Dedicated to Monique Dauge on her 60^{th} birthday

DIFFERENTIAL OPERATORS ON DOMAINS WITH CONICAL POINTS: PRECISE UNIFORM REGULARITY ESTIMATES

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Communicated by Vasile Brînzănescu

We study families of strongly elliptic, second order differential operators with singular coefficients on domains with conical points. We obtain estimates on the norms of the inverses of the operators in the family and on the regularity of the solutions to the associated Poisson problems with mixed boundary conditions. The coefficients and the solutions belong to (suitable) weighted Sobolev spaces. The space of coefficients is a Banach space that contains, in particular, the space of smooth functions. Hence, our results extend the classical well-posedness result for a strongly elliptic equation in a domain with conical points to *families* of such problems and to *singular* coefficients. The main point is that we obtain concrete and precise estimates on the norm of the inverse of an operator in our family in terms of its coercivity constant and the norms of its coefficients. Moreover, we show that the solutions depend analytically on the coefficients of the operators and on the forcing terms.

AMS 2010 Subject Classification: Primary: 35J25; Secondary: 35R05, 65N30, 58J32, 52B70.

Key words: polyhedral domain, strongly elliptic equations, mixed boundary conditions, weighted Sobolev spaces, well-posedness, parametric family, Legendre approximation.

1. INTRODUCTION

We consider families of mixed boundary value problems on a bounded, domain $\Omega \subset \mathbb{R}^d$ with conical points ($d \geq 2$). The associated differential operators belong to suitable families of strongly elliptic, second order differential

C. Bacuta has been partially supported by NSF Grant DMS-1522454. H. Li has been partially supported by the National Science Foundation Grant DMS-1418853, by the Natural Science Foundation of China Grant 11628104, and by the Wayne State University Grants Plus Program. V. Nistor has been partially supported by ANR-14-CE25-0012-01 (SINGSTAR).
Manuscripts available from http://iecl.univ-lorraine.fr/Victor.Nistor/

operators with singular coefficients. In our method, it is necessary to consider certain singular coefficients even if one is interested only in the case of regular coefficients. Using appropriate weighted Sobolev spaces, we obtain concrete estimates on the norm and on the regularity of the solutions of our boundary value problems in terms of the norms of the coefficients of the operators and their coercivity constants. In addition, we provide weighted Sobolev space conditions on the coefficients that ensure an analytic dependence of the solution on the coefficients of the operators and on the forcing terms (the free term).

To better explain our results, it is useful to put them into perspective. A classical result in Partial Differential Equations states that a second order, strongly coercive, strongly elliptic partial differential operator P induces an isomorphism

$$P\,:\,H^{m+1}(G)\cap\{u|_{\partial\Omega}=0\}\,\rightarrow\,H^{m-1}(G)\,, \tag{1}$$

for all $m\in\mathbb{Z}_+:=\{0,1,\ldots\}$, provided that G is a smooth, bounded domain in some euclidean space. See, for example, [2, 28, 30, 39] and the references therein. This result has many applications and extensions. However, it does not extend directly to non-smooth domains. In fact, on non-smooth domains, the solution u of $Pu=F$ will have singularities, even if the right hand side F is smooth. See Kondratiev's fundamental 1967 paper [33] for the case of domain with conical points and Dauge's comprehensive Lecture Notes [25] for the case of polyhedral domains. See [8, 9, 12, 22, 29, 34–36, 43, 46] for a sample of related results. These theoretical results have been a critical ingredient in developing effective numerical methods approximating singular solutions. See for example [7, 14]. In addition, we mention that estimates for equations on conical manifolds can also be obtained using the method of layer potentials (see, for example, [17, 27, 32, 41, 45] and references therein).

For polygonal domains (and, more generally, for domains with conical points), Kondratiev's results mentioned above extend the isomorphism in (1) to polygonal domains by replacing the usual Sobolev spaces $H^m(\Omega)$ with the Kondratiev type Sobolev spaces. Let Ω be then a curvilinear polygonal domain (see Definition 3.1, in particular, the sides are not required to be straight), and $r_\Omega>0$ be a *smooth function on* Ω *that coincides with the distance to its vertices close to the vertices.* We let

$$\mathcal{K}^m_a(\Omega)\;:=\;\{\,u:\Omega\rightarrow\mathbb{C},\;r_\Omega^{|\alpha|-a}\partial^\alpha u\in L^2(\Omega),\;|\alpha|\leq m\,\}\,, \tag{2}$$

where $\partial_i\,:=\,\frac{\partial}{\partial x_i}$, $i\,=\,1,\ldots,d$, and $\partial^\alpha\,:=\,\partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_d^{\alpha_d}$. Kondratiev's results [33] (see also [22, 34]) give that the Laplacian $\Delta:=\sum_{i\leq d}\partial_i^2$ induces an isomorphism of weighted Sobolev spaces. More precisely,

