ADDITIVITY OF MAPS ACTING IN A SUM OF TRIPLE PRODUCTS ON TRIANGULAR ALGEBRAS

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In this paper, we prove that surjective maps acting in a sum of triple products on triangular algebras are automatically additive.

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1. INTRODUCTION

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a commutative ring $\mathcal{R}$ and $\mathcal{X}$ a faithful $(\mathcal{A}, \mathcal{B})$-bimodule (see [1]). The $\mathcal{R}$-algebra

$$\text{Tri}(\mathcal{A}, \mathcal{X}, \mathcal{B}) = \left\{ \begin{pmatrix} a & x \\ b & 0 \end{pmatrix} : a \in \mathcal{A}, \ x \in \mathcal{X}, \ b \in \mathcal{B} \right\}$$

under the usual matrix addition and formal matrix multiplication will be called a Triangular algebra. Let $\mathcal{I} = \text{Tri}(\mathcal{A}, \mathcal{X}, \mathcal{B})$ be a Triangular algebra. We set

$$\mathcal{I}_{11} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathcal{A} \right\},$$

$$\mathcal{I}_{12} = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathcal{X} \right\},$$

and

$$\mathcal{I}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in \mathcal{B} \right\}.$$

Then we can may write $\mathcal{I} = \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{22}$, and every element $c \in \mathcal{I}$ can be written as $c = c_{11} + c_{12} + c_{22}$, where $c_{ij} \in \mathcal{I}_{ij}$ ($1 \leq i \leq j \leq 2$). Note that $a_{ij}b_{kl} = 0$ if $j \neq k$.

Ji in [1] studied the additivity of maps $M : \mathcal{I} \to \mathcal{G}$ and $M^* : \mathcal{G} \to \mathcal{I}$, where $\mathcal{G}$ is an arbitrary ring, acting on sums of double products of type $M^*(x)a + aM^*(x)$ and $M(a)x + xM(a)$, for all $a \in \mathcal{I}$ and $x \in \mathcal{G}$, respectively. He proved the following theorem.
Theorem 1.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$, $\mathfrak{X}$ a faithful $(\mathfrak{A}, \mathfrak{B})$-bimodule and let $\mathfrak{T}$ be the Triangular algebra $\text{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B})$ satisfying the following conditions:

(i) If $a \in \mathfrak{A}$ is such that $aa_1 + a_1a = 0$ for all $a_1 \in \mathfrak{A}$, then $a = 0$;
(ii) If $b \in \mathfrak{B}$ is such that $bb_1 + b_1b = 0$ for all $b_1 \in \mathfrak{B}$, then $b = 0$;
(iii) If $x \in \mathfrak{X}$ is such that $ax = 0$, for all $a \in \mathfrak{A}$, or $xb = 0$, for all $b \in \mathfrak{B}$, then $x = 0$.

Let $\mathfrak{S}$ be an arbitrary ring. If $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ are surjective maps such that

$$M(M^*(x)a + aM^*(x)) = xM(a) + M(a)x,$$

$$M^*(M(a)x + xM(a) = aM^*(x) + M^*(x)a,$$

for all $a, b \in \mathfrak{T}$ and $x, y \in \mathfrak{S}$, then both $M$ and $M^*$ are additives.

In this paper, we investigate the additivity of maps $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$, where $\mathfrak{S}$ is an arbitrary ring, acting on sums of triple products of type $M^*(x)ab + M^*(x)ba + aM^*(x)b + bM^*(x)a + abM^*(x) + baM^*(x)$ and $M(a)xy + M(a)yx + xM(a)y + yM(a)x + xyM(a) + yxM(a)$, for all $a, b \in \mathfrak{T}$ and $x, y \in \mathfrak{S}$, respectively.

2. THE MAIN RESULT

Our main result is the following theorem.

Theorem 2.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras over a commutative ring $\mathfrak{R}$, $\mathfrak{X}$ a faithful $(\mathfrak{A}, \mathfrak{B})$-bimodule and let $\mathfrak{T}$ be the Triangular algebra $\text{Tri}(\mathfrak{A}, \mathfrak{X}, \mathfrak{B})$ satisfying the following conditions:

(i) If $a \in \mathfrak{A}$ is such that $aa_1a_2 + aa_2a_1 + a_1aa_2 + a_2aa_1 + a_1a_2a + a_2a_1a = 0$ for all $a_1, a_2 \in \mathfrak{A}$, then $a = 0$;
(ii) If $b \in \mathfrak{B}$ is such that $bb_1b_2 + bb_2b_1 + b_1bb_2 + b_2bb_1 + b_1b_2b + b_2b_1b = 0$ for all $b_1, b_2 \in \mathfrak{B}$, then $b = 0$;
(iii) If $x \in \mathfrak{X}$ is such that $ax = 0$, for all $a \in \mathfrak{A}$, or $xb = 0$, for all $b \in \mathfrak{B}$, then $x = 0$.

Let $\mathfrak{S}$ be an arbitrary ring. If $M : \mathfrak{T} \to \mathfrak{S}$ and $M^* : \mathfrak{S} \to \mathfrak{T}$ are surjective maps such that

$$M(M^*(x)ab + M^*(x)ba + aM^*(x)b + bM^*(x)a + abM^*(x) + baM^*(x)) = xM(a)M(b) + xM(b)M(a) + M(a)xM(b) + M(b)xM(a)$$

(1) $$+ M(a)M(b)x + M(b)M(a)x,$$

$$M^*(M(a)xy + M(a)yx + xM(a)y + yM(a)x + xyM(a) + yxM(a))$$
\[ aM^*(x)M^*(y) + aM^*(y)M^*(x) + M^*(x) aM^*(y) + M^*(y) aM^*(x) \]
\[ + M^*(x)M^*(y)a + M^*(y)M^*(x)a, \]

for all \( a, b \in \mathfrak{S} \) and \( x, y \in \mathcal{S} \), then both \( M \) and \( M^* \) are additives.

