ADDITIVITY OF MAPS ACTING IN A SUM OF TRIPLE PRODUCTS ON TRIANGULAR ALGEBRAS

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In this paper, we prove that surjective maps acting in a sum of triple products on triangular algebras are automatically additives.

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1. INTRODUCTION

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$ and $\mathfrak{X}$ a faithful $(\mathfrak{A},\mathfrak{B})$-bimodule (see [1]). The $\mathfrak{R}$-algebra

$$\mathrm{Tri}(\mathfrak{A},\mathfrak{X},\mathfrak{B}) = \left\{ \begin{pmatrix} a & x \\ & b \end{pmatrix} \ : \ a \in \mathfrak{A},\ x \in \mathfrak{X},\ b \in \mathfrak{B} \right\}$$

under the usual matrix addition and formal matrix multiplication will be called a *Triangular algebra*. Let $\mathfrak{T} = \mathrm{Tri}(\mathfrak{A},\mathfrak{X},\mathfrak{B})$ be a Triangular algebra. We set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} \ : \ a \in \mathfrak{A} \right\},$$

$$\mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} \ : \ x \in \mathfrak{X} \right\},$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} \ : \ b \in \mathfrak{B} \right\}.$$

Then we can may write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$, and every element $c \in \mathfrak{T}$ can be written as $c = c_{11} + c_{12} + c_{22}$, where $c_{ij} \in \mathfrak{T}_{ij}$ $(1 \le i \le j \le 2)$. Note that $a_{ij}b_{kl} = 0$ if $j \neq k$.

Ji in [1] studied the additivity of maps $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$, where $\mathfrak{S}$ is an arbitrary ring, acting on sums of double products of type $M^*(x)a + aM^*(x)$ and $M(a)x + xM(a)$, for all $a \in \mathfrak{T}$ and $x \in \mathfrak{S}$, respectively. He proved the following theorem.

THEOREM 1.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$, $\mathfrak{X}$ a faithful $(\mathfrak{A},\mathfrak{B})$-bimodule and let $\mathfrak{T}$ be the Triangular algebra* $\mathrm{Tri}(\mathfrak{A},\mathfrak{X},\mathfrak{B})$ *satisfying the following conditions:*

(i) If $a \in \mathfrak{A}$ is such that $aa_1 + a_1a = 0$ for all $a_1 \in \mathfrak{A}$, then $a = 0$;

(ii) If $b \in \mathfrak{B}$ is such that $bb_1 + b_1b = 0$ for all $b_1 \in \mathfrak{B}$, then $b = 0$;

(iii) If $x \in \mathfrak{X}$ is such that $ax = 0$, for all $a \in \mathfrak{A}$, or $xb = 0$, for all $b \in \mathfrak{B}$, then $x = 0$.

Let $\mathfrak{S}$ be an arbitrary ring. If $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ are surjective maps such that

$$M\big(M^*(x)a + aM^*(x)\big) = xM(a) + M(a)x,$$
$$M^*\big(M(a)x + xM(a) = aM^*(x) + M^*(x)a,$$

for all $a, b \in \mathfrak{T}$ and $x, y \in \mathfrak{S}$, then both M and M^* are additives.

In this paper, we investigate the additivity of maps $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$, where $\mathfrak{S}$ is an arbitrary ring, acting on sums of triple products of type $M^*(x)ab + M^*(x)ba + aM^*(x)b + bM^*(x)a + abM^*(x) + baM^*(x)$ and $M(a)xy + M(a)yx + xM(a)y + yM(a)x + xyM(a) + yxM(a)$, for all $a, b \in \mathfrak{T}$ and $x, y \in \mathfrak{S}$, respectively.

2. THE MAIN RESULT

Our main result is the following theorem.

THEOREM 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$, $\mathfrak{X}$ a faithful $(\mathfrak{A},\mathfrak{B})$-bimodule and let $\mathfrak{T}$ be the Triangular algebra* $\mathrm{Tri}(\mathfrak{A},\mathfrak{X},\mathfrak{B})$ *satisfying the following conditions:*

(i) If $a \in \mathfrak{A}$ is such that $aa_1a_2 + aa_2a_1 + a_1aa_2 + a_2aa_1 + a_1a_2a + a_2a_1a = 0$ for all $a_1, a_2 \in \mathfrak{A}$, then $a = 0$;

(ii) If $b \in \mathfrak{B}$ is such that $bb_1b_2 + bb_2b_1 + b_1bb_2 + b_2bb_1 + b_1b_2b + b_2b_1b = 0$ for all $b_1, b_2 \in \mathfrak{B}$, then $b = 0$;

(iii) If $x \in \mathfrak{X}$ is such that $ax = 0$, for all $a \in \mathfrak{A}$, or $xb = 0$, for all $b \in \mathfrak{B}$, then $x = 0$.

Let $\mathfrak{S}$ be an arbitrary ring. If $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ are surjective maps such that

$$\begin{aligned} &M\big(M^*(x)ab + M^*(x)ba + aM^*(x)b + bM^*(x)a + abM^*(x) + baM^*(x)\big) \\ &= xM(a)M(b) + xM(b)M(a) + M(a)xM(b) + M(b)xM(a) \\ &+M(a)M(b)x + M(b)M(a)x, \end{aligned} \tag{1}$$

$$M^*\big(M(a)xy + M(a)yx + xM(a)y + yM(a)x + xyM(a) + yxM(a)\big)$$

$$= aM^*(x)M^*(y) + aM^*(y)M^*(x) + M^*(x)aM^*(y) + M^*(y)aM^*(x)$$
$$+M^*(x)M^*(y)a + M^*(y)M^*(x)a, \tag{2}$$

for all $a, b \in \mathfrak{T}$ and $x, y \in \mathfrak{S}$, then both M and M^* are additives.

Based on the techniques presented in [1] and [2], we shall organize the proof of Theorem 2.1 in a series of lemmas. We begin with the following lemma

LEMMA 2.1. *$M(0) = 0$ and $M^*(0) = 0$.*

Proof. From a direct calculation we have

$$\begin{aligned}
&M(0)\\
&= M\big(M^*(0)00 + M^*(0)00 + 0M^*(0)0 + 0M^*(0)0 + 00M^*(0)\\
&+00M^*(0)\big)\\
&= 0M(0)M(0) + 0M(0)M(0) + M(0)0M(0) + M(0)0M(0)\\
&+M(0)M(0)0 + M(0)M(0)0\\
&= 0.
\end{aligned}$$

Similarly, we prove $M^*(0) = 0$. □

LEMMA 2.2. *M and M^* are bijective.*

Proof. It suffices to prove that M and M^* are injective. First show that M is injective. Let c_1 and c_2 be in $\mathfrak{T}$ and suppose that $M(c_1) = M(c_2)$. From the equality (2), we have

$$\begin{aligned}
&M^*\big(M(c_i)xy + M(c_i)yx + xM(c_i)y + yM(c_i)x + xyM(c_i) + yxM(c_i)\big)\\
&= c_iM^*(x)M^*(y) + c_iM^*(y)M^*(x) + M^*(x)c_iM^*(y) + M^*(y)c_iM^*(x)\\
&+M^*(x)M^*(y)c_i + M^*(y)M^*(x)c_i
\end{aligned}$$

for all $x, y \in \mathfrak{S}$ and $i = 1, 2$. Hence, we have

$$\begin{aligned}
&c_1M^*(x)M^*(y) + c_1M^*(y)M^*(x) + M^*(x)c_1M^*(y) + M^*(y)c_1M^*(x)\\
&+M^*(x)M^*(y)c_1 + M^*(y)M^*(x)c_1 = c_2M^*(x)M^*(y) + c_2M^*(y)M^*(x)\\
&+M^*(x)c_2M^*(y) + M^*(y)c_2M^*(x) + M^*(x)M^*(y)c_2 + M^*(y)M^*(x)c_2.
\end{aligned}$$

