ARTIN APPROXIMATION PROPERTY AND THE GENERAL NÉRON DESINGULARIZATION

DORIN POPESCU

Communicated by Vasile Brînzănescu

This is an exposition on the General Néron Desingularization and its applications. We end with a recent constructive form of this desingularization in dimension one.

AMS 2010 Subject Classification: Primary 1302; Secondary 13B40, 13H05, 13H10, 13J05, 13J10, 13J15, 14B07, 14B12, 14B25.

Key words: Artin approximation, Néron Desingularization, Bass-Quillen Conjecture, Quillen's Question, smooth morphisms, regular morphisms, smoothing ring morphisms.

INTRODUCTION

Let K be a field and $R = K\langle x\rangle$, $x = (x_1, \dots, x_m)$ be the ring of algebraic power series in x over K, that is the algebraic closure of the polynomial ring $K[x]$ in the formal power series ring $\hat{R} = K[[x]]$. Let $f = (f_1, \dots, f_q)$ be a system of polynomials in $Y = (Y_1, \dots, Y_n)$ over R and $\hat{y}$ be a solution of f in the completion $\hat{R}$ of R.

THEOREM 1 (M. Artin [3]). *For any $c \in \mathbf{N}$ there exists a solution $y^{(c)}$ in R such that $y^{(c)} \equiv \hat{y}$ mod $(x)^c$.*

In general, we say that a local ring $(A, \mathfrak{m})$ has the *Artin approximation property* if for every system of polynomials $f = (f_1, \dots, f_q) \in A[Y]^q$, $Y = (Y_1, \dots, Y_n)$, a solution $\hat{y}$ of f in the completion $\hat{A}$ and $c \in \mathbf{N}$ there exists a solution $y^{(c)}$ in A of f such that $y^{(c)} \equiv \hat{y}$ mod $\mathfrak{m}^c$. In fact A has the Artin approximation property if every finite system of polynomial equations over A has a solution in A if and only if it has a solution in the completion $\hat{A}$ of A. We should mention that M. Artin proved already in [2] that the ring of convergent power series with coefficients in $\mathbf{C}$ has the Artin approximation property as it was later called.

We gratefully acknowledge the support from the project ID-PCE-2011-1023, granted by the Romanian National Authority for Scientific Research, CNCS – UEFISCDI.

A ring morphism $u : A \to A'$ of Noetherian rings has *regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e. its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P \in \operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K \otimes_{A/P} A'/PA'$ is regular.

A flat morphism of Noetherian rings u is *regular* if its fibers are geometrically regular. If u is regular of finite type then u is called *smooth*. A localization of a smooth algebra is called *essentially smooth*.

A Henselian Noetherian local ring A is *excellent* if the completion map $A \to \hat{A}$ is regular. For example, a Henselian discrete valuation ring V is excellent if the completion map $V \to \hat{V}$ induces a separable fraction field extension.

THEOREM 2 (M. Artin [3]). *Let V be an excellent Henselian discrete valuation ring and $V\langle x\rangle$ the ring of algebraic power series in x over V, that is the algebraic closure of the polynomial ring $V[x]$ in the formal power series ring $V[[x]]$. Then $V\langle x\rangle$ has the Artin approximation property.*

The proof used the so called Néron Desingularization, which says that an unramified extension $V \subset V'$ of valuation rings inducing separable field extensions on the fraction and residue fields, is a filtered inductive union of essentially finite type subextensions $V \subset A$, which are regular local rings, even essentially smooth V-subalgebras of V'.

Néron Desingularization is extended by the following theorem.

THEOREM 3 (General Néron Desingularization, Popescu [27,28,30], André [1], Teissier [42], Swan [41], Spivakovski [39]). *Let $u : A \to A'$ be a regular morphism of Noetherian rings and B an A-algebra of finite type. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$.*

The smooth A-algebra C given for B by the above theorem is called a *General Néron Desingularization*. Note that C is not uniquely associated to B and so we better speak about a General Néron Desingularization.

The above theorem gives a positive answer to a conjecture of M. Artin [4].

THEOREM 4 ([28, 31]). *An excellent Henselian local ring has the Artin approximation property.*

This paper is a survey on the Artin approximation property, the General Néron Desingularization and their applications. It relies mainly on some lectures given by the author within the special semester On Artin Approximation of the Chaire Jean Morlet at CIRM, Luminy, Spring 2015 (see http://hlombardi.free.fr/Popescu-Luminy2015.pdf).

We owe thanks to the Referee for very useful comments on our paper.

1. ARTIN APPROXIMATION PROPERTIES

First we show how one recovers Theorem 4 from Theorem 3. Indeed, let f be a finite system of polynomial equations over A in $Y=(Y_1,\ldots,Y_n)$ and $\hat{y}$ a solution of f in $\hat{A}$. Set $B=A[Y]/(f)$ and let $v:B\to\hat{A}$ be the morphism given by $Y\to\hat{y}$. By Theorem 3, v factors through a smooth A-algebra C, that is v is a composite A-morphism $B\to C\to\hat{A}$. Thus changing B by C we may reduce the problem to the case when B is smooth over A. Since $\hat{A}$ is local, changing B by B_b for some $b\in B\setminus v^{-1}(\mathfrak{m}\hat{A})$ we may assume that $1\in\big((g):I\big)MB$ for some polynomials $g=(g_1,\ldots,g_r)$ from (f) and a $r\times r$-minor M of the Jacobian matrix $\left(\dfrac{\partial g}{\partial Y}\right)$. Thus $g(\hat{y})=0$ and $M(\hat{y})$ is invertible. By the Implicit Function Theorem there exists $y\in A$ such that $y\equiv\hat{y}$ modulo $\mathfrak{m}\hat{A}$.

The following consequence of Theorem 3 was noticed and hinted by N. Radu to M. André. This was the origin of André's interest to read our theorem and to write later [1].

Corollary 5. *Let $u:A\to A'$ be a regular morphism of Noetherian rings. Then the differential module $\Omega_{A'/A}$ is flat.*

For the proof, note that by Theorem 3 it follows that A' is a filtered inductive limit of some smooth A-algebras C and so $\Omega_{A'/A}$ is a filtered inductive limit of $A'\otimes_C\Omega_{C/A}$, the last modules being free modules.

Definition 6. A Noetherian local ring $(A,\mathfrak{m})$ has the *strong Artin approximation property* if for every finite system of polynomial equations f in $Y=(Y_1,\ldots,Y_n)$ over A there exists a map $\nu:\mathbb{N}\to\mathbb{N}$ with the following property:

> If $y'\in A^n$ satisfies $f(y')\equiv 0$ modulo $\mathfrak{m}^{\nu(c)}$, $c\in\mathbf{N}$, then there exists a solution $y\in A^n$ of f with $y\equiv y'$ modulo $\mathfrak{m}^c$.

M. Greenberg [14] proved that excellent Henselian discrete valuation rings have the strong Artin approximation property and ν is linear in this case.

Theorem 7 (M. Artin [3]). *The algebraic power series ring over a field has the strong Artin approximation property.*

Note that in general ν is not linear as it is showed in [37]. The following theorem was conjectured by M. Artin in [4].

Theorem 8 ([25]). *Let A be an excellent Henselian discrete valuation ring and $A\langle x\rangle$ the ring of algebraic power series in x over A. Then $A\langle x\rangle$ has the strong Artin approximation property.*

Theorem 9 (Pfister-Popescu [21], see also [17,26]). *The Noetherian complete local rings have the strong Artin approximation property. In particular, a Noetherian local ring A has the strong Artin approximation property if and only if it has the Artin approximation property.*

Thus Theorem 8 is a consequence of Theorem 2 and the above theorem. Together with Theorem 4 the above theorem says that excellent Henselian local rings have the strong Artin approximation property. An easy direct proof of the last fact is given in [28, Corollary 4.5] using Theorem 3 and the ultrapower methods.

