SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH PRAJAPAT OPERATOR

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In this paper, we obtain the Fekete-Szegö inequalities for the functions of complex order associated with Prajapat operator. Also, find upper bounds of the second Hankel determinant $\left|a_2a_4 - a_3^2\right|$ for functions belonging to the class $S_{\gamma}^{b}(m, \lambda, \ell; A, B)$.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$, which are univalent functions. Furthermore, let $\mathcal{P}$ be the family of functions $p(z) \in \mathcal{A}$ (class of analytic function in $\mathbb{U}$) satisfying $p(0) = 1$ and $\Re(p(z)) > 0$.

If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [6] and [16]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In [21, with $p = 1$] Prajapat defined a generalized multiplier transformation operator $J^m(\lambda, \ell) : \mathcal{A} \to \mathcal{A}$, as follows (see also [25, with $p = 1$]):

$$J^m(\lambda, \ell) f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\ell + 1 + \lambda(k-1)}{1 + \ell}\right)^m a_k z^k \tag{2}$$
$$(\lambda \geq 0;\ \ell > -1;\ m \in \mathbb{Z} = \{0, \pm 1, ...\};\ z \in \mathbb{U}).$$

It is readily verified from (2) that (see [21, with $p=1$])

$$\text{(3)}\quad \lambda z\left(J^{m}\left(\lambda,\ell\right)f(z)\right)' = \left(\ell+1\right)J^{m+1}\left(\lambda,\ell\right)f(z) - \left[\ell+1-\lambda\right]J^{m}_{p,n}\left(\lambda,\ell\right)f(z) \quad (\lambda>0).$$

By specializing the parameters m, λ and ℓ, we obtain the following operators studied by various authors:

(i) $J^{m}\left(\lambda,\ell\right)f(z) = I^{m}(\lambda,\ell)f(z)$ ($\ell>-1$, $\lambda\geq 0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}=\{1,2,...\}$) (see [7]);

(ii) $J^{m}\left(1,\ell\right)f(z) = I^{m}_{\ell}f(z)$ ($\ell\geq 0$ and $m\in\mathbb{N}_0$) (see [8, 9]);

(iii) $J^{m}\left(\lambda,0\right)f(z) = D^{m}_{\lambda}f(z)$ ($\lambda\geq 0$ and $m\in\mathbb{N}_0$) (see [1]);

(iv) $J^{m}\left(1,0\right)f(z) = D^{m}f(z)$ ($m\in\mathbb{N}_0$) (see [24]);

(v) $J^{-m}\left(\lambda,0\right)f(z) = I^{-m}_{\lambda}f(z)$ ($\lambda\geq 0$ and $m\in\mathbb{N}_0$) (see [3, 20]);

(vi) $J^{-m}\left(1,1\right)f(z) = I^{m}f(z)$ ($m\in\mathbb{N}_0$) (see [11]).

In 1976, Noonan and Thomas [19] discussed the q^{th} Hankel determinant of a locally univalent analytic function $f(z)$ for $q\geq 1$ and $n\geq 1$ which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ .. & .. & \cdots & .. \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For our present discussion, we consider the Hankel determinant in the case $q=2$ and $n=2$, i.e. $H_2(2)=a_2a_4-a_3^2$. This is popularly known as the second Hankel determinant of f.

In this paper, we define the following class $S^{b}_{\gamma}\left(m,\lambda,\ell;A,B\right)$ as follows:

Definition 1. Let $0\leq\gamma\leq 1$, $\lambda\geq 0$, $\ell>-1$, $m\in\mathbb{Z}$, $b\in\mathbb{C}^{*}=\mathbb{C}\backslash\{0\}$ and $n\in\mathbb{N}_0$. A function $f(z)\in\mathcal{A}$ is said to be in the class $S^{b}_{\gamma}\left(m,\lambda,\ell;A,B\right)$ if

$$\text{(4)}\quad 1+\frac{1}{b}\left(\left(1-\gamma\right)\frac{J^{m}\left(\lambda,\ell\right)f(z)}{z}+\gamma\left(J^{m}\left(\lambda,\ell\right)f(z)\right)'-1\right)\prec\frac{1+Az}{1+Bz}$$