The above equality leads to the conclusion that

$$\begin{aligned}
&(c_1 - c_2)M^*(x)M^*(y) + (c_1 - c_2)M^*(y)M^*(x) + M^*(x)(c_1 - c_2)M^*(y)\\
&+M^*(y)(c_1 - c_2)M^*(x) + M^*(x)M^*(y)(c_1 - c_2) + M^*(y)M^*(x)(c_1 - c_2)\\
&= 0.
\end{aligned}$$

From the surjectivity of M^* we conclude that

$$\begin{aligned}
&(c_1 - c_2)ab + (c_1 - c_2)ba + a(c_1 - c_2)b\\
&\qquad + b(c_1 - c_2)a + ab(c_1 - c_2) + ba(c_1 - c_2) = 0
\end{aligned}$$

for all $a, b \in \mathfrak{T}$. Let us write $c_1 - c_2 = c_{11} + c_{12} + c_{22}$, where $c_{ij} \in \mathfrak{T}_{ij}$ $(1 \leq i \leq j \leq 2)$. Then, for arbitrary elements $a_{11}, b_{11} \in \mathfrak{T}_{11}$ we have $c_{11}a_{11}b_{11} + c_{11}b_{11}a_{11} + a_{11}c_{11}b_{11} + b_{11}c_{11}a_{11} + a_{11}b_{11}c_{11} + b_{11}a_{11}c_{11} = 0$ which implies $c_{11} = 0$, by condition (i) of the Theorem. Also, for arbitrary elements $a_{22}, b_{22} \in \mathfrak{T}_{22}$ we have $c_{22}a_{22}b_{22} + c_{22}b_{22}a_{22} + a_{22}c_{22}b_{22} + b_{22}c_{22}a_{22} + a_{22}b_{22}c_{22} + b_{22}a_{22}c_{22} = 0$ which yields $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $a_{11} \in \mathfrak{T}_{11}$ and $b_{22} \in \mathfrak{T}_{22}$ we have $c_{12}a_{11}b_{22} + c_{12}b_{22}a_{11} + a_{11}c_{12}b_{22} + b_{22}c_{12}a_{11} + a_{11}b_{22}c_{12} + b_{22}a_{11}c_{12} = 0$ which results in $a_{11}c_{12}b_{22} = 0$. Thus, $c_{12} = 0$, by condition (iii) of the Theorem. It follows that $c_1 - c_2 = 0$ and therefore $c_1 = c_2$.

Now, to prove the injectivity of M^* let x_1 and x_2 be in $\mathfrak{S}$ and suppose $M^*(x_1) = M^*(x_2)$. From the equality (1), we have

$$\begin{aligned}
&M^*M\big(M^*(x_i)ab + M^*(x_i)ba + aM^*(x_i)b + bM^*(x_i)a + abM^*(x_i)\\
&+baM^*(x_i)\big) = M^*\big(x_iM(a)M(b) + x_iM(b)M(a) + M(a)x_iM(b)\\
&+M(b)x_iM(a) + M(a)M(b)x_i + M(b)M(a)x_i\big)\\
&= M^*\big(MM^{-1}(x_i)M(a)M(b) + MM^{-1}(x_i)M(b)M(a)\\
&+M(a)MM^{-1}(x_i)M(b) + M(b)MM^{-1}(x_i)M(a)\\
&+M(a)M(b)MM^{-1}(x_i) + M(b)M(a)MM^{-1}(x_i)\big)\\
&= M^{-1}(x_i)M^*M(a)M^*M(b) + M^{-1}(x_i)M^*M(b)M^*M(a)\\
&+M^*M(a)M^{-1}(x_i)M^*M(b) + M^*M(b)M^{-1}(x_i)M^*M(a)\\
&+M^*M(a)M^*M(b)M^{-1}(x_i) + M^*M(b)M^*M(a)M^{-1}(x_i)
\end{aligned}$$

for all $a, b \in \mathfrak{T}$ and $i = 1, 2$. It follows that

$$\begin{aligned}
&M^{-1}(x_1)M^*M(a)M^*M(b) + M^{-1}(x_1)M^*M(b)M^*M(a)\\
&+M^*M(a)M^{-1}(x_1)M^*M(b) + M^*M(b)M^{-1}(x_1)M^*M(a)\\
&+M^*M(a)M^*M(b)M^{-1}(x_1) + M^*M(b)M^*M(a)M^{-1}(x_1)\big)\\
&= M^{-1}(x_2)M^*M(a)M^*M(b) + M^{-1}(x_2)M^*M(b)M^*M(a)\\
&+M^*M(a)M^{-1}(x_2)M^*M(b) + M^*M(b)M^{-1}(x_2)M^*M(a)\\
&+M^*M(a)M^*M(b)M^{-1}(x_2) + M^*M(b)M^*M(a)M^{-1}(x_2)
\end{aligned}$$

which results in

$$\begin{aligned}
&\big(M^{-1}(x_1) - M^{-1}(x_2)\big)M^*M(a)M^*M(b)\\
&+\big(M^{-1}(x_1) - M^{-1}(x_2)\big)M^*M(b)M^*M(a)\\
&+M^*M(a)\big(M^{-1}(x_1) - M^{-1}(x_2)\big)M^*M(b)\\
&+M^*M(b)\big(M^{-1}(x_1) - M^{-1}(x_2)\big)M^*M(a)\\
&+M^*M(a)M^*M(b)\big(M^{-1}(x_1) - M^{-1}(x_2)\big)\\
&+M^*M(b)M^*M(a)\big(M^{-1}(x_1) - M^{-1}(x_2)\big)\big)\\
&= 0.
\end{aligned}$$

Noting that M^*M is also surjective and using a similar argument as applied in the previous case, we can show that $M^{-1}(u_1) = M^{-1}(u_2)$. Consequently $u_1 = u_2$. $\square$

LEMMA 2.3. *The maps* $M^{*-1} : \mathfrak{T} \to \mathfrak{S}$ *and* $M^{-1} : \mathfrak{S} \to \mathfrak{T}$ *satisfy:*

(i)

$$\begin{aligned}&M^{*-1}\big(M^{-1}(x)ab + M^{-1}(x)ba + aM^{-1}(x)b + bM^{-1}(x)a\\&+abM^{-1}(x) + baM^{-1}(x)\big) = xM^{*-1}(a)M^{*-1}(b)\\&+xM^{*-1}(b)M^{*-1}(a) + M^{*-1}(a)xM^{*-1}(b) + M^{*-1}(b)xM^{*-1}(a)\\&+M^{*-1}(a)M^{*-1}(b)x + M^{*-1}(b)M^{*-1}(a)x,\end{aligned}$$

(ii)

$$\begin{aligned}&M^{-1}\big(M^{*-1}(a)xy + M^{*-1}(a)yx + xM^{*-1}(a)y + yM^{*-1}(a)x\\&+xyM^{*-1}(a) + yxM^{*-1}(a)\big) = aM^{-1}(x)M^{-1}(y)\\&+aM^{-1}(y)M^{-1}(x) + M^{-1}(x)aM^{-1}(y) + M^{-1}(y)aM^{-1}(x)\\&+M^{-1}(x)M^{-1}(y)a + M^{-1}(y)M^{-1}(x)a,\end{aligned}$$

for all $a, b \in \mathfrak{T}$ *and* $x, y \in \mathfrak{S}$.