What about the converse implication in Theorem 4? It is clear that A is Henselian if it has the Artin approximation property. On the other hand, if A is reduced and it has the Artin approximation property, then $\hat{A}$ is reduced, too. Indeed, if $\hat{z} \in \hat{A}$ is nonzero and satisfies $\hat{z}^r = 0$ then choosing $c \in \mathbf{N}$ such that $\hat{z} \notin \mathfrak{m}^c \hat{A}$ we get a $z \in A$ such that $z^r = 0$ and $z \equiv \hat{z}$ modulo $\mathfrak{m}^c \hat{A}$. It follows that $z \neq 0$, which contradicts our hypothesis. It is easy to see that a local ring B which is finite as a module over A has the Artin approximation property if A has it. It follows that if A has the Artin approximation property, then it has the so called reduced formal fibers. In particular, A must be a so called universally japanese ring.

Using the strong Artin approximation property we may prove that given a system of polynomial equations $f \in A[Y]^r$, $Y = (Y_1, \ldots, Y_n)$ and another one $g \in A[Y, Z]^t$, $Z = (Z_1, \ldots, Z_s)$ then the sentence

$$L_A := \text{ there exists } y \in A^n \text{ such that } f(y) = 0 \text{ and } g(y, z) \neq 0 \text{ for all } z \in A^s$$

holds in A if and only if $L_{\hat{A}}$ holds in $\hat{A}$ provided that A has the Artin approximation property. In this way it was proved in [8] that if A has the Artin approximation property, then A is a normal domain if and only if $\hat{A}$ is a normal domain, too (this was actually the starting point of the quoted paper). Later, Cipu and myself [10] used this fact to show that the formal fibers of A are the so called geometrically normal domains if A has the Artin approximation property. Finally, Rotthaus [38] proved that A is excellent if A has the Artin approximation property.

Next, let $(A, \mathfrak{m})$ be an excellent Henselian local ring, $\hat{A}$ its completion and MCM(A) (resp. MCM($\hat{A}$)) be the set of isomorphism classes of maximal Cohen Macaulay modules over A (resp. $\hat{A}$). Assume that A is an isolated singularity. Then a maximal Cohen-Macaulay module is free on the punctured spectrum. Since $\hat{A}$ is also an isolated singularity we see that the map φ :MCM(A) $\to$MCM($\hat{A}$) given by $M \to \hat{A} \otimes_A M$ is surjective by a theorem of Elkik [13, Theorem 3].

Theorem 10 (Popescu-Roczen [35]). *φ is bijective.*

Proof. Let M, N be two finite A-modules. We may suppose that $M = A^n/(u)$, $N = A^n/(v)$, $u_k = \sum_{j\in[n]} u_{kj}e_j$, $k \in [t]$, $v_r = \sum_{j\in[n]} v_{rj}e_j$, $r \in [p]$, where $u_{kj}, v_{rj} \in A$ and (e_j) is the canonical basis of A^n. Let $f : A^n \to A^n$ be an A-linear map defined by an invertible $n \times n$-matrix (x_{ij}) with respect to (e_j). Then f induces a bijection $M \to N$ if and only if f maps (u) onto (v), that is there exist $y_{kr}, z_{rk} \in A$, $k \in [t]$, $r \in [p]$ such that

1) $f(u_k) = \sum_{r\in[p]} y_{kr}v_r$, $k \in [t]$ and

2) $f(\sum_{k\in[t]} z_{rk}u_k) = v_r$, $r \in [p]$.

Note that 1), 2) are equivalent to

1′) $\sum_{i\in[n]} u_{ki}x_{ij} = \sum_{r\in[p]} y_{kr}v_{rj}$, $k \in [t]$, $j \in [n]$,

2′) $\sum_{k\in[t]} z_{rk}(\sum_{i\in[n]} u_{ki}x_{ij}) = v_{rj}$, $r \in [p]$, $j \in [n]$.

Therefore, if $\hat{A} \otimes_A M \cong \hat{A} \otimes_A N$ there exist $(\hat{x}_{ij})$, $(\hat{y}_{kr})$, $(\hat{z}_{rk})$ in $\hat{A}$ such that $\det(\hat{x}_{ij}) \notin \mathfrak{m}$ and

1″) $\sum_{i\in[n]} u_{ki}\hat{x}_{ij} = \sum_{r\in[p]} \hat{y}_{kr}v_{rj}$, $k \in [t]$, $j \in [n]$,

2″) $\sum_{k\in[t]} \hat{z}_{rk}(\sum_{i\in[n]} u_{ki}\hat{x}_{ij}) = v_{rj}$, $r \in [p]$, $j \in [n]$.

Then by the Artin approximation property there exists a solution of 1″), 2″), let us say (x_{ij}), (y_{kr}), (z_{rk}) in A, such that $x_{ij} \equiv \hat{x}_{ij}$, $y_{kr} \equiv \hat{y}_{kr}$, $z_{rk} \equiv \hat{z}_{rk}$ modulo $\mathfrak{m}\hat{A}$. It follows that $\det(x_{ij}) \equiv \det(\hat{x}_{ij}) \not\equiv 0$ modulo $\mathfrak{m}\hat{A}$ and so $M \cong N$. □

COROLLARY 11. *In the hypothesis of the above theorem if $M \in MCM(A)$ is indecomposable, then $\hat{A} \otimes_A M$ is indecomposable, too.*

Proof. Assume that $\hat{A} \otimes_A M = \hat{N}_1 \oplus \hat{N}_2$. Then $\hat{N}_i \in \mathrm{MCM}(\hat{A})$ and by the surjectivity of φ we get $\hat{N}_i = \hat{A} \otimes_A N_i$ for some $N_i \in \mathrm{MCM}(A)$. Then $\hat{A} \otimes_A M \cong (\hat{A} \otimes_A N_1) \oplus (\hat{A} \otimes_A N_2)$ and the injectivity of φ gives $M \cong N_1 \oplus N_2$. □

Remark 12. If A is not Henselian then the above corollary is false. For example, let $A = \mathbf{C}[X, Y]_{(X,Y)}/(Y^2 - X^2 - X^3)$. Then $M = (X, Y)A$ is indecomposable in $\mathrm{MCM}(A)$ but $\hat{A} \otimes_A M$ is decomposable. Indeed, for $\hat{u} = \sqrt{1 + X} \in \hat{A}$ we have $\hat{A} \otimes_A M = (Y - \hat{u}X)\hat{A} \oplus (Y - \hat{u}X)\hat{A}$.

Remark 13. Let $\Gamma(A)$, $\Gamma(\hat{A})$ be the so called AR-quivers of A, $\hat{A}$. Then φ induces also an inclusion $\Gamma(A) \subset \Gamma(\hat{A})$ (see [35]).

Remark 14. It is known that $\mathrm{MCM}(\hat{A})$ is finite if and only if $\hat{A}$ is a simple singularity. What about a complex unimodal singularity R? Certainly in this case $\mathrm{MCM}(R)$ is infinite but maybe there exists a special property which characterizes the unimodal singularities. For this purpose it would be necessary to describe somehow $\mathrm{MCM}(R)$ at least in some special cases. Small attempts are done by Andreas Steenpass [40].

For most of the cases when we need the Artin approximation property, it is enough to apply Artin's Theorem 1. Sometimes we might need a special kind of Artin approximation, the so called *Artin approximation in nested subring condition*, namely the following result which was also considered as possible by M. Artin in [4].

