HARMONIC BERGMAN KERNELS AND TOEPLITZ OPERATORS ON THE BALL WITH RADIAL MEASURES*

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We consider harmonic Bergman spaces and Toeplitz operators on the ball. In this note, we deal with radial measures as weight and symbol. For two radial measures, we introduce an averaging function, to discuss conditions for corresponding Toeplitz operators to be bounded or compact. We also discuss the boundary behavior of the harmonic Bergman kernels.

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1. INTRODUCTION

Bergman spaces were introduced as function spaces of square integrable holomorphic functions on the unit disk in the complex plane. By the Cauchy integral formula, *i.e.*, mean value properties, Bergman spaces are Hilbert spaces with reproducing kernel. Since mean value properties also hold for harmonic functions, spaces of harmonic functions have similar properties and are studied by many mathematicians.

We begin with a brief introduction to harmonic Bergman spaces.

1.1. Harmonic Bergman spaces

Let Ω be a domain in the n-dimensional Euclidean space $\mathbb{R}^n$ and put

$$b^2(\Omega) := \mathrm{Harm}(\Omega) \cap L^2(\Omega, \mathrm{d}V),$$

where $\mathrm{Harm}(\Omega)$ is the totality of (real-valued) harmonic functions on Ω, and $\mathrm{d}V$ denotes the usual n-dimensional Lebesgue measure. We call $b^2(\Omega)$ the harmonic Bergman space on Ω. Traditionally, from connections with spaces of holomorphic functions, complex-valued harmonic functions are usually considered. Nevertheless, since the Laplacian is a real operator, in this note,

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we assume that harmonic functions are always real-valued. This makes no difference.

First, we remark the following mean value property. Let $\mathbb{B}$ be the open unit ball in the n-dimensional Euclidean space $\mathbb{R}^n$ and ρ be a fixed C^∞ function of $|x|$ on $\mathbb{R}^n$ such that $\mathrm{supp}(\rho) \subset \mathbb{B}$ and $\int \rho \mathrm{d}V = 1$.

LEMMA 1.1. *For $\tau > 0$, we put $\rho_\tau(x) := \tau^{-n}\rho(\tau^{-1}x)$ and $\Omega_\tau := \{x \in \Omega | \delta_\Omega(x) > \tau\}$, where $\delta_\Omega(x)$ is the distance from x to the boundary $\partial\Omega$. Then we have*

$$u = u * \rho_\tau \quad \textit{on} \quad \Omega_\tau$$

for every $u \in \mathrm{Harm}(\Omega)$, where $*$ denotes the convolution on $\mathbb{R}^n$.

From the above mean value property, follows the boundedness of the point evaluation on harmonic Bergman spaces.

PROPOSITION 1.1. *There exists a constant C such that*

$$|u(x)| \leq C\delta_\Omega(x)^{-n/2}\|u\|_{L^2(\Omega)}$$

for every $u \in b^2(\Omega)$.

Proof. Let $x \in \Omega$. For $\tau < \delta_\Omega(x)$, by the Schwarz inequality, we have

$$\begin{aligned} |u(x)| &= \Big| \int u(x-y)\rho_\tau(y)\mathrm{d}V(y) \Big| \leq \|u\|_{L^2(\Omega)} \Big(\int \big(\tau^{-N}|\rho(\tau^{-1}y)|\big)^2 \mathrm{d}V(y) \Big)^{\frac{1}{2}} \\ &= \tau^{-n/2}\|u\|_{L^2(\Omega)} \cdot \|\rho\|_{L^2(\Omega)}. \end{aligned}$$

Letting $\tau \to \delta_\Omega(x)$, we have the proposition. $\square$

The above boundedness of the point evaluation implies that:

1. $b^2(\Omega)$ is a closed subspace of $L^2(\Omega)$,
2. for each $x \in \Omega$, there uniquely exists $R(x, \cdot) \in b^2(\Omega)$ such that

$$u(x) = \int_\Omega R(x,y)u(y)\mathrm{d}V(y)$$

   for $u \in b^2(\Omega)$, and
3. the integral operator Q defined by the kernel R is the orthogonal projection from $L^2(\Omega)$ onto $b^2(\Omega)$.

In fact, since by Proposition 1.1, L^2-convergence implies uniform convergence on compact sets on $b^2(\Omega)$, the assertion 1 holds. The assertion 2 follows from the fundamental Riesz representation theorem for Hilbert spaces. Concerning the assertion 3, we have only to remark that for $f \in L^2(\Omega)$ in the orthogonal complement of $b^2(\Omega)$, $Qf(x) = \langle R(x, \cdot), f\rangle = 0$.

The kernel R is called the harmonic Bergman kernel. We list some properties of R, which follow easily from the above.

1. $R(x,y) = \int R(y,z)R(x,z)\mathrm{d}V(z) = R(y,x)$,
2. $R(x,x) = \int R(x,z)R(x,z)\mathrm{d}V(z) = \|R(x,\cdot)\|^2_{L^2\Omega)} \geq 0$, and
3. $R(x,y) = \sum_j e_j(x)e_j(y)$, where $(e_j)_j$ is a complete orthonormal system in $b^2(\Omega)$.