Proof. From the equality (2) we have

$$\begin{aligned}&M^*\big(xM^{*-1}(a)M^{*-1}(b) + xM^{*-1}(b)M^{*-1}(a) + M^{*-1}(a)xM^{*-1}(b)\\&+M^{*-1}(b)xM^{*-1}(a) + M^{*-1}(a)M^{*-1}(b)x + M^{*-1}(b)M^{*-1}(a)x\big)\\&= M^*\big(MM^{-1}(x)M^{*-1}(a)M^{*-1}(b) + MM^{-1}(x)M^{*-1}(b)M^{*-1}(a)\\&+M^{*-1}(a)MM^{-1}(x)M^{*-1}(b) + M^{*-1}(b)MM^{-1}(x)M^{*-1}(a)\\&+M^{*-1}(a)M^{*-1}(b)MM^{-1}(x) + M^{*-1}(b)M^{*-1}(a)MM^{-1}(x)\big)\\&= M^{-1}(x)ab + M^{-1}(x)ba + aM^{-1}(x)b + bM^{-1}(x)a + abM^{-1}(x)\\&+baM^{-1}(x)\end{aligned}$$

for all $a, b \in \mathfrak{T}$ and $x \in \mathfrak{S}$. Applying M^{*-1} in the above equality, we obtain the equality (i).

The second equality follows in a similar way. $\square$

LEMMA 2.4. *Let* $a, b, c \in \mathfrak{T}$ *such that* $M(c) = M(a) + M(b)$. *Then*

(i)

$$\begin{aligned}&M\big(cst + cts + sct + tcs + stc + tsc\big)\\&\qquad = M\big(ast + ats + sat + tas + sta + tsa\big)\\&\qquad\qquad + M\big(bst + bts + sbt + tbs + stb + tsb\big)\end{aligned}$$

for all $s, t \in \mathfrak{T}$;

(ii)

$$\begin{aligned} M^{*-1}\big(cst + cts + sct + tcs + stc + tsc\big) \\ = M^{*-1}\big(ast + ats + sat + tas + sta + tsa\big) \\ + M^{*-1}\big(bst + bts + sbt + tbs + stb + tsb\big) \end{aligned}$$

for all $s, t \in \mathfrak{T}$.

Proof. From the equality (1), for arbitrary elements $s, t \in \mathfrak{T}$ we have

$$\begin{aligned}
&M\big(cst + cts + sct + tcs + stc + tsc\big) \\
&= M\big(cM^*M^{*-1}(s)t + ctM^*M^{*-1}(s) + M^*M^{*-1}(s)ct \\
&+tcM^*M^{*-1}(s) + M^*M^{*-1}(s)tc + tM^*M^{*-1}(s)c\big) \\
&= M(c)M^{*-1}(s)M(t) + M(c)M(t)M^{*-1}(s) + M^{*-1}(s)M(c)M(t) \\
&+M(t)M(c)M^{*-1}(s) + M^{*-1}(s)M(t)M(c) + M(t)M^{*-1}(s)M(c) \\
&= \big(M(a) + M(b)\big)M^{*-1}(s)M(t) + \big(M(a) + M(b)\big)M(t)M^{*-1}(s) \\
&+M^{*-1}(s)\big(M(a) + M(b)\big)M(t) + M(t)\big(M(a) + M(b)\big)M^{*-1}(s) \\
&+M^{*-1}(s)M(t)\big(M(a) + M(b)\big) + M(t)M^{*-1}(s)\big(M(a) + M(b)\big) \\
&= M(a)M^{*-1}(s)M(t) + M(a)M(t)M^{*-1}(s) + M^{*-1}(s)M(a)M(t) \\
&+M(t)M(a)M^{*-1}(s) + M^{*-1}(s)M(t)M(a) + M(t)M^{*-1}(s)M(a) \\
&+M(b)M^{*-1}(s)M(t) + M(b)M(t)M^{*-1}(s) + M^{*-1}(s)M(b)M(t) \\
&+M(t)M(b)M^{*-1}(s) + M^{*-1}(s)M(t)M(b) + M(t)M^{*-1}(s)M(b) \\
&= M\big(aM^*M^{*-1}(s)t + atM^*M^{*-1}(s) + M^*M^{*-1}(s)at \\
&+taM^*M^{*-1}(s) + M^*M^{*-1}(s)ta + tM^*M^{*-1}(s)a\big) \\
&+M\big(bM^*M^{*-1}(s)t + btM^*M^{*-1}(s) + M^*M^{*-1}(s)bt \\
&+tbM^*M^{*-1}(s) + M^*M^{*-1}(s)tb + tM^*M^{*-1}(s)b\big) \\
&= M\big(ast + ats + sat + tas + sta + tsa\big) + M\big(bst + bts + sbt + tbs \\
&+stb + tsb\big).
\end{aligned}$$

By a similar argument, we prove the equality (ii), from the equality (i) in the Lemma 2.3. □

LEMMA 2.5. *Let* $a_{11} \in \mathfrak{T}_{11}$ *and* $b_{12} \in \mathfrak{T}_{12}$. *Then*

(i) $M(a_{11} + b_{12}) = M(a_{11}) + M(b_{12})$;

(ii) $M^{*-1}(a_{11} + b_{12}) = M^{*-1}(a_{11}) + M^{*-1}(b_{12})$.

Proof. From the surjectivity of M, there exists $c \in \mathfrak{T}$ such that $M(c) = M(a_{11}) + M(b_{12})$, where $c = c_{11} + c_{12} + c_{22}$. Hence, for arbitrary elements

$s_{11} \in \mathfrak{T}_{11}$ and $t_{22} \in \mathfrak{T}_{22}$ we have

$$\begin{aligned}
&M\big(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c\big)\\
&= M\big(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11}\\
&+t_{22}s_{11}a_{11}\big) + M\big(b_{12}s_{11}t_{22} + b_{12}t_{22}s_{11} + s_{11}b_{12}t_{22} + t_{22}b_{12}s_{11}\\
&+s_{11}t_{22}b_{12} + t_{22}s_{11}b_{12}\big)\\
&= M\big(s_{11}b_{12}t_{22}\big),
\end{aligned}$$

by Lemma 2.4(i). It follows that $s_{11}ct_{22} + t_{22}cs_{11} = s_{11}b_{12}t_{22}$ which implies $s_{11}c_{12}t_{22} = s_{11}b_{12}t_{22}$. Hence, $c_{12} = b_{12}$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$ we have

$$\begin{aligned}
&M\big(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c\big)\\
&= M\big(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11}\\
&+t_{22}s_{22}a_{11}\big) + M\big(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + s_{22}b_{12}t_{22} + t_{22}b_{12}s_{22}\\
&+s_{22}t_{22}b_{12} + t_{22}s_{22}b_{12}\big)\\
&= M\big(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}\big),
\end{aligned}$$

by Lemma 2.4(i) again. This shows that $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}$ which results in $c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} = 0$. Thus $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $s_{12} \in \mathfrak{T}_{12}$ and $t_{22} \in \mathfrak{T}_{22}$ we have

$$\begin{aligned}
&M\big(cs_{12}t_{22} + ct_{22}s_{12} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c + t_{22}s_{12}c\big)\\
&= M\big(a_{11}s_{12}t_{22} + a_{11}t_{22}s_{12} + s_{12}a_{11}t_{22} + t_{22}a_{11}s_{12} + s_{12}t_{22}a_{11}\\
&+t_{22}s_{12}a_{11}\big) + M\big(b_{12}s_{12}t_{22} + b_{12}t_{22}s_{12} + s_{12}b_{12}t_{22} + t_{22}b_{12}s_{12}\\
&+s_{12}t_{22}b_{12} + t_{22}s_{12}b_{12}\big)\\
&= M\big(a_{11}s_{12}t_{22}\big).
\end{aligned}$$