$$\Delta:\mathcal{K}^{m+1}_{a+1}(\Omega)\cap\{u|_{\partial\Omega}=0\}\,\rightarrow\,\mathcal{K}^{m-1}_{a-1}(\Omega) \tag{3}$$

is a continuous bijection with continuous inverse for $m \in \mathbb{Z}_+ := \{0, 1, \ldots\}$ and $|a| < \pi/\alpha_{MAX}$, where α_{MAX} is the maximum angle of Ω. One can extend this result by interpolation to the usual range of values for m [39]. A similar result holds also for more general strongly elliptic operators [34]. In [12], this result of Kondratiev was extended to three dimensional polyhedral domains and in [10] it was extended to general d-dimensional polyhedral domains using as a main ingredient a suitable generalization of Hardy's inequality. In three dimensions and higher, this type of results is not enough for numerical methods. Thus, in [13], an anisotropic regularity and well-posedness result was proved for three dimensional polyhedral domains, building on previous results of Babuška and Guo [6] and Buffa, Costabel, and Dauge [15]. See also [22] for further references and for related results, including analytic regularity.

In this paper, we generalize Kondratiev's result by allowing families of operators, by allowing low-regularity coefficients, and by studying the quantitative and qualitative dependence of the solution on these coefficients. To state our main result, let us fix some notation. Let $\beta := (a_{ij}, b_i, c)$ denote the coefficients of

$$p_\beta u := -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j u) + \sum_{i=1}^{d} b_i \partial_i u - \sum_{i=1}^{d} \partial_i(b_{d+i}u) + cu\,, \tag{4}$$

a second order differential operator in divergence form on our domain $\Omega \subset \mathbb{R}^d$. Many concepts discussed in the paper make sense for any dimension $d \geq 1$. Nevertheless, the main results we prove are for $d = 2$. Thus, we assume for the rest of this introduction that Ω is a two-dimensional curvilinear polygonal domain. The coefficients β of the operator p_β are obtained using weighted $\mathcal{W}^{m,\infty}$-type space defined by

$$\mathcal{W}^{m,\infty}(\Omega) := \{\, u : \Omega \to \mathbb{C} \mid r_\Omega^{|\alpha|}\partial^\alpha u \in L^\infty(\Omega),\ |\alpha| \leq m\,\}\,, \tag{5}$$

where r_Ω is as in Equation (2) (that is, it is equal to the distance function to the conical points when close to those points). We fix for the rest of the introduction $m \in \mathbb{Z}_+ := \{0, 1, \ldots\}$ and we assume that $a_{ij}, r_\Omega b_i, r_\Omega^2 c \in \mathcal{W}^{m,\infty}(\Omega)$. We let

$$\|\beta\|_{Z_m} := \max\{\|a_{ij}\|_{\mathcal{W}^{m,\infty}(\Omega)},\ \|r_\Omega b_i\|_{\mathcal{W}^{m,\infty}(\Omega)},\ \|r_\Omega^2 c\|_{\mathcal{W}^{m,\infty}(\Omega)}\,\}\,, \tag{6}$$

(notice the factors involving r_Ω!), and for $P = p_\beta$ and $V = H_0^1(\Omega)$, define

$$\rho(\beta) := \rho(P) := \inf \frac{\Re(Pv, v)}{\|v\|^2_{H^1(\Omega)}}\,, \quad v \in V\,,\ v \neq 0\,, \tag{7}$$

where $\Re(z) = \Re z$ denotes the real part of z. Our main result *for Dirichlet boundary conditions in two dimensions* is as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygonal domain, $\eta_0 > 0$, $m \geq 0$, and $N_m = 2^{m+2} - m - 3 \geq 0$. Then there exist γ and C_m with the following property. For any $\beta \in Z_m$ and any $|a| < \eta := \min\{\eta_0, \gamma^{-1}\|\beta\|_{Z_0}^{-1}\rho(\beta)\}$, the operator p_β defined in Equation* (4) *induces an isomorphism*

$$p_\beta : \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{ u|_{\partial\Omega} = 0 \} \to \mathcal{K}_{a-1}^{m-1}(\Omega) \tag{8}$$

such that $p_\beta^{-1} : \mathcal{K}_{a-1}^{m-1}(\Omega) \to \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\}$ depends analytically on the coefficients $\beta := (a_{ij}, b_i, c)$ and has norm

$$\|p_\beta^{-1}\| \leq C_m(\rho(\beta) - \gamma|a|\|\beta\|_{Z_0})^{-N_m-1}\|\beta\|_{Z_m}^{N_m}.$$

The parameter η_0 has to role of ensuring that a belongs to a fixed bounded set, so we can bound a^2 with $|a|$ in the estimates involving $\beta(a)$ (see Theorem 4.4). The bounds γ_1 and C_m depend only on m, $(\Omega, \partial_D\Omega)$, and η_0.

