THE DISTRIBUTION OF RATIONAL NUMBERS AND ERGODIC THEORY

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Rational numbers, or equivalently roots of unity on the unit circle, are not randomly distributed when they are enumerated by the size of denominators (Farey), or by the sum of digits in their continued fraction expansion (Stern-Brocot). There are four types of measure-preserving transformations (Gauss, Farey, BCZ, Newman), with very different ergodic behaviour, that play a role in gathering information about this distribution. Some of their ergodic properties and applications will be surveyed in this paper.

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1. INTRODUCTION

Continued fractions provide a natural way of encoding real numbers into sequences of positive integers, called digits. Any irrational number in the interval $[0,1]$ can be uniquely represented as a regular continued fraction (RCF):

$$x = [a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, \tag{1.1}$$

with $a_1, a_2, a_3, \ldots$ positive integers.

The metrical theory of continued fractions is closely connected with ergodic theory through two types of (non-invertible) transformations of $[0,1]$: the *Gauss map* G and the *Farey map* F. The former acts on $x \in [0,1] \setminus \mathbb{Q}$ given by (1.1) by deleting the first digit:

$$G\big([a_1, a_2, a_3, \ldots]\big) = [a_2, a_3, a_4, \ldots], \tag{1.2}$$

or equivalently by $G(0) = 0$ and $G(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right]$ if $x \in (0,1)$. The n^{th} digit $a_n = a_n(x)$ of x is determined by the iterates of G as follows:

$$a_n(x) = \frac{1}{G^{n-1}(x)} - G^n(x) = \left[\frac{1}{G^{n-1}(x)}\right], \qquad \forall n \geq 1.$$

An additional important connection was found by Gauss, who discovered that the probability measure $d\mu(x) = \frac{1}{\log 2} \cdot \frac{dx}{1+x}$ is G-invariant, i.e. $\mu(G^{-1}A) = \mu(A)$ for any measurable set $A \subseteq [0,1]$. This shows that $([0,1), \mathcal{B}, G, \mu)$ defines a measure-preserving transformation (*m.p.t.*).

The Farey map, a "slow-down" of the Gauss map, acts by subtracting one from the first digit:

$$F\big([a_1, a_2, a_3, \ldots]\big) = \begin{cases} [a_1 - 1, a_2, a_3, \ldots] & \text{if } a_1 \geq 2 \\ [a_2, a_3, a_4, \ldots] & \text{if } a_1 = 1, \end{cases} \tag{1.3}$$

or equivalently

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{1-x}{x} & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{1.4}$$

It is plain to check that the infinite measure $\frac{\mathrm{d}x}{x}$ is F-invariant. Furthermore, the map $\phi : [0, \frac{1}{2}] \to [0,1]$, $\phi(x) = \frac{1-x}{x}$, provides an explicit isomorphism between the m.p.t. $([0,1), \mathcal{B}, G, \mu)$ and the first return m.p.t. $(\mathcal{C}, \mathcal{B}, R, \nu)$ of F with respect to the subset $\mathcal{C} := [\frac{1}{2}, 1]$, where $\mathrm{d}\nu(x) = \frac{1}{\log 2} \cdot \frac{\mathrm{d}x}{x}$ and $R(x) = F^{n(x)}(x)$ for $x = [1, a_2, a_3, \ldots]$ with $n(x) = \min\{n \geq 1 : F^n(x) \in \mathcal{C}\} = a_2$, in the sense that $\phi \circ R \circ \phi^{-1} = G$ and $\mu \circ \phi = \nu$.

The Gauss map gives a measure of the distribution of rationals approximating an irrational number in $[0,1]$, in particular of the convergents. The Farey map is suitable for studying the distribution of rationals in $[0,1]$ when ordered by the sum of their RCF digits. Analogues of the pair (G, F) are defined for other types of continued fraction expansions (for instance even CF, odd CF, Nakada's α-expansions) and play a fundamental role in their metric theory. These types of transformations usually arise as Poincaré sections or factors of the geodesic flow on modular surfaces.

A different way of enumerating the rationals in $[0,1]$ is through the size of their denominators. Consider the set $\mathcal{F}_Q$ of *Farey fractions of order* Q, consisting of those rational numbers $\gamma = \frac{a}{q}$ in the interval $(0,1]$ with $1 \leq q \leq Q$ and $\gcd(a,q) = 1$. The *BCZ map* defined by:

$$T(x,y) = \left(y, \left[\frac{1+x}{y}\right] y - x\right) = \left(y, 1 - yG\Big(\frac{y}{1+x}\Big)\right) \tag{1.5}$$

defines an area-preserving bijection of the *Farey triangle*

$$\mathcal{T} = \big\{(x,y) \in (0,1]^2 : x + y > 1\big\},$$

with the feature that, for every Q, the elements of $\mathcal{F}_Q$ are generated, in increasing order, by applying iterates of T on the point $\big(\frac{1}{Q}, 1\big)$. In contrast with the Gauss and Farey maps, which are closely related with the geodesic flow on the modular surface, the BCZ map is related with the horocycle flow, which can be realized as a suspension flow over this transformation.

There is yet another way of counting the rationals, related to the Stern-Brocot sequence and to the Calkin-Wilf tree, provided by the Newman function

$$T : [0,\infty) \to [0,\infty), \qquad T(x) = \frac{1}{2[x] - x + 1}. \tag{1.6}$$

2. CONTINUED FRACTIONS

2.1. General properties of continued fractions and the Gauss map

Next we review some basic properties of the regular continued fractions (RCF), which can be found, for instance, in [23]. A *RCF* is an expression of form:

$$C = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} =: [a_1, a_2, \ldots],$$

where $a_1, a_2, \ldots$ are commuting variables, called the *digits* of C. Here we shall take $a_1, a_2, \ldots \in \mathbb{N}$. Regular continued fractions establish a one-to-one correspondence between the set $\mathbb{I} := [0,1] \setminus \mathbb{Q}$ of irrational numbers in $(0,1)$ and the set of sequences of admissible digits $(a_n) \in \mathbb{N}^{\mathbb{N}}$.

In one direction, consider the *convergents* of C, defined by:

$$C_n = [a_1, a_2, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = \frac{P_n(a_1, \ldots, a_n)}{Q_n(a_1, \ldots, a_n)}. \tag{2.1}$$

Note that $[a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1]$ if $a_n \geq 2$. One plainly computes $P_1(a_1) = 1$, $Q_1(a_1) = a_1$, $P_2(a_1, a_2) = a_2$, $Q_2(a_1, a_2) = a_1a_2 + 1$, $P_3(a_1, a_2, a_3) = a_2a_3 + 1$, $Q_3(a_1, a_2, a_3) = a_1a_2a_3 + a_1 + a_3$. Equality (2.1) yields

$$\frac{P_{n-1}(a_2, \ldots, a_n)}{Q_{n-1}(a_2, \ldots, a_n)} = \frac{1}{\frac{P_n(a_1, \ldots, a_n)}{Q_n(a_1, \ldots, a_n)}} - a_1 = \frac{Q_n(a_1, \ldots, a_n) - a_1P_n(a_1, \ldots, a_n)}{P_n(a_1, \ldots, a_n)},$$

and thus,

$$\begin{pmatrix} Q_n(a_1,\ldots,a_n) & P_n(a_1,\ldots,a_n) \\ Q_{n-1}(a_1,\ldots,a_{n-1}) & P_{n-1}(a_1,\ldots,a_{n-1}) \end{pmatrix} = \begin{pmatrix} Q_{n-1}(a_2,\ldots,a_n) & P_{n-1}(a_2,\ldots,a_n) \\ Q_{n-2}(a_2,\ldots,a_{n-1}) & P_{n-2}(a_2,\ldots,a_{n-1}) \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This leads inductively to

$$\begin{aligned} \begin{pmatrix} Q_n(a_1,\ldots,a_n) & P_n(a_1,\ldots,a_n) \\ Q_{n-1}(a_1,\ldots,a_{n-1}) & P_{n-1}(a_1,\ldots,a_{n-1}) \end{pmatrix} &= \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Q_{n-1}(a_1,\ldots,a_{n-1}) & P_{n-1}(a_1,\ldots,a_{n-1}) \\ Q_{n-2}(a_1,\ldots,a_{n-2}) & P_{n-2}(a_1,\ldots,a_{n-2}) \end{pmatrix}. \end{aligned}$$