Based on the techniques presented in [1] and [2], we shall organize the proof of Theorem 2.1 in a series of lemmas. We begin with the following lemma

**Lemma 2.1.** \( M(0) = 0 \) and \( M^*(0) = 0 \).

**Proof.** From a direct calculation we have

\[
M(0) = M(M^*(0)00 + M^*(0)00 + 0M^*(0)0 + 0M^*(0)0 + 00M^*(0)) \\
= 0M(0)M(0) + 0M(0)M(0) + M(0)0M(0) + M(0)0M(0) \\
+ M(0)M(0)0 + M(0)M(0)0 \\
= 0.
\]

Similarly, we prove \( M^*(0) = 0 \). \( \square \)

**Lemma 2.2.** \( M \) and \( M^* \) are bijective.

**Proof.** It suffices to prove that \( M \) and \( M^* \) are injective. First show that \( M \) is injective. Let \( c_1 \) and \( c_2 \) be in \( \mathfrak{S} \) and suppose that \( M(c_1) = M(c_2) \). From the equality (2), we have

\[
m^*(M(c_i)xy + M(c_i)yx + xM(c_i)y + yM(c_i)x + xyM(c_i) + yxM(c_i)) \\
= c_i M^*(x)M^*(y) + c_i M^*(y)M^*(x) + M^*(x)c_i M^*(y) + M^*(y)c_i M^*(x) \\
+ M^*(x)M^*(y)c_i + M^*(y)M^*(x)c_i
\]

for all \( x, y \in \mathcal{S} \) and \( i = 1, 2 \). Hence, we have

\[
c_1 M^*(x)M^*(y) + c_1 M^*(y)M^*(x) + M^*(x)c_1 M^*(y) + M^*(y)c_1 M^*(x) \\
+ M^*(x)M^*(y)c_1 + M^*(y)M^*(x)c_1 = c_2 M^*(x)M^*(y) + c_2 M^*(y)M^*(x) \\
+ M^*(x)c_2 M^*(y) + M^*(y)c_2 M^*(x) + M^*(x)M^*(y)c_2 + M^*(y)M^*(x)c_2.
\]

The above equality leads to the conclusion that

\[
(c_1 - c_2) M^*(x)M^*(y) + (c_1 - c_2) M^*(y)M^*(x) + M^*(x)(c_1 - c_2) M^*(y) \\
+ M^*(y)(c_1 - c_2)M^*(x) + M^*(x)M^*(y)(c_1 - c_2) + M^*(y)M^*(x)(c_1 - c_2) = 0.
\]

From the surjectivity of \( M^* \) we conclude that

\[
(c_1 - c_2)ab + (c_1 - c_2)ba + a(c_1 - c_2)b \\
+ b(c_1 - c_2)a + ab(c_1 - c_2) + ba(c_1 - c_2) = 0
\]

\]
for all $a, b \in \mathfrak{S}$. Let us write $c_1 - c_2 = c_{11} + c_{12} + c_{22}$, where $c_{ij} \in \mathfrak{S}_{ij}$ ($1 \leq i \leq j \leq 2$). Then, for arbitrary elements $a_{11}, b_{11} \in \mathfrak{S}_{11}$ we have $c_{11}a_{11}b_{11} + c_{11}b_{11}a_{11} + a_{11}c_{11}b_{11} + b_{11}c_{11}a_{11} + a_{11}b_{11}c_{11} + b_{11}a_{11}c_{11} = 0$ which implies $c_{11} = 0$, by condition (i) of the Theorem. Also, for arbitrary elements $a_{22}, b_{22} \in \mathfrak{S}_{22}$ we have $c_{22}a_{22}b_{22} + c_{22}b_{22}a_{22} + a_{22}c_{22}b_{22} + b_{22}c_{22}a_{22} + a_{22}b_{22}c_{22} + b_{22}a_{22}c_{22} = 0$ which yields $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $a_{11} \in \mathfrak{S}_{11}$ and $b_{22} \in \mathfrak{S}_{22}$ we have $c_{12}a_{11}b_{22} + c_{12}b_{22}a_{11} + a_{11}c_{12}b_{22} + b_{22}c_{12}a_{11} + a_{11}b_{22}c_{12} + b_{22}a_{11}c_{12} = 0$ which results in $a_{11}c_{12}b_{22} = 0$. Thus, $c_{12} = 0$, by condition (iii) of the Theorem. It follows that $c_1 - c_2 = 0$ and therefore $c_1 = c_2$.

Now, to prove the injectivity of $M^*$ let $x_1$ and $x_2$ be in $\mathfrak{S}$ and suppose $M^*(x_1) = M^*(x_2)$. From the equality (1), we have

$$
M^*M(M^*(x_i)ab + M^*(x_i)ba + aM^*(x_i)b + bM^*(x_i)a + abM^*(x_i) + baM^*(x_i)) = M^*(x_iM(a)M(b) + x_iM(b)M(a) + M(a)x_iM(b) + M(b)x_iM(a)) = M^*(MM^{-1}(x_i)M(a)M(b) + MM^{-1}(x_i)M(b)M(a) + M(a)MM^{-1}(x_i)M(b) + M(b)MM^{-1}(x_i)M(a)) = M^{-1}(x_i)M^*M(a)M^*M(b) + M^{-1}(x_i)M^*M(b)M^*M(a) + M^*M(a)M^{-1}(x_i)M^*M(b) + M^*M(b)M^{-1}(x_i)M^*M(a) + M^*M(a)M^*M(b)M^{-1}(x_i) + M^*M(b)M^*M(a)M^{-1}(x_i)
$$

for all $a, b \in \mathfrak{S}$ and $i = 1, 2$. It follows that

$$
M^{-1}(x_1)M^*M(a)M^*M(b) + M^{-1}(x_1)M^*M(b)M^*M(a) + M^*M(a)M^*M(b)M^{-1}(x_1) + M^*M(b)M^*M(a)M^{-1}(x_1)
$$

which results in

$$
(M^{-1}(x_1) - M^{-1}(x_2))M^*M(a)M^*M(b) + (M^{-1}(x_1) - M^{-1}(x_2))M^*M(b)M^*M(a) + M^*M(a)(M^{-1}(x_1) - M^{-1}(x_2))M^*M(b) + M^*M(b)(M^{-1}(x_1) - M^{-1}(x_2))M^*M(a) + M^*M(a)M^*M(b)(M^{-1}(x_1) - M^{-1}(x_2)) + M^*M(b)M^*M(a)(M^{-1}(x_1) - M^{-1}(x_2))
$$

$$
= 0.
$$
Noting that $M^*M$ is also surjective and using a similar argument as applied in the previous case, we can show that $M^{-1}(u_1) = M^{-1}(u_2)$. Consequently $u_1 = u_2$. □

**Lemma 2.3.** The maps $M^{*-1} : \mathcal{S} \to \mathcal{S}$ and $M^{-1} : \mathcal{S} \to \mathcal{S}$ satisfy:

(i) \[ M^{*-1}(M^{-1}(x)ab + M^{-1}(x)ba + aM^{-1}(x)b + bM^{-1}(x)a) \]
\[ + abM^{-1}(x) + baM^{-1}(x)) = xM^{*-1}(a)M^{*-1}(b) \]
\[ + xM^{*-1}(b)M^{*-1}(a) + M^{*-1}(a)xM^{*-1}(b) + M^{*-1}(b)xM^{*-1}(a) \]
\[ + M^{*-1}(a)M^{*-1}(b)x + M^{*-1}(b)M^{*-1}(a)x, \]

(ii) \[ M^{-1}(M^{*-1}(a)xy + M^{*-1}(a)yx + xM^{*-1}(a)y + yM^{*-1}(a)x) \]
\[ + xyM^{*-1}(a) + yxM^{*-1}(a)) = aM^{-1}(x)M^{-1}(y) \]
\[ + aM^{-1}(y)M^{-1}(x) + M^{-1}(x)aM^{-1}(y) + M^{-1}(y)aM^{-1}(x) \]
\[ + M^{-1}(x)M^{-1}(y)a + M^{-1}(y)M^{-1}(x)a, \]

for all $a, b \in \mathcal{S}$ and $x, y \in \mathcal{S}$.