Since the solution u of the equation $p_\beta u = F$, $u = 0$ on the boundary, is in $\mathcal{K}_{a+1}^{m+1}(\Omega)$ for $F \in \mathcal{K}_{a-1}^{m-1}(\Omega)$, $|a| < \eta$, we obtain the usual applications to the Finite Element Method on straight polygonal domains for $m \geq 1$ and $a > 0$ [1, 11, 36].

Theorem 1.1 is a consequence of Theorem 4.4, which deals with the mixed boundary value problem

$$\begin{cases} p_\beta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial_D\Omega \\ \partial_\nu^\beta u = h & \text{on } \partial_N\Omega\,, \end{cases} \tag{9}$$

where $(\partial_\nu^\beta v) := \sum_{i=1}^d \nu_i(\sum_{j=1}^d a_{ij}\partial_j v + b_{d+i}v)$. An exotic example to which Theorem 4.4 applies is that of the Schrödinger operator $H := -\Delta + cr_\Omega^{-2}$ on Ω with *pure Neumann* boundary conditions (and suitable positivity conditions on c), see Theorem 5.4.

The main novelties of Theorem 4.4 (and of the paper in general) are the following:

(i) The precise estimate on the norm of the inverse of P^β seems to be new even in the smooth case.

(ii) We deal with singular coefficients of a type that has not been systematically considered in the literature on non-smooth domains. Thus our coefficients have both singular parts at the corners of the form r_Ω^{-j} $(j \leq 2)$ and have limited regularity away from the corners.

(iii) We provide a new method to obtain higher regularity in weighted Sobolev spaces using divided differences; a method that is, in fact, closer to the one used in the classical case of smooth domains.

(iv) Our method of obtaining higher regularity for the solution also yields regularity for the dependence on the coefficients, more precisely, in this case, an analytic dependence on the coefficients and the free term (f in Problem (9)).

The paper is organized as follows. In Section 2, we introduce the notation and necessary preliminary results for our problem in the usual Sobolev spaces. In particular, an enhanced Lax-Milgram Lemma (Lemma 2.5) provides uniform estimates for the solution of our problem (9) and analytic dependence of this solution on the coefficients β. In Section 3, we first define curvilinear polygonal domains (Definition 3.1). We then provide several equivalent definitions of the weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ and the form of our differential operators. Then, in Section 4, using local coordinate transformations, we derive our main result, the analytic dependence of the solution on the coefficients in high-order weighted Sobolev spaces (Theorem 4.4). Finally, Section 5 contains some consequences of Theorem 4.4 and some extensions. In particular, we consider a framework for the pure Neumann problem with inverse square potentials at vertices. For notational simplicity, we do not consider systems, although many of the techniques below apply to this more general setting.

2. COERCIVITY IN CLASSICAL SOBOLEV SPACES

In this section, we recall some needed results on coercive operators, on (uniformly) strongly elliptic operators, and on analytic functions defined on open subsets of Banach spaces.

2.1. Function spaces and boundary conditions

Throughout the paper, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denotes a connected, bounded domain. Further conditions on Ω will be imposed in the next section. As usual, $H^m(\Omega)$ denotes the space of (equivalence classes of) functions on Ω with m derivatives in $L^2(\Omega)$. When we write $A \subset B$, we allow also $A = B$. In what follows, $\partial_D\Omega$ is a suitable closed subset of the boundary $\partial\Omega$, where we impose *Dirichlet* boundary conditions.

We shall rely heavily on the weak formulation of Problem (9). Thus, let us recall that $H^{-1}(\Omega)$ is defined as the dual space of

$$H_0^1(\Omega) := \{\, u \in H^1(\Omega) \mid u|_{\partial\Omega} = 0 \,\}, \tag{10}$$

with pivot $L^2(\Omega)$. We introduce homogeneous essential boundary conditions abstractly, by considering a subspace V,

$$H_0^1(\Omega) \subset V \subset H^1(\Omega), \tag{11}$$

such that V has the norm induced either from $H^1(\Omega)$ or from $\mathcal{K}^1_1(\Omega)$ and $H^1_0(\Omega)$ is a closed subspace of V. In many applications, V is closed in $H^1(\Omega)$, but this is not the case in our application to the Neumann problem with inverse square potentials at vertices (see Theorem 5.4). Let V^* be *the dual of V with pivot space $L^2(\Omega)$*. Therefore, by $(\, , \,)$ we shall denote *both* the inner product $(f,g) = \int_\Omega f(x)\overline{g(x)}\, dx$ on $L^2(\Omega)$, and by continuous extension, also the duality pairing between V^* and V. Thus, $V^* = H^{-1}(\Omega)$ if $V = H^1_0(\Omega)$; otherwise, V^* will incorporate also non-homogeneous natural boundary conditions.