If we denote $p_n = P_n(a_1, \dots, a_n), q_n = Q_n(a_1, \dots, a_n)$, then $\frac{p_n}{q_n} = [a_1, \dots, a_n]$ with $0 < p_n \leq q_n$ and the equality above becomes

$$\begin{pmatrix} q_n & p_n \\ q_{n-1} & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{n-1} & p_{n-1} \\ q_{n-2} & p_{n-2} \end{pmatrix}, \tag{2.2}$$

or in a more familiar form

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, & p_0 = 0,\ p_1 = 1, \\ q_n = a_n q_{n-1} + q_{n-2}, & q_0 = 1,\ q_1 = a_1. \end{cases}$$

Comparing determinants of both sides in (2.2) we find the fundamental equalities:

$$\begin{aligned} p_{n-1} q_n - p_n q_{n-1} &= (-1)^n, \\ p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-2}(a_n q_{n-1} + q_{n-2}) \\ &= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^n a_n. \end{aligned}$$

Therefore, with (F_n) denoting the Fibonacci sequence, we have

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}} \leq \frac{1}{F_n F_{n+1}} \qquad \text{and} \qquad \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n}{q_n q_{n-2}}.$$

In particular the sequence $\left(\frac{p_n}{q_n}\right)$ is Cauchy and we can plainly prove:

PROPOSITION 2.1. (i) *There exists* $x = [a_1, a_2, \dots] := \lim_n \frac{p_n}{q_n} \in [0, 1]$.
(ii) $\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+2}}{q_{2k+2}} < x < \frac{p_{2k+1}}{q_{2k+1}} < \frac{p_{2k-1}}{q_{2k-1}}$.

The rational number $\frac{p_n}{q_n}$ is called the n^{th} convergent of x. The limit x is an irrational number, as $x = \frac{a}{q}$ with $a, q \in \mathbb{N}$ would imply

$$\frac{1}{q q_{2k}} \leq x - \frac{p_{2k}}{q_{2k}} \leq \frac{p_{2k+1}}{q_{2k+1}} - \frac{p_{2k}}{q_{2k}} = \frac{1}{q_{2k+1} q_{2k}},$$

and therefore $q > q_{2k+1}$ for every k, which is not possible.

In the opposite direction, consider the *Gauss map*

$$G : [0, 1) \to [0, 1), \qquad G(x) = \begin{cases} \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Given a sequence (a_n) of positive integers, consider $x = [a_1, a_2, \dots] \in \mathbb{I}$ as above. Clearly $\frac{1}{a_1+1} < x < \frac{1}{a_1}$ and $G(x) = \frac{1}{x} - a_1$, showing that equality (1.1) holds. Inductively we now infer:

$$G^{n-1}(x) = [a_n, a_{n+1}, \dots] = \frac{1}{a_n + G^n(x)},$$

showing that the n^{th} digit $a_n = a_n(x)$ of x is determined by G as follows:

$$a_n(x) = \frac{1}{G^{n-1}(x)} - G^n(x) = \left[\frac{1}{G^{n-1}(x)}\right], \qquad \forall n \geq 1.$$

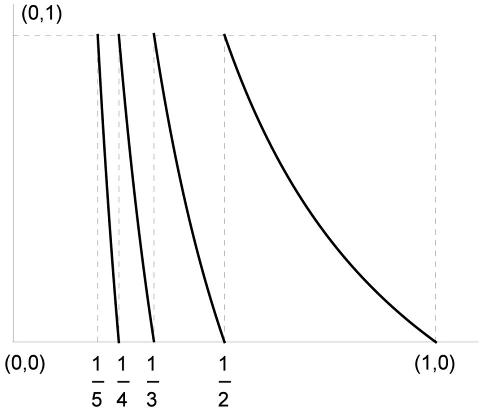

Fig. 1. The Gauss map.

The n^{th} iterate of G can be expressed by the following useful formula:

$$G^n(x) = \frac{p_n - q_n x}{q_{n-1}x - p_{n-1}}, \tag{2.3}$$

which leads in turn to

$$|q_n x - p_n| = xG(x)G^2(x)\cdots G^n(x). \tag{2.4}$$

To prove (2.3), first notice that $G(x) = \frac{1-a_1 x}{x} = \frac{p_1 - q_1 x}{q_0 x - p_0}$. Assuming that (2.3) holds for some $n \geq 1$, we infer

$$\begin{aligned}\frac{p_{n+1}-q_{n+1}x}{q_n x - p_n} &= \frac{a_{n+1}p_n + p_{n-1} - (a_{n+1}q_n + q_{n-1})x}{q_n x - p_n} = -a_{n+1} + \frac{p_{n-1}x - q_{n-1}x}{q_n x - p_n} \\ &= -\Big[\frac{1}{G^n(x)}\Big] + \frac{1}{G^n(x)} = \Big\{\frac{1}{G^n(x)}\Big\} = G^{n+1}(x),\end{aligned}$$

showing that (2.3) holds for $n+1$, as desired.

2.2. **Ergodic properties of the Gauss map**

To check that the Gauss measure $\mathrm{d}\mu(x) = \frac{1}{\log 2}\cdot\frac{\mathrm{d}x}{1+x}$ is G-invariant, it suffices to check that $\mu(G^{-1}[0,\theta]) = \mu([0,\theta])$ for every $\theta \in [0,1)$. Using $G^{-1}[0,\theta] = \bigcup\limits_{k=1}^{\infty}\big[\frac{1}{k+\theta},\frac{1}{k}\big]$ we can write

$$\begin{aligned}\mu\big(G^{-1}[0,\theta]\big) &= \frac{1}{\log 2}\int_{G^{-1}[0,\theta]}\frac{\mathrm{d}x}{1+x} = \frac{1}{\log 2}\sum_{k=1}^{\infty}\log\left(\frac{1+1/k}{1+1/(k+\theta)}\right) \\ &= \frac{1}{\log 2}\lim_n \log\left(\frac{(n+1)(1+\theta)}{n+\theta+1}\right) = \frac{\log(1+\theta)}{\log 2} = \mu\big([0,\theta]\big).\end{aligned}$$

Recall that a measure-preserving transformation $(X, \mathcal{M}, T, \mu)$ is called *ergodic* if for any measurable set B which is T-invariant (*i.e.* $T^{-1}B = B$) one has either $\mu(B) = 0$ or $\mu(B^c) = 0$. It is well-known that the measure-preserving transformation $([0,1), \mathcal{B}, G, \mu)$ is ergodic. Clearly $L^p(\mu) = L^p(\lambda)$, where λ denotes the Lebesgue measure on $[0,1]$. The pointwise ergodic theorem then yields, for every $f \in L^1(\mu)$ and almost every $x \in [0,1]$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_0^1 f\,\mathrm{d}\mu. \tag{2.5}$$

This property has important consequences on the distribution of digits in the RCF expansion. For instance, taking $f(x) = \log x \in L^1(\lambda)$ in (2.5) and using some elementary RCF-properties, one can show that

$$\lim_n \frac{\log q_n(x)}{n} = \frac{\pi^2}{12\log 2}, \qquad \text{a.e. } x \in \mathbb{I},$$

where $q_n(x)$ denotes the denominator of the n^{th} convergent of x. Other interesting properties regarding the distribution of RCF-digits are obtained taking $f(x) = \left[\frac{1}{x}\right]^{1-\delta}$ with $\delta \searrow 0$, and respectively $f(x) = \log\left[\frac{1}{x}\right]$, leading to:

$$\lim_n \frac{a_1(x) + \cdots + a_n(x)}{n} = \infty, \qquad \text{a.e. } x \in \mathbb{I},$$

and respectively

$$\lim_n \sqrt[n]{a_1(x)\cdots a_n(x)} = C \approx 2.6854, \qquad \text{a.e. } x \in \mathbb{I}.$$

The Gauss map enjoys two other important ergodic properties, namely

$$\lim_n \lambda(G^{-n}A) = \mu(A) \quad \text{and} \quad \lim_n \mu(G^{-n}A \cap B) = \mu(A)\mu(B) = 1,$$

for every $A, B \in \mathcal{B}$. The later is the familiar mixing property. The difference $\lambda(G^{-n}A) - \mu(A)$ was first estimated by Kuzmin in 1928. A complete answer was provided by Wirsing and by Babenko by analyzing the spectral theory of the corresponding Perron-Frobenius operator on $C[0,1]$ or on a certain Hilbert space of functions (see [23] and references therein).