THEOREM 15 ([28], [31, Theorem 3.6]). *Let K be a field, $A = K\langle x\rangle$, $x = (x_1, \dots, x_m)$, $f = (f_1, \dots, f_r) \in K\langle x, Y\rangle^r$, $Y = (Y_1, \dots, Y_n)$ and $0 \le s_1 \le \dots \le s_n \le m$, c be some non-negative integers. Suppose that f has a solution $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$ in $K[[x]]$ such that $\hat{y}_i \in K[[x_1, \dots, x_{s_i}]]$ for all $1 \le i \le n$. Then there exists a solution $y = (y_1, \dots, y_n)$ of f in A such that $y_i \in K\langle x_1, \dots, x_{s_i}\rangle$ for all $1 \le i \le n$ and $y \equiv \hat{y} \mod (x)^c K[[x]]$.*

COROLLARY 16. *The Weierstrass Preparation Theorem holds for the ring of algebraic power series over a field.*

Proof. Let $f \in K\langle x\rangle$, $x = (x_1, \dots, x_m)$ be an algebraic power series such that $f(0, \dots, 0, x_m) \ne 0$. By Weierstrass Preparation Theorem f is associated in divisibility with a monic polynomial $\hat{g} = x_m^p + \sum_{i=0}^{p-1} \hat{z}_i x_m^i \in K[[x_1, \dots, x_{m-1}]][x_m]$ for some $p \in \mathbf{N}$, $\hat{z}_i \in (x_1, \dots, x_{m-1})K[[x_1, \dots, x_{m-1}]]$. Thus the system $F_1 = f - Y(x_m^p + \sum_{i=0}^{p-1} Z_i x_m^i)$, $F_2 = YU - 1$ has a solution $\hat{y}$, $\hat{u}$, $\hat{z}_i$ in $K[[x]]$ such that $\hat{z}_i \in K[[x_1, \dots, x_{m-1}]]$. By Theorem 15 there exists a solution y, u, z_i in $K\langle x\rangle$ such that $z_i \in K\langle x_1, \dots, x_{m-1}\rangle$ and is congruent modulo (x) with the previous one. Thus y is invertible and $f = yg$, where $g = x_m^p + \sum_{i=0}^{p-1} z_i x_m^i \in K\langle x_1, \dots, x_{m-1}\rangle[x_m]$. By the unicity of the (formal) Weierstrass Preparation Theorem it follows that $y = \hat{y}$ and $g = \hat{g}$. □

Now, we see that Theorem 15 is useful to get algebraic versal deformations (see [5]). Let $D = K\langle Z\rangle$, $A = K\langle T\rangle/J$, $Z = (Z_1, \dots, Z_s)$, $T = (T_1, \dots, T_n)$ and $N = D/(f_1, \dots, f_d)$. A deformation of N over A is a $P = K\langle T, Z\rangle/(J) \cong ((A \otimes_K D)_{(T,Z)})^h$-module L such that

1) $L \otimes_A K \cong N$,

2) L is flat over A,

where above B^h denotes the Henselization of a local ring B. The condition 1) says that L has the form $P/(F_1, \dots, F_d)$ with $F_i \in K\langle T, Z\rangle$, $F_i \cong f_i$ modulo (T) and 2) says that

2′) $\mathrm{Tor}_1^A(L, K) = 0$

by the Local Flatness Criterion, since L is (T)-adically ideal separated because P is local Noetherian. Let

$$P^e \xrightarrow{\nu} P^d \to P \to L \to 0$$

be part of a free resolution of L over P, where the map $P^d \to P$ is given by $(F_1, \dots, F_d)$. Then 2′) says that tensorizing with $K \otimes_A -$ the above sequence we get an exact sequence

$$D^e \to D^d \to D \to N \to 0,$$

because P is flat over A. Therefore, $2'$) is equivalent to

$2''$) For all $g \in D^d$ with $\sum_{i=1}^d g_i f_i = 0$ there exists $G \in K\langle T, Z\rangle^d$ with $G \equiv g$ modulo (T) such that G modulo $J \in \operatorname{Im} \nu$, that is $\sum_{i=1}^d G_i F_i \in (J)$.

We would like to construct a versal deformation L (see [17, pages 157–159]), that is for any $A' = K\langle U\rangle/J'$, $U = (U_1, \ldots, U_{n'})$, $P' = ((A' \otimes_K D)_{(U,Z)})^h$ and $L' = K\langle U\rangle/(F')$ a deformation of N to A' there exists a morphism $\alpha : A \to A'$ such that $P' \otimes_P L \cong L'$, where the structural map of P' over P is given by α. If we replace above the algebraic power series with formal power series then this problem is solved by Schlessinger in the infinitesimal case followed by some theorems of Elkik and M. Artin. Set $\hat{A} = K[[T]]/(J)$, $\hat{P} = ((\hat{A} \otimes_K D)_{(T,Z)})^h$. We will assume that we have already L such that $\hat{L} = \hat{P} \otimes_P L$ is versal in the frame of complete local rings. How to get the versal property for L in the frame of algebraic power series?

Let A', P', L' be as above. Since $\hat{L}$ is versal in the frame of complete local rings, there exists $\hat{\alpha} : \hat{A} \to \hat{A}'$ such that $\hat{P}' \otimes_{\hat{P}} \hat{L} \cong \hat{L}' = \hat{P}' \otimes_{P'} L'$, where the structure of $\hat{P}'$ as a $\hat{P}$-algebra is given by $\hat{\alpha}$. Assume that $\hat{\alpha}$ is given by $T \to \hat{t} \in (U)K[[U]]^n$. Then we have

i) $J(\hat{t}) \equiv 0$ modulo (J').

On the other hand, we may suppose that $\hat{\alpha}$ induces an isomorphism $\hat{P}' \otimes_{\hat{P}} \hat{L} \to \hat{L}'$ which is given by $(T, Z) \to (\hat{t}, \hat{z})$ for some $\hat{z} \in (U, Z)K[[U, Z]]^s$ with $\hat{z} \equiv Z$ modulo $(U, Z)^2$ and the ideals $(F(\hat{t}, \hat{z}))$, (F') of $K[[U, Z]]$ coincide. Thus there exists an invertible $d \times d$-matrix $\hat{C} = (\hat{C}_{ij})$ over $K[[U, Z]]$ with

ii) $F_i' = \sum_{j=1}^d \hat{C}_{ij} F_j(\hat{t}, \hat{z})$).

By Theorem 15 we may find $t \in (U)K\langle U\rangle^n$ and $z \in (U, Z)K\langle U, Z\rangle^s$, $C_{ij} \in K\langle U, Z\rangle$ satisfying i), ii) and such that $t \equiv \hat{t}$, $z \equiv \hat{z}$, $C_{ij} \equiv \hat{C}_{ij}$ modulo $(U, Z)^2$. Note that $\det(C_{ij}) \equiv \det \hat{C}$ modulo $(U, Z)^2$ and so (C_{ij}) is invertible. It follows that $\alpha : A \to A'$ given by $T \to t$ is the wanted one, that is $P' \otimes_P L \cong L'$, where the structure of P' as a P-algebra is given by α.

Next we give an idea of the proof of Theorem 15 in a particular, but essential case.