For Problem (9), we choose

$$V \;=\; H^1_D(\Omega) \;:=\; \{\, u \in H^1(\Omega) \mid u = 0 \text{ on } \partial_D\Omega \,\}, \tag{12}$$

and assume that the Neumann part of the boundary contains no adjacent edges.

2.2. **The weak formulation**

Recall from Equation (4) the differential operator $p_\beta u := -\sum_{i,j=1}^{d} \partial_i\big(a_{ij}\partial_j u\big) + \sum_{i=1}^{d} b_i\partial_i u - \sum_{i=1}^{d} \partial_i(b_{d+i}u) + cu$, which is used in Problem (9), where $a_{ij}, b_i, c : \Omega \to \mathbb{C}$ denote measurable *complex valued* functions as in (4) and β denotes the coefficients (a_{ij}, b_i, c). We shall make suitable further assumptions on these coefficients below.

Equation (9), makes sense as formulated only if u is regular enough (at least in $H^{3/2+\epsilon}$, to validate the Neumann derivatives at the boundary). In order to use the Lax-Milgram Lemma for the problem (9), we formulate our problem in a more general way that allows $u \in V$. To this end, let us introduce the Dirichlet form B^β associated to (9), that is, the sesquilinear form

$$\begin{aligned} B^\beta(u,v) \;:=\; & \sum_{i,j=1}^{d} \big(a_{ij}\partial_j u, \partial_i v\big) + \sum_{i=1}^{d} \big(b_i\partial_i u, v\big) + \sum_{i=1}^{d} \big(b_{d+i}u, \partial_i v\big) + \big(cu, v\big) \\ = & \int_\Omega \Big[\sum_{i=1}^{d} \big(\sum_{j=1}^{d} a_{ij}(x)\partial_j u(x) + b_{d+i}(x)u(x)\big)\partial_i \overline{v(x)} \\ & \qquad + \Big(\sum_{i=1}^{d} b_i(x)\partial_i u(x) + c(x)u(x)\Big)\overline{v(x)}\Big]\, \mathrm{d}x \,, \end{aligned}$$

where $\mathrm{d}x$ denotes the volume element in the Lebesgue integral on $\Omega \subset \mathbb{R}^d$.

Remark 2.1. Let $F(v) = \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\partial_N \Omega} h(x)v(x)dS$. Then the weak variational formulation of Equation (9) is: Find $u \in V$, such that

$$B^\beta(u, v) \;=\; F(\overline{v}), \quad \text{for all } \; v \in V. \tag{13}$$

We then define $P^\beta : V \to V^*$ by

$$(P^\beta u, v) := B^\beta(u, v), \quad \text{for all } \; u, v \in V. \tag{14}$$

Thus, the weak formulation of Equation (9) is equivalent to

$$P^\beta u \;=\; F \in V^*. \tag{15}$$

We are interested in the dependence of u on F and on the coefficients $\beta := (a_{ij}, b_i, c)$ of P^β. We notice that if the Neumann part of the boundary $\partial_N\Omega$ is empty, then p_β and P^β can be identified, but this is not possible in general. In fact, we are looking for an analytic dependence of the solutions on the coefficients. For this reason, it is useful to consider complex Banach spaces and complex valued coefficients.

2.3. **Bounded forms and operators**

For two Banach spaces X and Y, let $\mathcal{L}(X; Y)$ denote the Banach space of continuous, linear maps $T : X \to Y$ endowed with the operator norm

$$\|T\|_{\mathcal{L}(X;Y)} \;:=\; \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}. \tag{16}$$

We write $\mathcal{L}(X) := \mathcal{L}(X; X)$.

Let us define Z to be the set of coefficients $\beta = (a_{ij}, b_i, c)$ such that the form B^β is defined and continuous on $V \times V$, and we give Z the induced norm. Thus Z is given the induced topology from $\mathcal{L}(V; V^*)$.

It will be convenient to use a slightly enhanced version of the well-known Lax-Milgram Lemma stressing the analytic dependence on the operator and on the data. We thus first review a few basic definitions and results on analytic functions [26].