2.3. **Generalized Gauss-Kuzmin statistics for regular, even and odd continued fractions**

For any integer n-tuple $(a_1, \ldots, a_n) \in \mathbb{N}^n$, consider the corresponding cylinder

$$\begin{aligned}\Delta_{[a_1,\ldots,a_n]} :=& \{x \in [0,1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\} \\ =& \Delta_{[a_1]} \cap G^{-1}\Delta_{[a_2]} \cap \cdots \cap G^{-(n-1)}\Delta_{[a_n]}.\end{aligned}$$

Clearly $(\Delta_{[a_1,\dots,a_n]})_{a_1,\dots,a_n=1}^{\infty}$ is a partition of $[0,1)$. Furthermore, it is elementary to check that, with $\frac{p_n}{q_n}=[a_1,\dots,a_n]$, we have

$$\Delta_{[a_1,\dots,a_n]}=\begin{cases}\left[\frac{p_n}{q_n},\frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if } n \text{ is even}\\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}},\frac{p_n}{q_n}\right] & \text{if } n \text{ is odd.}\end{cases}$$

This is useful to quantify the frequency of appearances of fixed strings of digits. Employing $a_{i+j}(x)=a_j(G^i x)$, we can write for every fixed string of digits $(a_1,\dots,a_h)$:

$$\#\{i<N : a_{i+1}(x)=a_1,\dots,a_{i+h}(x)=a_h\}=\sum_{i=0}^{N-1}\chi_{\Delta_{[a_1,\dots,a_h]}}(G^i x).$$

Taking $f=\chi_{\Delta_{[a_1,\dots,a_h]}}$ in (2.5) we retrieve the following Gauss-Kuzmin type result that holds for almost every $x\in\mathbb{I}$:

$$\lim_N \frac{\#\{i<N : a_{i+1}(x)=a_1,\dots,a_{i+h}(x)=a_h\}}{N}=\mu(\Delta_{[a_1,\dots,a_h]}).$$

In particular for almost every $x\in\mathbb{I}$ one has:

$$\lim_N \frac{\#\{i<N : a_i(x)=k\}}{N}=\frac{1}{\log 2}\cdot\log\left(1+\frac{1}{k(k+2)}\right).$$

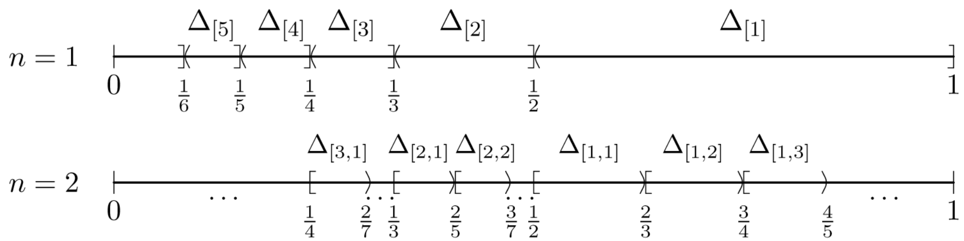

Fig. 2. The RCF partition of the unit interval.

A related but different problem, concerning the probability that a random number $x\in\mathbb{I}$ has a prescribed string of digits in its RCF expansion, at the first place where the denominator of the convergent is larger than some (large) R, was investigated in [37]. Considering the *renewal time* $n_R=n_R(x):=\min\{n : q_n(x)>R\}$, so that $q_{n_R-1}\le R<q_{n_R}$, Sinai and Ulcigrai proved the existence of the joint limiting distribution of

$$\left(\frac{q_{n_R-1}}{R},\frac{R}{q_{n_R}},a_{n_R-K},\dots,a_{n_R+K}\right)\quad \text{as } R\to\infty,$$

with fixed integer $K \geq 0$, where $a_n = a_n(x)$, $q_n = q_n(x)$ are viewed as random variables on $\mathbb{I}$. The mixing property of the suspension flow of the natural extension $\overline{G}$, acting on $\mathbb{I} \times \mathbb{I}$ by

$$\overline{G}\big([a_1, a_2, \ldots], [a_0, a_{-1}, \ldots]\big) = \big([a_2, a_3, \ldots], [a_1, a_0, a_{-1}, \ldots]\big),$$

under the roof function $r\left((a_n)_{-\infty}^{\infty}\right) = -\log(a_1 + [a_0, a_{-1}, \ldots])$, was the main technical ingredient in that proof. A direct proof, that also led to the exact calculation of the limiting distribution, was given by Ustinov [39] through an elementary number theoretical approach. His key observation was the following characterization of consecutive convergents in the RCF expansion of irrational numbers:

Proposition 2.2 ([39]). *For $x \in \mathbb{I}$, the following are equivalent:*

(i) $\frac{P}{Q}$, $\frac{P'}{Q'}$ *successive convergents in* RCF(x).

(ii) $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}$ *and* $0 < \frac{Q'x - P'}{-Qx + P} < 1$, *where*

$$\mathcal{R} := \left\{ \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) : 0 \leq P \leq Q,\ 1 \leq P' \leq Q' \right\}.$$

The main results and approach from [39] had been extended by Cellarosi [13] to even CF (*ECF*). Ustinov's approach had been further extended to the situation of ECF, odd CF (*OCF*), and to Nakada's α-expansions in [7]. Following [35], every number $x \in \mathbb{I}$ can be uniquely expanded in ECF, respectively in OCF, as

$$x = [[(a_1, e_1), (a_2, e_2), \ldots]] = \cfrac{1}{a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{a_3 + \cfrac{e_3}{\ddots}}}},$$

with $e_n \in \{\pm 1\}$ and $a_n \in 2\mathbb{N}$, respectively $a_n \in 2\mathbb{N} - 1$ and $a_n + e_n \geq 2$. In both of the ECF and OCF situations, the corresponding measure-preserving transformation is given by the Gauss type map defined by:

$$G_D(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in D \\ 1 - \left\{\frac{1}{x}\right\} & \text{if } x \in (0,1) \setminus D \end{cases} = e_D(x)\Big(\frac{1}{x} - a_D(x)\Big), \qquad x \in \mathbb{I},$$

where

$$e_D(x) := \begin{cases} 1 & \text{if } x \notin D \\ -1 & \text{if } x \in D, \end{cases} \qquad a_D(x) := \begin{cases} \left[\frac{1}{x}\right] & \text{if } x \notin D \\ 1 + \left[\frac{1}{x}\right] & \text{if } x \in D, \end{cases}$$

$D_{\mathrm{even}} = [\frac{1}{2}, 1) \cup [\frac{1}{4}, \frac{1}{3}) \cup \cdots$, and respectively $D_{\mathrm{odd}} = [\frac{1}{3}, \frac{1}{2}) \cup [\frac{1}{5}, \frac{1}{4}) \cup \cdots$ Furthermore, we also have

$$G_D^n[[(a_1, e_1), (a_2, e_2), \ldots]] = [[(a_{n+1}, e_{n+1}), (a_{n+2}, e_{n+2}), \ldots]],$$

with digits (a_n, e_n) given by $a_n(x) = a_D\big(G_D^{n-1}(x)\big)$ and $e_n(x) = e_D\big(G_D^{n-1}(x)\big)$.

