insures that $\mathscr{V}_s(\mathscr{A})$ is a finite union of translated subtori. Moreover, as shown by Cohen–Suciu [11] and Libgober–Yuzvinsky [33], and, in a broader context, by Dimca–Papadima–Suciu [22] and Dimca–Papadima [21], the tangent cone at the origin to $\mathscr{V}_s(\mathscr{A})$ coincides with the resonance variety $\mathscr{R}_s(\mathscr{A})$, for all $s \geqslant 1$.

More explicitly, let $\exp \colon H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*)$ be the coefficient homomorphism induced by the exponential map $\mathbb{C} \to \mathbb{C}^*$. Then, if $P \subset H^1(M, \mathbb{C})$ is one of the linear subspaces comprising $\mathscr{R}_s(\mathscr{A})$, its image under the exponential map, $\exp(P) \subset H^1(M, \mathbb{C}^*)$, is one of the subtori comprising $\mathscr{V}_s(\mathscr{A})$. Furthermore, the correspondence $P \rightsquigarrow \exp(P)$ yields a bijection between the components of $\mathscr{R}_s(\mathscr{A})$ and the components of $\mathscr{V}_s(\mathscr{A})$ passing through the origin $\mathbf{1}$.

Now recall that each positive-dimensional component of $\mathscr{R}_1(\mathscr{A})$ is obtained by pullback along a pencil $f \colon M \to S$, where $S = \mathbb{CP}^1 \setminus \{k \text{ points}\}$ and

$k \geqslant 3$. Thus, each positive-dimensional component of $\mathscr{V}_1(\mathscr{A})$ containing the origin is of the form $\exp(P) = f^*(H^1(S, \mathbb{C}^*))$, for some pencil f. An easy computation shows that $\mathscr{V}_s^1(S) = H^1(S, \mathbb{C}^*) = (\mathbb{C}^*)^{k-1}$ for all $s \leqslant k - 2$. Hence, the subtorus $f^*(H^1(S, \mathbb{C}^*))$ is a positive-dimensional component of $\mathscr{V}_1(\mathscr{A})$ that contains the origin and lies inside $\mathscr{V}_{k-2}(\mathscr{A})$.

As shown in [50], the (depth 1) characteristic variety of an arrangement may have irreducible components not passing through the origin. A general combinatorial machine for producing such translated subtori has been recently given in [14]. Namely, suppose $\mathscr{A}$ admits a *pointed multinet, i.e.*, a multinet $\mathcal{N}$ and a hyperplane $H \in \mathscr{A}$ for which $m_H > 1$, and $m_H \mid n_X$ for each flat X in the base locus such that $X \subset H$. Letting $\mathscr{A}' = \mathscr{A} \setminus \{H\}$ be the deletion of $\mathscr{A}$ with respect to H, it turns out that $\mathscr{V}_1(\mathscr{A}')$ has a component which is a 1-dimensional subtorus of $H^1(M(\mathscr{A}'), \mathbb{C}^*)$, translated by a character of order m_H.

For instance, if $\mathscr{A}$ is the reflection arrangement of type B_3 and $\mathcal{N}$ is the $(4, 3)$-multinet depicted in the middle of Figure 3.1, then choosing H to be one of the hyperplanes with multiplicity $m_H = 2$ leads to a translated torus in the first characteristic variety of the deleted B_3 arrangement, $\mathscr{A}' = \mathscr{A} \setminus \{H\}$. Whether all positive-dimensional translated subtori in the (degree 1, depth 1) characteristic varieties of arrangements occur in this fashion is an open problem.

5. THE ALGEBRAIC MONODROMY OF THE MILNOR FIBER

5.1. The homology of the Milnor fiber

Using the interpretation of the Milnor fiber of a hyperplane arrangement as a finite cyclic cover of the projectivized complement, we may compute the homology groups of the Milnor fiber and the characteristic polynomial of the algebraic monodromy in terms of the characteristic varieties of the arrangement.