We have a simple estimate of the harmonic Bergman kernel.

Proposition 1.2.

$$R(x,x) \leq C\delta_\Omega(x)^{-n}.$$

Proof. Since $R(x,\cdot) \in b^2(\Omega)$, we have

$$R(x,x) \leq C\delta_\Omega(x)^{-n/2}\|R(x,\cdot)\|_{L^2(\Omega)} = C\delta_\Omega(x)^{-n/2}R(x,x)^{1/2},$$

which shows the proposition. □

We give estimates of derivatives of harmonic Bergman functions.

Proposition 1.3. *Let $\beta \in \mathbb{N}_0^n$ be a multi-index, where $\mathbb{N}_0$ is the set of all nonnegative integers. Then we have for $u \in b^2(\Omega)$*

1. $|\partial^\beta u(x)| \leq C\delta_\Omega(x)^{-n/2-|\beta|}\|u\|_{L^2(\Omega)}$,

and

2. $\|\delta_\Omega^{|\beta|}\partial^\beta u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}$

with some constant C.

Proof. We have the assertion 1 in the similar way to Proposition 1.1. To show the assertion 2, we take the Whitney decomposition $\Omega = \cup_j Q_j$, *i.e.*, Q_j are cubes whose edges are parallel with some axis, the length of edges is comparable with the distance from the boundary and the intersection of cubes is included in the boundary of their cubes. Take any j, and put $\tau_j = \mathrm{dist}(Q_j, \partial\Omega)$. Then, for each $x \in Q_j$,

$$|\partial^\beta u(x)| \leq C\tau_j^{-n/2-|\beta|}\|u\|_{L^2(\widetilde{Q_j})}$$

because of the assertion 1, where $\widetilde{Q_j}$ is the cube with the same center and double size of Q_j. Hence,

$$\int_{Q_j} |\delta_\Omega(x)^{|\beta|}\partial^\beta u(x)|^2\mathrm{d}V(x) \leq C\|u\|^2_{L^2(\widetilde{Q_j})}\tau_j^{-n}V(Q_j) \leq C\int_{\widetilde{Q_j}} |u|^2\mathrm{d}V(x).$$

Summing up in j, we have

$$\begin{aligned}\int_\Omega |\delta_\Omega(x)^{|\beta|}\partial^\beta u(x)|^2\mathrm{d}V(x) &= \sum_j \int_{Q_j} |\delta_\Omega(x)^{|\beta|}\partial^\beta u(x)|^2\mathrm{d}V(x)\\ &\leq C\sum_j \int_{\widetilde{Q_j}} |u|^2\mathrm{d}V \leq C\int_\Omega |u|^2\mathrm{d}V,\end{aligned}$$

which completes the proof. □

Next, let Ω be a bounded domain surrounded by C^∞-smooth surfaces in $\mathbb{R}^n$. Then Kang and Koo obtain a sharp estimate of harmonic Bergman kernels in 2001.

Theorem A (Kang-Koo, [5]).

$$|R(x,y)| \le C\frac{1}{(\delta_\Omega(x)+\delta_\Omega(y)+|x-y|)^n}$$
$$R(x,x) \ge C^{-1}\frac{1}{\delta_\Omega(x)^n}$$

with some constant C.

Englis [3] gives precise boundary behavior of harmonic Bergman kernels for smooth bounded domain.

Now, we go back to the case of ball $\Omega=\mathbb{B}$. Then, the explicit closed form of the harmonic Bergman kernel are known (for example, see [1]):

$$R(x,y)=\frac{n-(2n+4-8x\cdot y)|x|^2|y|^2+(n-4)|x|^4|y|^4}{nV(\mathbb{B})(1-2x\cdot y+|x|^2|y|^2)^{n/2+1}}.$$

1.2. **Toeplitz operators**

For $\varphi\in L^\infty(\mathbb{B})$, the Toeplitz operator T_φ is defined by $T_\varphi u=Q(uf)$, *i.e.*,

$$T_\varphi u(x)=\int_{\mathbb{B}}R(x,y)u(y)\varphi(y)\mathrm{d}V(y)$$

for $u\in b^2(\mathbb{B})$. By the recognition that it is not necessary for symbol φ to be bounded, it is natural to discuss Toeplitz operators of measure symbol μ

$$T_\mu u(x)=\int_{\mathbb{B}}R(x,y)u(y)\mathrm{d}\mu(y).$$

The problem is to obtain relations between properties of T_φ and those of φ. For example, J. Miao gives a characterization for T_φ to be compact in 1997.

Theorem B (Miao, [9]). *Let $\varphi\in L^\infty(\mathbb{B})$ be a radial function, i.e., $\varphi(x)=\varphi_r(|x|)$ with some function φ_r on $[0,1)$. Then the following conditions are equivalent:*

1. *T_φ is compact;*
2. *$\tilde\varphi(x)\to 0$ as $|x|\to 1$;*
3. *$\frac{1}{1-r}\int_r^1\varphi_r(t)\mathrm{d}t\to 0$ as $|x|\to 1$.*

Here,

$$\tilde\varphi(x):=\frac{\int_{\mathbb{B}}R(x,y)^2\varphi(y)\mathrm{d}V(y)}{\int_{\mathbb{B}}R(x,y)^2\mathrm{d}V(y)}$$

is the Berezin transform of φ.