**Proof.** From the equality (2) we have
\[ M^*(xM^{*-1}(a)M^{*-1}(b) + xM^{*-1}(b)M^{*-1}(a) + M^{*-1}(a)xM^{*-1}(b) \]
\[ + M^{*-1}(b)xM^{*-1}(a) + M^{*-1}(a)xM^{*-1}(b) + M^{*-1}(b)xM^{*-1}(a) \]
\[ = M^*(MM^{-1}(x)M^{*-1}(a)M^{*-1}(b) + MM^{-1}(x)M^{*-1}(b)M^{*-1}(a) \]
\[ + M^{*-1}(a)MM^{-1}(x)M^{*-1}(b) + M^{*-1}(b)MM^{-1}(x)M^{*-1}(a) \]
\[ + M^{*-1}(a)MM^{-1}(b)MM^{-1}(x) + M^{*-1}(b)MM^{-1}(a)MM^{-1}(x) \]
\[ = M^{-1}(x)ab + M^{-1}(x)ba + aM^{-1}(x)b + bM^{-1}(x)a + abM^{-1}(x) \]
\[ + baM^{-1}(x) \]

for all $a, b \in \mathcal{S}$ and $x \in \mathcal{S}$. Applying $M^{*-1}$ in the above equality, we obtain the equality (i).

The second equality follows in a similar way. □

**Lemma 2.4.** Let $a, b, c \in \mathcal{S}$ such that $M(c) = M(a) + M(b)$. Then

(i) \[ M(cst + cts + sc + tcs + stc + tsc) \]
\[ = M(ast + ats + sat + tas + sta + tsa) \]
\[ + M(bst + bts + sbt + tbs + stb + tsb) \]

for all $s, t \in \mathcal{S}$;
(ii)
\[ M^{*^{-1}}(cst + cts + sct + tcs + stc + tsc) = M^{*^{-1}}(ast + ats + sat + tas + sta + tsa) + M^{*^{-1}}(bst + bts + sbt + tbs + stb + tsb) \]

for all \( s, t \in \mathcal{F} \).

Proof. From the equality (1), for arbitrary elements \( s, t \in \mathcal{F} \) we have

\[
\begin{align*}
M(cst + cts + sct + tcs + stc + tsc) &= M(cM^{*^{-1}}(s)t + ctM^{*^{-1}}(s) + M^{*^{-1}}(s)ct) \\
+ tcM^{*^{-1}}(s) + M^{*^{-1}}(s)tc + tM^{*^{-1}}(s)c) &= M(a)M^{*^{-1}}(s)M(t) + M(a)M(t)M^{*^{-1}}(s) + M^{*^{-1}}(s)M(a)M(t) \\
+ M(t)M(a)M^{*^{-1}}(s) + M^{*^{-1}}(s)M(t)M(a) + M(t)M^{*^{-1}}(s)M(a) \\
+ M(b)M^{*^{-1}}(s)M(t) + M(b)M(t)M^{*^{-1}}(s) + M^{*^{-1}}(s)M(b)M(t) \\
+ M(t)M(b)M^{*^{-1}}(s) + M^{*^{-1}}(s)M(t)M(b) + M(t)M^{*^{-1}}(s)M(b) \\
= M(aM^{*^{-1}}(s)t + atM^{*^{-1}}(s) + M^{*^{-1}}(s)at \\
+ taM^{*^{-1}}(s) + M^{*^{-1}}(s)ta + tM^{*^{-1}}(s)a) \\
+ M(bM^{*^{-1}}(s)t + btM^{*^{-1}}(s) + M^{*^{-1}}(s)bt \\
+ tbM^{*^{-1}}(s) + M^{*^{-1}}(s)tb + tM^{*^{-1}}(s)b) &= M(ast + ats + sat + tas + sta + tsa) + M(bst + bts + sbt + tbs + stb + tsb). \\
\end{align*}
\]

By a similar argument, we prove the equality (ii), from the equality (i) in the Lemma 2.3. \( \square \)

Lemma 2.5. Let \( a_{11} \in \mathcal{F}_{11} \) and \( b_{12} \in \mathcal{F}_{12} \). Then

(i) \( M(a_{11} + b_{12}) = M(a_{11}) + M(b_{12}) \);

(ii) \( M^{*^{-1}}(a_{11} + b_{12}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{12}) \).

Proof. From the surjectivity of \( M \), there exists \( c \in \mathcal{F} \) such that \( M(c) = M(a_{11}) + M(b_{12}) \), where \( c = c_{11} + c_{12} + c_{22} \). Hence, for arbitrary elements
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$s_{11} \in \mathfrak{T}_{11}$ and $t_{22} \in \mathfrak{T}_{22}$ we have

\[
M(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c) \\
= M(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11} \\
+ t_{22}s_{11}a_{11}) + M(b_{12}s_{11}t_{22} + b_{12}t_{22}s_{11} + s_{11}b_{12}t_{22} + t_{22}b_{12}s_{11} \\
+ s_{11}t_{22}b_{12} + t_{22}s_{11}b_{12}) \\
= M(s_{11}b_{12}t_{22}),
\]

by Lemma 2.4(i). It follows that $s_{11}ct_{22} + t_{22}cs_{11} = s_{11}b_{12}t_{22}$ which implies $s_{11}c_{12}t_{22} = s_{11}b_{12}t_{22}$. Hence, $c_{12} = b_{12}$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$ we have

\[
M(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c) \\
= M(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11} \\
+ t_{22}s_{22}a_{11}) + M(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + s_{22}b_{12}t_{22} + t_{22}b_{12}s_{22} \\
+ s_{22}t_{22}b_{12} + t_{22}s_{22}b_{12}) \\
= M(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}),
\]

by Lemma 2.4(i) again. This shows that $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}$ which results in $c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} = 0$. Thus $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $s_{12} \in \mathfrak{T}_{12}$ and $t_{22} \in \mathfrak{T}_{22}$ we have

\[
M(cs_{12}t_{22} + ct_{22}s_{12} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c + t_{22}s_{12}c) \\
= M(a_{11}s_{12}t_{22} + a_{11}t_{22}s_{12} + s_{12}a_{11}t_{22} + t_{22}a_{11}s_{12} + s_{12}t_{22}a_{11} \\
+ t_{22}s_{12}a_{11}) + M(b_{12}s_{12}t_{22} + b_{12}t_{22}s_{12} + s_{12}b_{12}t_{22} + t_{22}b_{12}s_{12} \\
+ s_{12}t_{22}b_{12} + t_{22}s_{12}b_{12}) \\
= M(a_{11}s_{12}t_{22}).
\]

We can thus conclude that $cs_{12}t_{22} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c = a_{11}s_{12}t_{22}$ which yields $c_{11}s_{12}t_{22} = a_{11}s_{12}t_{22}$. Therefore, $c_{11} = a_{11}$, by condition (iii) of the Theorem and the hypothesis that $\mathfrak{T}$ is a faithful module. □

Similarly, we prove the following lemma.