The numerators and denominators of D-convergents satisfy the relations:

$$\begin{cases} p_n = a_n p_{n-1} + e_{n-1} p_{n-2}, & p_0 = 0,\ p_1 = 1, \\ q_n = a_n q_{n-1} + e_{n-1} q_{n-2}, & q_0 = 1,\ q_1 = a_1, \\ p_{n-1} q_n - p_n q_{n-1} = (-1)^n e_1 \cdots e_{n-1}. \end{cases}$$

The maps $G_{D_{\mathrm{even}}}$ and $G_{D_{\mathrm{odd}}}$ are ergodic with respect to the invariant measures $\mathrm{d}\nu_{\mathrm{even}} = \frac{\mathrm{d}x}{1-x^2}$, and respectively $\mathrm{d}\nu_{\mathrm{odd}} = \frac{2G}{3\log G} \cdot \frac{\mathrm{d}x}{(G-1+x)(G+1-x)}$, where $G = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.

The approach in [7] consisted in estimating (for $\xi_1, \xi_2, \xi_3, \xi_4 > 0$ fixed) the Lebesgue measure $\mathcal{L}^{E/O,\pm}_{\xi_1,\xi_2,\xi_3,\xi_4}$ of the set of those $x \in \mathbb{I}$ for which there exist $\frac{P}{Q}$, $\frac{P'}{Q'}$ successive convergents in $\mathrm{ECF}(x)$, respectively in $\mathrm{OCF}(x)$, such that

$$\frac{Q}{R} \le \xi_1, \quad \frac{R}{Q'} \le \xi_2, \quad \frac{Q}{Q'} \le \xi_3 \quad \text{and} \quad 0 \le \pm \frac{Q'x - P'}{-Qx + P} \le \xi_4.$$

Theorem 2.3 ([7]). *The joint distributions $\mathcal{L}^{E/O,\pm}_{x_1,x_2,x_3,x_4}(R)$ exist as $R \to \infty$ and*

$$\begin{aligned} \mathcal{L}^{E,\pm}_{x_1,x_2,x_3,x_4}(R) &= \frac{2F_\pm}{3\zeta(2)} + O_\varepsilon\big(R^{-1+\varepsilon}\big), \\ \mathcal{L}^{O,+}_{x_1,x_2,x_3,x_4}(R) &= \frac{F_+ - D_1}{\zeta(2)} + O_\varepsilon\big(R^{-1/2+\varepsilon}\big), \\ \mathcal{L}^{O,-}_{x_1,x_2,x_3,x_4}(R) &= \frac{F_- - D_2 - D_3}{\zeta(2)} + O_\varepsilon\big(R^{-1/2+\varepsilon}\big) \\ &= \frac{F_-(x_1,x_2,\min\{x_3,g^2\},x_4) - D_3}{\zeta(2)} + O_\varepsilon\big(R^{-1/2+\varepsilon}\big), \end{aligned}$$

where $F_\pm = F_\pm(x_1,x_2,x_3,x_4)$ and $D_i = D_i(x_1,x_2,x_3,x_4)$ are given by

$$\begin{aligned} F_\pm &= \mp \begin{cases} \mathrm{Li}_2(\mp x_1 x_2 x_4) & \text{if } x_3 > x_1 x_2, \\ \mathrm{Li}_2(\mp x_3 x_4) - \log(1 \pm x_3 x_4) \log \frac{x_1 x_2}{x_3} & \text{if } x_3 \le x_1 x_2, \end{cases} \\ D_2 &= F_-(x_1,x_2,x_3,x_4) - F_-(x_1,x_2,\min\{x_3,g^2\},x_4), \\ D_1 &= \sum_{\ell \ge 1} I_\ell^+, \quad D_3 = \sum_{\ell \ge 2} I_\ell^-, \quad I_\ell^\pm = \int_{1/x_2}^{A_\ell} \mathrm{d}x \int_{x/(2\ell+g)}^{B_\ell(x)} \frac{x_4\,\mathrm{d}y}{y(y \pm x_4 x)}, \end{aligned}$$

with $g = \frac{1}{G} = \frac{\sqrt{5}-1}{2}$, the convention that $\int_a^b = 0$ when $a \ge b$, and

$$A_\ell = (2\ell + g)x_1, \quad B_\ell(x) = B_{\ell,x_2,x_3}(x) = \min\Big\{x_3 x, x_1, \frac{x}{2\ell}, \frac{x-1}{2\ell-1}\Big\}.$$

The integrals $I_\ell^\pm$ can be written explicitly as a combination of logarithms and dilogarithms.

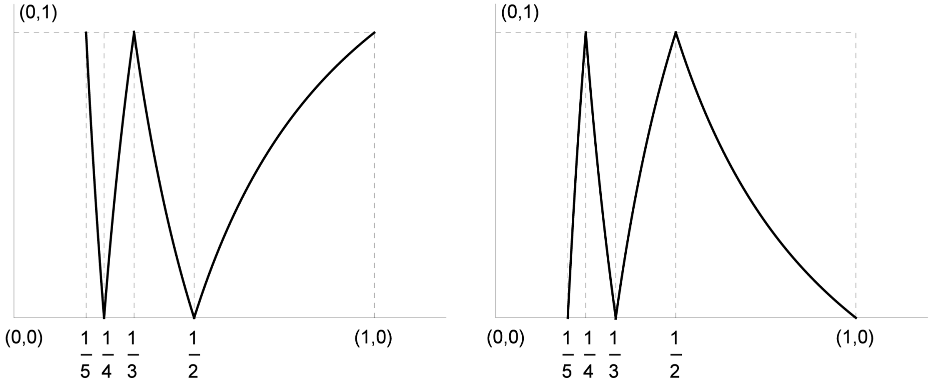

Fig. 3. The ECF and OCF Gauss maps.

The following characterizations of pairs of consecutive ECF convergents, respectively of OCF convergents, play an important role in the proof:

PROPOSITION 2.4 ([7]). *For $x \in \mathbb{I}$, the following are equivalent:*

(i) $\frac{P}{Q}$, $\frac{P'}{Q'}$ *successive convergents in* ECF(x).

(ii) $\begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}$, $\begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \pmod 2$, $1 \le Q \le Q'$, *and* $0 < \left| \frac{Q'x - P'}{-Qx + P} \right| < 1$.

PROPOSITION 2.5 ([7]). *For $x \in \mathbb{I}$, the following are equivalent:*

(i) $\frac{P}{Q}$, $\frac{P'}{Q'}$ *successive convergents in* OCF(x).

(ii) $\begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}$, $\begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, *or* $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \pmod 2$, $\frac{Q'}{Q} > g$, *and one of the following holds:*

($*$) $\frac{Q'}{Q} > G$, *and* $0 < \left| \frac{Q'x - P'}{-Qx + P} \right| < 1$.

($**$) $g < \frac{Q'}{Q} < G$ *and* $0 < \frac{Q'x - P'}{-Qx + P} < 1$.

Note that in the OCF case the sequence (q_n) of denominators of consecutive convergents is not necessarily increasing.

2.4. **The Pascal triangle with memory**

Given $p, p', q, q' \in \mathbb{Z}$, $q, q' > 0$ with $p'q - pq' = 1$, define the "mediant sum" $\dfrac{p}{q} \boxplus \dfrac{p'}{q'} := \dfrac{p + p'}{q + q'}$. This is the correct addition in the lattice $\mathbb{Z}v_1 + \mathbb{Z}v_2$ where $\vec{v}_1 = (p, q)$ and $\vec{v}_2 = (p', q')$ with $\|\vec{v}_1 \times \vec{v}_2\| = |p'q - pq'| = 1$. Starting with $(p, q) = (0, 1)$ and $(p', q') = (1, 1)$ and iterating this construction, one gets the following Pascal triangle $\mathcal{D}$ with memory, which resembles the Stern-Brocot tree [12,38]. The vertices of $\mathcal{D}$ are labeled by rational numbers $r(n, k) = \frac{p(n,k)}{q(n,k)}$ in lowest terms, $0 \le k \le 2^n$, $n \ge 0$.