To see how that works, let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. Without loss of generality, we may assume $d = 2$. Let M be the complement of the arrangement, and let U be its projectivization. Recall that the Milnor fiber $F = F(\mathscr{A})$ may be viewed as the regular, $\mathbb{Z}_n$-cover of U, classified by the homomorphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ taking each meridian loop x_H to 1.

For each divisor r of n, let $\rho_r \colon \pi_1(U) \to \mathbb{C}^*$ be the character defined by $\rho_r(x_H) = e^{2\pi i/r}$. It follows from formula (17) that

$$(19) \qquad H_1(F(\mathscr{A}), \mathbb{C}) = H_1(U, \mathbb{C}) \oplus \bigoplus_{1 < r \mid n} (\mathbb{C}[t]/\Phi_r(t))^{e_r(\mathscr{A})},$$

as modules over $\mathbb{C}[\mathbb{Z}_n]$, where the integers $e_r(\mathscr{A}) := \operatorname{depth}(\rho_r)$ depend on the position of the diagonal characters $\rho_r \in (\mathbb{C}^*)^{n-1}$ with respect to the characteris-

tic varieties $\mathscr{V}_s(U)$. Note that only the essential components of these varieties may contribute to the sum. Indeed, if a component lies on a proper coordinate subtorus C, then the diagonal subtorus, $D = \{(t, \ldots, t) \mid t \in \mathbb{C}^*\}$, intersects C only at the origin. In particular, components arising from multinets supported on proper sub-arrangements of $\mathscr{A}$, do not produce jumps in the first Betti number of $F(\mathscr{A})$.

Let $h_*\colon H_1(F, \mathbb{C}) \to H_1(F, \mathbb{C})$ be the degree 1 algebraic monodromy of the Milnor fibration, and let $\Delta_{\mathscr{A}}(t) = \det(t \cdot \mathrm{id} - h_*)$ be its characteristic polynomial. Formula (19) may be interpreted as saying that

$$(20) \qquad \Delta_{\mathscr{A}}(t) = (t-1)^{n-1} \cdot \prod_{1 < r \mid n} \Phi_r(t)^{e_r(\mathscr{A})}.$$

Consequently, if $\varphi(r)$ denotes the Euler totient function, then

$$(21) \qquad b_1(F(\mathscr{A})) = n - 1 + \sum_{1 < r \mid n} \varphi(r) e_r(\mathscr{A}).$$

In the above expressions, not all the divisors r of n appear. For instance, as shown by Libgober [35, Prop. 2.1] and Măcinic–Papadima [38, Thm. 3.13], the following holds: if there is no flat $X \in L_2(\mathscr{A})$ of multiplicity $q \geqslant 3$ such that $r \mid q$, then $e_r(\mathscr{A})$ vanishes. In particular, if the lines of $\mathscr{A}$ intersect only in points of multiplicity 2 and 3, then only $e_3(\mathscr{A})$ may be non-zero, whereas if points of multiplicity 4 occur, then $e_2(\mathscr{A})$ and $e_4(\mathscr{A})$ may also be non-zero. For more combinatorial conditions that lead to the vanishing of the exponents $e_r(\mathscr{A})$ we refer to [3, 5, 9, 10].

In [9, Thm. 13], Cohen, Dimca, and Orlik give combinatorial upper bounds on the exponents of the cyclotomic polynomials appearing in (19). The next result provides lower bounds for those exponents, in the presence of reduced multinets on the arrangement.

THEOREM 5.1 ([47]). *Suppose that an arrangement $\mathscr{A}$ admits a reduced k-multinet, and let $f\colon M(\mathscr{A}) \to S$ denote the associated pencil. Then:*
  (1) *The character ρ_k belongs to $f^*(H^1(S, \mathbb{C}^*))$, and $e_k(\mathscr{A}) \geqslant k - 2$.*
  (2) *If $k = p^s$, then $\rho_{p^r} \in f^*(H^1(S, \mathbb{C}^*))$ and $e_{p^r}(\mathscr{A}) \geqslant k-2$, for all $1 \leqslant r \leqslant s$.*