**Lemma 2.6.** Let $a_{22} \in \mathfrak{T}_{22}$ and $b_{12} \in \mathfrak{T}_{12}$. Then

(i) $M(a_{22} + b_{12}) = M(a_{22}) + M(b_{12})$;

(ii) $M^{*-1}(a_{22} + b_{12}) = M^{*-1}(a_{22}) + M^{*-1}(b_{12})$.

**Lemma 2.7.** Let $t_{11} \in \mathfrak{T}_{11}$, $a_{12}, b_{12} \in \mathfrak{T}_{12}$ and $c_{22} \in \mathfrak{T}_{22}$. Then

(i) $M(t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22}) = M(t_{11}a_{12}c_{22}) + M(t_{11}b_{12}c_{22})$;

(ii) $M^{*-1}(t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22}) = M^{*-1}(t_{11}a_{12}c_{22}) + M^{*-1}(t_{11}b_{12}c_{22})$. 
Proof. First of all, we note that the following identity is valid
\[
t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22} = t_{11}(a_{12} + c_{22})(b_{12} + c_{22}) + t_{11}(b_{12} + c_{22})(a_{12} + c_{22})
+ (a_{12} + c_{22})t_{11}(b_{12} + c_{22}) + (b_{12} + c_{22})t_{11}(a_{12} + c_{22})
+ (a_{12} + c_{22})(b_{12} + c_{22})t_{11} + (b_{12} + c_{22})(a_{12} + c_{22})t_{11}.
\]
Hence, from the equality (1) we compute
\[
M(t_{11}a_{12}c_{22} + t_{11}b_{12}c_{22})
= M(t_{11}(a_{12} + c_{22})(b_{12} + c_{22}) + t_{11}(b_{12} + c_{22})(a_{12} + c_{22})
+ (a_{12} + c_{22})t_{11}(b_{12} + c_{22}) + (b_{12} + c_{22})t_{11}(a_{12} + c_{22})
+ (a_{12} + c_{22})(b_{12} + c_{22})t_{11} + (b_{12} + c_{22})(a_{12} + c_{22})t_{11})
= M(M^*M^{-1}(t_{11}))(a_{12} + c_{22})(b_{12} + c_{22})
+ M^*M^{-1}(t_{11})(b_{12} + c_{22})(a_{12} + c_{22})
+ (a_{12} + c_{22})M^*M^{-1}(t_{11})(b_{12} + c_{22})
+ (b_{12} + c_{22})M^*M^{-1}(t_{11})(a_{12} + c_{22})
+ (a_{12} + c_{22})(b_{12} + c_{22})M^*M^{-1}(t_{11})
+ (b_{12} + c_{22})(a_{12} + c_{22})M^*M^{-1}(t_{11})
= M^{-1}(t_{11})(M(a_{12}) + M(c_{22}))M(b_{12} + c_{22})
+ M^{-1}(t_{11})M(b_{12} + c_{22})(M(a_{12}) + M(c_{22}))
+ (M(a_{12}) + M(c_{22}))M^{-1}(t_{11})M(b_{12} + c_{22})
+ M(b_{12} + c_{22})M^{-1}(t_{11})(M(a_{12}) + M(c_{22}))
+ (M(a_{12}) + M(c_{22}))M(b_{12} + c_{22})M^{-1}(t_{11})
+ M(b_{12} + c_{22})(M(a_{12}) + M(c_{22}))M^{-1}(t_{11})
= M^{-1}(t_{11})M(a_{12})M(b_{12} + c_{22}) + M^{-1}(t_{11})M(b_{12} + c_{22})M(a_{12})
+ M(a_{12})M^{-1}(t_{11})M(b_{12} + c_{22}) + M(b_{12} + c_{22})M^*-1(t_{11})M(a_{12})
+ M(a_{12})M(b_{12} + c_{22})M^*-1(t_{11}) + M(b_{12} + c_{22})M(a_{12})M^*-1(t_{11})
+ M^*-1(t_{11})M(c_{22})M(b_{12} + c_{22}) + M^*-1(t_{11})M(b_{12} + c_{22})M(c_{22})
\[ +M(c_{22})M^{-1}(t_{11})M(b_{12} + c_{22}) + M(b_{12} + c_{22})M^{-1}(t_{11})M(c_{22}) \\
+M(c_{22})M(b_{12} + c_{22})M^{-1}(t_{11}) + M(b_{12} + c_{22})M(c_{22})M^{-1}(t_{11}) \]
\[ = M(t_{11}a_{12}(b_{12} + c_{22}) + t_{11}(b_{12} + c_{22})a_{12} + a_{12}t_{11}(b_{12} + c_{22}) \\
+(b_{12} + c_{22})t_{11}a_{12} + a_{12}(b_{12} + c_{22})t_{11} + (b_{12} + c_{22})a_{12}t_{11} \\
+M(t_{11}c_{22}(b_{12} + c_{22}) + t_{11}(b_{12} + c_{22})c_{22} + c_{22}t_{11}(b_{12} + c_{22}) \\
+(b_{12} + c_{22})t_{11}c_{22} + c_{22}(b_{12} + c_{22})t_{11} + (b_{12} + c_{22})c_{22}t_{11} \\
= M(t_{11}a_{12}c_{22}) + M(t_{11}b_{12}c_{22}). \]

Similarly, we prove (ii), using the Lemma 2.3(i). □

**Lemma 2.8.** Let $a_{11}, b_{11} \in \mathfrak{T}_{11}$. Then

(i) $M(a_{11} + b_{11}) = M(a_{11}) + M(b_{11})$;

(ii) $M^{-1}(a_{11} + b_{11}) = M^{-1}(a_{11}) + M^{-1}(b_{11})$.