Proposition 17. *Let K be a field, $A = K\langle x\rangle$, $x = (x_1, \ldots, x_m)$, $f = (f_1, \ldots, f_r) \in K\langle x, Y\rangle^r$, $Y = (Y_1, \ldots, Y_n)$ and $0 \le s \le m$, $1 \le q \le n$, c be some non-negative integers. Suppose that f has a solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)$ in $K[[x]]$ such that $\hat{y}_i \in K[[x_1, \ldots, x_s]]$ for all $1 \le i \le q$. Then there exists a solution $y = (y_1, \ldots, y_n)$ of f in A such that $y_i \in K\langle x_1, \ldots, x_s\rangle$ for all $1 \le i \le q$ and $y \equiv \hat{y} \mod (x)^c K[[x]]$.*

Proof. Note that $B = K[[x_1, \ldots, x_s]]\langle x_{s+1}, \ldots, x_m\rangle$ is excellent Henselian and so it has the Artin approximation property. Thus the system of polynomials $f((\hat{y}_i)_{1\leq i\leq q}, Y_{q+1}, \ldots, Y_n)$ has a solution $(\tilde{y}_j)_{q<j\leq n}$ in B with $\tilde{y}_j \equiv \hat{y}_j$ modulo $(x)^c$. Now it is enough to apply the following lemma for $A = K\langle x_1, \ldots, x_s\rangle$. □

An *etale neighborhood* of a local ring $(A, \mathfrak{m})$ is a local smooth A-algebra B, which is essentially finite (that is a localization of a finite A-algebra) and such that $\mathfrak{m}B$ is maximal in B with $A/\mathfrak{m} \cong B/\mathfrak{m}B$. The structure of an etale neighborhood is given for example in [41, Theorem 2.5].

LEMMA 18. *Let $(A, \mathfrak{m})$ be an excellent Henselian local ring, $\hat{A}$ its completion, $A[x]^h$, $x = (x_1, \ldots, x_m)$, $\hat{A}[x]^h$ be the Henselizations of $A[x]_{(\mathfrak{m},x)}$ respectively $\hat{A}[x]_{(\mathfrak{m},x)}$, $f = (f_1, \ldots, f_r)$ a system of polynomials in $Y = (Y_1, \ldots, Y_n)$ over $A[x]^h$ and $1 \leq q < n$, c be some positive integers. Suppose that f has a solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)$ in $\hat{A}[x]^h$ such that $\hat{y}_i \in \hat{A}$ for all $i \leq q$. Then there exists a solution $y = (y_1, \ldots, y_n)$ of f in $A[x]^h$ such that $y_i \in A$ for all $i \leq q$ and $y \equiv \hat{y} \mod \mathfrak{m}^c\hat{A}[x]^h$.*

Proof. $\hat{A}[x]^h$ is a filtered union of etale neighborhoods of $\hat{A}[x]$. Take an etale neighborhood B of $\hat{A}[x]_{(\mathfrak{m},x)}$ such that $\hat{y}_i \in B$ for all $q < i \leq n$. Then $B \cong (\hat{A}[x,T]/(\hat{g}))_{(\mathfrak{m},x,T)}$ for some monic polynomial $\hat{g}$ in T over $\hat{A}[x]$ with $\hat{g}(0) \in (\mathfrak{m}, x)$ and $\partial\hat{g}/\partial T(0) \notin (\mathfrak{m}, x)$, let us say

$$\hat{g} = T^e + \sum_{j=0}^{e-1}\Big(\sum_{k\in\mathbf{N}^m,|k|<u} \hat{z}_{jk}x^k\Big)T^j,$$

for some u high enough and $\hat{z}_{jk} \in \hat{A}$. Note that $\hat{z}_{00} \in \mathfrak{m}\hat{A}$ and $\hat{z}_{10} \notin \mathfrak{m}\hat{A}$. Changing if necessary u, we may suppose that

$$\hat{y}_i \equiv \sum_{j=0}^{e-1}\Big(\sum_{k\in\mathbf{N}^m,|k|<u} \hat{y}_{ijk}x^k\Big)T^j \mod \hat{g}$$

for some $\hat{y}_{ijk} \in \hat{A}$, $q < i \leq n$. Actually, we should take $\hat{y}_i$ as a fraction but for an easier expression we will skip the denominator. Substitute Y_i, $q < i \leq n$ by

$$Y_i^+ = \sum_{j=0}^{e-1}\Big(\sum_{k\in\mathbf{N}^m,|k|<u} Y_{ijk}x^k\Big)T^j$$

in f and divide by the monic polynomial

$$G = T^e + \sum_{j=0}^{e-1}\Big(\sum_{k\in\mathbf{N}^m,|k|<u} Z_{jk}x^k\Big)T^j$$

in $\hat{A}[x, T, Y_1, \ldots, Y_t, (Y_{ij}), (Z_j)]$, where $(Y_{ijk}), (Z_{jk})$ are new variables.

We get

$$f_p(Y_1,\ldots,Y_q,Y^+)\equiv\sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^m,|k|<u}F_{pjk}(Y_1,\ldots,Y_q,(Y_{ijk}),(Z_{jk}))x^kT^j \mod G,$$

$1\leq p\leq r$. Then $\hat{y}$ is a solution of f in B if and only if $(\hat{y}_1,\ldots,\hat{y}_q,(\hat{y}_{ijk}),(\hat{z}_{jk}))$ is a solution of (F_{pjk}) in $\hat{A}$. As A has the Artin approximation property we may choose a solution $(y_1,\ldots,y_q,(y_{ijk}),(z_{jk}))$ of (F_{pjk}) in A which coincides modulo $\mathfrak{m}^c\hat{A}$ with the former one. Then

$$y_i=\sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^m,|k|<u}(y_{ijk}))x^kT^j,$$

$q<i\leq n$ together with y_i, $1\leq i\leq q$ form a solution of f in the etale neighborhood $B'=(A[x,T]/(g))_{(\mathfrak{m},x,T)}$,

$$g=T^e+\sum_{i=0}^{e-1}(\sum_{k\in\mathbf{N}^m,|k|<u}z_{jk}x^k)T^j$$

of $A[x]_{(\mathfrak{m},x)}$, which is contained in $A[x]^h$. Clearly, y is the wanted solution. □

2. APPLICATIONS TO THE BASS-QUILLEN CONJECTURE

Let $R[T]$, $T=(T_1,\ldots,T_n)$ be a polynomial algebra in T over a regular local ring $(R,\mathfrak{m})$. An extension of Serre's Problem proved by Quillen and Suslin is the following

CONJECTURE 19 (Bass-Quillen). *Every finitely generated projective module over $R[T]$ is free.*

THEOREM 20 (Lindel [19]). *The Bass-Quillen Conjecture holds if R is essentially of finite type over a field.*

Swan's unpublished notes on Lindel's paper (see [29, Proposition 2.1]) contain two interesting remarks.

1) Lindel's proof works also when R is essentially of finite type over a DVR A such that its local parameter $p\not\in\mathfrak{m}^2$.

2) The Bass-Quillen Conjecture holds if $(R,\mathfrak{m})$ is a regular local ring containing a field, or $p=$char $R/\mathfrak{m}\not\in\mathfrak{m}^2$ providing that the following question has a positive answer.

Question 21 (Swan). Is a regular local ring a filtered inductive limit of regular local rings essentially of finite type over $\mathbf{Z}$?

Indeed, suppose for example that R contains a field and R is a filtered inductive limit of regular local rings R_i essentially of finite type over a prime field P. A finitely generated projective $R[T]$-module M is an extension of a finitely generated projective $R_i[T]$-module M_i for some i, that is $M \cong R[T] \otimes_{R_i[T]} M_i$. By Theorem 20 we get M_i free and so M is free, too.

THEOREM 22 ([29]). *Swan's Question 21 holds for regular local rings $(R, \mathfrak{m}, k)$ which are in one of the following cases:*
(1) *R contains a field,*
(2) *the characteristic p of k is not in $\mathfrak{m}^2$,*
(3) *R is excellent Henselian.*

Proof. (1) Suppose that R contains a field k. We may assume that k is the prime field of R and so a perfect field. Then the inclusion $u : k \to R$ is regular and by Theorem 3 it is a filtered inductive limit of smooth k morphisms $k \to R_i$. Thus R_i is a regular ring of finite type over k and so over $\mathbf{Z}$. Therefore R is a filtered inductive limit of regular local rings essentially of finite type over $\mathbf{Z}$. Similarly we may treat (2).