We can thus conclude that $cs_{12}t_{22} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c = a_{11}s_{12}t_{22}$ which yields $c_{11}s_{12}t_{22} = a_{11}s_{12}t_{22}$. Therefore, $c_{11} = a_{11}$, by condition (iii) of the Theorem and the hypothesis that $\mathfrak{X}$ is a faithful module. $\square$

Similarly, we prove the following lemma.

LEMMA 2.6. *Let $a_{22} \in \mathfrak{T}_{22}$ and $b_{12} \in \mathfrak{T}_{12}$. Then*

(i) $M(a_{22} + b_{12}) = M(a_{22}) + M(b_{12})$;

(ii) $M^{*-1}(a_{22} + b_{12}) = M^{*-1}(a_{22}) + M^{*-1}(b_{12})$.

LEMMA 2.7. *Let $t_{11} \in \mathfrak{T}_{11}$, $a_{12}, b_{12} \in \mathfrak{T}_{12}$ and $c_{22} \in \mathfrak{T}_{22}$. Then*

(i) $M(t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22}) = M(t_{11}a_{12}c_{22}) + M(t_{11}b_{12}c_{22})$;

(ii) $M^{*-1}(t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22}) = M^{*-1}(t_{11}a_{12}c_{22}) + M^{*-1}(t_{11}b_{12}c_{22})$.

Proof. First of all, we note that the following identity is valid

$$\begin{aligned}
&t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22} \\
&= t_{11}(a_{12}+c_{22})(b_{12}+c_{22}) + t_{11}(b_{12}+c_{22})(a_{12}+c_{22}) \\
&+(a_{12}+c_{22})t_{11}(b_{12}+c_{22}) + (b_{12}+c_{22})t_{11}(a_{12}+c_{22}) \\
&+(a_{12}+c_{22})(b_{12}+c_{22})t_{11} + (b_{12}+c_{22})(a_{12}+c_{22})t_{11}.
\end{aligned}$$

Hence, from the equality (1) we compute

$$\begin{aligned}
&M\big(t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22}\big) \\
&= M\big(t_{11}(a_{12}+c_{22})(b_{12}+c_{22}) + t_{11}(b_{12}+c_{22})(a_{12}+c_{22}) \\
&+(a_{12}+c_{22})t_{11}(b_{12}+c_{22}) + (b_{12}+c_{22})t_{11}(a_{12}+c_{22}) \\
&+(a_{12}+c_{22})(b_{12}+c_{22})t_{11} + (b_{12}+c_{22})(a_{12}+c_{22})t_{11}\big) \\
&= M\big(M^*M^{*-1}(t_{11})(a_{12}+c_{22})(b_{12}+c_{22}) \\
&+M^*M^{*-1}(t_{11})(b_{12}+c_{22})(a_{12}+c_{22}) \\
&+(a_{12}+c_{22})M^*M^{*-1}(t_{11})(b_{12}+c_{22}) \\
&+(b_{12}+c_{22})M^*M^{*-1}(t_{11})(a_{12}+c_{22}) \\
&+(a_{12}+c_{22})(b_{12}+c_{22})M^*M^{*-1}(t_{11}) \\
&+(b_{12}+c_{22})(a_{12}+c_{22})M^*M^{*-1}(t_{11})\big) \\
&= M^{*-1}(t_{11})M(a_{12}+c_{22})M(b_{12}+c_{22}) \\
&+M^{*-1}(t_{11})M(b_{12}+c_{22})M(a_{12}+c_{22}) \\
&+M(a_{12}+c_{22})M^{*-1}(t_{11})M(b_{12}+c_{22}) \\
&+M(b_{12}+c_{22})M^{*-1}(t_{11})M(a_{12}+c_{22}) \\
&+M(a_{12}+c_{22})M(b_{12}+c_{22})M^{*-1}(t_{11}) \\
&+M(b_{12}+c_{22})M(a_{12}+c_{22})M^{*-1}(t_{11}) \\
&= M^{*-1}(t_{11})\big(M(a_{12})+M(c_{22})\big)M(b_{12}+c_{22}) \\
&+M^{*-1}(t_{11})M(b_{12}+c_{22})\big(M(a_{12})+M(c_{22})\big) \\
&+\big(M(a_{12})+M(c_{22})\big)M^{*-1}(t_{11})M(b_{12}+c_{22}) \\
&+M(b_{12}+c_{22})M^{*-1}(t_{11})\big(M(a_{12})+M(c_{22})\big) \\
&+\big(M(a_{12})+M(c_{22})\big)M(b_{12}+c_{22})M^{*-1}(t_{11}) \\
&+M(b_{12}+c_{22})\big(M(a_{12})+M(c_{22})\big)M^{*-1}(t_{11})\big) \\
&= M^{*-1}(t_{11})M(a_{12})M(b_{12}+c_{22}) + M^{*-1}(t_{11})M(b_{12}+c_{22})M(a_{12}) \\
&+M(a_{12})M^{*-1}(t_{11})M(b_{12}+c_{22}) + M(b_{12}+c_{22})M^{*-1}(t_{11})M(a_{12}) \\
&+M(a_{12})M(b_{12}+c_{22})M^{*-1}(t_{11}) + M(b_{12}+c_{22})M(a_{12})M^{*-1}(t_{11})\big) \\
&+M^{*-1}(t_{11})M(c_{22})M(b_{12}+c_{22}) + M^{*-1}(t_{11})M(b_{12}+c_{22})M(c_{22})
\end{aligned}$$

$$
\begin{aligned}
&+M(c_{22})M^{*-1}(t_{11})M(b_{12}+c_{22})+M(b_{12}+c_{22})M^{*-1}(t_{11})M(c_{22})\\
&+M(c_{22})M(b_{12}+c_{22})M^{*-1}(t_{11})+M(b_{12}+c_{22})M(c_{22})M^{*-1}(t_{11})\big)\\
&=M\big(t_{11}a_{12}(b_{12}+c_{22})+t_{11}(b_{12}+c_{22})a_{12}+a_{12}t_{11}(b_{12}+c_{22})\\
&+(b_{12}+c_{22})t_{11}a_{12}+a_{12}(b_{12}+c_{22})t_{11}+(b_{12}+c_{22})a_{12}t_{11}\big)\\
&+M\big(t_{11}c_{22}(b_{12}+c_{22})+t_{11}(b_{12}+c_{22})c_{22}+c_{22}t_{11}(b_{12}+c_{22})\\
&+(b_{12}+c_{22})t_{11}c_{22}+c_{22}(b_{12}+c_{22})t_{11}+(b_{12}+c_{22})c_{22}t_{11}\big)\\
&=M\big(t_{11}a_{12}c_{22}\big)+M\big(t_{11}b_{12}c_{22}\big).
\end{aligned}
$$

Similarly, we prove (ii), using the Lemma 2.3(i). □

LEMMA 2.8. *Let* $a_{11}, b_{11} \in \mathfrak{T}_{11}$. *Then*

(i) $M(a_{11}+b_{11}) = M(a_{11}) + M(b_{11})$;

(ii) $M^{*-1}(a_{11}+b_{11}) = M^{*-1}(a_{11}) + M^{*-1}(b_{11})$.