Let X and Y be Banach spaces. In what follows, $\mathcal{L}_i(Y; X)$ will denote the space of continuous, multi-linear functions $L : Y \times Y \times \ldots \times Y \to X$, where i denotes the number of copies of Y. The norm on the space $\mathcal{L}_i(Y; X)$ is

$$\|L\|_{\mathcal{L}_i(Y;X)} \;:=\; \sup_{\|y_j\| \leq 1} \|L(y_1, y_2, \ldots, y_i)\|_X.$$

Of course, $\mathcal{L}_1(Y; X) = \mathcal{L}(Y; X)$, isometrically. We shall need analytic functions defined on open subsets of a Banach space. Let $U \subset Y$ and consider the spaces $\mathcal{C}^k(U; X)$, $k \in \mathbb{Z}_+ \cup \{\infty, \omega\}$ defined as follows. If $k \in \mathbb{Z}_+ \cup \{\infty\}$, then $\mathcal{C}^k(U; X)$

denotes the space of functions $v : U \to X$ with continuous (Fréchet) derivatives $D^i v : U \to \mathcal{L}_i(Y; X)$, $i \le k$. Similarly, $\mathcal{C}^k_b(U; X) \subset \mathcal{C}^k(U; X)$, $k \in \mathbb{Z}_+ \cup \{\infty\}$, denotes the subspace of those functions $v \in \mathcal{C}^k(U; X)$ for which the derivatives $D^i v$, $i \le k$, are bounded on U. For each finite j, we let

$$\|v\|_{\mathcal{C}^j_b(U;X)} := \sup_{i \le j,\, y \in U} \|D^i_y v\|_{\mathcal{L}_i(Y;X)} \tag{17}$$

denote the natural Banach space norm on $\mathcal{C}^j_b(U; X)$, with $D^i_a v \in \mathcal{L}_i(Y; X)$ denoting the value of $D^i v$ at a.

The space $\mathcal{C}^\omega(U; X)$ consists of the functions $f : U \to X$ that have, for any $a \in U$, an expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k_a f(x - a, x - a, \ldots, x - a)$$

that is uniformly convergent for x in a small, non-empty open ball centered at a. We let $\mathcal{C}^\omega_b(U; X) := \mathcal{C}^\omega(U; X) \cap \mathcal{C}^\infty_b(U; X)$. If k is not finite, that is, if $k = \infty$ or $k = \omega$, we endow $\mathcal{C}^k_b(U; X)$ with the Fréchet topology defined by the family of seminorms $\| \cdot \|_{\mathcal{C}^j_b(U;X)}$, $j \ge 1$. We shall use that multilinear functions are analytic. We shall need the following standard result.

Lemma 2.2. *Let Y_1, Y_2 be Banach spaces.*

(i) *The map $\mathcal{L}(Y_1; Y_2) \times Y_1 \ni (T, y) \to Ty \in Y_2$ is analytic.*

(ii) *The map $T \to T^{-1} \in \mathcal{L}(Y_1)$ is analytic on the open set $\mathcal{L}_{\mathrm{inv}}(Y_1)$ of invertible operators in $\mathcal{L}(Y_1) := \mathcal{L}(Y_1; Y_1)$.*

Proof. In (i), the given map is bilinear, and hence analytic. To prove (ii), we simply write the Neumann series $(T - R)^{-1} = \sum_{n=0}^{\infty} T^{-1}(RT^{-1})^n$, which is uniformly and absolutely convergent for $\|R\|\|T^{-1}\| \le 1 - \epsilon$, for any $1 \ge \epsilon > 0$. $\square$

2.4. An enhanced Lax-Milgram Lemma

We now recall the classical Lax-Milgram Lemma, in the form that we will need.

Definition 2.3. Let $H^1_0(\Omega) \subset V \subset H^1(\Omega)$. A continuous operator $P : V \to V^*$ is called *strongly coercive on V* (or simply *strongly coercive* when there is no danger of confusion) if

$$0 < \rho(P) := \inf_{v \in V \smallsetminus \{0\}} \frac{\Re(Pv, v)}{\|v\|^2_V}\,.$$

We shall usually write $\rho(\beta) = \rho(P^\beta)$, where $\rho(P^\beta)$ is as defined in Equation (7). For $P = P^\beta$, we thus have $\rho(\beta)\|v\|^2_{H^1(\Omega)} = \rho(P^\beta)\|v\|^2_{H^1(\Omega)} \le \Re B^\beta(v,v)$, for all $v \in V$. We shall need the following simple observation:

Remark 2.4. If $P : V \to V^*$ is strongly coercive on V and $P_1 : V \to V^*$ satisfies $\|P_1\| := \|P_1\|_{\mathcal{L}(V,V^*)} < \rho(P)$, then $P + P_1$ is also strongly coercive on V and $\rho(P + P_1) \ge \rho(P) - \|P_1\|$. Indeed,

$$\Re\big((P+P_1)u,u\big) \ \ge\ \Re(Pu,u) - \|P_1\|\|u\|_V^2 \ \ge\ (\rho(P) - \|P_1\|)\|u\|_V^2\,, \tag{18}$$

and hence the set $\mathcal{L}(V;V^*)_c$ of strongly coercive operators is open in $\mathcal{L}(V;V^*)$.