(3) First assume that R is complete. By the Cohen Structure Theorem we may also assume that R is a factor of a complete local ring of type $A = \mathbf{Z}_{(p)}[[x_1, \ldots, x_m]]$ for some prime integer p. By (2) we see that A is a filtered inductive limit of regular local rings A_i essentially of finite type over $\mathbf{Z}$. Since R, A are regular local rings we see that $R = A/(x)$ for a part x of a regular system of parameters of A. Then there exists a system of elements x' of a certain A_i which is mapped into x by the limit map $A_i \to A$. It follows that x' is part of a regular system of parameters of A_t for all $t \geq j$ for some $j > i$ and so $R_t = A_t/(x')$ are regular local rings. Now, it is enough to see that R is a filtered inductive limit of R_t, $t \geq j$.

Next assume that R is excellent Henselian and let $\hat{R}$ be its completion. Using [7], or [41] it is enough to show that given a finite type $\mathbf{Z}$-subalgebra E of R the inclusion $\alpha : E \to R$ factors through a regular local ring E' essentially of finite type over $\mathbf{Z}$, that is there exists $\beta : E' \to R$ such that α is the composite map $E \to E' \xrightarrow{\beta} R$.

As above, $\hat{R}$ is a filtered inductive limit of regular local rings and so the composite map $\hat{\alpha} : E \xrightarrow{\alpha} R \to \hat{R}$ factors through a regular local ring F essentially of finite type over $\mathbf{Z}$. We may choose a finite type $\mathbf{Z}$-subalgebra $D \subset F$ such that $F \cong D_q$ for some $q \in \operatorname{Spec} D$ and the map $E \to F$ factors through D, i.e. D is an E-algebra. As D is excellent, its regular locus Reg D is open and so there exists $d \in D \setminus q$ such that D_d is a regular ring. Changing D by D_d we may assume that D is regular.

Let $E = \mathbf{Z}[b_1, \ldots, b_n]$ for some $b_i \in E \subset R$ and let $D = \mathbf{Z}[Y]/(h)$, $Y = (Y_1, \ldots, Y_n)$ for some polynomials h. Since D is an E-algebra we may write

$b_i \equiv P_i(Y)$ modulo h, $i = 1, \ldots, n$ for some polynomials $P_i \in E[Y] \subset R[Y]$. Note that there exists $\hat{y} \in \hat{R}^n$ such that $b_i = P_i(\hat{y})$, $h(\hat{y}) = 0$ because $\hat{\alpha}$ factors through D. As R has the Artin approximation property, by Theorem 4 there exists $y \in R^n$ such that $b_i = P_i(y)$, $h(y) = 0$. Let $\rho : D \to R$ be the map given by $Y \to y$. Clearly, α factors through D and we may take $E' = D_{\rho^{-1}\mathfrak{m}}$. More precisely, we have the following diagram which is commutative except in the right square,

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[dr] & R \arrow[r] & \hat{R} \\ & D \arrow[u, "\rho"] \arrow[r] & F \arrow[u] \end{tikzcd}$$

□

Corollary 23 ([29]). *The Bass-Quillen Conjecture holds if R is a regular local ring in one of the cases (1), (2) of the above theorem.*

Remark 24. Theorem 22 is not a complete answer to Question 21, but (3) says that a positive answer is expected in general. Since there exists no result similar to Lindel's saying that the Bass-Quillen Conjecture holds for all regular local rings essentially of finite type over $\mathbf{Z}$ we decided to wait with our further research. So we have waited already for 25 years.

Another problem is to replace in the Bass-Quillen Conjecture the polynomial algebra $R[T]$ by other R-algebras. The tool is given by the following theorem.

Theorem 25 (Vorst [43]). *Let A be a ring, $A[x]$, $x = (x_1, \ldots, x_m)$ a polynomial algebra, $I \subset A[x]$ a monomial ideal and $B = A[x]/I$. Then every finitely generated projective B-module M is extended from a finitely generated projective A-module N, that is $M \cong B \otimes_A N$, if for all $n \in \mathbf{N}$ every finitely generated projective $A[T]$-module, $T = (T_1, \ldots, T_n)$ is extended from a finitely generated projective A-module.*

Corollary 26 ([34]). *Let R be a regular local ring in one of the cases (1), (2) of Theorem 22, $I \subset R[x]$ be a monomial ideal with $x = (x_1, \ldots, x_m)$ and $B = R[x]/I$. Then any finitely generated projective B-module is free.*

For the proof, apply the above theorem using Corollary 23.

The Bass-Quillen Conjecture could also hold when R is not regular as the following corollary shows.

Corollary 27 ([34]). *Let R be a regular local ring in one of the cases of Theorem 22, $I \subset R[x]$ be a monomial ideal with $x = (x_1, \ldots, x_m)$ and $B = R[x]/I$. Then every finitely generated projective $B[T]$-module is free, where $T = (T_1, \ldots, T_n)$.*

This result holds because $B[T]$ is a factor of $R[x,T]$ by the monomial ideal $IR[x,T]$.

Remark 28. If I is not monomial, then the Bass-Quillen Conjecture may fail when replacing R by B. Indeed, if $B = \mathbf{R}[x_1,x_2]/(x_1^2 - x_2^3)$ then there exist finitely generated projective $B[T]$-modules of rank one which are not free (see [18, (5.10)]).

Now, let $(R,\mathfrak{m})$ be a regular local ring and $f \in \mathfrak{m}\setminus\mathfrak{m}^2$.

Question 29 (Quillen [36]). Is free a finitely generated projective module over R_f?

A positive answer to this question could solve completely the Bass-Quillen Conjecture (see [36]). Indeed, let $(A,\mathfrak{m})$ be a regular local ring, P a finitely generated projective $A[T]$-module and $A(T)$ the localization of $A[T]$ with respect to all monic polynomials in T. If $P\otimes_{A[T]} A(T)$ is free over $A(T)$ then P is free over $A[T]$ as it was already known. Set $R = A[T]_{(\mathfrak{m},T)}$. A positive answer of Quillen's question will give that $P\otimes_{A[T]} R_T$ is free and so $P\otimes_{A[T]} A(T)$ is free, which is enough.

THEOREM 30 (Bhatwadeckar-Rao [9]). *Quillen's Question has a positive answer if R is essentially of finite type over a field.*

THEOREM 31 ([33]). *Quillen's Question has a positive answer if R contains a field.*

This goes similarly to Corollary 23 using Theorem 30 instead of Theorem 20.

Remark 32. Paper [33] was not accepted for publication in many journals since the referees said that it "relies on a theorem [that is Theorem 3] which is still not recognized by the mathematical community". Since our paper was quoted as an unpublished preprint in [41] we published it later in the Bulletin Mathématique de la Societé des Sciences Matématiques de Roumanie, and it was noticed and quoted by many people (see for instance [16]).

3. GENERAL NERON DESINGULARIZATION

Using Artin's methods from [2], Ploski gave the following theorem, which is the first form of a possible extension of Neron Desingularization in dim > 1.

THEOREM 33 ([23]). *Let $\mathbf{C}\{x\}$, $x = (x_1,\ldots,x_m)$, $f = (f_1,\ldots,f_s)$ be some convergent power series from $\mathbf{C}\{x,Y\}$, $Y = (Y_1,\ldots,Y_n)$ and $\hat{y}\in\mathbf{C}[[x]]^n$ with $\hat{y}(0) = 0$ be a solution of $f = 0$. Then the map $v : B = \mathbf{C}\{x,Y\}/(f) \to$*

$\mathbf{C}[[x]]$ given by $Y \to \hat{y}$ factors through an A-algebra of type $B' = \mathbf{C}\{x, Z\}$ for some variables $Z = (Z_1, \ldots, Z_s)$, that is v is a composite map $B \to B' \to \mathbf{C}[[x]]$.

Using Theorem 3 one can get an extension of the above theorem.