Before this, for holomorphic Bergman space on the unit disk, the corresponding result has been shown by B. Korenblum and K. Zhu [6] in 1995.

2. WEIGHTED HARMONIC BERGMAN KERNEL ON $\mathbb{B}$

The weighted harmonic Bergman spaces $b^2_\alpha(\mathbb{B})$, defined by

$$b^2_\alpha(\mathbb{B}) := \mathrm{Harm}(\mathbb{B}) \cap L^2(\mathbb{B}, \mathrm{d}V_\alpha),$$

are also studied by many mathematicians. In particular, the analysis of the Toeplitz operators on the harmonic Bergman space is a main topic of the harmonic Bergman space (for example, see [8, 9, 15]).

In this note, we discuss general measure weight. For a positive Radon measure ν on $\mathbb{B}$, we put

$$b^2_\nu := \mathrm{Harm}(\mathbb{B}) \cap L^2(\mathbb{B}, \mathrm{d}\nu)$$

with $L^2(\mathrm{d}\nu)$-norm $\|\cdot\|$. We mainly treat a radial (spherically symmetric) measure ν on $\mathbb{B}$, *i.e.*, ν is of form $\mathrm{d}\nu(x) = \mathrm{d}\nu_r(r)\mathrm{d}\sigma(\theta)$ with some measure ν_r on $[0, 1)$, where $x = r\theta$, $r \in [0, 1)$, $\theta \in \mathbb{S}$ and σ is the normalized surface measure on $\mathbb{S}$. Let $\mathcal{M}_{rad}$ be the set of all radial finite Radon measures $\mathrm{d}\nu = \mathrm{d}\nu_r\mathrm{d}\sigma$ on $\mathbb{B}$ satisfying $\nu_r([r, 1)) > 0$ for any $r \in [0, 1)$. In this section, we shall see when $\nu \in \mathcal{M}_{rad}$, b^2_ν is a Hilbert space with reproducing kernel, denoted by R_ν, and that the orthogonal projection Q_ν from $L^2(\mathbb{B}, \mathrm{d}\nu)$ to b^2_ν is an integral operator

$$Q_\nu f(x) = \int_{\mathbb{B}} R_\nu(x, y) f(y) \mathrm{d}\nu(y).$$

First, we recall some basic properties for harmonic functions on $\mathbb{B}$. We refer to [1] here. Let $\mathcal{H}_m = \mathcal{H}_m(\mathbb{R}^n)$ be the space of all harmonic homogeneous polynomials of degree m and for a set D in $\mathbb{R}^n$ denote by $\mathcal{H}_m(D)$ the space of their restriction to D. We put $B_r = \{|x| < r\}$ and $S_r = \partial B_r = \{|x| = r\}$. It is well-known that for $r > 0$, the Hilbert space $L^2(S_r) = L^2(S_r, \mathrm{d}\sigma_r)$ has an orthogonal decomposition

$$L^2(S_r, \mathrm{d}\sigma_r) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S_r), \tag{1}$$

where σ_r is the surface measure on S_r. We denote by $L^2(\mathbb{S}) = L^2(\mathbb{S}, \mathrm{d}\sigma)$, where σ is the normalized surface measure on $\mathbb{S}$, *i.e.*, $\mathrm{d}\sigma(\theta) = \mathrm{d}\sigma_1(\theta)/\sigma_1(\mathbb{S})$. The orthogonal projection from $L^2(\mathbb{S})$ to $\mathcal{H}_m(\mathbb{S})$ is an integral operator by a kernel $Z_m(\theta, \eta)$

$$Z_m(\theta, \eta) = \sum_{k=1}^{h_m} p_{k,m}(\theta) p_{k,m}(\eta),$$

where $\{p_{k,m}\}_{k=1}^{h_m}$ is an orthogonal basis on $\mathcal{H}_m(\mathbb{S}) \subset L^2(\mathbb{S})$. Here $h_m = \dim \mathcal{H}_m$. We remark that Z_m is real valued and $Z_m(\theta, \theta) = h_m$, because

$$Z_m(\theta,\theta) = \sum_{k=1}^{h_m} |p_{k,m}(\theta)|^2 = \|Z_m(\theta,\cdot)\|_{L^2(\mathrm{d}\sigma)}^2$$

is independent of θ. The kernel $Z_m(\theta, \eta)$, which is called the zonal harmonic, can be extended to $\mathbb{R}^n \times \mathbb{R}^n$ as follows:

$$Z_m(x,y) = \sum_{k=0}^{h_m} p_k(x)p_k(y) = \sum_{k=0}^{h_m} |x|^m |y|^m p_k(\theta) r_k(\eta)$$

for $x = |x|\theta, y = |y|\eta \in \mathbb{R}^n$. Then, we have

$$Z_m(x,x) = h_m |x|^{2m} \tag{2}$$

and

$$|Z_m(x,y)| \leq Z_m(x,x)^{\frac{1}{2}} Z_m(y,y)^{\frac{1}{2}} = h_m |x|^m |y|^m \tag{3}$$

for any $x, y \in \mathbb{R}^n$.