Recall now the standard way of solving Equation (13) using the Lax-Milgram Lemma for strongly coercive operators.

LEMMA 2.5 (Analytic Lax-Milgram Lemma). *Assume that $P : V \to V^*$ is strongly coercive. Then P is invertible and $\|P^{-1}\| \le \rho(P)^{-1}$. Moreover, the map $\mathcal{L}(V;V^*)_c \times V^* \ni (P,F) \to P^{-1}F \in V$ is analytic. Consequently,*

$$\big(Z \cap \mathcal{L}(V;V^*)_c\big) \times V^* \ \ni\ (\beta, F) \ \to\ (P^\beta)^{-1}F \ \in\ V$$

is analytic as well.

Proof. The first part is just the classical Lax-Milgram Lemma [16, 19, 42], which states that "coercivity implies invertibility" and gives the norm estimate. The second part follows from Lemma 2.2. Indeed, the map $\Phi : \mathcal{L}(V;V^*)_c \times V^* \to V$, $\Phi(\beta,F) := (P^\beta)^{-1}F$ is the composition of the maps

$$\mathcal{L}(V;V^*)_c \times V^* \times V^* \ \ni\ (\beta, F) \ \to\ (P^\beta, F) \ \in\ \mathcal{L}_{inv}(V,V^*) \times V^*\,,$$
$$\mathcal{L}_{inv}(V;V^*) \times V^* \ \ni\ (P,F) \ \to\ (P^{-1},F) \ \in\ \mathcal{L}(V^*;V) \times V^*\,,$$
$$\text{and}\quad \mathcal{L}(V^*;V) \times V^* \ \ni\ (P^{-1},F) \ \to\ P^{-1}F \ \in\ V\,.$$

The first of these three maps is well defined and linear by the classical Lax-Milgram Lemma. The other two maps are analytic by Lemma 2.2. Since the composition of analytic functions is analytic, the result follows. $\square$

Examples of strongly coercive operators are obtained using "uniformly strongly elliptic" operators, whose definition we recall next.

Definition 2.6. Let $\beta \in Z$. The operator P^β is called *uniformly strongly elliptic* if there exists $\gamma > 0$ such that

$$\sum_{ij=1}^{d} \Re\big(a_{ij}(x)\xi_i\xi_j\big) \ \ge\ \gamma\|\xi\|^2\,, \tag{19}$$

for all $\xi = (\xi_i) \in \mathbb{R}^d$ and all $x \in \overline{\Omega}$. Here $\|\cdot\|$ denote the standard euclidean norm on $\mathbb{R}^d$. The largest γ with the property in (19) will be denoted $\gamma_{use}(\beta)$ or $\gamma_{use}(P^\beta)$.

Then, we have the following standard example.

Example 2.7. Let $\beta \in Z$, as in Definition 2.6. We shall regard a matrix $X := [x_{ij}]$, $(X)_{ij} = x_{ij}$, as a linear operator acting on $\mathbb{C}^d$ by the formula $X\zeta = \xi$, where $\xi_i = \sum_j x_{ij}\zeta_j$. We consider the adjoint and positivity with respect to the usual inner product on $\mathbb{C}^d$. We thus have $X \geq 0$ if, and only if $(X\xi, \xi) = \sum_{ij} x_{ij}\xi_j\overline{\xi_i} \geq 0$ for all $\xi \in \mathbb{C}^d$. Also, recall that X^*, the adjoint of the matrix X, has entries $(X^*)_{ij} = \overline{x_{ji}}$. Then P^β is uniformly strongly elliptic if, and only if, there exists $\gamma > 0$ such that the matrix $a(x) := [a_{ij}(x)]$ of highest order coefficients of P^β satisfies

$$a(x) + a(x)^* \, \geq \, 2\gamma I_d\,, \quad \text{for all } x \in \Omega\,, \tag{20}$$

where I_d denotes the unit matrix on $\mathbb{C}^d$. Assume also that $b_i = c = 0$. Then,

$$\begin{aligned} 2\Re(P^\beta u, u) \, &:= \, 2\Re\Big(\int_\Omega \sum_{i,j=1}^{d} a_{ij}(x)\partial_j u(x)\partial_i\overline{u(x)}\,\mathrm{d}x \Big) \, = \, 2\Re(a\nabla u, \nabla u) \\ &= \, (a\nabla u, \nabla u) + (\nabla u, a\nabla u) \, = \, \big((a+a^*)\nabla u, \nabla u\big) \, \geq \, 2\gamma\|\nabla u\|^2_{L^2(\Omega)}\,, \end{aligned}$$

for $u \in H^1_D(\Omega)$. (Recall that $H^1_D(\Omega)$ was defined in Equation (12). In particular, $v = 0$ on $\partial_D\Omega$ if $v \in H^1_D(\Omega)$.)