**Proof.** Choose $c = c_{11} + c_{12} + c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{11}) + M(b_{11})$. For arbitrary elements $s_{11} \in \mathfrak{T}_{11}, t_{22} \in \mathfrak{T}_{22}$, we have

\[ M^{-1}(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c) \]
\[ = M^{-1}(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11} \\
+ t_{22}s_{11}a_{11}) + M^{-1}(b_{11}s_{11}t_{22} + b_{11}t_{22}s_{11} + s_{11}b_{11}t_{22} + t_{22}b_{11}s_{11} \\
+ s_{11}t_{22}b_{11} + t_{22}s_{11}b_{11}) = 0. \]

Hence, $s_{11}ct_{22} + t_{22}cs_{11} = 0$ which results $s_{11}c_{12}t_{22} = 0$. Therefore, $c_{12} = 0$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}, t_{22} \in \mathfrak{T}_{22}$ we have

\[ M^{-1}(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c) \]
\[ = M^{-1}(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11} \\
+ t_{22}s_{22}a_{11}) + M^{-1}(b_{11}s_{22}t_{22} + b_{11}t_{22}s_{22} + s_{22}b_{11}t_{22} + t_{22}b_{11}s_{22} \\
+ s_{22}t_{22}b_{11} + t_{22}s_{22}b_{11}) = 0 \]

which implies

\[ c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} = 0. \]

Thus, $c_{22} = 0$, by condition (ii) of the Theorem. Yet, for arbitrary elements $s_{12} \in \mathfrak{T}_{12}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

\[ M^{-1}(cs_{12}t_{22} + ct_{22}s_{12} + s_{12}ct_{22} + t_{22}cs_{12} + s_{12}t_{22}c + t_{22}s_{12}c) \]
\[ = M^{-1}(a_{11}s_{12}t_{22} + a_{11}t_{22}s_{12} + s_{12}a_{11}t_{22} + t_{22}a_{11}s_{12} + s_{12}t_{22}a_{11} \\
+ t_{22}s_{12}a_{11}) + M^{-1}(b_{11}s_{12}t_{22} + b_{11}t_{22}s_{12} + s_{12}b_{11}t_{22} + t_{22}b_{11}s_{12} \\
+ s_{12}t_{22}b_{11} + t_{22}s_{12}b_{11}) = 0 \]
Hence, from the equality (1) and Lemmas 2.6(i) and 2.9(i), we compute

\[ + s_{12} t_{22} b_{11} + t_{22} s_{12} b_{11} \]

\[ = M^{*-1}(a_{11} s_{12} t_{22}) + M^{*-1}(b_{11} s_{12} t_{22}) \]

\[ = M^{*-1}(a_{11} s_{12} t_{22} + b_{11} s_{12} t_{22}), \]

by Lemma 2.7(ii). It follows that \( c s_{12} t_{22} + s_{12} c t_{22} + t_{22} c s_{12} + s_{12} t_{22} c = a_{11} s_{12} t_{22} + b_{11} s_{12} t_{22} \) which yields \( c_{11} s_{12} t_{22} = a_{11} s_{12} t_{22} + b_{11} s_{12} t_{22}. \) So \( c_{11} = a_{11} + b_{11}, \) by condition (iii) of the Theorem and the hypothesis that \( \mathcal{X} \) is a faithful module. \( \square \)

Similarly, we prove the following Lemma.

**Lemma 2.9.** Let \( a_{22}, b_{22} \in \mathfrak{T}_{22}. \) Then

(i) \( M(a_{22} + b_{22}) = M(a_{22}) + M(b_{22}); \)

(ii) \( M^{*-1}(a_{22} + b_{22}) = M^{*-1}(a_{22}) + M^{*-1}(b_{22}). \)

**Lemma 2.10.** Let \( a_{12}, b_{12} \in \mathfrak{T}_{12} \) and \( s_{22}, t_{22} \in \mathfrak{T}_{22}. \) Then

(i) \( M(a_{12} s_{22} t_{22} + a_{12} t_{22} s_{22} + b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22}) = M(a_{12} s_{22} t_{22} + a_{12} t_{22} s_{22}) + M(b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22}); \)

(ii) \( M^{*-1}(a_{12} s_{22} t_{22} + a_{12} t_{22} s_{22} + b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22}) = M^{*-1}(a_{12} s_{22} t_{22} + a_{12} t_{22} s_{22}) + M^{*-1}(b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22}). \)

**Proof.** First of all, we note that the following identity is valid

\[ 2 s_{22}^2 t_{22} + 2 s_{22} t_{22} s_{22} + 2 t_{22}^2 s_{22} \]

\[ + a_{12} s_{22} t_{22} + a_{12} t_{22} s_{22} + b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22} \]

\[ = (a_{12} + s_{22})(b_{12} + s_{22}) t_{22} + (a_{12} + s_{22}) t_{22} (b_{12} + s_{22}) \]

\[ (b_{12} + s_{22})(a_{12} + s_{22}) t_{22} + t_{22}(a_{12} + s_{22})(b_{12} + s_{22}) \]

\[ (b_{12} + s_{22}) t_{22}(a_{12} + s_{22}) + t_{22}(b_{12} + s_{22})(a_{12} + s_{22}). \]

Hence, from the equality (1) and Lemmas 2.6(i) and 2.9(i), we compute

\[ M(2 s_{22}^2 t_{22} + 2 s_{22} t_{22} s_{22} + 2 t_{22}^2 s_{22}) \]

\[ + M(a_{12} s_{22} t_{22} + a_{12} t_{22} s_{22} + b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22}) \]

\[ = M(2 s_{22}^2 t_{22} + 2 s_{22} t_{22} s_{22} + 2 t_{22}^2 s_{22} + a_{12} s_{22} t_{22}) \]

\[ + a_{12} t_{22} s_{22} + b_{12} s_{22} t_{22} + b_{12} t_{22} s_{22} \]

\[ = M((a_{12} + s_{22})(b_{12} + s_{22}) t_{22} + (a_{12} + s_{22}) t_{22} (b_{12} + s_{22}) \]

\[ (b_{12} + s_{22})(a_{12} + s_{22}) t_{22} + t_{22}(a_{12} + s_{22})(b_{12} + s_{22}) \]

\[ (b_{12} + s_{22}) t_{22}(a_{12} + s_{22}) + t_{22}(b_{12} + s_{22})(a_{12} + s_{22})), \]

\[ = M((a_{12} + s_{22})(b_{12} + s_{22})) M^{*} M^{*-1}(t_{22}) \]

\[ + (a_{12} + s_{22}) M^{*} M^{*-1}(t_{22})(b_{12} + s_{22}) \]

\[ + (b_{12} + s_{22})(a_{12} + s_{22}) M^{*} M^{*-1}(t_{22}) \]