5.2. Aomoto–Betti numbers

Consider now a field $\mathbb{k}$ of characteristic p, and let $A = H^*(M, \mathbb{k})$ be the Orlik–Solomon algebra over $\mathbb{k}$. Recall that the $\mathbb{k}$-vector space $A^1 = \mathbb{k}^{\mathscr{A}}$ comes endowed with a preferred basis, $\{e_H\}_{H \in \mathscr{A}}$; let $\sigma = \sum_{H \in \mathscr{A}} e_H$ be the "diagonal" element. Following [47], we define the *Aomoto–Betti number* of $\mathscr{A}$ (over $\mathbb{k}$) as

$$(22) \qquad \beta_{\mathbb{k}}(\mathscr{A}) = \max\{s \mid \sigma \in \mathscr{R}_s(\mathscr{A}, \mathbb{k})\}.$$

Clearly, this integer depends only on the prime $p = \mathrm{char}(\Bbbk)$, and so we will write it simply as $\beta_p(\mathscr{A})$. The following result provides useful information about these combinatorial invariants of arrangements.

PROPOSITION 5.2 ([47]). *Let $\mathscr{A}$ be an arrangement, and p a prime.*
(1) *If $p \nmid |X|$, for any $X \in L_2(\mathscr{A})$ with $|X| > 2$, then $\beta_p(\mathscr{A}) = 0$.*
(2) *If $\mathscr{A}$ supports a k-net, then $\beta_p(\mathscr{A}) = 0$ if $p \nmid k$, and $\beta_p(\mathscr{A}) \geqslant k - 2$, otherwise.*

To a large extent, the β_p invariants control the (degree 1) algebraic monodromy of the Milnor fibration. More precisely, the "modular upper bound" (18) yields the following inequalities on the prime-power exponents,

$$(23) \qquad\qquad e_{p^s}(\mathscr{A}) \leqslant \beta_p(\mathscr{A}),$$

for all primes p and integers $s \geqslant 1$. In particular, if $\beta_p(\mathscr{A}) = 0$, then $e_{p^s}(\mathscr{A}) = 0$, for all $s \geqslant 1$.

5.3. Nets, multiplicities, and the Milnor fibration

Under suitable restrictions on the multiplicities of rank 2 flats, the above modular bounds are sharp, at least for the prime $p = 3$ and for $s = 1$.

THEOREM 5.3 ([47]). *Let $\mathscr{A}$ be a hyperplane arrangement, and suppose $L_2(\mathscr{A})$ has no flats of multiplicity $3r$, for any $r > 1$. Then $\beta_3(\mathscr{A}) \neq 0$ if and only if $\mathscr{A}$ admits a reduced 3-multinet, or, equivalently, a 3-net. Moreover, $\beta_3(\mathscr{A}) \leqslant 2$ and $e_3(\mathscr{A}) = \beta_3(\mathscr{A})$.*

Putting things together, we have the following immediate corollary.

COROLLARY 5.4 ([47]). *Suppose $L_2(\mathscr{A})$ has only flats of multiplicity 2 and 3. Then*
$$\Delta_{\mathscr{A}}(t) = (t - 1)^{|\mathscr{A}|-1} \cdot (t^2 + t + 1)^{\beta_3(\mathscr{A})},$$
where $\beta_3(\mathscr{A}) \in \{0, 1, 2\}$ is combinatorially determined.

For more information on the class of 'triple point' line arrangements, we refer to [17, 18, 20, 36]. When multiplicity 4 does occur, some further combinatorial restrictions lead to equalities in the modular bounds (23), at the prime $p = 2$ and for $s \leqslant 2$.

THEOREM 5.5 ([47]). *If $\mathscr{A}$ admits a 4-net, and if $\beta_2(\mathscr{A}) \leqslant 2$, then $e_2(\mathscr{A}) = e_4(\mathscr{A}) = \beta_2(\mathscr{A})$.*

The above results, and many other computations naturally lead to the following conjecture.