Proof. Choose $c = c_{11}+c_{12}+c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{11})+M(b_{11})$. For arbitrary elements $s_{11} \in \mathfrak{T}_{11}$, $t_{22} \in \mathfrak{T}_{22}$, we have

$$
\begin{aligned}
&M^{*-1}\big(cs_{11}t_{22}+ct_{22}s_{11}+s_{11}ct_{22}+t_{22}cs_{11}+s_{11}t_{22}c+t_{22}s_{11}c\big)\\
&=M^{*-1}\big(a_{11}s_{11}t_{22}+a_{11}t_{22}s_{11}+s_{11}a_{11}t_{22}+t_{22}a_{11}s_{11}+s_{11}t_{22}a_{11}\\
&+t_{22}s_{11}a_{11}\big)+M^{*-1}\big(b_{11}s_{11}t_{22}+b_{11}t_{22}s_{11}+s_{11}b_{11}t_{22}+t_{22}b_{11}s_{11}\\
&+s_{11}t_{22}b_{11}+t_{22}s_{11}b_{11}\big)=0.
\end{aligned}
$$

Hence, $s_{11}ct_{22}+t_{22}cs_{11}=0$ which results $s_{11}c_{12}t_{22}=0$. Therefore, $c_{12}=0$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$, $t_{22} \in \mathfrak{T}_{22}$ we have

$$
\begin{aligned}
&M^{*-1}\big(cs_{22}t_{22}+ct_{22}s_{22}+s_{22}ct_{22}+t_{22}cs_{22}+s_{22}t_{22}c+t_{22}s_{22}c\big)\\
&=M^{*-1}\big(a_{11}s_{22}t_{22}+a_{11}t_{22}s_{22}+s_{22}a_{11}t_{22}+t_{22}a_{11}s_{22}+s_{22}t_{22}a_{11}\\
&+t_{22}s_{22}a_{11}\big)+M^{*-1}\big(b_{11}s_{22}t_{22}+b_{11}t_{22}s_{22}+s_{22}b_{11}t_{22}+t_{22}b_{11}s_{22}\\
&+s_{22}t_{22}b_{11}+t_{22}s_{22}b_{11}\big)=0
\end{aligned}
$$

which implies

$$c_{22}s_{22}t_{22}+c_{22}t_{22}s_{22}+s_{22}c_{22}t_{22}+t_{22}c_{22}s_{22}+s_{22}t_{22}c_{22}+t_{22}s_{22}c_{22}=0.$$

Thus, $c_{22}=0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $s_{12} \in \mathfrak{T}_{12}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$
\begin{aligned}
&M^{*-1}\big(cs_{12}t_{22}+ct_{22}s_{12}+s_{12}ct_{22}+t_{22}cs_{12}+s_{12}t_{22}c+t_{22}s_{12}c\big)\\
&=M^{*-1}\big(a_{11}s_{12}t_{22}+a_{11}t_{22}s_{12}+s_{12}a_{11}t_{22}+t_{22}a_{11}s_{12}+s_{12}t_{22}a_{11}\\
&+t_{22}s_{12}a_{11}\big)+M^{*-1}\big(b_{11}s_{12}t_{22}+b_{11}t_{22}s_{12}+s_{12}b_{11}t_{22}+t_{22}b_{11}s_{12}
\end{aligned}
$$

$$
\begin{aligned}
&+s_{12}t_{22}b_{11} + t_{22}s_{12}b_{11}\big)\\
&= M^{*-1}\big(a_{11}s_{12}t_{22}\big) + M^{*-1}\big(b_{11}s_{12}t_{22}\big)\\
&= M^{*-1}\big(a_{11}s_{12}t_{22} + b_{11}s_{12}t_{22}\big),
\end{aligned}
$$

by Lemma 2.7(ii). It follows that $cs_{12}t_{22}+s_{12}ct_{22}+t_{22}cs_{12}+s_{12}t_{22}c = a_{11}s_{12}t_{22}+b_{11}s_{12}t_{22}$ which yields $c_{11}s_{12}t_{22} = a_{11}s_{12}t_{22} + b_{11}s_{12}t_{22}$. So $c_{11} = a_{11} + b_{11}$, by condition (iii) of the Theorem and the hypothesis that $\mathfrak{X}$ is a faithful module. $\square$

Similarly, we prove the following Lemma.

LEMMA 2.9. *Let* $a_{22}, b_{22} \in \mathfrak{T}_{22}$. *Then*

(i) $M(a_{22} + b_{22}) = M(a_{22}) + M(b_{22})$;

(ii) $M^{*-1}(a_{22} + b_{22}) = M^{*-1}(a_{22}) + M^{*-1}(b_{22})$.

LEMMA 2.10. *Let* $a_{12}, b_{12} \in \mathfrak{T}_{12}$ *and* $s_{22}, t_{22} \in \mathfrak{T}_{22}$. *Then*

(i) $M(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}+b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}) = M(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}) + M(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22})$;

(ii) $M^{*-1}(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) = M^{*-1}(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22}) + M^{*-1}(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22})$.

Proof. First of all, we note that the following identity is valid

$$
\begin{aligned}
&2s_{22}^2t_{22} + 2s_{22}t_{22}s_{22} + 2t_{22}s_{22}^2\\
&+a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}\\
&= (a_{12} + s_{22})(b_{12} + s_{22})t_{22} + (a_{12} + s_{22})t_{22}(b_{12} + s_{22})\\
&(b_{12} + s_{22})(a_{12} + s_{22})t_{22} + t_{22}(a_{12} + s_{22})(b_{12} + s_{22})\\
&(b_{12} + s_{22})t_{22}(a_{12} + s_{22}) + t_{22}(b_{12} + s_{22})(a_{12} + s_{22}).
\end{aligned}
$$