$$\left(b\in\mathbb{C}^{*},\ 0\leq\gamma\leq 1,\ \lambda\geq 0,\ \ell>-1,\ m\in\mathbb{Z},\ -1\leq B<A\leq 1,\ z\in\mathbb{U}\right),$$

which is equivalent to say that

$$\left|\frac{\left(1-\gamma\right)\frac{J^{m}(\lambda,\ell)f(z)}{z}+\gamma\left(J^{m}\left(\lambda,\ell\right)f(z)\right)'-1}{\left[B+(A-B)b\right]-B\left[\left(1-\gamma\right)\frac{J^{m}(\lambda,\ell)f(z)}{z}+\gamma\left(J^{m}\left(\lambda,\ell\right)f(z)\right)'\right]}\right|<1.$$

We note that for suitable choices of b, γ, A, B, λ, ℓ and m we obtain the following subclasses:

(i) $S^{b}_{\gamma}\left(0,1,0;A,B\right) = S^{b}_{\gamma}\left(A,B\right)$ $\left(0\leq\gamma\leq 1,\ b\in\mathbb{C}^{*},\ -1\leq B<A\leq 1\right)$ (see Bansal [5]);

(ii) $S_{\gamma}^{b}(m,1,0;1,-1) = G_m(\gamma,b)$ $(b\in\mathbb{C}^*,\ 0\le\gamma\le 1,\ m\in\mathbb{N}_0)$ (see Aouf [2]);

(iii) $S_{\gamma}^{b}(m,1,0;A,B) = G_m(\gamma,b,A,B)$ $(0\le\gamma\le 1,\ b\in\mathbb{C}^*,\ m\in\mathbb{N}_0,$ $-1\le B<A\le 1)$ (see Sivasubramanian *et al.* [26]).

Also, we note that:

(i) $S_{\gamma}^{b}(m,\lambda,0;A,B) = S_{\gamma}^{b}(\lambda,m;A,B)$

$$= \left\{ f(z)\in\mathcal{A} : 1+\frac{1}{b}\left((1-\gamma)\frac{D_{\lambda}^{m}f(z)}{z}+\gamma\left(D_{\lambda}^{m}f(z)\right)'-1\right) \prec \frac{1+Az}{1+Bz}, \right. \tag{5}$$
$$\left. (b\in\mathbb{C}^*;\ 0\le\gamma\le 1;\ m\in\mathbb{N}_0;\ \lambda\ge 0;\ -1\le B<A\le 1;\ z\in\mathbb{U}) \right\};$$

(ii) $S_{\gamma}^{b}(-n,\lambda,0;A,B) = G_{\gamma}^{b}(\lambda,n;A,B)$

$$= \left\{ f(z)\in\mathcal{A} : 1+\frac{1}{b}\left((1-\gamma)\frac{I_{\lambda}^{n}f(z)}{z}+\gamma(I_{\lambda}^{n}f(z))'-1\right) \prec \frac{1+Az}{1+Bz}\ , \right. \tag{6}$$
$$\left. (b\in\mathbb{C}^*;\ 0\le\gamma\le 1;\ n\in\mathbb{N}_0;\ \lambda\ge 0;\ -1\le B<A\le 1;\ z\in\mathbb{U}) \right\}.$$

(iii) $S_{\gamma}^{(1-\rho)\cos\eta e^{-i\eta}}(m,\lambda,\ell;A,B) = S^{\gamma}[m,\rho,\eta,A,B]$

$$= \left\{ f(z)\in\mathcal{A} : e^{i\eta}\left[(1-\gamma)\frac{J^{m}(\lambda,\ell)f(z)}{z}+\gamma\left(J^{m}(\lambda,\ell)f(z)\right)'\right] \right. \tag{7}$$
$$\prec (1-\rho)\cos\eta\cdot\tfrac{1+Az}{1+Bz}+\rho\cos\eta+i\sin\eta,$$
$$\big(|\eta|<\tfrac{\pi}{2};\ 0\le\gamma\le 1;\ 0\le\rho<1;\ \lambda\ge 0;\ \ell>-1;\ m\in\mathbb{Z};$$
$$\left. -1\le B<A\le 1; z\in\mathbb{U}\big) \right\}.$$

In this paper, we obtain the Fekete-Szegö inequalities for the functions in the class $S_{\gamma}^{b}(m,\lambda,\ell;A,B)$. We also obtain an upper bound to the functional $H_2(2)$ for $f(z)\in S_{\gamma}^{b}(m,\lambda,\ell;A,B)$. Earlier Janteng *et al.* [13], Mishra and Gochhayat [17], Mishra and Kund [18], Bansal [4] and many other authors have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions.