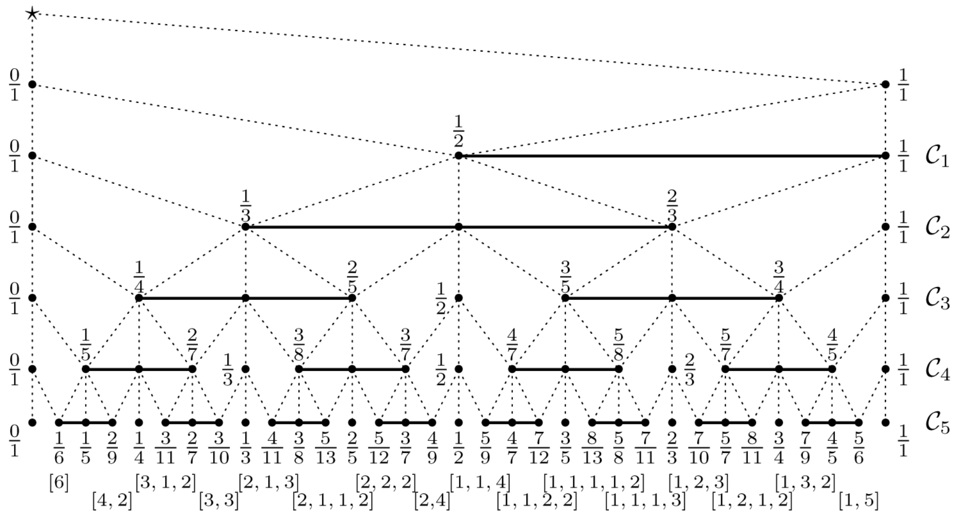

Fig. 4. The Pascal triangle with memory $\mathcal{D}$ and the sets $\mathcal{C}_{n+1} = F^{-n}\left[\frac{1}{2}, 1\right]$.

It is elementary to check the following properties of the Pascal triangle with memory:

(i) For consecutive neighbors $\frac{c}{d} \bullet \quad \bullet \frac{a}{b}$ at the same level one has $ad - bc = 1$.

(ii) $q(n, k)$ counts the number of paths connecting (n, k) and the top vertex $\star$, while $p(n, k)$ counts the number of paths connecting (n, k) and $\star$ after removing all vertices $(m, 0)$, $m \ge 0$.

(iii) The labels $r(n, k)$ with k odd of the "new stuff" at level n are given exactly by the rational numbers $[a_1, \ldots, a_t]$ with $a_1 + \cdots + a_t = n$ and $a_t \ge 2$. The set $\mathcal{L}_n = \{r(n, k) : 0 \le k \le 2^n, n \ge 0\}$ of all labels at level n coincides with the rationals with $a_1 + \cdots + a_t \le n$ and $a_t \ge 2$. In particular $(r(n, k))_{0 \le k \le 2^n, k \,\mathrm{odd}, n \ge 0}$ gives an enumeration of $\mathbb{Q} \cap (0, 1)$ without repetition.

(iv) Irrationals in $[0, 1]$ are in one-to-one correspondence with drunkard's walks on $\mathcal{D}$, encoded by infinite products in $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. More precisely, if $\frac{p_k}{q_k}$ denote the convergents of $[a_1, a_2, \ldots]$, then

$$B^{a_1}A^{a_2}\cdots B^{a_{2m-1}}A^{a_{2m}} = \begin{pmatrix} q_{2m} & q_{2m-1} \\ p_{2m} & p_{2m-1} \end{pmatrix},$$
$$B^{a_1}A^{a_2}\cdots A^{a_{2m}}B^{a_{2m+1}} = \begin{pmatrix} q_{2m} & q_{2m+1} \\ p_{2m} & p_{2m+1} \end{pmatrix}.$$

(v) $(q(n,k))_{n\geq 0, 0\leq k<2^n}$ gives in lexicographic order the Stern-Brocot sequence (θ_n) with generating function

$$\Theta(X) := \sum_{n=0}^{\infty} \theta_n X^n = \prod_{k=0}^{\infty} \big(1 + X^{2^k} + X^{2^{k+1}}\big),$$

and obeying the recursive relation $\theta_0 = 1$, $\theta_{2k+1} = \theta_k$, $\theta_{2k+2} = \theta_k + \theta_{k+1}$.

(vi) The map $r(n,k) \mapsto \frac{k}{2^n}$ extends to Minkowski's question mark function $? : [0,1] \to [0,1]$ given by $?\big(\frac{p}{q} \boxplus \frac{p'}{q'}\big) = \frac{1}{2}?\big(\frac{p}{q}\big) + \frac{1}{2}?\big(\frac{p'}{q'}\big)$, or by the general formula

$$?[a_1,a_2,a_3,\dots] = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \cdots \tag{2.6}$$

(vii) The function ? is singular, yet strictly increasing, continuous, and surjective, mapping rational numbers onto dyadics and quadratic irrationals onto rationals. In the Stern-Brocot enumeration, the rationals are uniformly distributed with respect to the probability measure $d?$ (thus not with respect to the Lebesgue measure):

$$\lim_{n\to\infty} \frac{\#\{k \in [0,2^n] : r(n,k) \leq x\}}{2^n+1} = ?(x), \qquad \forall x \in [0,1].$$

2.5. **Ergodic properties of the Farey map**

The Farey map defined in (1.4) and originally defined in [17] and [26] has two inverse branches, given by $F_1(y) = \frac{y}{1+y}$ and $F_2(y) = \frac{1}{1+y}$, that is:

$$F_1\big([a_1,a_2,\dots]\big) = [1+a_1,a_2,\dots], \qquad F_2\big([a_1,a_2,\dots]\big) = [1,a_1,a_2,\dots]. \tag{2.7}$$

The description of the set $\mathcal{L}_n$ described in property (iii) above shows that $F(\mathcal{L}_n) = \mathcal{L}_{n-1}$ and $F_1(\mathcal{L}_n) \cup F_2(\mathcal{L}_n) = \mathcal{L}_{n+1}$. Equalities (2.7) immediately lead to

$$\mathcal{C}_n := F^{-(n-1)}\Big[\frac{1}{2},1\Big] = \big\{[a_1,a_2,\dots] \in \mathbb{I} : \exists k,\ a_1+\cdots+a_k = n\big\}.$$

The dynamical system $([0,1],\mathcal{B},F,\nu)$ is conservative ergodic. Furthermore, the map F is exact [28], *i.e.*

$$A \in \mathfrak{T}(F) := \bigcap_{n=1}^{\infty} F^{-n}\mathcal{B} \implies \nu(A)\nu(A^c) = 0.$$

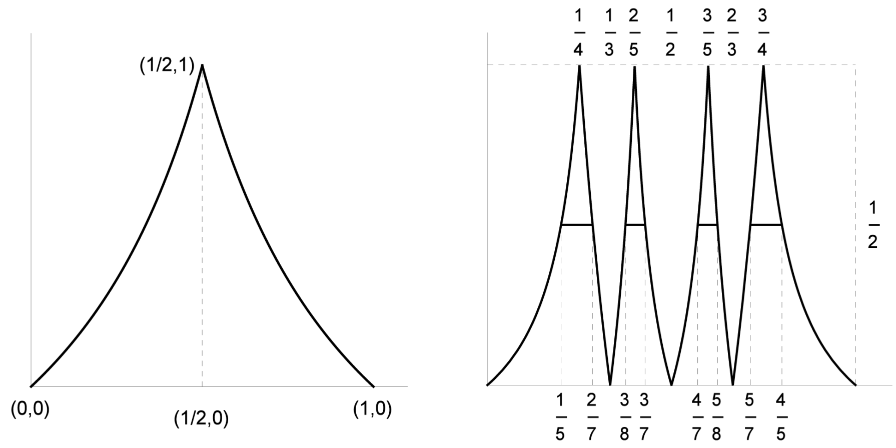

Fig. 5. The graphs of F and $F^3 = F \circ F \circ F$.