THEOREM 34 ([34]). *Let $(A, \mathfrak{m})$ be an excellent Henselian local ring, $\hat{A}$ its completion, B a finite type A-algebra and $v : B \to \hat{A}$ an A-morphism. Then v factors through an A-algebra of type $A[Z]^h$ for some variables $Z = (Z_1, \ldots, Z_s)$, where $A[Z]^h$ is the Henselization of $A[Z]_{(\mathfrak{m},Z)}$.*

Suppose that $B = A[Y]/I$, $Y = (Y_1, \ldots, Y_n)$. If $f = (f_1, \ldots, f_r)$, $r \leq n$ is a system of polynomials from I, then denote by Δ_f the ideal generated by all $r \times r-$minors of the Jacobian matrix $\left(\dfrac{\partial f_i}{\partial Y_j}\right)$. After Elkik [13], let $H_{B/A}$ be the radical of the ideal $\sum_f \big((f) : I\big)\Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. Then B_P, $P \in \operatorname{Spec} B$ is essentially smooth over A if and only if $P \not\supset H_{B/A}$ by the Jacobian criterion for smoothness. Thus $H_{B/A}$ measures the non smooth locus of B over A.

In the linear case we may easily get cases of Theorem 3 when $\dim A > 1$.

LEMMA 35 ([27, (4.1)]). *Let A be a ring and a_1, a_2 a weak regular sequence of A, that is a_1 is a non-zero divisor of A and a_2 is a non-zero divisor of $A/(a_1)$. Let A' be a flat A-algebra and set $B = A[Y_1, Y_2]/(f)$, where $f = a_1Y_1 + a_2Y_2$. Then $H_{B/A}$ is the radical of (a_1, a_2) and any A-morphism $B \to A'$ factors through a polynomial A-algebra in one variable.*

Proof. Note that all solutions of $f = 0$ in A are multiples of $(-a_2, a_1)$. By flatness, any solution of f in A' is a linear combination of some solutions of f in A and so again a multiple of $(-a_2, a_1)$. Let $h : B \to A'$ be a map given by $Y_i \to y_i \in A'$. Then $(y_1, y_2) = z(-a_2, a_1)$ and so h factors through $A[Z]$, that is h is the composite map $B \to A[Z] \to A'$, the first map being given by $(Y_1, Y_2) \to Z(-a_2, a_1)$ and the second one by $Z \to z$. □

PROPOSITION 36 ([27, Lemma 4.2]). *Let $f_i = \sum_{i=1}^n a_{ij}Y_j \in A[Y_1, \ldots, Y_n]$, $i \in [r]$ be a system of linear homogeneous polynomials and $y^{(k)} = (y_1^{(k)}, \ldots, y_n^{(k)})$, $k \in [p]$ be a complete system of solutions of $f = (f_1, \ldots, f_r) = 0$ in A. Let $b = (b_1, \ldots, b_r) \in A^r$ and c a solution of $f = b$ in A. Let A' be a flat A-algebra and $B = A[Y_1, \ldots, Y_n]/(f - b)$. Then any A-morphism $B \to A'$ factors through a polynomial A-algebra in p variables.*

Proof. Let $h : B \to A'$ be a map given by $Y \to y' \in A'^n$. Since A' is flat over A we see that $y' - h(c)$ is a linear combinations of $y^{(k)}$, that is there exists $z = (z_1, \ldots, z_p) \in A'^p$ such that $y' - h(c) = \sum_{k=1}^p z_k h(y^{(k)})$. Therefore,

h factors through $A[Z_1,\ldots,Z_p]$, that is h is the composite A-morphism $B \to A[Z_1,\ldots,Z_p] \to A'$, where the first map is given by $Y \to c + \sum_{k=1}^{p} Z_k y^{(k)}$ and the second one by $Z \to z$. $\square$

Another form of Theorem 3 is the following theorem which is a positive answer to a conjecture of M. Artin [6].

Theorem 37 (Cipu-Popescu [11]). *Let $u : A \to A'$ be a regular morphism of Noetherian rings, B an A-algebra of finite type, $v : B \to A'$ an A-morphism and $D \subset \operatorname{Spec} B$ the open smooth locus of B over A. Then there exist a smooth A-algebra C and two A-morphisms $t : B \to C$, $w : C \to A'$ such that $v = wt$ and C is smooth over B at $t^{*-1}(D)$, with $t^* : \operatorname{Spec} C \to \operatorname{Spec} B$ being induced by t.*

There exists also a form of Theorem 3 recalling us the strong Artin approximation property.

Theorem 38 ([32, 34]). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with the completion map $A \to \hat{A}$ regular, B an A-algebra of finite type and ν the Artin function over $\hat{A}$ associated to the system of polynomials f defining B. Then there exists a function $\lambda : \mathbf{N} \to \mathbf{N}$, $\lambda \geq \nu$ such that for every positive integer c and every morphism $v : B \to A/\mathfrak{m}^{\lambda(c)}$ there exists a smooth A-algebra C and two A-algebra morphisms $t : B \to C$, $w : C \to A/\mathfrak{m}^c$ such that wt is the composite map $B \xrightarrow{v} A/\mathfrak{m}^{\lambda(c)} \to A/\mathfrak{m}^c$.*

Sometimes, we may find some information about λ (and so about ν). Let A be a discrete valuation ring, x a local parameter of A, $A' = \hat{A}$ its completion and $B = A[Y]/I$, $Y = (Y_1,\ldots,Y_n)$ an A-algebra of finite type. If $f = (f_1,\ldots,f_r)$, $r \leq n$ is a system of polynomials from I then we consider a $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$. Let $c \in \mathbf{N}$. Suppose that there exists an A-morphism $v : B \to A'/(x^{2c+1})$ and $N \in ((f) : I)$ such that $v(NM) \notin (x)^c/(x^{2c+1})$, where for simplicity we write $v(NM)$ instead of $v(NM + I)$.

Theorem 39 ([34, Theorem 10]). *There exists a B-algebra C which is smooth over A such that every A-morphism $v' : B \to A'$ with $v' \equiv v$ modulo x^{2c+1} (that is $v'(Y) \equiv v(Y)$ modulo x^{2c+1}) factors through C.*

Corollary 40 ([34, Theorem 15]). *With the assumptions and notation of Theorem 39 there exists a canonical bijection*

$$A'^s \to \{v' \in \operatorname{Hom}_A(B, A') : v' \equiv v \textit{ modulo } x^{2c+1}\}$$

for some $s \in \mathbf{N}$.

Let k be a field and F a k-algebra of finite type, let us say $F = k[U]/J$, $U = (U_1, \dots, U_n)$. An arc $\operatorname{Spec} k[[x]] \to \operatorname{Spec} F$ is given by a k-morphism $F \to A' = k[[x]]$. Assume that $H_{F/k} \neq 0$ (this happens for example when F is reduced and k is perfect). Set $A = k[x]_{(x)}$, $B = A \otimes_k F$. Let $f = (f_1, \dots, f_r)$, $r \leq n$ be a system of polynomials from J and M a $r \times r$-minor of the Jacobian matrix $(\partial f_i/\partial U_j)$. Let $c \in \mathbf{N}$. Assume that there exists an A-morphism $g : F \to A'/(x^{2c+1})$ and $N \in ((f) : J)$ such that $g(NM) \notin (x)^c/(x^{2c+1})$. Note that $A \otimes_k -$ induces a bijection $\operatorname{Hom}_k(F, A') \to \operatorname{Hom}_A(B, A')$ by adjunction.