We begin by remarking the following (see, for example [1, p. 84]).

LEMMA 2.1. *Let f be a harmonic function on $\mathbb{B}$. Then there uniquely exist $\phi_m \in \mathcal{H}_m$ $(m = 0, 1, 2, \cdots)$ such that*

$$f = \sum_{m=0}^{\infty} \phi_m, \tag{4}$$

where the series converges uniformly on any compact subset of $\mathbb{B}$.

For the uniqueness, we remark the following (cf. [1, p.23]).

LEMMA 2.2. *Let $\phi_m \in \mathcal{H}_m$ $(m \geq 0)$. If*

$$\sum_{m=0}^{\infty} \phi_m(x) = 0$$

pointwise in a neighborhood of the origin, then $\phi_m = 0$ for every $m \geq 0$.

The following is a reason why we assume a weight ν is finite.

PROPOSITION 2.1. *Let ν be a radial positive Radon measure on $\mathbb{B}$. If $\nu(\mathbb{B}) = \infty$, then $b_\nu^2 = \{0\}$.*

Proof. We write $\mathrm{d}\nu(x) = \mathrm{d}\nu_r(r)\mathrm{d}\sigma(\theta)$ where $x = r\theta$, $r \in [0, 1)$ and $\theta \in \mathbb{S}$. First, we remark that $\nu(\mathbb{B}) = \infty$ implies $\displaystyle\int_{[\frac{1}{2},1)} r^{2m} \mathrm{d}\nu_r(r) = \infty$, because $\nu(B_{1/2}) < \infty$. Take an arbitrary $f \in b_\nu^2$. By Lemma 2.1, we have

$$f = \sum_{m=0}^{\infty} \phi_m$$

where $\phi_m \in \mathcal{H}_m$ and the series converges uniformly on any compact subset of $\mathbb{B}$. Remarking that $\phi_l \perp \phi_k$ in $L^2(\mathbb{S})$ for $l \neq k$, we have

$$\begin{aligned}
\infty > \int_{\mathbb{B}} |f(x)|^2 \mathrm{d}\nu(x) &= \lim_{\epsilon \to 1} \int_{[0,\epsilon)} \int_{\mathbb{S}} \left(\sum_{m=0}^{\infty} \phi_m(r\theta) \right)^2 \mathrm{d}\sigma(\theta) \mathrm{d}\nu_r(r) \\
&= \lim_{\epsilon \to 1} \int_{[0,\epsilon)} \sum_{m=0}^{\infty} \int_{\mathbb{S}} \phi_m(r\theta)^2 \mathrm{d}\sigma(\theta) \mathrm{d}\nu_r(r) \\
&= \int_{[0,1)} \sum_{m=0}^{\infty} r^{2m} \|\phi_m\|^2_{L^2(\mathbb{S})} \mathrm{d}\nu_r(r) \\
&\geq \|\phi_m\|^2_{L^2(\mathbb{S})} \int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r),
\end{aligned}$$

for any integer $m \geq 0$. Since ν is an infinite positive Radon measure,

$$\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r) \geq \int_{[\frac{1}{2},1)} r^{2m} \mathrm{d}\nu_r(r) = \infty.$$

Therefore, we have $\|\phi_m\|_{L^2(\mathbb{S})} = 0$ for every m, which shows $f = 0$. $\square$

We do not know whether the assumption that ν is radial is really necessary in the above proposition.

In general, b^2_ν is not always a normed space, ($\|f\|_\nu = 0$ does not always imply $f = 0$ on $\mathbb{B}$). A set $E \subset \mathbb{R}^n$ is said to be a uniqueness set for harmonic functions if E has the following property: For every harmonic function f defined on a domain $\Omega \supset E$, "$f = 0$ on E" implies "$f = 0$ on Ω". It is clear that if $\mathrm{supp}(\nu)$ is a uniqueness set for harmonic functions, then b^2_ν is a normed space. When ν is radial, $\mathrm{supp}(\nu)$ is a uniqueness set for harmonic functions except for the Dirac measure at the origin, then b^2_ν is a normed space.

Next, we consider the completeness of b^2_ν.

Proposition 2.2. *Let ν be a finite positive measure on $\mathbb{B}$. Suppose that* $\mathrm{supp}(\nu)$ *is a uniqueness set for harmonic functions. If* $\mathrm{supp}(\nu)$ *is compact on $\mathbb{B}$, then b^2_ν is not complete.*

Proof. We remark that b^2_ν is a normed space, because $\mathrm{supp}(\nu)$ is assumed to be a uniqueness set for harmonic functions. We give the proof by contradiction. Suppose that b^2_ν is complete. Take $0 < r_0 < 1$ with $\mathrm{supp}(\nu) \subset B_{r_0}$ and $\theta_0 \in \mathbb{S}$. For $m \geq 0$, we take $\phi_m \in \mathcal{H}_m$ such that $\phi_m(\theta_0) = \max_{\theta \in \mathbb{S}} \phi_m(\theta) = 1$. Then, we have

$$\begin{aligned}
\|\phi_m\|^2_\nu &= \int_{B_{r_0}} |\phi_m(x)|^2 \mathrm{d}\nu(x) \\
&\leq \sup_{x \in B_{r_0}} |\phi_m(x)|^2 \nu(B_{r_0}) = r_0^{2m} \nu(\mathbb{B}). \qquad (5)
\end{aligned}$$