2. PRELIMINARIES

To prove our results, we need the following lemmas.

LEMMA 1 ([23]). *Let*

$$h(z) = 1+_{n=1}^{\infty}c_nz^n \prec 1+_{n=1}^{\infty}C_nz^n = H(z)\quad (z\in\mathbb{U}). \tag{8}$$

If the function H is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is a convex set, then

$$|c_n| \leq |C_1|. \tag{9}$$

LEMMA 2 ([10]). *Let a function $p \in \mathcal{P}$ be given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \quad (z \in \mathbb{U}), \tag{10}$$

then, we have

$$|c_n| \leq 2 \quad (n \in \mathbb{N}). \tag{11}$$

The result is sharp.

LEMMA 3 ([14, 15]). *Let $p \in \mathcal{P}$ be given by the power series (10), then for any complex number ν, then*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1; |2\nu - 1|\}. \tag{12}$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z} \ (z \in \mathbb{U}).$$

LEMMA 4 ([12]). *Let a function $p \in \mathcal{P}$ be given by the power series (10), then*

$$2c_2 = c_1^2 + \varkappa(4 - c_1^2) \tag{13}$$

for some $\varkappa$, $|\varkappa| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1\varkappa - c_1(4 - c_1^2)\varkappa^2 + 2(4 - c_1^2)\left(1 - |\varkappa|^2\right) z, \tag{14}$$

for some z, $|z| \leq 1$.

3. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that : $b \in \mathbb{C}^*$, $0 \leq \gamma \leq 1$, $\lambda \geq 0$, $\ell > -1$, $m \in \mathbb{Z}$, $-1 \leq B < A \leq 1$ and $z \in \mathbb{U}$.

We give the following result related to the coefficient of $f(z) \in S_\gamma^b(m, \lambda, \ell; A, B)$.

THEOREM 1. *Let $f(z) \in \mathcal{A}$ given by (1) belongs to the class $S_\gamma^b(m, \lambda, \ell; A, B)$, then*

$$|a_k| \leq \frac{(A - B)(1 + \ell)^m |b|}{[1 + \gamma(k-1)](1 + \ell + \lambda(k-1))^m} \quad (k \in \mathbb{N} \setminus \{1\}). \tag{15}$$

Proof. If $f(z)$ of the form (1) belongs to the class $S_\gamma^b(m,\lambda,\ell;A,B)$, then

$$1+\frac{1}{b}\left((1-\gamma)\frac{J^m(\lambda,\ell)f(z)}{z}+\gamma\left(J^m(\lambda,\ell)f(z)\right)'-1\right)\prec\frac{1+Az}{1+Bz}=h(z)$$

where $h(z)$ is convex univalent in $\mathbb{U}$, we have

$$(16)\quad 1+\frac{1}{b}\left((1-\gamma)\tfrac{J^m(\lambda,\ell)f(z)}{z}+\gamma\left(J^m(\lambda,\ell)f(z)\right)'-1\right)$$
$$=1+\sum_{k=2}^{\infty}\tfrac{(1+(k-1)\gamma)}{b}\left(\tfrac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m a_k z^{k-1}$$
$$\prec 1+(A-B)z-B(A-B)z^2+\ldots.\,(z\in\mathbb{U}).$$

Now, by applying Lemma 1, we get the desired result. □

Remark 1. Putting $m=0$ in Theorem 1, we obtain the result obtained by Bansal [5,Theorem 2.1].

It is easy to derive a sufficient condition for $f(z)$ to be in the class $S_\gamma^b(m,\lambda,\ell;A,B)$ using standard techniques (see [22]). Hence we state the following result without proof.

THEOREM 2. *Let $f(z)\in\mathcal{A}$ given by (1), then a sufficient condition for $f(z)$ to be in the class $S_\gamma^b(m,\lambda,\ell;A,B)$ is*

$$(17)\qquad \sum_{k=2}^{\infty}[1+\gamma(k-1)]\left(\tfrac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m|a_k|\le\frac{(A-B)|b|}{1+B}.$$

Remark 2. Putting $m=0$ in Theorem 2, we obtain the result obtained by Bansal [5, Theorem 2.2].