Remark 2.8. We use the notation of the previous example. If, moreover, $\partial_D\Omega$ has positive measure, then there exists $c = c_{\Omega,\partial_D\Omega} > 0$ such that $\int_\Omega |\nabla v|^2 \mathrm{d}x \geq c\|v\|^2_{H^1(\Omega)}$ for all $v \in H^1_D(\Omega)$, and hence P^β is strongly coercive on $V = H^1_D(\Omega)$. If $\partial_N\Omega$ has not adjacent edges, then P^β is also strongly coercive on $V = \mathcal{K}^1_1(\Omega) \cap H^1_D(\Omega)$, with the norm induced from $\mathcal{K}^1_1(\Omega)$, in view of the Hardy inequality [12,34]. Moreover, we will have $\rho(P^\beta) \geq c\gamma$, with c depending only on the domain Ω.

We then have the following result that is standard for non-weighted spaces (see also [44]).

PROPOSITION 2.9. *If $\beta = (a_{ij}, b_i, c) \in Z$ is such that P^β is strongly coercive on V, $H^1_0(\Omega) \subset V \subset H^1(\Omega)$, with the norm induced from $H^1(\Omega)$ or from $\mathcal{K}^1_1(\Omega)$, then P^β is uniformly strongly elliptic, more precisely, the estimate* (19) *is satisfied for any $\gamma \leq \rho(\beta) := \rho(P^\beta)$. Moreover, $P^\beta : V \to V^*$ is a continuous bijection and $(P^\beta)^{-1}F$ depends analytically on the coefficients β and on $F \in V^*$.*

Proof. The second part is an immediate consequence of the analytic Lax-Milgram Lemma. Let us concentrate then on the first part. Let us assume that the norm on V is the one induced from $\mathcal{K}^1_1(\Omega)$, the case of $H^1(\Omega)$ being completely similar. Let us assume that P^β is strongly coercive and let $\xi =$

$(\xi_i) \in \mathbb{R}^d$. Also, let us choose an arbitrary smooth function ϕ with compact support D in Ω. We then define the function $\psi \in \mathcal{C}_c^\infty(\Omega) \subset V$ by the formula $\psi(x) := e^{\imath t\xi\cdot x}\phi(x) \in \mathbb{C}$, where $\imath := \sqrt{-1}$ and $\xi \cdot x = \sum_{k=1}^d \xi_k x_k$. Then $\partial_j\psi(x) = \imath t\xi_j e^{\imath t\xi\cdot x}\phi(x) + e^{\imath t\xi\cdot x}\partial_j\phi(x)$, and hence $\imath t\xi_j e^{\imath t\xi\cdot x}\phi(x)$ is the dominant term in $\partial_j\psi(x)$ as $t \to \infty$. Taking into account all the indices j and computing the squares of the L^2-norms, we obtain

$$\lim_{t\to\infty} t^{-2}\|\psi\|^2_{\mathcal{K}_1^1(D)} = \sum_{j=1}^d \xi_j^2 \int_D |\phi(x)|^2\,\mathrm{d}x = \|\xi\|^2 \int_D |\phi(x)|^2\,\mathrm{d}x\,. \tag{21}$$

Similarly, the coefficients a_{ij} of P^β, are estimated using "oscillatory testing"

$$\lim_{t\to\infty} t^{-2}(P^\beta\psi,\psi) = \int_D \sum_{i,j=1}^d a_{ij}(x,y)\xi_i\xi_j|\phi(x)|^2\,\mathrm{d}x\,. \tag{22}$$

We then use Definition 2.3 for $v = \psi$ and we pass to the limit as $t \to \infty$. By coercivity and the definition of $\rho(\beta) := \rho(P^\beta)$, we have that $\rho(\beta)\|\psi\|^2_{\mathcal{K}_1^1(D)} \le \Re(P^\beta\psi,\psi)$. Dividing this inequality by t^{-2} and taking the limit as $t \to \infty$, we obtain from Equations (21) and (22) that

$$\rho(\beta)\|\xi\|^2 \int_D |\phi(x)|^2\,\mathrm{d}x \le \Re \int_D \sum_{ij} a_{ij}(x,y)\xi_i\overline{\xi}_j|\phi(x)|^2\,\mathrm{d}x\,.$$

Since ϕ is an arbitrary compactly supported smooth function on D, it follows that, for all $x \in D$,

$$\rho(\beta)\|\xi\|^2 \le \Re \sum_{ij} a_{ij}(x)\xi_i\xi_j\,.$$

Since ξ is arbitrary, we obtain Equation (19) with $\gamma = \rho(P)$. $\square$

An immediate corollary of Proposition 2.9 is

COROLLARY 2.10. *We have* $\rho(\beta) \le \gamma_{use}(\beta)$.