\[ + (b_{12} + s_{22})(a_{12} + s_{22}) M^{*} M^{*-1}(t_{22}). \]
\[\begin{align*} &+ M^* M^{-1}(t_{22})(a_{12} + s_{22})(b_{12} + s_{22}) \\
&\quad + (b_{12} + s_{22})M^* M^{-1}(t_{22})(a_{12} + s_{22}) \\
&\quad + M^* M^{-1}(t_{22})(b_{12} + s_{22})(a_{12} + s_{22}) \\
&= M(a_{12} + s_{22})M(b_{12} + s_{22})M^*^{-1}(t_{22}) \\
&\quad + M(a_{12} + s_{22})M^*^{-1}(t_{22})M(b_{12} + s_{22}) \\
&\quad + M(b_{12} + s_{22})M(a_{12} + s_{22})M^*^{-1}(t_{22}) \\
&\quad + M^*^{-1}(t_{22})M(a_{12} + s_{22})M(b_{12} + s_{22}) \\
&\quad + M(b_{12} + s_{22})M^*^{-1}(t_{22})M(a_{12} + s_{22}) \\
&\quad + M^*^{-1}(t_{22})M(b_{12} + s_{22})M(a_{12} + s_{22}) \\
&\quad + (M(a_{12}) + M(s_{22}))M(b_{12} + s_{22})M^*^{-1}(t_{22}) \\
&\quad + (M(a_{12}) + M(s_{22}))M^*^{-1}(t_{22})M(b_{12} + s_{22}) \\
&\quad + M(b_{12} + s_{22})(M(a_{12}) + M(s_{22}))M^*^{-1}(t_{22}) \\
&\quad + M^*^{-1}(t_{22})(M(a_{12}) + M(s_{22}))M(b_{12} + s_{22}) \\
&\quad + M(b_{12} + s_{22})M^*^{-1}(t_{22})(M(a_{12}) + M(s_{22})) \\
&\quad + M^*^{-1}(t_{22})M(b_{12} + s_{22})(M(a_{12}) + M(s_{22})) \\
&\quad = M(a_{12})M(b_{12} + s_{22})M^*^{-1}(t_{22}) + M(a_{12})M^*^{-1}(t_{22})M(b_{12} + s_{22}) \\
&\quad + M(b_{12} + s_{22})M(a_{12})M^*^{-1}(t_{22}) + M^*^{-1}(t_{22})M(a_{12})M(b_{12} + s_{22}) \\
&\quad + M(b_{12} + s_{22})M^*^{-1}(t_{22})M(a_{12}) + M^*^{-1}(t_{22})M(b_{12} + s_{22})M(a_{12}) \\
&\quad + M(s_{22})M(b_{12} + s_{22})M^*^{-1}(t_{22}) + M(s_{22})M^*^{-1}(t_{22})M(b_{12} + s_{22})M(a_{12}) \\
&\quad + M(b_{12} + s_{22})M(s_{22})M^*^{-1}(t_{22}) + M^*^{-1}(t_{22})M(s_{22})M(b_{12} + s_{22})M(a_{12}) \\
&\quad + M(b_{12} + s_{22})M^*^{-1}(t_{22})M(s_{22}) + M^*^{-1}(t_{22})M(b_{12} + s_{22})M(s_{22}) \\
&\quad = M(a_{12}(b_{12} + s_{22})M^* M^*^{-1}(t_{22}) + a_{12}M^* M^*^{-1}(t_{22})(b_{12} + s_{22}) \\
&\quad + (b_{12} + s_{22})a_{12}M^* M^*^{-1}(t_{22}) + M^* M^*^{-1}(t_{22})a_{12}(b_{12} + s_{22}) \\
&\quad + (b_{12} + s_{22})M^* M^*^{-1}(t_{22})a_{12} + M^* M^*^{-1}(t_{22})M(b_{12} + s_{22})a_{12} \\
&\quad + M(s_{22}(b_{12} + s_{22})M^* M^*^{-1}(t_{22}) + s_{22}M^* M^*^{-1}(t_{22})(b_{12} + s_{22}) \\
&\quad + (b_{12} + s_{22})s_{22}M^* M^*^{-1}(t_{22}) + M^* M^*^{-1}(t_{22})s_{22}(b_{12} + s_{22}) \\
&\quad + (b_{12} + s_{22})M^* M^*^{-1}(t_{22})s_{22} + M^* M^*^{-1}(t_{22})(b_{12} + s_{22})s_{22} \\
&\quad = M(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22}) + M(s_{22}^2t_{22} + s_{22}t_{22}s_{22} + b_{12}s_{22}t_{22} + s_{22}^2t_{22} \\
&\quad + t_{22}s_{22}^2 + b_{12}t_{22}s_{22} + s_{22}t_{22}s_{22} + t_{22}s_{22}^2) \\
&\quad = M(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22}) + M(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) \\
&\quad + M(2s_{22}^2t_{22} + 2s_{22}t_{22}s_{22} + 2t_{22}s_{22}^2).\end{align*}\]
It follows that
\[ M(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) = M(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22}) + M(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}). \]

Similarly, we prove (ii), using the Lemma 2.3(i). \( \square \)

**Lemma 2.11.** Let \( a_{12}, b_{12} \in \mathfrak{T}_{12} \). Then

(i) \( M(a_{12} + b_{12}) = M(a_{12}) + M(b_{12}) \);

(ii) \( M^{-1}(a_{12} + b_{12}) = M^{-1}(a_{12}) + M^{-1}(b_{12}) \).

**Proof.** By the surjectivity of \( M \), there exists \( c \in \mathfrak{T} \) such that \( M(c) = M(a_{12}) + M(b_{12}) \), where \( c = c_{11} + c_{12} + c_{22} \). Hence, for arbitrary elements \( s_{11} \in \mathfrak{T}_{11}, t_{22} \in \mathfrak{T}_{22} \) we have

\[ M^{-1}(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22} + t_{22}s_{11}) = M^{-1}(a_{12}s_{11}t_{22} + a_{12}t_{22}s_{11} + s_{11}a_{12}t_{22} + t_{22}s_{11}a_{12} + s_{11}t_{22} + t_{22}s_{11}) = M^{-1}(s_{11}a_{12}t_{22}) + M^{-1}(s_{11}b_{12}t_{22}) = M^{-1}(s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}), \]

by Lemma 2.7(ii). It follows that \( s_{11}ct_{22} + t_{22}cs_{11} = s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22} \) which implies \( s_{11}c_{12}t_{22} = s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22} \). Therefore, \( c_{12} = a_{12} + b_{12} \), by condition (iii) of the Theorem. Also, for arbitrary elements \( s_{22} \in \mathfrak{T}_{22} \) and \( t_{22} \in \mathfrak{T}_{22} \), we have

\[ M^{-1}(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}) = M^{-1}(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + s_{22}a_{12}t_{22} + t_{22}s_{22}a_{12} + s_{22}t_{22} + t_{22}s_{22}b_{12}) = M^{-1}(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22}) + M^{-1}(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) = M^{-1}(a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}). \]

by Lemma 2.10(ii). It follows that \( cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = a_{12}s_{22}t_{22} + a_{12}t_{22}s_{22} + b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} \) which implies \( c_{22} = 0 \), by condition (ii) of the Theorem.