Hence, from the equality (1) and Lemmas 2.6(i) and 2.9(i), we compute

$$
\begin{aligned}
&M(2s_{22}^2t_{22} + 2s_{22}t_{22}s_{22} + 2t_{22}s_{22}^2)\\
&+M(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22})\\
&= M(2s_{22}^2t_{22} + 2s_{22}t_{22}s_{22} + 2t_{22}s_{22}^2 + a_{12}s_{22}t_{22}\\
&+a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22})\\
&= M\big((a_{12} + s_{22})(b_{12} + s_{22})t_{22} + (a_{12} + s_{22})t_{22}(b_{12} + s_{22})\\
&(b_{12} + s_{22})(a_{12} + s_{22})t_{22} + t_{22}(a_{12} + s_{22})(b_{12} + s_{22})\\
&(b_{12} + s_{22})t_{22}(a_{12} + s_{22}) + t_{22}(b_{12} + s_{22})(a_{12} + s_{22})\big)\\
&= M\big((a_{12} + s_{22})(b_{12} + s_{22})M^*M^{*-1}(t_{22})\\
&+(a_{12} + s_{22})M^*M^{*-1}(t_{22})(b_{12} + s_{22})\\
&+(b_{12} + s_{22})(a_{12} + s_{22})M^*M^{*-1}(t_{22})
\end{aligned}
$$

$$
\begin{aligned}
&+M^*M^{*-1}(t_{22})(a_{12}+s_{22})(b_{12}+s_{22})\\
&(b_{12}+s_{22})M^*M^{*-1}(t_{22})(a_{12}+s_{22})\\
&+M^*M^{*-1}(t_{22})(b_{12}+s_{22})(a_{12}+s_{22})\big)\\
&=M(a_{12}+s_{22})M(b_{12}+s_{22})M^{*-1}(t_{22})\\
&+M(a_{12}+s_{22})M^{*-1}(t_{22})M(b_{12}+s_{22})\\
&M(b_{12}+s_{22})M(a_{12}+s_{22})M^{*-1}(t_{22})\\
&+M^{*-1}(t_{22})M(a_{12}+s_{22})M(b_{12}+s_{22})\\
&+M(b_{12}+s_{22})M^{*-1}(t_{22})M(a_{12}+s_{22})\\
&+M^{*-1}(t_{22})M(b_{12}+s_{22})M(a_{12}+s_{22})\\
&=\big(M(a_{12})+M(s_{22})\big)M(b_{12}+s_{22})M^{*-1}(t_{22})\\
&+\big(M(a_{12})+M(s_{22})\big)M^{*-1}(t_{22})M(b_{12}+s_{22})\\
&M(b_{12}+s_{22})\big(M(a_{12})+M(s_{22})\big)M^{*-1}(t_{22})\\
&+M^{*-1}(t_{22})\big(M(a_{12})+M(s_{22})\big)M(b_{12}+s_{22})\\
&+M(b_{12}+s_{22})M^{*-1}(t_{22})\big(M(a_{12})+M(s_{22})\big)\\
&+M^{*-1}(t_{22})M(b_{12}+s_{22})\big(M(a_{12})+M(s_{22})\big)\\
&=M(a_{12})M(b_{12}+s_{22})M^{*-1}(t_{22})+M(a_{12})M^{*-1}(t_{22})M(b_{12}+s_{22})\\
&M(b_{12}+s_{22})M(a_{12})M^{*-1}(t_{22})+M^{*-1}(t_{22})M(a_{12})M(b_{12}+s_{22})\\
&M(b_{12}+s_{22})M^{*-1}(t_{22})M(a_{12})+M^{*-1}(t_{22})M(b_{12}+s_{22})M(a_{12})\\
&+M(s_{22})M(b_{12}+s_{22})M^{*-1}(t_{22})+M(s_{22})M^{*-1}(t_{22})M(b_{12}+s_{22})\\
&M(b_{12}+s_{22})M(s_{22})M^{*-1}(t_{22})+M^{*-1}(t_{22})M(s_{22})M(b_{12}+s_{22})\\
&M(b_{12}+s_{22})M^{*-1}(t_{22})M(s_{22})+M^{*-1}(t_{22})M(b_{12}+s_{22})M(s_{22})\\
&=M\big(a_{12}(b_{12}+s_{22})M^*M^{*-1}(t_{22})+a_{12}M^*M^{*-1}(t_{22})(b_{12}+s_{22})\\
&+(b_{12}+s_{22})a_{12}M^*M^{*-1}(t_{22})+M^*M^{*-1}(t_{22})a_{12}(b_{12}+s_{22})\\
&+(b_{12}+s_{22})M^*M^{*-1}(t_{22})a_{12}+M^*M^{*-1}(t_{22})M(b_{12}+s_{22})a_{12}\big)\\
&+M(s_{22}(b_{12}+s_{22})M^*M^{*-1}(t_{22})+s_{22}M^*M^{*-1}(t_{22})(b_{12}+s_{22})\\
&+(b_{12}+s_{22})s_{22}M^*M^{*-1}(t_{22})+M^*M^{*-1}(t_{22})s_{22}(b_{12}+s_{22})\\
&+(b_{12}+s_{22})M^*M^{*-1}(t_{22})s_{22}+M^*M^{*-1}(t_{22})(b_{12}+s_{22})s_{22}\big)\\
&=M(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22})+M(s_{22}^2t_{22}+s_{22}t_{22}s_{22}+b_{12}s_{22}t_{22}+s_{22}^2t_{22}\\
&+t_{22}s_{22}^2+b_{12}t_{22}s_{22}+s_{22}t_{22}s_{22}+t_{22}s_{22}^2)\\
&=M(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22})+M(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22})\\
&+M(2s_{22}^2t_{22}+2s_{22}t_{22}s_{22}+2t_{22}s_{22}^2).
\end{aligned}
$$

It follows that

$$\begin{aligned}&M(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}+b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22})\\&=M(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22})+M(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}).\end{aligned}$$

Similarly, we prove (ii), using the Lemma 2.3(i). □

LEMMA 2.11. *Let* $a_{12}, b_{12} \in \mathfrak{T}_{12}$. *Then*

(i) $M(a_{12}+b_{12}) = M(a_{12}) + M(b_{12})$;

(ii) $M^{*-1}(a_{12}+b_{12}) = M^{*-1}(a_{12}) + M^{*-1}(b_{12})$.

Proof. By the surjectivity of M, there exists $c \in \mathfrak{T}$ such that $M(c) = M(a_{12}) + M(b_{12})$, where $c = c_{11} + c_{12} + c_{22}$. Hence, for arbitrary elements $s_{11} \in \mathfrak{T}_{11}$, $t_{22} \in \mathfrak{T}_{22}$ we have

$$\begin{aligned}&M^{*-1}\big(cs_{11}t_{22}+ct_{22}s_{11}+s_{11}ct_{22}+t_{22}cs_{11}+s_{11}t_{22}c+t_{22}s_{11}c\big)\\&=M^{*-1}\big(a_{12}s_{11}t_{22}+a_{12}t_{22}s_{11}+s_{11}a_{12}t_{22}+t_{22}a_{12}s_{11}+s_{11}t_{22}a_{12}\\&+t_{22}s_{11}a_{12}\big)+M^{*-1}\big(b_{12}s_{11}t_{22}+b_{12}t_{22}s_{11}+s_{11}b_{12}t_{22}+t_{22}b_{12}s_{11}\\&+s_{11}t_{22}b_{12}+t_{22}s_{11}b_{12}\big)\\&=M^{*-1}\big(s_{11}a_{12}t_{22}\big)+M^{*-1}\big(s_{11}b_{12}t_{22}\big)\\&=M^{*-1}\big(s_{11}a_{12}t_{22}+s_{11}b_{12}t_{22}\big),\end{aligned}$$