Next, take $r_1 \in (r_0, 1)$ and put $a_m = 1/r_1^m$. Then, by (5), a series

$$f = \sum_{m=0}^{\infty} a_m \phi_m \tag{6}$$

converges in b_ν^2 by the completeness of b_ν^2. Since f is harmonic on $\mathbb{B}$, by Lemma 2.1, there exist $\psi_m \in \mathcal{H}_m$ such that

$$f(x) = \sum_{m=0}^{\infty} \psi_m(x)$$

which converges uniformly on any compact subset of $\mathbb{B}$. On the other hand, by the construction of ϕ_m, we have

$$\sum_{m=0}^{\infty} a_m |\phi_m(x)| \le \sum_{m=0}^{\infty} \frac{1}{r_1^m} \max_{x \in B_{r_0}} |\phi_m(x)| \le \sum_{m=0}^{\infty} \left(\frac{r_0}{r_1}\right)^m < \infty$$

for $x \in B_{r_0}$, *i.e.*, the series (6) converges uniformly on B_{r_0}. By Lemma 2.2, we have $a_m \phi_m = \psi_m$ for $m \ge 0$. Therefore, we can write

$$f(x) = \sum_{m=0}^{\infty} a_m \phi_m(x)$$

for $x \in \mathbb{B}$. On the other hand, $f(r_1\theta_0) = \infty$, which contradicts the harmonicity of f. This completes the proof. $\square$

In this way, it is a natural condition that ν belongs to $\mathcal{M}_{rad}$. Next, we can show that point evaluation maps are bounded, b_ν^2 is complete and that b_ν^2 has the reproducing kernel for $\nu \in \mathcal{M}_{rad}$.

LEMMA 2.3. *Let $\nu \in \mathcal{M}_{rad}$. Then, for any $x \in \mathbb{B}$ the point evaluation map $f \mapsto f(x)$ from b_ν^2 to $\mathbb{C}$ is bounded. Moreover, for any $r \in [0, 1)$ there exists a positive constant $C(r, \nu) > 0$ such that*

$$|f(x)| \le C(r, \nu) \|f\|_\nu$$

for any $x \in B_r$.

Proof. Let $r \in [0, 1)$ and $x \in B_r$ and take $r_0 \in (r, 1)$. By using the Poisson integral, we have

$$f(x)\nu_r([r_0, 1)) = \int_{[r_0,1)} \int_{S_t} P_t(x, y) f(y) \mathrm{d}\sigma_t(y) \mathrm{d}\nu_r(t)$$

for $f \in b_\nu^2$, where

$$P_t(x, y) = \frac{1}{\sigma_1(\mathbb{S})} \frac{r^2 - |x|^2}{t|x - y|^n},$$

is the Poisson kernel on B_t. Since there exists a constant $C(r_0)$ such that

$$|P_t(x,y)| \leq C(r_0)$$

for $t \in [r_0, 1)$, $x \in B_r$ and $y \in S_t$, we have

$$|f(x)|^2 \nu_r([r_0,1)) \leq C(r_0) \int_{\mathbb{B}\setminus B_{r_0}} |f(x)|^2 \mathrm{d}\nu(x) \leq C(r_0) \|f\|_\nu^2.$$

This implies that the point evaluation map is bounded for $x \in \mathbb{B}$. □

LEMMA 2.4. *Let $\nu \in \mathcal{M}_{rad}$. Then b_ν^2 is complete.*

Proof. Let $\{f_n\} \subset b_\nu^2$ be a Cauchy sequence, which converges to some f in $L^2(\mathbb{B}, \mathrm{d}\nu)$. By Lemma 2.3, the sequence $\{f_n\}$ converges to f uniformly on any compact subset of $\mathbb{B}$. Therefore, f is harmonic on $\mathbb{B}$ which shows b_ν^2 is complete. □

In this way, we have

PROPOSITION 2.3. *Let ν be a radial finite positive Radon measure on $\mathbb{B}$. Then b_ν^2 is a Hilbert space if and only if $\nu \in \mathcal{M}_{rad}$.*

In what follow, we mainly consider $\nu \in \mathcal{M}_{rad}$ which is the set of all radial finite positive measure on $\mathbb{B}$ satisfying $\nu_r([r,1)) > 0$ for any $r \in [0,1)$. By Lemmas 2.3 and 2.4, b_ν^2 is the reproducing kernel Hilbert space if $\nu \in \mathcal{M}_{rad}$. We denote by $R_\nu(x,y)$ the reproducing kernel, which is called the harmonic Bergman kernel. We denote by Q_ν the orthogonal projection from $L^2(\mathbb{B}, \mathrm{d}\nu)$ to b_ν^2, which is an integral operator by the kernel R_ν:

$$Q_\nu f(x) = \int_{\mathbb{B}} R_\nu(x,y) f(y) \mathrm{d}\nu(y),$$

for $x \in \mathbb{B}$ and $f \in L^2(\mathbb{B}, \mathrm{d}\nu)$. Q_m^ν be the orthogonal projection from b_ν^2 to $\mathcal{H}_m(\mathbb{B})$. Since $\mathcal{H}_m$ is of finite dimension, the orthogonal projection Q_m^ν is given by an integral operator

$$Q_m^\nu f(x) = \int_{\mathbb{B}} Z_m^\nu(x,y) f(y) \mathrm{d}\nu(y),$$

for $x \in \mathbb{B}$ and $f \in b_\nu^2$.