Now, for arbitrary elements \( s_{11}, t_{11} \in \mathfrak{T}_{11} \), we have

\[ M^{-1}(cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c) = M^{-1}(a_{12}s_{11}t_{11} + a_{12}t_{11}s_{11} + s_{11}a_{12}t_{11} + t_{11}a_{12}s_{11} + s_{11}t_{11}a_{12} + t_{11}a_{12}s_{11}). \]
$+t_{11}s_{11}a_{12}) + M^*^{-1}(b_{12}s_{11}t_{11} + b_{12}t_{11}s_{11} + s_{11}b_{12}t_{11}$
$+t_{11}b_{12}s_{11} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12})$

$= M^*^{-1}(s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12}) + M^*^{-1}(s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12})$
$= M^*^{-1}(s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12})$.

This shows that $cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c = s_{11}t_{11}a_{12} + t_{11}s_{11}a_{12} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}$ which results in $c_{11}s_{11}t_{11} + c_{11}t_{11}s_{11} + s_{11}c_{11}t_{11} + t_{11}c_{11}s_{11} + s_{11}t_{11}c_{11} + t_{11}s_{11}c_{11} = 0$. So $c_{11} = 0$, by condition (i) of the Theorem.

The proof of (ii) is similar. \( \square \)

**Lemma 2.12.** Let $a_{11} \in \mathfrak{T}_{11}$ and $b_{22} \in \mathfrak{T}_{22}$. Then

(i) $M(a_{11} + b_{22}) = M(a_{11}) + M(b_{22})$;

(ii) $M^*^{-1}(a_{11} + b_{22}) = M^*^{-1}(a_{11}) + M^*^{-1}(b_{22})$.

**Proof.** Choose $c = c_{11} + c_{12} + c_{22} \in \mathfrak{T}$ such that $M(c) = M(a_{11}) + M(b_{22})$. For arbitrary elements $s_{11} \in \mathfrak{T}_{11}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$M^*^{-1}(cs_{11}t_{22} + ct_{22}s_{11} + s_{11}ct_{22} + t_{22}cs_{11} + s_{11}t_{22}c + t_{22}s_{11}c)$

$= M^*^{-1}(a_{11}s_{11}t_{22} + a_{11}t_{22}s_{11} + s_{11}a_{11}t_{22} + t_{22}a_{11}s_{11} + s_{11}t_{22}a_{11} + t_{22}s_{11}a_{11})$

$+ M^*^{-1}(b_{22}s_{11}t_{22} + b_{22}t_{22}s_{11} + s_{11}b_{22}t_{22} + t_{22}b_{22}s_{11} + s_{11}t_{22}b_{22} + t_{22}s_{11}b_{22}) = 0.$

It follows that $s_{11}ct_{22} + t_{22}cs_{11} = 0$ which implies $s_{11}c_{12}t_{22} = 0$. Hence, $c_{12} = 0$, by condition (iii) of the Theorem. Also, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$M^*^{-1}(cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c)$

$= M^*^{-1}(a_{11}s_{22}t_{22} + a_{11}t_{22}s_{22} + s_{22}a_{11}t_{22} + t_{22}a_{11}s_{22} + s_{22}t_{22}a_{11} + t_{22}s_{22}a_{11})$

$+ M^*^{-1}(b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22})$

$= M^*^{-1}(b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}).$

We can then conclude that $cs_{22}t_{22} + ct_{22}s_{22} + s_{22}ct_{22} + t_{22}cs_{22} + s_{22}t_{22}c + t_{22}s_{22}c = b_{22}s_{22}t_{22} + b_{22}t_{22}s_{22} + s_{22}b_{22}t_{22} + t_{22}b_{22}s_{22} + s_{22}t_{22}b_{22} + t_{22}s_{22}b_{22}$ which yields $(c_{22} - b_{22})s_{22}t_{22} + (c_{22} - b_{22})t_{22}s_{22} + s_{22}(c_{22} - b_{22})t_{22} + t_{22}(c_{22} - b_{22})s_{22} + s_{22}t_{22}(c_{22} - b_{22}) + t_{22}s_{22}(c_{22} - b_{22}) = 0$. Thus, $c_{22} = b_{22}$, by condition (ii) of the Theorem. Yet, for arbitrary elements $s_{11} \in \mathfrak{T}_{11}$ and $t_{11} \in \mathfrak{T}_{11}$, we have

$M^*^{-1}(cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c)$

$= M^*^{-1}(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11})$

$+ M^*^{-1}(b_{22}s_{11}t_{11} + b_{22}t_{11}s_{11} + s_{11}b_{22}t_{11} + t_{11}b_{22}s_{11} + s_{11}t_{11}b_{22} + t_{11}s_{11}b_{22})$

$= M^*^{-1}(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}).$
This allows us to conclude again that $cs_{11}t_{11} + ct_{11}s_{11} + s_{11}ct_{11} + t_{11}cs_{11} + s_{11}t_{11}c + t_{11}s_{11}c = a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}$ which results in $c_{11} = a_{11}$, by condition (i) of the Theorem.

**Lemma 2.13.** Let $a_{11} \in \mathfrak{T}_{11}$, $b_{12} \in \mathfrak{T}_{12}$ and $c_{22} \in \mathfrak{T}_{22}$. Then

(i) $M(a_{11} + b_{12} + c_{22}) = M(a_{11}) + M(b_{12}) + M(c_{22})$;

(ii) $M^{-1}(a_{11} + b_{12} + c_{22}) = M^{-1}(a_{11}) + M^{-1}(b_{12}) + M^{-1}(c_{22})$.

**Proof.** Choose $d = d_{11} + d_{12} + d_{22} \in \mathfrak{T}$ such that $M(d) = M(a_{11}) + M(b_{12}) + M(c_{22})$ and write $M(d) = M(a_{11} + b_{12} + c_{22})$. For arbitrary elements $s_{11} \in \mathfrak{T}_{11}$, $t_{11} \in \mathfrak{T}_{11}$, we have

$$M^{-1}(fs_{11}t_{11} + ft_{11}s_{11} + s_{11}ft_{11} + t_{11}fs_{11} + s_{11}t_{11}f + t_{11}s_{11}f)$$

$$= M^{-1}((a_{11} + b_{12})s_{11}t_{11} + (a_{11} + b_{12})t_{11}s_{11} + s_{11}(a_{11} + b_{12})t_{11}$$

$$+ t_{11}(a_{11} + b_{12})s_{11} + s_{11}t_{11}(a_{11} + b_{12}) + t_{11}s_{11}(a_{11} + b_{12}))$$

$$+ M^{-1}(c_{22}s_{11}t_{11} + c_{22}t_{11}s_{11} + s_{11}c_{22}t_{11} + t_{11}c_{22}s_{11} + s_{11}t_{11}c_{22}$$

$$+ t_{11}s_{11}c_{22}) = M^{-1}(a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11}$$