The associated Perron-Frobenius operator $\widehat{F} : C[0,1] \to C[0,1]$ is given by

$$(\widehat{F}g)(y) = F_1(y)g\big(F_2(y)\big) + F_2(y)g\big(F_1(y)\big),$$

and satisfies, with $\phi_0(x) = x$, the equality

$$\lambda(F^{-n}[u,1]) = \int_u^1 (\widehat{F}^n \phi_0)(x)\mathrm{d}\nu(x), \quad \forall u \in [0,1].$$

Another useful property of $\widehat{F}$ is that the linear space $\{g \in C^2[0,1] : g' \geq 0, g'' \leq 0\}$ is $\widehat{F}$-invariant [28].

In 2006 Fiala and Kleban conjectured that

$$\lim_n \lambda(\mathcal{C}_n) = 0.$$

This question was positively answered by Kesseböhmer and Stratmann:

Theorem 2.6 ([28]). *The sequence* $\big(\lambda(\mathcal{C}_n)\big)$ *is strictly decreasing and for every* $0 < \alpha \leq \beta \leq 1$ *one has*

$$\lambda\big(F^{-n}[\alpha,\beta]\big)\big) \sim \frac{\log(\beta/\alpha)}{\log n} \qquad \textit{as } n \to \infty.$$

The proof employed some infinite ergodic theory results of Aaronson [1], specifically the inverse relationship between the wandering rate of a uniformly returning set of F and the decay of the iterates $\widehat{F}^n$. For a detailed discussion see also [24].

More recently, an effective asymptotic formula was proved by Heersink, using an adaptation of Freud's effective version [19] of Karamata's tauberian theorem.

THEOREM 2.7 ([22]). *For every $0 < \alpha \leq \beta \leq 1$ one has*

$$\lambda\big(F^{-n}[\alpha,\beta]\big) = \frac{\log(\beta/\alpha)}{\log n}\bigg(1 + O_{\alpha,\beta}\Big(\frac{1}{\log n}\Big)\bigg).$$

The monotonicity of the sequence $\big(\lambda(F^{-n}[\alpha,1])\big)$ does play an important role in both proofs. The analysis of the spectrum of the Perron-Frobenius operator $\widehat{F}$ acting on various function spaces is complicated. It turns out that these operators do not have a spectral gap [11,25], which makes the problem of obtaining an optimal error term delicate. This is in contrast with the situation of the Gauss map G, whose corresponding Perron-Frobenius operator $\widehat{G}$, defined algebraically by (2.8) below, can be realized as a compact self-adjoint operator on a certain space of functions (see [23] and references therein), providing a complete answer to the original Gauss problem of estimating the Lebesgue measure $\lambda\big(G^{-n}[\alpha,\beta]\big)$ as $n \to \infty$.

2.6. **A noncommutative analogue of the interval** $[0,1]$ **and of the Gauss map**

Considering the Pascal triangle with memory $\mathcal{D}$ as a Bratteli diagram, one gets an AF C^*-algebra $\mathfrak{A}$, with center $Z(\mathfrak{A}) = C[0,1]$ and interesting number theoretical properties [9, 33, 34]. The structure and the Jacobson topology of its primitive spectrum, together with the nature of the isomorphism between its space of traces and the space of probability measures on $[0,1]$, turn the C^*-algebra $\mathfrak{A}$ into a noncommutative analogue of the interval $[0,1]$. In particular, this C^*-algebra captures the continued fraction decomposition of every number $\theta \in \mathbb{I}$ through the dimension group of a corresponding primitive of $\mathfrak{A}/I_\theta$, which is isomorphic to the Effros-Shen AF-algebra $\mathcal{C}_\theta$ with dimension group isomorphic to the ordered group $\mathbb{Z} + \mathbb{Z}\theta$. A genuinely noncommutative extension of the classical Gauss map G, or rather of the Perron-Frobenius operator $\widehat{G} : C[0,1] = Z(\mathfrak{A}) \to C[0,1] = Z(\mathfrak{A})$,

$$(\widehat{G}f)(x) = \sum_{k=1}^{\infty} \frac{x+1}{(x+k)(x+k+1)} f\bigg(\frac{1}{x+k}\bigg), \tag{2.8}$$

was proposed by Eckhardt [14], in the form of a unital completely positive map $\Phi : \mathfrak{A} \to \mathfrak{A}$ that extends $\widehat{G}$ and acts on each primitive ideal I_θ in a way that is compatible with the action of the Gauss map G on $\theta \in \mathbb{I}$.

3. FAREY FRACTIONS, THE BCZ MAP AND APPLICATIONS

3.1. Elementary properties

The number of elements of the set $\mathcal{F}_Q$ of Farey fractions of order Q is given by

$$N_Q = \sum_{q=1}^{Q} \varphi(q) = \frac{3Q^2}{\pi^2} + O(Q \log Q).$$

Arranging the elements of $\mathcal{F}_Q$ in increasing order, it is well-known that two reduced rational numbers $\gamma = \frac{a}{q} < \gamma' = \frac{a'}{q'}$ are *adjacent* in $\mathcal{F}_Q$ if and only if the following conditions hold:

$$\begin{cases} 0 < a \leq q \leq Q, \quad 0 < a' \leq q' \leq Q, \\ q + q' > Q, \quad a'q - aq' = 1. \end{cases}$$

Three Farey fractions of order Q, $\gamma_j = \frac{a_j}{q_j} < \gamma_{j+1} = \frac{a_{j+1}}{q_{j+1}} < \gamma_{j+2} = \frac{a_{j+2}}{q_{j+2}}$, are adjacent in $\mathcal{F}_Q$ if and only if

$$q_{j+2} = \left[\frac{Q + q_j}{q_{j+1}}\right] q_{j+1} - q_j. \tag{3.1}$$

3.2. The BCZ map

Consider the map $T : \mathcal{T} \to \mathcal{T}$ defined in (1.5). Equality (3.1) is actually equivalent to

$$\left(\frac{q_{j+1}}{Q}, \frac{q_{j+2}}{Q}\right) = T^j\left(\frac{q_1}{Q}, \frac{q_2}{Q}\right), \tag{3.2}$$

where $\gamma_j = \frac{a_j}{q_j}$ denote the elements of $\mathcal{F}_Q$ written in increasing order, and T is the bijective area-preserving map introduced in [5] and defined by (1.5). The sets $\mathcal{T}_k = \{(x,y) \in \mathcal{T} : [\frac{1+x}{y}] = k\}$, $k = 1, 2, \ldots$, give a partition of the triangle $\mathcal{T}$. On each set $\mathcal{T}_k$ the map T is linear and $T(x,y) = (y, ky - x)$, but $\mathcal{T}_k$ is not T-invariant (see Fig. 6). The pair $(T, \mathcal{T})$ is a fibred system in the sense of [36].