by Lemma 2.7(ii). It follows that $s_{11}ct_{22}+t_{22}cs_{11} = s_{11}a_{12}t_{22}+s_{11}b_{12}t_{22}$ which implies $s_{11}c_{12}t_{22} = s_{11}a_{12}t_{22}+s_{11}b_{12}t_{22}$. Therefore, $c_{12} = a_{12}+b_{12}$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$\begin{aligned}&M^{*-1}\big(cs_{22}t_{22}+ct_{22}s_{22}+s_{22}ct_{22}+t_{22}cs_{22}+s_{22}t_{22}c+t_{22}s_{22}c\big)\\&=M^{*-1}\big(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}+s_{22}a_{12}t_{22}+t_{22}a_{12}s_{22}+s_{22}t_{22}a_{12}\\&+t_{22}s_{22}a_{12}\big)+M^{*-1}\big(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}+s_{22}b_{12}t_{22}+t_{22}b_{12}s_{22}\\&+s_{22}t_{22}b_{12}+t_{22}s_{22}b_{12}\big)\\&=M^{*-1}\big(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}\big)+M^{*-1}\big(b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}\big)\\&=M^{*-1}\big(a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}+b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}\big),\end{aligned}$$

by Lemma 2.10(ii). It follows that $cs_{22}t_{22}+ct_{22}s_{22}+s_{22}ct_{22}+t_{22}cs_{22}+s_{22}t_{22}c+t_{22}s_{22}c = a_{12}s_{22}t_{22}+a_{12}t_{22}s_{22}+b_{12}s_{22}t_{22}+b_{12}t_{22}s_{22}$ which implies $c_{22}s_{22}t_{22}+c_{22}t_{22}s_{22}+s_{22}c_{22}t_{22}+t_{22}c_{22}s_{22}+s_{22}t_{22}c_{22}+t_{22}s_{22}c_{22} = 0$. Thus $c_{22} = 0$, by condition (ii) of the Theorem.

Now, for arbitrary elements $s_{11}, t_{11} \in \mathfrak{T}_{11}$, we have

$$\begin{aligned}&M^{*-1}\big(cs_{11}t_{11}+ct_{11}s_{11}+s_{11}ct_{11}+t_{11}cs_{11}+s_{11}t_{11}c+t_{11}s_{11}c\big)\\&=M^{*-1}\big(a_{12}s_{11}t_{11}+a_{12}t_{11}s_{11}+s_{11}a_{12}t_{11}+t_{11}a_{12}s_{11}+s_{11}t_{11}a_{12}\end{aligned}$$

$$
\begin{aligned}
&+t_{11}s_{11}a_{12}\big) + M^{*-1}\big(b_{12}s_{11}t_{11} + b_{12}t_{11}s_{11} + s_{11}b_{12}t_{11}\\
&+t_{11}b_{12}s_{11} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}\big)\\
&= M^{*-1}\big(s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12}\big) + M^{*-1}\big(s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}\big)\\
&= M^{*-1}\big(s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}\big).
\end{aligned}
$$

This shows that $cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c = s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}$ which results in $c_{11}s_{11}t_{11} + c_{11}t_{11}s_{11} + s_{11}c_{11}t_{11} + t_{11}c_{11}s_{11} + s_{11}t_{11}c_{11} + t_{11}s_{11}c_{11} = 0$. So $c_{11} = 0$, by condition (i) of the Theorem.

The proof of (ii) is similar. $\square$

LEMMA 2.12. *Let $a_{11} \in \mathfrak{T}_{11}$ and $b_{22} \in \mathfrak{T}_{22}$. Then*

(i) $M(a_{11} + b_{22}) = M(a_{11}) + M(b_{22})$;

(ii) $M^{*-1}(a_{11} + b_{22}) = M^{*-1}(a_{11}) + M^{*-1}(b_{22})$.

Proof. Choose $c = c_{11} + c_{12} + c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{11}) + M(b_{22})$. For arbitrary elements $s_{11} \in \mathfrak{T}_{11}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$
\begin{aligned}
&M^{*-1}\big(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c\big)\\
&= M^{*-1}\big(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11} + t_{22}s_{11}a_{11}\big)\\
&+M^{*-1}\big(b_{22}s_{11}t_{22} + b_{22}t_{22}s_{11} + s_{11}b_{22}t_{22} + t_{22}b_{22}s_{11} + s_{11}t_{22}b_{22} + t_{22}s_{11}b_{22}\big) = 0.
\end{aligned}
$$

It follows that $s_{11}ct_{22} + t_{22}cs_{11} = 0$ which implies $s_{11}c_{12}t_{22} = 0$. Hence, $c_{12} = 0$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$
\begin{aligned}
&M^{*-1}\big(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c\big)\\
&= M^{*-1}\big(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11} + t_{22}s_{22}a_{11}\big)\\
&+ M^{*-1}\big(b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}\big)\\
&= M^{*-1}\big(b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}\big).
\end{aligned}
$$

We can then conclude that $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}$ which yields $(c_{22} - b_{22})s_{22}t_{22} + (c_{22} - b_{22})t_{22}s_{22} + s_{22}(c_{22} - b_{22})t_{22} + t_{22}(c_{22} - b_{22})s_{22} + s_{22}t_{22}(c_{22} - b_{22}) + t_{22}s_{22}(c_{22} - b_{22}) = 0$. Thus, $c_{22} = b_{22}$, by condition (ii) of the Theorem. Yet, for arbitrary elements $s_{11} \in \mathfrak{T}_{11}$ and $t_{11} \in \mathfrak{T}_{11}$, we have

$$
\begin{aligned}
&M^{*-1}\big(cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c\big)\\
&= M^{*-1}\big(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}\big)\\
&+ M^{*-1}\big(b_{22}s_{11}t_{11} + b_{22}t_{11}s_{11} + s_{11}b_{22}t_{11} + t_{11}b_{22}s_{11} + s_{11}t_{11}b_{22} + t_{11}s_{11}b_{22}\big)\\
&= M^{*-1}\big(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}\big).
\end{aligned}
$$

This allows us to conclude again that $cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c = a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}$ which results in $c_{11} = a_{11}$, by condition (i) of the Theorem. $\square$

LEMMA 2.13. *Let $a_{11} \in \mathfrak{T}_{11}$, $b_{12} \in \mathfrak{T}_{12}$ and $c_{22} \in \mathfrak{T}_{22}$. Then*

(i) $M(a_{11} + b_{12} + c_{22}) = M(a_{11}) + M(b_{12}) + M(c_{22})$;

(ii) $M^{*-1}(a_{11} + b_{12} + c_{22}) = M^{*-1}(a_{11}) + M^{*-1}(b_{12}) + M^{*-1}(c_{22})$.