$$+ s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}).$$

It follows that $fs_{11}t_{11} + ft_{11}s_{11} + s_{11}ft_{11} + t_{11}fs_{11} + s_{11}t_{11}f + t_{11}s_{11}f = a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11} + s_{11}t_{11}b_{12} + t_{11}s_{11}b_{12}$ which implies $f_{11}s_{11}t_{11} + f_{11}t_{11}s_{11} + s_{11}f_{11}t_{11} + t_{11}f_{11}s_{11} + s_{11}t_{11}f_{11} + t_{11}s_{11}f_{11} = a_{11}s_{11}t_{11} + a_{11}t_{11}s_{11} + s_{11}a_{11}t_{11} + t_{11}a_{11}s_{11} + s_{11}t_{11}a_{11} + t_{11}s_{11}a_{11}$. Hence, $f_{11} = a_{11}$, by condition (i) of the Theorem. Also, for arbitrary elements $s_{11} \in \mathfrak{T}_{11}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$M^{-1}(fs_{11}t_{22} + ft_{22}s_{11} + s_{11}ft_{22} + t_{22}fs_{11} + s_{11}t_{22}f + t_{22}s_{11}f)$$

$$= M^{-1}((a_{11} + b_{12})s_{11}t_{22} + (a_{11} + b_{12})t_{22}s_{11} + s_{11}(a_{11} + b_{12})t_{22}$$

$$+ t_{22}(a_{11} + b_{12})s_{11} + s_{11}t_{22}(a_{11} + b_{12}) + t_{22}s_{11}(a_{11} + b_{12}))$$

$$+ M^{-1}(c_{22}s_{11}t_{22} + c_{22}t_{22}s_{11} + s_{11}c_{22}t_{22} + t_{22}c_{22}s_{11} + s_{11}t_{22}c_{22}$$

$$+ t_{22}s_{11}c_{22}) = M^{-1}(s_{11}b_{12}t_{22}).$$

This shows that $s_{11}ft_{22} + t_{22}fs_{11} = s_{11}b_{12}t_{22}$ which yields $s_{11}f_{12}t_{22} = s_{11}b_{12}t_{22}$ Therefore, $f_{12} = b_{12}$, by condition (iii) of the Theorem. Yet, for arbitrary elements $s_{22} \in \mathfrak{T}_{22}$ and $t_{22} \in \mathfrak{T}_{22}$, we have

$$M^{-1}(fs_{22}t_{22} + ft_{22}s_{22} + s_{22}ft_{22} + t_{22}fs_{22} + s_{22}t_{22}f + t_{22}s_{22}f)$$

$$= M^{-1}((a_{11} + b_{12})s_{22}t_{22} + (a_{11} + b_{12})t_{22}s_{22} + s_{22}(a_{11} + b_{12})t_{22}$$

$$+ t_{22}(a_{11} + b_{12})s_{22} + s_{22}t_{22}(a_{11} + b_{12}) + t_{22}s_{22}(a_{11} + b_{12}))$$

$$+ M^{-1}(c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22})$$
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\[ M^{*-1}(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22}) + M^{*-1}(c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}) = M^{*-1}(b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22}), \]

by Lemma 2.6(ii). This allows us to conclude that \( f_{22}s_{22}t_{22} + ft_{22}s_{22} + s_{22}ft_{22} + t_{22}fs_{22} + s_{22}t_{22}f + t_{22}s_{22}f = b_{12}s_{22}t_{22} + b_{12}t_{22}s_{22} + c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} \) which results in \( f_{22}s_{22}t_{22} + f_{22}t_{22}s_{22} + s_{22}f_{22}t_{22} + t_{22}f_{22}s_{22} + s_{22}t_{22}f_{22} + t_{22}s_{22}f_{22} = c_{22}s_{22}t_{22} + c_{22}t_{22}s_{22} + s_{22}c_{22}t_{22} + t_{22}c_{22}s_{22} + s_{22}t_{22}c_{22} + t_{22}s_{22}c_{22} \). Thus, \( f_{22} = c_{22} \), by condition (ii) of the Theorem.

Now we are able to prove the Theorem 2.1. Our proof is similar to those presented by Ji [1].

**Proof of Theorem.** Let \( a = a_{11} + a_{12} + a_{22} \) and \( b = b_{11} + b_{12} + b_{22} \) be arbitrary elements of \( \mathfrak{T} \). From lemmas 2.8, 2.9, 2.11 and 2.13, we compute

\[
M(a + b) = M((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{22} + b_{22})) = M(a_{11} + b_{11}) + M(a_{12} + b_{12}) + M(a_{22} + b_{22}) = M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{22}) + M(b_{22}) = M(a_{11} + a_{12} + a_{22}) + M(b_{11} + b_{12} + b_{22}) = M(a) + M(b).
\]

This shows that the map \( M \) is additive.

Now, we prove that \( M^* \) is additive. For any \( x, y \in \mathfrak{S} \), there exist elements \( c \) and \( d \) in \( \mathfrak{T} \) such that \( c = M^*(x + y) \) and \( d = M^*(x) + M^*(y) \). Hence, for arbitrary \( s, t \in \mathfrak{T} \), by the additivity of \( M \), we have

\[
M(cst + cts + sct + tcs + stc + tsc)
= M(M^*(x + y)st + M^*(x + y)ts + sM^*(x + y)t + tM^*(x + y)s + sM^*(x + y) + tsM^*(x + y))
= (x + y)M(s)M(t) + (x + y)M(t)M(s) + M(s)(x + y)M(t)
+ M(t)(x + y)M(s) + M(s)M(t)(x + y) + M(t)M(s)(x + y)
= xM(s)M(t) + xM(t)M(s) + M(s)xM(t) + M(t)xM(s)
+ M(s)M(t)x + M(t)M(s)x + yM(s)M(t) + yM(t)M(s)
+ M(s)yM(t) + M(t)yM(s) + M(s)M(t)y + M(t)M(s)y
= M(M^*(x)st + M^*(x)ts + sM^*(x)t + tM^*(x)s + stM^*(x) + tsM^*(x))
\]
\[ +M(M^*(y)st + M^*(y)ts + sM^*(y)t + tM^*(y)s + stM^*(y) + tsM^*(y)) \\
= M((M^*(x) + M^*(y))st + (M^*(x) + M^*(y))ts + s(M^*(x) + M^*(y))t \\
+ t(M^*(x) + M^*(y))s + st(M^*(x) + M^*(y)) + ts(M^*(x) + M^*(y))) \\
= M(dst + dts + st + tds + std + tsd). \]

Therefore, \(cst + cts + sc + tcs + stc + tsc = dst + dts + st + tds + std + tsd\) and by a similar argument used in the proof of Lemma 2.2 we can conclude that \(c = d\). Thus, \(M^*(x + y) = M^*(x) + M^*(y)\). The Theorem is proved. \(\square\)

REFERENCES