The area-preserving T has a number of interesting features:

(i) The T-orbit of any rational point in $\mathcal{T}$ is finite and

$$\#\left\{T^j\left(\frac{r}{n}, \frac{s}{n}\right) : j \in \mathbb{N}\right\} \leq \frac{N_Q}{\gcd(r,s)}.$$

(ii) According to (3.2), T can be used to generate the Farey fractions of ANY order: if $\gamma_1 = \frac{1}{Q} < \gamma_2 = \frac{1}{Q-1} < \cdots < \gamma_{N_Q} = 1$ denote the elements of $\mathcal{F}_Q$, $\gamma_0 = \gamma_{N_Q} = 1$, then $q_0 = 1$, $q_1 = Q$, $q_2 = Q - 1$, and

$$(q_j, q_{j+1}) = QT^j\left(\frac{1}{Q}, 1\right), \qquad 1 \leq j \leq N_Q - 1.$$

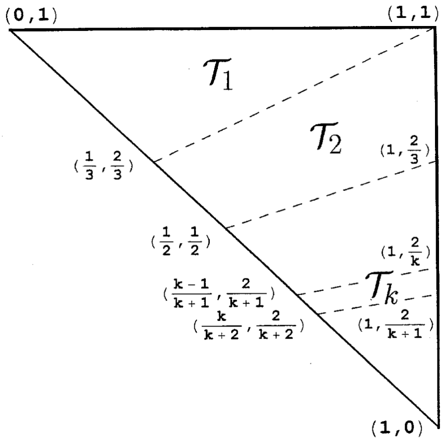

Fig. 6. The decomposition $\mathcal{T} = \cup_{k=1}^{\infty} \mathcal{T}_k$.

The fractions γ_j can be retrieved from the pair (q_j, q_{j+1}) by solving the equation $q_j a_{j+1} - q_{j+1} a_j = 1$, with $0 < a_j \leq q_j$ and $0 < a_{j+1} \leq q_{j+1}$.

(iii) The function $r(x,y) = \frac{1}{xy}$ is integrable on $\mathcal{T}$ and

$$\iint_{\mathcal{T}} \frac{\mathrm{d}x\mathrm{d}y}{xy} = \zeta(2).$$

(iv) Points in the orbit of T cannot spend a long time near the corners $(0,1)$ and $(1,0)$ of $\mathcal{T}$. Concretely, if we write $T^n = (L_n, L_{n+1})$ and assume that $\max\{L_i(x,y), L_j(x,y)\} \leq \frac{1}{4r+2}$ for some $i \neq j$ and $(x,y) \in \mathcal{T}$, then $|i-j| > r+1$.

(v) The (periodic) orbit of the point $\left(\frac{1}{Q}, 1\right)$ is uniformly distributed in $\mathcal{T}$ with respect to the Lebesque measure λ_2 on $\mathcal{T}$, that is:

$$w^* - \lim_{Q\to\infty} \frac{1}{\#(\mathcal{F}_Q \cap I)} \sum_{\gamma_i \in \mathcal{F}_Q \cap I} \delta_{T^i(\frac{1}{Q},1)} = 2\lambda_2,$$

for any interval $I = I(Q) \subset [0,1]$ with $|I| \gg_\varepsilon Q^{-1/2+\varepsilon}$ for some $\varepsilon > 0$. This can be proved using the Weil bound on Kloosterman sums as in Section 2 of [6]. When $|I| = [0,1]$ the result, first shown in [27], can also be proved directly using Möbius inversion.

(vi) The periodic points of T coincide exactly with the points $(x, tx) \in \mathcal{T}$ of rational slope t (cf. [2]).

(vii) The iterates of T describe the limiting joint distribution $\nu^{(h)}$ of h consecutive gaps in $\mathcal{F}_Q$, which was shown [3] to be given, for any rectangular box $\mathcal{B} \subset \mathbb{R}_+^h$, by

$$\nu^{(h)}(\mathcal{B}) = 2\text{Area}\, \Phi_h^{-1}(\mathcal{B}),$$

where

$$\Phi_h : \mathcal{T} \to \mathbb{R}_+^h, \quad \Phi_h = \frac{3}{\pi^2}\Big(\frac{1}{L_0 L_1}, \frac{1}{L_1 L_2}, \dots, \frac{1}{L_{h-1} L_h}\Big).$$

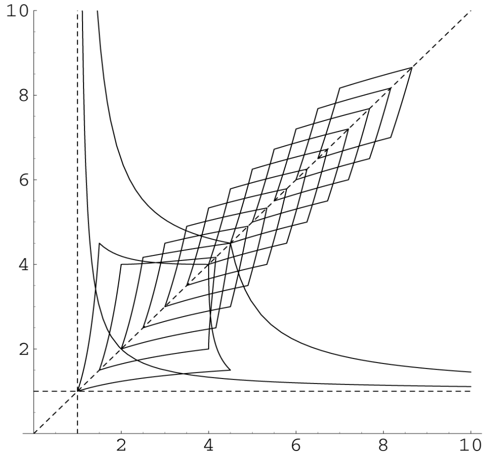

Fig. 7. The support of the measure $\nu^{(2)}$.

3.3. **Two applications of the BCZ map**

It was conjectured by Hall in [20] that for every positive integer h there exists a constant $C(h)$ such that

$$\sum_{i=1}^{N(Q)-h} (\gamma_{i+h} - \gamma_i)^2 = \frac{2(2h-1)}{\zeta(2)} \cdot \frac{\log Q}{Q^2} + \frac{C(h)}{Q^2} + o_h\Big(\frac{1}{Q^2}\Big). \tag{3.3}$$

Hall was able to prove that (3.3) is valid for $h = 1$ and $h = 2$. The main term was shown by Huxley to be always as predicted. The general conjecture was solved in [5] employing the BCZ map and some of its properties, together with asymptotic estimates for the number of primitive lattice points in plane regions, as follows:

Theorem 3.1 ([5]). *Hall's conjecture is valid for every $h \geq 3$ with*

$$C(h) = \frac{1}{\zeta(2)}\bigg(2(2h-1)\Big(\gamma - \frac{\zeta'(2)}{\zeta(2)}\Big) + h + 2(h-1)B + \sum_{n=2}^{h-1}(h-n)I_n\bigg),$$

where $I_n = \iint_{\mathcal{T}} r(\omega)(r \circ T^n)(\omega)\, \mathrm{d}\omega.$

In [15] Erdős, Szüsz and Turán raised the problem of existence, for every two fixed real numbers $A > 0$, $c > 1$, of the limit $f(A, c) = \lim_N \lambda(S(N, A, c))$, where

$$S(N, A, c) = \Big\{\theta \in (0, 1) : \exists\, a, q, \gcd(a, q) = 1, N < q \le cN, \Big|\theta - \frac{a}{q}\Big| \le \frac{A}{q^2}\Big\}.$$

Kesten showed that in the range $Ac \le 1$ the quantity $f(A, c)$ exists and computed it explicitly [29]. The existence of $f(A, c)$ was further proved, using probabilistic arguments but without an explicit formula, by Kesten and Sós [30]. A new approach relying on the above properties of the BCZ map T, pursued by Xiong and Zaharescu, and independently by this author, led to the following result:

THEOREM 3.2 ([8, 40]). *The limit $f(A, c)$ exists for every $A > 1$ and $c > 0$ and can be explicitly expressed in terms of iterates of the map T. Furthermore $S(A, N, c)$ is uniformly distributed when θ is restricted to intervals of* $[0, 1]$.

3.4. **The horocycle flow as a suspension flow over the BCZ map**

Consider the group $G = \mathrm{PSL}_2(\mathbb{R})$ acting on the upper half plane $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$. Consider also the discrete subgroup $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The group G acts naturally on the unit tangent bundle $T^1(\Gamma\backslash\mathbb{H}) \cong \Gamma\backslash T^1\mathbb{H} \cong \Gamma\backslash G$. In this representation the geodesic flow on $T^1(\Gamma\backslash\mathbb{H})$ acts as $\Gamma g \mapsto \Gamma g g_t$ with $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, while the (unstable and stable) horocyclic flows acts as $\Gamma g \mapsto \Gamma g h_s$ and respectively $\Gamma g \mapsto \Gamma g h_s$, with $u_s = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$ and $h_s = \left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right)$. These matrices satisfy the fundamental relations

$$g_t h_s g_{-t} = h_{se^t} \qquad \text{and} \qquad J h_s J = u_s,$$

with $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The Howe-Moore theorem states [4] that the geodesic and horocyclic flows on $T^1(\Gamma\backslash\mathbb{H})$ are mixing, that is $\mu(T_t A \cap B) \overset{t\to\pm\infty}{\longrightarrow} \mu(A)\mu(B)$ for any measurable sets A and B, where $\mu = \mu_{\Gamma\backslash G}$ is the right G-invariant measure on $T^1(\Gamma\backslash\mathbb{H}) = \Gamma\backslash G$ (see [4] for a short proof in this situation). In particular these flows are ergodic, *i.e.* if $\mu(T_t A) = \mu(A)$, $\forall t \in \mathbb{R}$, then $\mu(A)\mu(A^c) = 0$.