Next, we remark the orthogonal decomposition of b_ν^2.

PROPOSITION 2.4. *Let $\nu \in \mathcal{M}_{rad}$. Then, we have the following orthogonal decomposition*

$$b_\nu^2 = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{B})$$

Proof. First, clearly, $\{\mathcal{H}_m(\mathbb{B})\}$ are mutually orthogonal closed subspace of the Hilbert space b^2_ν. Then, we have only to show that $b^2_\nu \subset \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{B})$. Let $f \in b^2_\nu$. Then $\sum_{m=0}^{\infty} Q^\nu_m f$ converges to some g in b^2_ν. By Lemma 2.2, the above series converges uniformly on compact subsets of $\mathbb{B}$. On the other hand, by Lemma 2.1,

$$f(x) = \sum_{m=0}^{\infty} \phi_m(x)$$

for some $\phi_m \in \mathcal{H}_m$, which converges uniformly on any compact subset of $\mathbb{B}$. Hence, we have

$$\begin{aligned} Q^\nu_m f(x) &= \int_{\mathbb{B}} Z^\nu_m(x,y) f(y) \mathrm{d}\nu(y) = \lim_{r\to 1} \int_{B_r} Z^\nu_m(x,y) \sum_{k=0}^{\infty} \phi_k(y) \mathrm{d}\nu(y) \\ &= \lim_{r\to 1} \sum_{k=0}^{\infty} \int_{[0,r)} t^{m+k} \int_{\mathbb{S}} Z^\nu_m(x,\theta)\phi_k(\theta) \mathrm{d}\sigma(\theta) d\nu_r(t) \\ &= \lim_{r\to 1} \int_{B_r} \phi_m(y) Z^\nu_m(x,y) \mathrm{d}\nu(y) = \phi_m(x). \end{aligned}$$

Therefore, $g(x) = f(x)$ for $x \in \mathbb{B}$, which implies that

$$f = \sum_{m=0}^{\infty} Q^\nu_m f \in \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

This completes the proof. □

Finally, we state some properties of R_ν. The following follows from Proposition 2.4.

PROPOSITION 2.5. *Let $\nu \in \mathcal{M}_{rad}$. Then, we have*

$$R_\nu(x,y) = \sum_{m=0}^{\infty} Z^\nu_m(x,y) = \sum_{m=0}^{\infty} \frac{Z_m(x,y)}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)},$$

which converges uniformly on any compact set of $\mathbb{B} \times \mathbb{B}$.

PROPOSITION 2.6. *Let $\nu \in \mathcal{M}_{rad}$. Then, we have*

1. *$R_\nu(x,x)$ is an increasing function of $|x|$.*
2. *$R_\nu(x,x) \to \infty$ as $|x| \to 1$.*
3. *For any $r_0 \in (0,1)$, R_ν can be extended continuously on $\overline{B_{r_0}} \times \overline{\mathbb{B}}$.*

Proof. By Proposition 2.5 and (2), we have

$$R_\nu(x,x) = \sum_{m=0}^{\infty} \frac{Z_m(x,x)}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)} = \sum_{m=0}^{\infty} \frac{h_m |x|^{2m}}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)}, \tag{7}$$

where h_m is the dimension of $\mathcal{H}_m$. This implies that $R_\nu(x,x)$ is an increasing function of $|x|$. Letting $|x| \to 1$, we have

$$\lim_{|x|\to 1} R_\nu(x,x) = \sum_{m=0}^{\infty} \frac{h_m}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)} \geq \frac{1}{\nu_r([0,1))} \sum_{m=0}^{\infty} h_m = \infty.$$

because $h_m \geq 1$ for $m \geq 0$. Finally, let $r_0 \in (0,1)$. By (3), we have

$$\sum_{m=0}^{\infty} \left| \frac{Z_m(x,y)}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)} \right|$$

$$\leq \sum_{m=0}^{\infty} \frac{h_m |x|^m |y|^m}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)} \leq \sum_{m=0}^{\infty} \frac{h_m r_0^m}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)} = R_\nu(r_0\theta_0, r_0\theta_0) < \infty.$$

for $x \in \overline{B_{r_0}}$, $y \in \overline{\mathbb{B}}$ and $\theta_0 \in \mathbb{S}$. □

3. TOEPLITZ OPERATORS OF RADIAL MEASURE SYMBOL

For a Radon measure μ on $\mathbb{B}$, the Toeplitz operator $T_{\mu,\nu}$ is formally defined by

$$T_{\mu,\nu} f(x) = \int_{\mathbb{B}} R_\nu(x,y) f(y) \mathrm{d}\mu(y). \tag{8}$$

We make a remark on the definition of Toeplitz operators.