Corollary 41 ([34, Corollary 16]). *The set $\{g' \in \operatorname{Hom}_k(F, A') : g' \equiv g$ modulo $x^{2c+1}\}$ is in bijection with an affine space A'^s over A' for some $s \in \mathbf{N}$.*

Next we give a possible extension of Greenberg's result on the strong Artin approximation property [14]. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring (for example a reduced ring) of dimension one, $A' = \hat{A}$ the completion of A, $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$ an A-algebra of finite type and $c, e \in \mathbb{N}$. Suppose that there exists $f = (f_1, \dots, f_r)$ in I, a $r \times r$-minor M of the Jacobian matrix $(\partial f/\partial Y)$, $N \in ((f) : I)$ and an A-morphism $v : B \to A/\mathfrak{m}^{2e+c}$ such that $(v(MN)) \supset \mathfrak{m}^e/\mathfrak{m}^{2e+c}$. Then we may construct a General Neron Desingularization in the idea of Theorem 39, which could be used to get the following theorem.

Theorem 42 (A. Popescu-D. Popescu [24]). *There exists an A-morphism $v' : B \to \hat{A}$ such that $v' \equiv v$ modulo $\mathfrak{m}^c\hat{A}$, that is $v'(Y + I) \equiv v(Y + I)$ modulo $\mathfrak{m}^c\hat{A}$. Moreover, if A is also excellent Henselian then there exists an A-morphism $v' : B \to A$ such that $v' \equiv v$ modulo $\mathfrak{m}^c$.*

Remark 43. The above theorem could be extended for Noetherian local rings of dimension one (see [22]). In this case, the statement depends also on a reduced primary decomposition of (0) in A.

Using [24] we end this section with an algorithmic attempt to explain the proof of Theorem 3 in the frame of Noetherian local domains of dimension one. Let $u : A \to A'$ be a flat morphism of Noetherian local domains of dimension 1. Suppose that $A \supset \mathbb{Q}$ and the maximal ideal $\mathfrak{m}$ of A generates the maximal ideal of A'. Then u is a regular morphism. Moreover, we suppose that there exist canonical inclusions $k = A/\mathfrak{m} \to A$, $k' = A'/\mathfrak{m}A' \to A'$ such that $u(k) \subset k'$.

If A is essentially of finite type over $\mathbb{Q}$, then the ideal $H_{B/A}$ can be computed in Singular by following its definition but it is easier to describe only the ideal $\sum_f \big((f) : I\big)\Delta_f B$ defined above. This is the case considered in our algorithmic part, let us say $A \cong k[x]/(F)$ for some variables $x = (x_1, \dots, x_m)$, and the completion of A' is $k'[\![x]\!]/(F)$. When v is defined by polynomials y from $k'[x]$ then our problem is easy. Let L be the field obtained by adjoining to k all coefficients of y. Then $R = L[x]/(F)$ is a subring of A' containing $\operatorname{Im} v$ which is essentially smooth over A. Then we may take B' as a standard smooth A-algebra such that R is a localization of B'. Consequently, we suppose usually that y is not in $k'[x]$.

We may suppose that $v(H_{B/A}) \neq 0$. Indeed, if $v(H_{B/A}) = 0$ then v induces an A-morphism $v' : B' = B/H_{B/A} \to A'$ and we may replace (B, v) by (B', v'). Applying this trick several times we reduce to the case $v(H_{B/A}) \neq 0$. However, the fraction field of $\operatorname{Im} v$ is essentially smooth over A by separability, that is $H_{\operatorname{Im} v/A} A' \neq 0$ and in the worst case our trick will change B by $\operatorname{Im} v$ after several steps.

Choose $P' \in \big(\Delta_f((f) : I)\big) \setminus I$ for some system of polynomials $f = (f_1, \ldots, f_r)$ from I and $d' \in \big(v(P')A'\big) \cap A$, $d' \neq 0$. Moreover, we may choose P' to be from $M\big((f) : I\big)$ where M is a $r \times r$-minor of $\left(\dfrac{\partial f}{\partial Y}\right)$. Then $d' = v(P')z \in \big(v(H_{B/A})\big) \cap A$ for some $z \in A'$. Set $B_1 = B[Z]/(f_{r+1})$, where $f_{r+1} = -d' + P'Z$ and let $v_1 : B_1 \to A'$ be the map of B-algebras given by $Z \to z$. It follows that $d' \in \big((f, f_{r+1}) : (I, f_{r+1})\big)$ and $d' \in \Delta_f$, $d' \in \Delta_{f_{r+1}}$. Then $d = d'^2 \equiv P$ modulo (I, f_{r+1}) for $P = P'^2 Z^2 \in H_{B_1/A}$. Replace B by B_1 and the Jacobian matrix $J = (\partial f/\partial Y)$ will be now the new J given by $\begin{pmatrix} J & 0 \\ * & P' \end{pmatrix}$. Thus we reduce to the case when $d \in H_{B/A} \cap A$.

But how to get d with a computer if y is not polynomial defined over k'? Then the algorithm is complicated because we are not able to tell the computer who is y and so how to get d'. We may choose an element $a \in \mathfrak{m}$ and find a minimal $c \in \mathbb{N}$ such that $a^c \in (v(M)) + (a^{2c})$ (this is possible because $\dim A = 1$). Set $d' = a^c$. It follows that $d' \in (v(M)) + (d'^2) \subset (v(M)) + (d'^4) \subset \ldots$ and so $d' \in (v(M))$, that is $d' = v(M)z$ for some $z \in A'$. Certainly we cannot find precisely z, but later it is enough to know just a kind of truncation of it modulo d'^6.

Thus we may suppose that there exist $f = (f_1, \ldots, f_r)$, $r \leq n$ a system of polynomials from I, a $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$, $N \in ((f) : I)$ such that $0 \neq d \equiv P = MN$ modulo I. We may assume that $M = \det((\partial f_i/\partial Y_j)_{i,j \in [r]})$. Set $\bar{A} = A/(d^3)$, $\bar{A}' = A'/d^3 A'$, $\bar{u} = \bar{A} \otimes_A u$, $\bar{B} = B/d^3 B$, $\bar{v} = \bar{A} \otimes_A v$. Clearly, $\bar{u}$ is a regular morphism of Artinian local rings and it is easy to find a General Neron Desingularization in this frame. Thus there exists a $\bar{B}$-algebra C, which is smooth over $\bar{A}$ such that $\bar{v}$ factors through C. Moreover, we may suppose that $C \cong (\bar{A}[U]/(\omega))_\tau$ for some polynomials $\omega, \tau \in k[U]$ which are not in $m(\bar{A}[U]/(\omega))$ (note that $k \subset A$). Then $D \cong (A[U]/(\omega))_\tau$ is smooth over A and u factors through D. Usually, v does not factor through D, though $\bar{v}$ factors through $C \cong \bar{A} \otimes_A D$.

Let $y' \in D^n$ be such that the composite map $\bar{B} \to C \to \bar{D}$ is given by $Y \to y' + d^3D$. Thus $I(y') \equiv 0$ modulo d^3D. We have $d \equiv P$ modulo I and so $P(y') \equiv d$ modulo d^3 . Thus $P(y') = ds$ for a certain $s \in D$ with $s \equiv 1$ modulo d. Let H be the $n \times n$-matrix obtained by adding down to $(\partial f/\partial Y)$ as a border the block $(0|\mathrm{Id}_{n-r})$. Let G' be the adjoint matrix of H and $G = NG'$. We have

$$GH = HG = NM\mathrm{Id}_n = P\mathrm{Id}_n$$

and so

$$ds\mathrm{Id}_n = P(y')\mathrm{Id}_n = G(y')H(y').$$

Set $h = s(Y - y') - dG(y')T$, where $T = (T_1, \ldots, T_n)$ are new variables. Since

$$s(Y - y') \equiv dG(y')T \text{ modulo } h$$

and

$$f(Y) - f(y') \equiv \sum_j \frac{\partial f}{\partial Y_j}(y')(Y_j - y'_j)$$

modulo higher order terms in $Y_j - y'_j$, by Taylor's formula we see that for $p = \max_i \deg f_i$ we have

$$s^p f(Y) - s^p f(y') \equiv s^{p-1} dP(y')T + d^2 Q$$

modulo h, where $Q \in T^2 D[T]^r$. This is because $(\partial f/\partial Y)G = (P\mathrm{Id}_r|0)$. We have $f(y') = d^2 b$ for some $b \in dD^r$. Set $g_i = s^p b_i + s^p T_i + Q_i$, $i \in [r]$. Then we may take B' to be a localization of $(D[Y, T]/(I, h, g))_s$.