Given a finite measure space $(\Omega, \mathcal{M}, \mu)$, an invertible μ-preserving transformation $T : \Omega \to \Omega$, and a "roof" function $r : \Omega \to (c, \infty)$, $c > 0$, consider the product measure $\widetilde{\mu} = \mu \times \lambda$ and the $\widetilde{\mu}$-preserving flow $T_t(\omega, s) = (\omega, s + t)$ on the space $X = \Omega \times \mathbb{R}$. Using the identification $\big(\omega, r(\omega)\big) \sim \big(T(\omega), 0\big)$, the flow (T_t) induces a flow on $\widetilde{\Omega} = \big\{(\omega, t) : \omega \in \Omega, 0 \leqslant t < r(\omega)\big\}$, also denoted (T_t) and called the *suspension flow* (or *special flow*) for T under r. The induced measure $\widetilde{\mu}$ is finite on $\widetilde{\Omega}$ if and only if $r \in L^1(\mu)$, and in this case the transformation T

is ergodic on Ω if and only if the flow (T_t) is ergodic on $\widetilde{\Omega}$. For more details see, *e.g.*, [16].

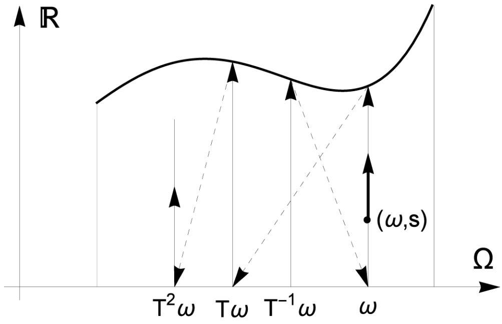

Fig. 8. The horocycle flow as a suspension flow.

The geodesic flow on $T^1(\Gamma\backslash\mathbb{H})$ is measurably isomorphic to a suspension flow over a two-sheeted extension of the natural extension of the RCF Gauss map (see, *e.g.*, [16]). In particular, this gives a geometric proof for the ergodicity of the Gauss map. More recently, Athreya and Cheung proved the following result relating the horocycle flow on $T^1(\Gamma\backslash\mathbb{H})$ with the BCZ map:

THEOREM 3.3 ([2]). *The horocycle flow is the suspension flow of the BCZ transformation T under the roof function $r(a,b)=\frac{1}{ab}$.*

It is useful to identify $T^1(\Gamma\backslash G)=\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ with the space X_2 of unimodular lattices via $\Gamma\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \longleftrightarrow \mathbb{Z}^2 g=\mathbb{Z}(a,b)+\mathbb{Z}(c,d)$. Consider the lattices $\Lambda_{a,b}=\Gamma\left(\begin{smallmatrix} a & 0 \\ b & 1/a\end{smallmatrix}\right)$ and the set $\Omega=\{\Lambda_{a,b}:(a,b)\in\mathcal{T}\}\subset\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$. A unimodular lattice λ is called *horizontally short* if it contains a vector of the form $(a,0)\in\Lambda$ with $0<a\le 1$, and is called *vertically long* if it contains a vector $(0,b)$ with $0<b\le 1$. Theorem 3.3 was proved in [2] by assembling the following three elementary lemmas:

LEMMA 3.4. *A lattice Λ is horizontally short if and only if $\Lambda\in\Omega$.*

LEMMA 3.5. *Let $(a,b)\in\mathcal{T}$ and $s_0=R(a,b)=\frac{1}{ab}$. Then*
(i) $h_{s_0}(\Lambda_{a,b})=\Lambda_{T(a,b)}\in\Omega$.
(ii) $h_s(\Lambda_{a,b})\notin\Omega$ *whenever* $0<s<s_0$.

LEMMA 3.6. *If Λ is not vertically short, then there exists $s_1\in\mathbb{R}$ such that $h_{s_1}(\Lambda)\in\Omega$.*

Theorem 3.3 has two important consequences, showing that the BCZ map is ergodic and has zero measurable entropy. It is not known at this time whether T is mixing. It would also be interesting to establish these two properties in a more direct way, without appealing to the properties of the horocycle flow.

The situation where $\Gamma = PSL_2(\mathbb{Z})$ is replaced by a finite index subgroup was subsequently analyzed by Heersink [21], and led to new results concerning the gap distribution of Farey fractions and to a solution of the Erdős-Szüsz-Turán problem when congruence conditions are imposed.

3.5. **A concrete way of enumerating the rational numbers. The Newman map**

Perhaps the most explicit way of enumerating the rational numbers is provided by the Newman map [31] defined by (1.6). The map T is a bijection and has the surprising feature that the function

$$n \mapsto T^n(0) \in \mathbb{Q} \cap (0, \infty)$$

realizes an explicit bijection between the sets $\mathbb{N}$ and $\mathbb{Q} \cap (0, \infty)$.

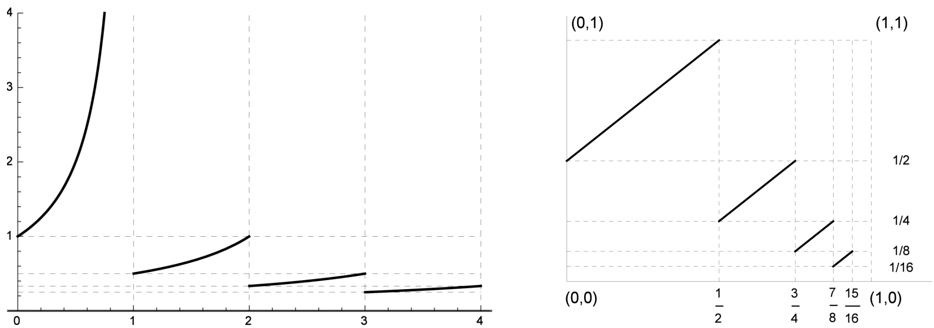

Fig. 9. The Newman map and the Kakutani-von Neumann odometer.

Furthermore, T is conjugated to the Kakutani-von Neumann odometer ϕ, which is the Lebesgue-measure, piecewise linear, invertible, uniquely ergodic transformation on $[0,1)$ defined by $\phi(x) = x - 1 + \frac{3}{2^{n+1}}$ if $x \in [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$. More precisely, taking $h : [0,\infty) \to [0,1)$, $h(x) = \frac{x}{x+1}$, one has [10]

$$T = (? \circ h)^{-1} \circ \phi \circ (? \circ h).$$

It is well-known that the question mark function can also be used to linearize the Farey and the Gauss maps. Employing formula (2.6), (1.3) and (1.2) it is plain to check that

$$F = ?^{-1} \circ F_0 \circ ? \qquad \text{and} \qquad G = ?^{-1} \circ G_0 \circ ?,$$

where F_0 and G_0 are the piecewise linear maps on $[0,1]$ given by $F_0(x) = 2x$ if $x \in [0, \frac{1}{2}]$, $F_0(x) = 1 - 2x$ if $x \in [\frac{1}{2}, 1]$, and respectively $G_0(x) = 2(1 - 2^n x)$ if $x \in [2^{-n-1}, 2^{-n})$, $n \in \mathbb{Z}_{\geq 0}$.

The Newman map is closely related to the Pascal triangle with memory by the following formulas:

$$T\Big(\frac{q(n,k)}{q(n,k+1)}\Big) = \frac{q(n,k+1)}{q(n,k+2)} \qquad \text{and} \qquad T\Big(\frac{p(n,k)}{p(n,k+1)}\Big) = \frac{p(n,k+1)}{q(p,k+2)}.$$

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