(Compact). First, if $\mathrm{supp}(\mu)$ is a compact set in $\mathbb{B}$, then the operator $T_{\mu,\nu}$ defined by (8) can be expressed as

$$T_{\mu,\nu} f(x) = \int_{\mathbb{B}} B_{\mu,\nu}(x,y) f(y) \mathrm{d}V(y)$$

by Proposition 2.6.3. and the Fubini theorem, where

$$B_{\mu,\nu}(x,y) = \int_{\mathbb{B}} R_\nu(x,z) R_\nu(z,y) \mathrm{d}\mu(z).$$

Hence, we see that $T_{\mu,\nu}$ is a compact operator on b^2_ν, because

$$\int_{\mathbb{B}} \int_{\mathbb{B}} |B_{\mu,\nu}(x,y)|^2 \mathrm{d}V(x) \mathrm{d}V(y) < \infty.$$

PROPOSITION 3.1. *Let $\nu \in \mathcal{M}_{rad}$. If* $\mathrm{supp}(\mu)$ *is compact in $\mathbb{B}$, then $T_{\mu,\nu}$ is a Hilbert-Schmidt operator on b^2_ν.*

We also remark that

$$\langle T_{\mu,\nu} f, g \rangle_\nu = \int_{\mathbb{B}} f(y) g(y) \mathrm{d}\mu(y) \tag{9}$$

for any $f, g \in b^2_\nu$, where $\langle \cdot, \cdot \rangle_\nu$ denote the inner product in b^2_ν.

(Positive). Next, we consider finite positive Radon measures μ on $\mathbb{B}$. We put

$$\mathcal{E}(f,g)=\int_{\mathbb{B}} f(x)g(x)\mathrm{d}\mu(x) \tag{10}$$

for $f,g\in\mathcal{F}:=b^2_\nu\cap L^2(\mathbb{B},\mathrm{d}\mu)$. Then, $(\mathcal{E},\mathcal{F})$ is a densely-defined closed form on b^2_ν, *i.e.*, $\mathcal{F}$ is complete with respect to a norm

$$\|f\|^2=\mathcal{E}(f,f)+\langle f,f\rangle^2_\nu. \tag{11}$$

By the basic theory of quadratic forms (for example, see [2] p. 82 Theorem 4.4.2), there exists a unique densely-defined positive self-adjoint operator $T_{\mu,\nu}$ such that for $f,g\in\mathrm{Dom}(T_{\mu,\nu})\subset\mathcal{F}$

$$\int_{\mathbb{B}} f(x)g(x)\mathrm{d}\mu(x)=\mathcal{E}(f,g)=\langle\sqrt{T_{\mu,\nu}}f,\sqrt{T_{\mu,\nu}}g\rangle_\nu=\langle T_{\mu,\nu}f,g\rangle_\nu. \tag{12}$$

We define the Toeplitz operator by (12), which is natural because of (8).

(Radial). Finally, we consider a radial measure on $\mathbb{B}$. For a radial positive measure on $\mathbb{B}$, we easily find the spectrum of the Toeplitz operator $T_{\mu,\nu}$ on b^2_ν.

Proposition 3.2. *Let a measure $\nu\in\mathcal{M}_{rad}$ and μ be a radial positive measure. Then, the eigenvalues of the Toeplitz operator $T_{\mu,\nu}$ corresponding to the eigenspace $\mathcal{H}_m(\mathbb{B})$ are the following:*

$$\lambda_m=\lambda_m(T_{\mu,\nu})=\frac{\int_{[0,1)}r^{2m}\mathrm{d}\mu_r(r)}{\int_{[0,1)}r^{2m}\mathrm{d}\nu_r(r)}. \tag{13}$$

Moreover, $T_{\mu,\nu}$ can be decomposed as

$$T_{\mu,\nu}=\sum_{m=0}^{\infty}\lambda_m Q^\nu_m. \tag{14}$$

Proof. We put $T=\sum_{m=0}^{\infty}\lambda_m Q^\nu_m$. Take an orthonormal basis $\{p^m_k\}_{k=1}^{h_m}$ in $\mathcal{H}_m(S)\subset L^2(\mathbb{S})$, and put

$$e^m_k(x)=\frac{p^m_k(x)}{(\int_{[0,1)}r^{2m}\mathrm{d}\nu_r(r))^{\frac{1}{2}}}\in\mathcal{H}_m(\mathbb{B})\ (\ 1\le k\le h_m\ )$$

for $x\in\mathbb{B}$. Then, $\{e^m_k\}_{k=1}^{h_m}$ is considered as an orthonormal basis of $\mathcal{H}_m(\mathbb{B})\subset b^2_\nu$. We have

$$\begin{aligned}\langle Te^m_k,e^m_j\rangle_\nu&=\langle\lambda_m e^m_k,e^m_j\rangle_\nu\\&=\frac{\int_{[0,1)}r^{2m}\mathrm{d}\mu_r(r)\int_{[0,1)}\int_{\mathbb{S}}e^m_k(r\theta)e^m_j(r\theta)\mathrm{d}\sigma(\theta)\mathrm{d}\nu_r(r)}{\int_{[0,1)}r^{2m}\mathrm{d}\nu_r(r)}\end{aligned}$$

$$= \int_{\mathbb{B}} e_k^m(x) e_j^m(x) \mathrm{d}\mu(x) = \mathcal{E}(e_k^m, e_j^m).$$