CONJECTURE 5.6 ([47]). *The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathscr{A}$ of rank at least 3 is given by the following combinatorial formula:*

$$(24) \qquad \Delta_{\mathscr{A}}(t) = (t-1)^{|\mathscr{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathscr{A})}(t^2+t+1)^{\beta_3(\mathscr{A})}.$$

This conjecture has been verified for several large classes of arrangements, including

(1) all sub-arrangements of non-exceptional Coxeter arrangements [38];
(2) all complex reflection arrangements [19, 23, 39];
(3) certain types of complexified real arrangements [4, 54, 55].

6. FURTHER TOPOLOGICAL INVARIANTS OF THE MILNOR FIBER

6.1. Torsion in the homology of the Milnor fiber

A long-standing question, raised by Randell and Dimca–Némethi among others, asks whether the Milnor fiber of a complex hyperplane arrangement can have non-trivial torsion in (integral) homology.

As a first step towards answering this question, it was shown by Cohen, Denham, and Suciu [8] that the first homology of the Milnor fiber of a multi-arrangement may have torsion. In recent work of Denham and Suciu [14], these examples were recast in a more general framework, leading to hyperplane arrangements $\mathscr{B}$ for which $H_q(F(\mathscr{B}), \mathbb{Z})$ has torsion, in some degree $q > 1$. The precise result reads as follows.

THEOREM 6.1 ([14]). *Suppose $\mathscr{A}$ admits a pointed multinet, with distinguished hyperplane H. Let p be a prime dividing the multiplicity m_H. There is then a choice of multiplicities m' on the deletion $\mathscr{A}' = \mathscr{A} \backslash \{H\}$ such that $H_q(F(\mathscr{B}), \mathbb{Z})$ has p-torsion, where $\mathscr{B}$ is the arrangement obtained from the multi-arrangement $(\mathscr{A}', m')$ by a process of polarization, and $q = 1 + |\{K \in \mathscr{A}' : m'_K \geqslant 3\}|$.*

For instance, if $\mathscr{A}'$ is the deleted B_3 arrangement mentioned in §4.5, then a suitable choice of multiplicities m' produces an arrangement $\mathscr{B}$ of 27 hyperplanes in $\mathbb{C}^8$ such that $H_6(F(\mathscr{B}), \mathbb{Z})$ has 2-torsion. Nevertheless, it is still not known whether there is a hyperplane arrangement $\mathscr{A}$ (without multiplicities) such that $H_1(F(\mathscr{A}), \mathbb{Z})$ has non-trivial torsion. For more on this topic, we refer to [15].

6.2. The homology of the boundary of the Milnor fiber

A detailed study of the boundary of the Milnor fiber of a non-isolated surface singularity was done by Némethi and Szilárd in [42]. When applied to arrangements in $\mathbb{C}^3$, their work yields the following result.

THEOREM 6.2 ([42]). *Let $\mathscr{A}$ be an arrangement of n planes in $\mathbb{C}^3$, and let $\partial \overline{F}$ be the boundary of its Milnor fiber. The characteristic polynomial of the algebraic monodromy acting on $H_1(\partial \overline{F}, \mathbb{C})$ is equal to the product*

$$\prod_{X \in L_2(\mathscr{A})} (t-1)(t^{\gcd(|\mathscr{A}_X|,n)} - 1)^{|\mathscr{A}_X|-2}.$$

In particular, the Betti number $b_1(\partial \overline{F})$ is determined by very simple combinatorial data associated to the arrangement, namely, the multiplicities of the rank 2 flats. In general, torsion can occur in the first homology of $\partial \overline{F}$. For instance, as noted in [42], if $\mathscr{A}$ is an arrangement of 4 planes in general position in $\mathbb{C}^3$, then $H_1(\partial \overline{F}, \mathbb{Z}) = \mathbb{Z}^6 \oplus \mathbb{Z}_4$. For a generic arrangement of n planes in $\mathbb{C}^3$, it is conjectured in [52] that

$$(25) \qquad H_1(\partial \overline{F}, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2} \oplus \mathbb{Z}_n^{(n-2)(n-3)/2}.$$

For an arbitrary arrangement $\mathscr{A}$ in $\mathbb{C}^3$, it is an open question whether all the torsion in $H_1(\partial \overline{F}, \mathbb{Z})$ consists of $\mathbb{Z}_n$-summands, where $n = |\mathscr{A}|$. Likewise, it is an open question whether such torsion is combinatorially determined.