In the next two theorems, we obtain the result concerning Fekete-Szegö inequality and upper bound on second Hankel determinant for the class $S_\gamma^b(m,\lambda,\ell;A,B)$.

THEOREM 3. *Let $f(z)\in\mathcal{A}$ given by (1) belongs to the class $S_\gamma^b(m,\lambda,\ell;A,B)$, then*

$$(18)\quad \left|a_3-\mu a_2^2\right|\le\frac{(A-B)(1+\ell)^m|b|}{(1+2\gamma)(1+\ell+2\lambda)^m}$$
$$\cdot\max\left\{1,\left|B+\frac{\mu b(A-B)(1+2\gamma)\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^m}{(1+\gamma)^2\left(\frac{1+\ell+\lambda}{1+\ell}\right)^{2m}}\right|\right\}.$$

This result is sharp.

Proof. Let $f(z) \in S_{\gamma}^{b}(m,\lambda,\ell;A,B)$, then there is a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{U}$ and such that
(19)
$$1+\frac{1}{b}\left((1-\gamma)\frac{J^{m}(\lambda,\ell)f(z)}{z}+\gamma\left(J^{m}(\lambda,\ell)f(z)\right)'-1\right)=\Phi(w(z)) \quad (z\in\mathbb{U}),$$
where

$$\text{(20)}\quad \begin{aligned}\Phi(z) &= \tfrac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^{2}+B^{2}(A-B)z^{3}-...\\ &= 1+B_{1}z+B_{2}z^{2}+B_{3}z^{3}+.......(z\in\mathbb{U}).\end{aligned}$$

If the function $p_1(z)$ is analytic and has positive real part in $\mathbb{U}$ and $p_1(0)=1$, then

$$\text{(21)}\qquad p_{1}(z)=\frac{1+w(z)}{1-w(z)}=1+c_{1}z+c_{2}z^{2}+......(z\in\mathbb{U}).$$

Since $w(z)$ is a Schwarz function. Define

$$\text{(22)}\quad \begin{aligned}p(z)=1+\frac{1}{b}\left((1-\gamma)\frac{J^{m}(\lambda,\ell)f(z)}{z}+\gamma\left(J^{m}(\lambda,\ell)f(z)\right)'-1\right)\\ =1+d_{1}z+d_{2}z^{2}+.......(z\in\mathbb{U}).\end{aligned}$$

In view of the equations (19) and (21), we have

$$p(z)=\Phi\left(\frac{p_{1}(z)-1}{p_{1}(z)+1}\right).$$

Since

$$\text{(23)}\quad \frac{p_{1}(z)-1}{p_{1}(z)+1}=\frac{1}{2}\left[c_{1}z+\left(c_{2}-\frac{c_{1}^{2}}{2}\right)z^{2}+\left(c_{3}+\frac{c_{1}^{3}}{4}-c_{1}c_{2}\right)z^{3}+.......\right].$$

Therefore, we have

$$\text{(24)}\quad \begin{aligned}\Phi\left(\frac{p_{1}(z)-1}{p_{1}(z)+1}\right)=1+\frac{1}{2}B_{1}c_{1}z+\left[\frac{1}{2}B_{1}\left(c_{2}-\frac{c_{1}^{2}}{2}\right)+\frac{1}{4}B_{2}c_{1}^{2}\right]z^{2}+\\ +\left[\frac{B_{1}}{2}\left(c_{3}-c_{1}c_{2}+\frac{c_{1}^{3}}{4}\right)+\frac{B_{2}c_{1}}{2}\left(c_{2}-\frac{c_{1}^{2}}{2}\right)+\frac{B_{3}c_{1}^{3}}{8}\right]z^{3}+......,\end{aligned}$$

and from this equation and (22), we obtain

$$\text{(25)}\qquad d_{1}=\frac{1}{2}B_{1}c_{1},\quad d_{2}=\frac{1}{2}B_{1}\left(c_{2}-\frac{c_{1}^{2}}{2}\right)+\frac{1}{4}B_{2}c_{1}^{2},$$

and

$$\text{(26)}\qquad d_{3}=\frac{B_{1}}{2}\left(c_{3}-c_{1}c_{2}+\frac{c_{1}^{3}}{4}\right)+\frac{B_{2}c_{1}}{2}\left(c_{2}-\frac{c_{1}^{2}}{2}\right)+\frac{B_{3}c_{1}^{3}}{8}.$$