REFERENCES

[1] M. André, *Cinq exposes sur la desingularisation.* Handwritten manuscript Ecole Polytechnique Federale de Lausanne, 1991.

[2] M. Artin, *On the solutions of analytic equations.* Invent. Math. **5** (1968), 277–291.

[3] M. Artin, *Algebraic approximation of structures over complete local rings.* Publ. Math. Inst. Hautes Études Sci. **36** (1969), 23–58.

[4] M. Artin, *Constructions techniques for algebraic spaces.* Actes Congres. Intern. Math. **1** (1970), 419–423–291.

[5] M. Artin, *Versal deformations and algebraic stacks.* Invent. Math. **27** (1974), 165–189.

[6] M. Artin, *Algebraic structure of power series rings.* Contemp. Math. AMS, Providence, 1982, pp. 223–227.

[7] M. Artin and J. Denef, *Smoothing of a ring homomorphism along a section.* Arithmetic and Geometry **II**, Birkhäuser, Boston, 1983, pp. 5–32.

[8] S. Basarab, V. Nica and D. Popescu, *Approximation properties and existential completeness for ring morphisms.* Manuscripta Math. **33** (1981), 227–282.

[9] S.M. Bhatwadeckar and R.A. Rao, *On a question of Quillen.* Trans. Amer. Math. Soc. **279** (1983), 801–810.

[10] M. Cipu and D. Popescu, *Some extensions of Neron's p-desingularization and approximation.* Rev. Roum. Math. Pures Appl. **26** (1981), 1299–1304.

[11] M. Cipu and D. Popescu, *A desingularization theorem of Neron type.* Ann. Univ. Ferrara **30** (1984), 63–76.

[12] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, *Singular 3-1-6 – A computer algebra system for polynomial computations.* http://www.singular.uni-kl.de (2012).

[13] R. Elkik, *Solutions d'equations a coefficients dans un anneaux henselien.* Ann. Sci. Éc. Norm. Supèr. **6** (1973), 553–604.

[14] M. Greenberg, *Rational points in henselian discrete valuation rings.* Publ. Math. Inst. Hautes Études Sci. **31** (1966), 59–64.

[15] A. Grothendieck and J. Dieudonne, *Elements de geometrie algebrique, IV, Part 1.* Publ. Math. Inst. Hautes Études Sci., 1966.

[16] M. Kashiwara and K. Vilonen, *Microdifferential systems and the codimension-three conjecture.* Ann. of Math. **(2) 180** (2014), 573–620.

[17] H. Kurke, T. Mostowski, G. Pfister, D. Popescu and M. Roczen, *Die Approximationseigenschaft lokaler Ringe.* Springer Lect. Notes in Math. **634**, Springer-Verlag, Berlin-New York, 1978.

[18] T.Y. Lam, *Serre's Conjecture.* Springer Lect. Notes in Math. **635**, Berlin, 1978.

[19] H. Lindel, *On the Bass-Quillen conjecture concerning projective modules over polynomial rings.* Invent. Math. **65** (1981), 319–323.

[20] A. Néron, *Modeles minimaux des varietes abeliennes sur les corps locaux et globaux.* Publ. Math. Inst. Hautes Études Sci. **21** (1964), 5–128.

[21] G. Pfister and D. Popescu, *Die strenge Approximationseigenschaft lokaler Ringe.* Invent. Math. **30** (1975), 145–174.

[22] G. Pfister and D. Popescu, *Constructive General Neron Desingularization for one dimensional local rings.* J. Symbolic Comput. **80** (2017), 570–580. arXiv:AC/1512.08435v1.

[23] A. Ploski, *Note on a theorem of M. Artin.* Bull. Acad. Polon. des Sci. **XXII** (1974), *11*, 1107–1110.

[24] A. Popescu and D. Popescu, *A method to compute the General Neron Desingularization in the frame of one dimensional local domains.* In: W. Decker, G. Pfister and M. Schulze (Eds.), *Singularities and Computer Algebra-Festschrift for Gert-Martin Greuel On the Occasion of his 70th Birthday.* Springer Monograph. DOI 10.1007/978-319-28829-1, arXiv:AC/1508.05511.

[25] D. Popescu, *A strong approximation theorem over discrete valuation rings.* Rev. Roum. Math. Pures Appl. **20** (1975), 659–692.

[26] D. Popescu, *Algebraically pure morphisms.* Rev. Roum. Math. Pures Appl. **24** (1979), 947–977.

[27] D. Popescu, *General Neron Desingularization.* Nagoya Math. J. **100** (1985), 97–126.

[28] D. Popescu, *General Neron Desingularization and approximation.* Nagoya Math. J. **104** (1986), 85–115.

[29] D. Popescu, *Polynomial rings and their projective modules.* Nagoya Math. J. **113** (1989), 121–128.

[30] D. Popescu, *Letter to the Editor. General Neron Desingularization and approximation.* Nagoya Math. J. **118** (1990), 45–53.

[31] D. Popescu, *Artin Approximation.* In: M. Hazewinkel (Ed.), *Handbook of Algebra*, **2**, Elsevier, 2000, pp. 321–355.

[32] D. Popescu, *Variations on Néron desingularization.* In: *Sitzungsberichte der Berliner Mathematischen Gesselschaft*, Berlin, 2001, pp. 143–151.

[33] D. Popescu, *On a question of Quillen.* Bull. Math. Soc. Sci. Math. Roumanie **45(93)** (2002), 209–212.

[34] D. Popescu, *Around General Neron Desingularization*. J. Algebra Appl. **16** (2017), *2*. DOI: 10.1142/SO219498817500724, arXiv:AC/1504.06938.

[35] D. Popescu and M. Roczen, *Indecomposable Cohen-Macaulay modules and irreducible maps*. Compos. Math. **76** (1990), 277–294.

[36] D. Quillen, *Projective modules over polynomial rings*. Invent. Math. **36** (1976), 167–171.

[37] G. Rond, *Sur la linearité de la fonction de Artin*. Ann. Sci. Éc. Norm. Supér. (4) **38** (2005), 979–988.

[38] C. Rotthaus, *Rings with the property of approximation*. Math. Ann. **287** (1990), 455–466.

[39] M. Spivakovski, *A new proof of D. Popescu's theorem on smoothing of ring homomorphisms*. J. Amer. Math. Soc. **294** (1999), 381–444.

[40] A. Steenpass, *Algorithms in SINGULAR: Parallelization, Syzygies, and Singularities*, PhD Thesis. Kaiserslautern, 2014.

[41] R. Swan, *Neron-Popescu desingularization*. In: M. Kang (Ed.), *Algebra and Geometry*. International Press, Cambridge, 1998, pp. 135–192.

[42] B. Teissier, *Résultats récents sur l'approximation des morphismes en algèbre commutative* [*d'après Artin, Popescu et Spivakovski*], Sem. Bourbaki **784** (1994), 259–282.

[43] T. Vorst, *The Serre Problem for discrete Hodge algebras*. Math. Z. **184** (1983), 425–433.

Received 1 August 2016

University of Bucharest,
Simion Stoilow Institute of Mathematics
of the Romanian Academy,
Research unit 5,
P.O.Box 1-764, Bucharest 014700,
Romania
dorin.popescu@imar.ro