6.3. Complement, boundary, and intersection lattice

Once again, let $\mathscr{A}$ be an arrangement in $\mathbb{C}^3$, with intersection lattice $L(\mathscr{A})$, and let U be its projectivized complement. Recall that the boundary manifold, $\partial \overline{U}$, is a closed graph manifold, with underlying graph Γ the bipartite graph whose vertices correspond to the lines and the intersection points of the projectivized line arrangement $\bar{\mathscr{A}}$.

Now suppose $\mathscr{A}'$ is another arrangement in $\mathbb{C}^3$, and that U is homeomorphic to U'. It follows that $\partial \overline{U}$ is homeomorphic to $\partial \overline{U}'$, or, equivalently (since both graph manifolds are either S^3 or have positive first Betti number), $\partial \overline{U}$ is homotopy equivalent to $\partial \overline{U}'$. Using Waldhausen's classification of graph manifolds, Jiang and Yau [30,31] conclude that the underlying graphs, Γ and Γ' must be isomorphic, and thus the corresponding intersection lattices, $L(\mathscr{A})$ and $L(\mathscr{A}')$, must also be isomorphic.

Example 6.3. Let $\mathscr{A}$ and $\mathscr{A}'$ be the pair of arrangements in $\mathbb{C}^3$ whose projectivizations are depicted in Fig. 5. Both $\bar{\mathscr{A}}$ and $\bar{\mathscr{A}}'$ have 2 triple points
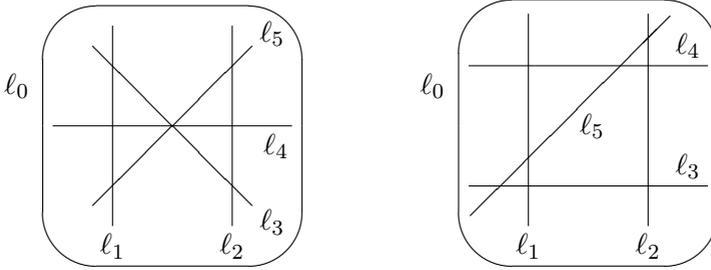
Fig. 5. The arrangements $\mathscr{A}$ and $\mathscr{A}'$.

and 9 double points, yet the two intersection lattices are non-isomorphic: the two triple points of $\mathscr{A}$ lie on a common line, whereas the two triple points of $\mathscr{A}'$ don't. Nevertheless, as first noted by L. Rose and H. Terao in an unpublished note, the corresponding Orlik–Solomon algebras are isomorphic. In fact, as shown by Falk in [25], the two projective complements, U and U', are homotopy equivalent.

Now, since $L(\mathscr{A}) \not\cong L(\mathscr{A}')$, we know from [30, 31] that the corresponding boundary manifolds, $\partial\overline{U}$ and $\partial\overline{U}'$, are not homotopy equivalent, even though $b_1(\partial\overline{U}) = b_1(\partial\overline{U}') = 13$. In fact, as noted in [13, Ex. 5.3], the two manifolds may be distinguished by their (multi-variable) Alexander polynomials: $\Delta_{\partial\overline{U}}(t)$ has 7 distinct factors, whereas $\Delta_{\partial\overline{U}'}(t)$ has 8 distinct factors. The characteristic varieties $\mathscr{V}_1(\partial\overline{U})$ and $\mathscr{V}_1(\partial\overline{U}')$ are the zero sets of these polynomials. Hence, the first variety consists of 7 codimension-1 subtori in $(\mathbb{C}^*)^{13}$, while the second one consists of 8 such subtori. This shows, once again, that $\partial\overline{U} \not\simeq \partial\overline{U}'$.