Then, from (19), we see that

$$d_1 = \frac{(1+\gamma)\left(\frac{1+\ell+\lambda}{1+\ell}\right)^m a_2}{b}, \quad d_2 = \frac{(1+2\gamma)\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^m a_3}{b}$$

(27) $$\text{and} \quad d_3 = \frac{(1+3\gamma)\left(\frac{1+\ell+3\lambda}{1+\ell}\right)^m a_4}{b}.$$

Now from (21), (22) and (27), we have

(28) $$a_2 = \frac{(A-B)\,bc_1}{2(1+\gamma)\left(\frac{1+\ell+\lambda}{1+\ell}\right)^m}, \quad a_3 = \frac{b(A-B)}{4(1+2\gamma)\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^m}\left\{2c_2 - c_1^2(1+B)\right\}$$

and

(29) $$a_4 = \frac{b(A-B)}{8(1+3\gamma)\left(\frac{1+\ell+3\lambda}{1+\ell}\right)^m}\left\{4c_3 - 4c_1c_2(1+B) + c_1^3(1+B)^2\right\}.$$

Therefore, we have

(30) $$a_3 - \mu a_2^2 = \frac{b(A-B)}{2(1+2\gamma)\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^m}\left\{c_2 - \nu c_1^2\right\},$$

where

(31) $$\nu = \frac{1}{2}\left[1 + B + \frac{\mu b(A-B)(1+2\gamma)\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^m}{(1+\gamma)^2\left(\frac{1+\ell+\lambda}{1+\ell}\right)^{2m}}\right].$$

Our result now follows by an application of Lemma 3. The result is sharp for the functions

(32) $$1 + \frac{1}{b}\left((1-\gamma)\frac{J^m(\lambda,\ell)f(z)}{z} + \gamma\left(J^m(\lambda,\ell)f(z)\right)' - 1\right) = \Phi(z^2)$$

and

(33) $$1 + \frac{1}{b}\left((1-\gamma)\frac{J^m(\lambda,\ell)f(z)}{z} + \gamma\left(J^m(\lambda,\ell)f(z)\right)' - 1\right) = \Phi(z).$$

This completes the proof of Theorem 3. □

Remark 3. Putting $m = 0$ in Theorem 3, we obtain the result obtained by Bansal [5, Theorem 2.3].

THEOREM 4. *Let $f(z) \in \mathcal{A}$ given by (1) belongs to the class $S_\gamma^b(m,\lambda,\ell; A,B)$, then*

(34) $$\left|a_2a_4 - a_3^2\right| \le \frac{(A-B)^2|b|^2}{(1+2\gamma)^2\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^{2m}}.$$

Proof. Using (28) and (29), we have

$$(35)\quad \left|a_2a_4-a_3^2\right| = \frac{(A-B)^2\,|b|^2}{16\,(1+\gamma)\,(1+3\gamma)\left(\frac{1+\ell+\lambda}{1+\ell}\right)^m\left(\frac{1+\ell+3\lambda}{1+\ell}\right)^m} \Big|4c_1c_3-4c_1^2c_2(1+B)+c_1^4(1+B)^2 - \frac{(1+\gamma)\,(1+3\gamma)\,(1+\ell+\lambda)^m\,(1+\ell+3\lambda)^m}{(1+2\gamma)^2\,(1+\ell+2\lambda)^{2m}} \left[4c_2^2-4c_1^2c_2(1+B)+c_1^4(1+B)^2\right]\Big|$$
$$= M\left|4c_1c_3-4c_1^2c_2(1+B)+c_1^4(1+B)^2 - N\left[4c_2^2-4c_1^2c_2(1+B)+c_1^4(1+B)^2\right]\right|,$$

where

$$(36)\quad M = \frac{(A-B)^2\,|b|^2}{16\,(1+\gamma)\,(1+3\gamma)\left(\frac{1+\ell+\lambda}{1+\ell}\right)^m\left(\frac{1+\ell+3\lambda}{1+\ell}\right)^m}$$
$$\text{and}\quad N = \frac{(1+\gamma)\,(1+3\gamma)\,(1+\ell+\lambda)^m\,(1+\ell+3\lambda)^m}{(1+2\gamma)^2\,(1+\ell+2\lambda)^{2m}}.$$