Since $\mathcal{E}(f_m, f_l) = 0$ for $f_m \in \mathcal{H}_m(\mathbb{B})$, $f_m \in \mathcal{H}_l(\mathbb{B})$ and $m \neq l$, we have $T = T_{\mu,\nu}$. $\square$

We may consider a radial signed measure. Let $\mu = \mu_+ - \mu_-$ be a radial signed measure where μ_+ and μ_- are positive measures. By Theorem 3.2, we can define the Toeplitz operator $T_{\mu,\nu}$ by

$$T_{\mu,\nu} := T_{\mu_+,\nu} - T_{\mu_-,\nu} = \sum_{m=0}^{\infty} \lambda_m Q_m^\nu, \tag{15}$$

where

$$\lambda_m = \lambda_m(T_{\mu_+,\nu}) - \lambda_m(T_{\mu_-,\nu}) = \frac{\int_{[0,1)} r^{2m} \mathrm{d}\mu_r(r)}{\int_{[0,1)} r^{2m} \mathrm{d}\nu_r(r)}. \tag{16}$$

We define the averaging function $a_{\mu,\nu}(r)$ and the Berezin transform $b_{\mu,\nu}(x)$ by

$$a_{\mu,\nu}(r) := \frac{\mu(\mathbb{B} \setminus B_r)}{\nu(\mathbb{B} \setminus B_r)} = \frac{\mu_r([r,1))}{\nu_r([r,1))}$$

and

$$b_{\mu,\nu}(x) := \frac{\int_{\mathbb{B}} R_\nu(x,y)^2 \mathrm{d}\mu(y)}{R_\nu(x,x)},$$

respectively. We can estimate the norm of Toeplitz operators by those functions. Here, we state some results in [13].

Theorem 1. *Let $\nu \in \mathcal{M}_{rad}$ and μ be a radial finite Radon measure on $\mathbb{B}$. Then, we have*

$$\sup_{x \in \mathbb{B}} |b_{\mu,\nu}(x)| \leq \|T_{\mu,\nu}\| \leq \sup_{0 \leq r < 1} |a_{\mu,\nu}(r)|$$

and

$$\limsup_{|x| \to 1} |b_{\mu,\nu}(x)| \leq \|T_{\mu,\nu}\|_e \leq \limsup_{r \to 1} |a_{\mu,\nu}(r)|,$$

where $\|T\|_e = \inf\{\|T - K\| : K$ is compact on $b_\nu^2\}$. In particular, if $a_{\mu,\nu}$ is a bounded function on $[0,1)$, then $T_{\mu,\nu}$ is bounded and if $\limsup_{r \to 1} |a_{\mu,\nu}(r)| = 0$, $T_{\mu,\nu}$ is compact.

We give a relation of reproducing kernels with Toeplitz operators.

Theorem 2. *Let $\mu, \nu \in \mathcal{M}_{rad}$. Then, we have*

$$\frac{R_\nu(x,x)}{R_\mu(x,x)} \leq b_{\mu,\nu}(x)$$

for $x \in \mathbb{B}$. Moreover, if the limit $\lim_{r \to 1} a_{\mu,\nu}(r)$ exists, then we have

$$\lim_{|x| \to 1} \frac{R_\nu(x,x)}{R_\mu(x,x)} = \lim_{|x| \to 1} b_{\mu,\nu}(x) = \|T_{\mu,\nu}\|_e = \lim_{r \to 1} a_{\mu,\nu}(r).$$

We refer to previous works of Bergman spaces with measure weight on the unit disc in the complex plane. Nakazi and Yamada consider analytic Bergman spaces with general measure weight, and give the necessary and sufficient condition that an analytic Bergman space with general measure weight is a Hilbert space [10]. Nakazi and Yoneda deal with radial measure weights and characterize compact Toeplitz operators of continuous function symbols [11]. For harmonic Bergman space, T. Le [7] obtains the similar results to those of Nakazi and Yoneda [11]. In our paper, we shall discuss the boundedness and the compactness of Toeplitz operators with radial measure symbols.

Finally, we remark that we do not know whether the boundedness of the Toeplitz operator $T_{\mu,\nu}$ implies the boundedness of the averaging function $a_{\mu,\nu}(r)$.

Example. Let ν be the normalized Lebesgue measure on $\mathbb{B}$, *i.e.*, $\mathrm{d}\nu_r = nr^{n-1}dr$. Consider a singular radial measure μ such that

$$\mu_r = \frac{6n}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \delta_{s_k},$$

where δ_{s_k} denotes the Dirac measure at a point s_k and

$$s_k := \frac{6}{\pi^2} \sum_{l=1}^{k} \frac{1}{l^2}.$$

Then, we have

$$\lim_{|x|\to 1} \frac{R_\nu(x,x)}{R_\mu(x,x)} = 1, \tag{17}$$

although $\nu, \mu \in \mathcal{M}_{rad}$ are mutually singular. In fact, the assertion follows from Theorem 2 since we can see that $\lim_{r\to 1} a_{\mu,\nu}(r) = 1$ by calculation.

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