In [30], Jiang and Yau conjecture that the homeomorphism type of $U(\mathscr{A})$ is determined by isomorphism type of $L(\mathscr{A})$, for any arrangement $\mathscr{A}$ in $\mathbb{C}^3$. Motivated by the above considerations, we propose a more precise conjecture.

CONJECTURE 6.4. *Let $\mathscr{A}$ and $\mathscr{A}'$ be two central arrangements in $\mathbb{C}^3$. The following conditions are equivalent:*
  (1) $U(\mathscr{A}) \cong U(\mathscr{A}')$.
  (2) $\partial\overline{U}(\mathscr{A}) \simeq \partial\overline{U}(\mathscr{A}')$.
  (3) $\Delta_{\partial\overline{U}(\mathscr{A})}(t) = \Delta_{\partial\overline{U}(\mathscr{A}')}(t)$.
  (4) $\Gamma(\mathscr{A}) \cong \Gamma(\mathscr{A}')$.
  (5) $L(\mathscr{A}) \cong L(\mathscr{A}')$.

6.4. Milnor fiber and intersection lattice

We now show that there are invariants which can tell apart homologically equivalent Milnor fibers of arrangements.

Example 6.5. Let $\mathscr{A}$ and $\mathscr{A}'$ be the two arrangements from Example 6.3, and let F and F' be the corresponding Milnor fibers. It is readily seen that neither of the two arrangements supports an essential multinet. Since both $\bar{\mathscr{A}}$ and $\bar{\mathscr{A}}'$ have only double and triple points, Corollary 5.4 shows that, in both cases, the characteristic polynomial of the algebraic monodromy acting on the Milnor fiber is $(t-1)^5$. By the same token, Theorem 6.2 shows that, in both cases, the characteristic polynomial of the algebraic monodromy acting on the boundary of the Milnor fiber is $(t-1)^{13}(t^2+t+1)^2$.

It can also be verified that $H_1(F,\mathbb{Z}) = H_1(F',\mathbb{Z}) = \mathbb{Z}^5$. Nevertheless, the two Milnor fibers are *not* homotopy equivalent. In fact, we claim that $\pi_1(F) \not\cong \pi_1(F')$. To establish this claim, we consider the characteristic varieties of F and F', lying in the character torus $(\mathbb{C}^*)^5$. A computation shows that

$$\mathscr{V}_1(F) = \{t_1 = t_4 = t_5 = 1\} \cup \{t_3^3 t_5^{-1} = t_3^6 t_4^{-1} = t_2 t_3^{-1} = 1\},$$
$$\mathscr{V}_2(F) = \{\mathbf{1}, (1,\omega,\omega,1,1), (1,\omega^2,\omega^2,1,1)\},$$

where $\omega = \exp(2\pi\mathrm{i}/3)$, while

$$\mathscr{V}_1(F') = \{t_1 t_4 = t_1 t_5 = t_3 t_5^{-3} = 1\} \cup \{t_2 t_4 = t_3 t_5^2 = t_4 t_5 = 1\},$$
$$\mathscr{V}_2(F') = \{\mathbf{1}\}.$$

Note that the two, 2-dimensional components of $\mathscr{V}_1(F)$ meet at the three characters of order 3 comprising $\mathscr{V}_2(F)$. The variety $\mathscr{V}_1(F')$ also consists of two, 2-dimensional subtori, but these subtori only meet at the origin, which is the only point comprising $\mathscr{V}_2(F')$.

In view of these considerations, we conclude with a (rather optimistic) conjecture, which can be viewed as a Milnor fiber analogue of Conjecture 6.4.

CONJECTURE 6.6. *Let $\mathscr{A}$ and $\mathscr{A}'$ be two central arrangements in $\mathbb{C}^3$. The following conditions are equivalent:*
  (1) $F(\mathscr{A}) \cong F(\mathscr{A}')$.
  (2) $\partial\overline{F}(\mathscr{A}) \simeq \partial\overline{F}(\mathscr{A}')$.
  (3) $L(\mathscr{A}) \cong L(\mathscr{A}')$.

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