The above equation (35) is equivalent to

$$(37)\quad \left|a_2a_4-a_3^2\right| = M\left|4c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|,$$

where

$$(38)\quad d_1=4,\quad d_2=-4(1+B)(1-N),\quad d_3=-4N,\quad d_4=(1-N)(1+B)^2.$$

Since the functions $p(z)$ and $p(re^{i\theta})$ $(\theta\in\mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1>0$. For convenience of notation, we take $c_1=c$ ($c\in[0,2]$, see (11)). Also, substituting the values of c_2 and c_3, respectively, from (13) and (14) in (37), we have

$$\left|a_2a_4-a_3^2\right| = \frac{M}{4}\Big|c^4(d_1+2d_2+d_3+4d_4)+2\varkappa c^2(4-c^2)(d_1+d_2+d_3) +(4-c^2)\varkappa^2(-d_1c^2+d_3(4-c^2))+2d_1c(4-c^2)\left(1-|\varkappa|^2 z\right)\Big|.$$

An application of triangle inequality, replacement of $|\varkappa|$ by ν and substituting the values of $d_1,\ d_2,\ d_3$ and d_4 from (38), we have

$$(39)\quad \left|a_2a_4-a_3^2\right| \le \frac{M}{4}\Big[4c^4(1-N)B^2+8\,|B|\,(1-N)\nu c^2(4-c^2)+ (4-c^2)\nu^2\left(4c^2+4N(4-c^2)\right)+8c(4-c^2)\left(1-\nu^2\right)\Big],$$

$$= M\left[c^4(1-N)B^2+2c(4-c^2)+2\nu\,|B|\,(1-N)c^2(4-c^2)\right.$$

$$+\nu^2(4-c^2)\left(c^2(1-N)-2c+4N\right)\Big]$$
$$(40) \quad = \quad F(c,\nu).$$

Next, we assume that the upper bound for (40) occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Differentiating $F(c,\nu)$ in (40) partially with respect to ν, we have

(41)
$$\frac{\partial F(c,\nu)}{\partial \nu} = M\left[2|B|(1-N)c^2(4-c^2)+2\nu(4-c^2)\left(c^2(1-N)-2c+4N\right)\right].$$

For $0<\nu<1$ and for any fixed c with $0<c<2$, from (41), we observe that $\frac{\partial F}{\partial \nu}>0$. Therefore $F(c,\nu)$ is an increasing function of ν, which contradicts our assumption that the maximum value of $F(c,\nu)$ occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$,

$$(42) \qquad Max\ F(c,\nu) = F(c,1) = G(c).$$

Thus

(43)
$$G(c) = M\left[c^4(1-N)\left(B^2-2|B|-1\right)+4c^2(2|B|(1-N)+1-2N)+16N\right].$$

Next,

$$(44) \quad G^{'}(c) = 4Mc\left[c^2(1-N)\left(B^2-2|B|-1\right)+2(2|B|(1-N)+1-2N\right]$$
$$= 4Mc\left[c^2(1-N)\left(B^2-2|B|-1\right)+2\left\{(1-N)(2|B|+1)-N\right\}\right].$$

So $G^{'}(c)<0$ for $0<c<2$ and has real critical point at $c=0$. Also $G(c)>G(2)$. Therefore, maximum of $G(c)$ occurs at $c=0$. Therefore, the upper bound of $F(c,\nu)$ corresponds to $\nu=1$ and $c=0$. Hence,

$$\left|a_2a_4-a_3^2\right| \le 16MN = \frac{(A-B)^2|b|^2}{(1+2\gamma)^2\left(\frac{1+\ell+2\lambda}{1+\ell}\right)^{2m}}.$$

This completes the proof of the Theorem 4. □

Remark 4. Putting $m=0$ in Theorem 4, we obtain the result obtained by Bansal [5, Theorem 2.4].

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