Proof. Choose $d = d_{11} + d_{12} + d_{22} \in \mathfrak{T}$ such that $M(d) = M(a_{11}) + M(b_{12}) + M(c_{22})$ and write $M(d) = M(a_{11} + b_{12}) + M(c_{22})$. For arbitrary elements $s_{11} \in \mathfrak{T}_{11}$, $t_{11} \in \mathfrak{T}_{11}$, we have

$$\begin{aligned}
&M^{*-1}\big(fs_{11}t_{11} + ft_{11}s_{11} + s_{11}ft_{11} + t_{11}fs_{11} + s_{11}t_{11}f + t_{11}s_{11}f\big)\\
&= M^{*-1}\big((a_{11} + b_{12})s_{11}t_{11} + (a_{11} + b_{12})t_{11}s_{11} + s_{11}(a_{11} + b_{12})t_{11}\\
&+t_{11}(a_{11} + b_{12})s_{11} + s_{11}t_{11}(a_{11} + b_{12}) + t_{11}s_{11}(a_{11} + b_{12})\big)\\
&+M^{*-1}\big(c_{22}s_{11}t_{11} + c_{22}t_{11}s_{11} + s_{11}c_{22}t_{11} + t_{11}c_{22}s_{11} + s_{11}t_{11}c_{22}\\
&+t_{11}s_{11}c_{22}\big) = M^{*-1}\big(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11}\\
&+s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}\big).
\end{aligned}$$

It follows that $fs_{11}t_{11} + ft_{11}s_{11} + s_{11}ft_{11} + t_{11}fs_{11} + s_{11}t_{11}f + t_{11}s_{11}f = a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}$ which implies $f_{11}s_{11}t_{11} + f_{11}t_{11}s_{11} + s_{11}f_{11}t_{11} + t_{11}f_{11}s_{11} + s_{11}t_{11}f_{11} + t_{11}s_{11}f_{11} = a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}$. Hence, $f_{11} = a_{11}$, by condition (i) of the Theorem. Also, for arbitrary elements $s_{11} \in \mathfrak{T}_{11}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$\begin{aligned}
&M^{*-1}\big(fs_{11}t_{22} + ft_{22}s_{11} + s_{11}ft_{22} + t_{22}fs_{11} + s_{11}t_{22}f + t_{22}s_{11}f\big)\\
&= M^{*-1}\big((a_{11} + b_{12})s_{11}t_{22} + (a_{11} + b_{12})t_{22}s_{11} + s_{11}(a_{11} + b_{12})t_{22}\\
&+t_{22}(a_{11} + b_{12})s_{11} + s_{11}t_{22}(a_{11} + b_{12}) + t_{22}s_{11}(a_{11} + b_{12})\big)\\
&+M^{*-1}\big(c_{22}s_{11}t_{22} + c_{22}t_{22}s_{11} + s_{11}c_{22}t_{22} + t_{22}c_{22}s_{11} + s_{11}t_{22}c_{22}\\
&+t_{22}s_{11}c_{22}\big)\\
&= M^{*-1}\big(s_{11}b_{12}t_{22}\big).
\end{aligned}$$

This shows that $s_{11}ft_{22} + t_{22}fs_{11} = s_{11}b_{12}t_{22}$ which yields $s_{11}f_{12}t_{22} = s_{11}b_{12}t_{22}$ Therefore, $f_{12} = b_{12}$, by condition (iii) of the Theorem. Yet, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$\begin{aligned}
&M^{*-1}\big(fs_{22}t_{22} + ft_{22}s_{22} + s_{22}ft_{22} + t_{22}fs_{22} + s_{22}t_{22}f + t_{22}s_{22}f\big)\\
&= M^{*-1}\big((a_{11} + b_{12})s_{22}t_{22} + (a_{11} + b_{12})t_{22}s_{22} + s_{22}(a_{11} + b_{12})t_{22}\\
&+t_{22}(a_{11} + b_{12})s_{22} + s_{22}t_{22}(a_{11} + b_{12}) + t_{22}s_{22}(a_{11} + b_{12})\big)\\
&+M^{*-1}\big(c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}\big)
\end{aligned}$$

$$= M^{*-1}\big(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}\big) + M^{*-1}\big(c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22}$$
$$+t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}\big)$$
$$= M^{*-1}\big(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22}$$
$$+t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}\big),$$

by Lemma 2.6(ii). This allows us to conclude that $fs_{22}t_{22} + ft_{22}s_{22} + s_{22}ft_{22} + t_{22}fs_{22} + s_{22}t_{22}f + t_{22}s_{22}f = b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}$ which results in $f_{22}s_{22}t_{22} + f_{22}t_{22}s_{22} + s_{22}f_{22}t_{22} + t_{22}f_{22}s_{22} + s_{22}t_{22}f_{22} + t_{22}s_{22}f_{22} = c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}$. Thus, $f_{22} = c_{22}$, by condition (ii) of the Theorem.

Now we are able to prove the Theorem 2.1. Our proof is similar to those presented by Ji [1].

Proof of Theorem. Let $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$ be arbitrary elements of $\mathfrak{T}$. From lemmas 2.8, 2.9, 2.11 and 2.13, we compute

$$\begin{aligned}
& M(a+b) \\
= \; & M\big((a_{11}+b_{11}) + (a_{12}+b_{12}) + (a_{22}+b_{22})\big) \\
= \; & M(a_{11}+b_{11}) + M(a_{12}+b_{12}) + M(a_{22}+b_{22}) \\
= \; & M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{22}) + M(b_{22}) \\
= \; & M(a_{11}) + M(a_{12}) + M(a_{22}) + M(b_{11}) + M(b_{12}) + M(b_{22}) \\
= \; & M(a_{11}+a_{12}+a_{22}) + M(b_{11}+b_{12}+b_{22}) \\
= \; & M(a) + M(b).
\end{aligned}$$

This shows that the map M is additive.

Now, we prove that M^* is additive. For any $x, y \in \mathfrak{S}$, there exist elements c and d in $\mathfrak{T}$ such that $c = M^*(x+y)$ and $d = M^*(x) + M^*(y)$. Hence, for arbitrary $s, t \in \mathfrak{T}$, by the additivity of M, we have

$$M(cst + cts + sct + tcs + stc + tsc)$$
$$= M\big(M^*(x+y)st + M^*(x+y)ts + sM^*(x+y)t + tM^*(x+y)s$$
$$+stM^*(x+y) + tsM^*(x+y)\big)$$
$$= (x+y)M(s)M(t) + (x+y)M(t)M(s) + M(s)(x+y)M(t)$$
$$+M(t)(x+y)M(s) + M(s)M(t)(x+y) + M(t)M(s)(x+y)$$
$$= xM(s)M(t) + xM(t)M(s) + M(s)xM(t) + M(t)xM(s)$$
$$+M(s)M(t)x + M(t)M(s)x + yM(s)M(t) + yM(t)M(s)$$
$$+M(s)yM(t) + M(t)yM(s) + M(s)M(t)y + M(t)M(s)y$$
$$= M\big(M^*(x)st + M^*(x)ts + sM^*(x)t + tM^*(x)s + stM^*(x) + tsM^*(x)\big)$$

$$
\begin{aligned}
&+M\big(M^*(y)st + M^*(y)ts + sM^*(y)t + tM^*(y)s + stM^*(y) + tsM^*(y)\big)\\
&= M\big(\big(M^*(x) + M^*(y)\big)st + \big(M^*(x) + M^*(y)\big)ts + s\big(M^*(x) + M^*(y)\big)t\\
&+t\big(M^*(x) + M^*(y)\big)s + st\big(M^*(x) + M^*(y)\big) + ts\big(M^*(x) + M^*(y)\big)\big)\\
&= M\big(dst + dts + sdt + tds + std + tsd\big).
\end{aligned}
$$

Therefore, $cst + cts + sct + tcs + stc + tsc = dst + dts + sdt + tds + std + tsd$ and by a similar argument used in the proof of Lemma 2.2 we can conclude that $c = d$. Thus, $M^*(x + y) = M^*(x) + M^*(y)$. The Theorem is proved. $\square$

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