In the following sections, this inequality will be used in the form $\gamma_{use}^{-1}(P^\beta) = \gamma_{use}^{-1}(\beta) \le \rho(\beta)^{-1} := \rho(P^\beta)^{-1}$.

3. POLYGONAL DOMAINS, OPERATORS, AND WEIGHTED SOBOLEV SPACES

In this section, we introduce the domains for our boundary value problems, the weighted Sobolev spaces, and the differential operators that we shall use. We also provide equivalent definitions of the needed weighted Sobolev spaces and prove some intermediate results.

3.1. **Polygonal domains and defining local coordinates**

In this section, we let Ω be a *curvilinear polygonal domain,* although our method works without significant change for domains with conical points.

Let us describe in detail our domain Ω as a Dauge-type corner domain, with the purpose of fixing the notation and of introducing some useful local coordinate systems – called "defining coordinates" – that will be used in the proofs below. Let B_j denote the open unit ball in $\mathbb{R}^j$. Thus $B_0 := \{0\}$ is reduced to one point, $B_1 = (-1, 1)$, and $B_2 = \{(x, y) \subset \mathbb{R}^2, x^2 + y^2 < 1\}$.

Definition 3.1. A *curvilinear polygonal domain* $\Omega \subset \mathbb{R}^2$ is an open, bounded subset of $\mathbb{R}^2$ with the property that, for every point $p \in \overline{\Omega}$, there exists $j \in \{0, 1, 2\}$, a neighborhood U_p of p in $\mathbb{R}^2$, and a smooth map $\phi_p : \mathbb{R}^2 \to \mathbb{R}^2$ that defines a diffeomorphism $\phi_p : U_p \to B_j \times B_{2-j} \subset \mathbb{R}^2$, $\phi_p(p) = 0$, satisfying the following conditions:

(i) If $j = 2$, then $U_p \subset \Omega$;
(ii) If $j = 1$, then $\phi_p(U_p \cap \Omega) = B_1 \times (0, 1)$ or $\phi_p(U_p \cap \Omega) = B_1 \times (B_1 \smallsetminus \{0\})$;
(iii) If $j = 0$, then

$$\phi_p(U_p \cap \Omega) \;=\; \{\, (r\cos\theta, r\sin\theta),\ \text{ with } r \in (0, 1),\, \theta \in I_p \,\}$$

for some finite union I_p of open intervals in S^1.

For $p \in \overline{\Omega}$, we let j_p the *largest* j for which p satisfies one of the above properties.

These are (essentially) the *corner domains* in [25]. The definition above was generalized to arbitrary dimensions in [10]. See also [34, 35, 40, 43]. The second case in (ii) corresponds to cracks in the domain. We continue with some remarks.

Remark 3.2. We notice that in the two cases (i) and (iii) of Definition 3.1 ($j = 2$ and $j = 0$), the spaces $\phi_p(U_p) = B_j \times B_{2-j}$ will be the same (up to a canonical diffeomorphism), but the spaces $\phi_p(U_p \cap \Omega)$ will not be diffeomorphic.

Remark 3.3. Let Ω be a curvilinear polygonal domain and $p \in \overline{\Omega}$. Then p satisfies the conditions of the definition for *exactly* one value of j, *except* the case when p is on a smooth part of the boundary, when a choice of $j = 1$ or $j = 0$ is possible. This is the case exactly when $j_p = 1$. If $j = 0$ is chosen, then I_p is half a circle.

Remark 3.4. The set $\mathcal{V}_g := \{p \in \overline{\Omega} |\, j_p = 0\}$ is finite and is contained in the boundary of Ω. It is the set of *geometric vertices.*

Let us choose for each point $p \in \overline{\Omega}$ a value $j = i_p$ that satisfies the conditions of the definition. If $j_p = 1$, we choose $i_p = j_p = 1$, except possibly for finitely many points $p \in \overline{\Omega}$. These points will be called *artificial vertices*. The set of all vertices (geometric and artificial) is finite, which will be denoted by $\mathcal{V}$, and *will be fixed in what follows*. We assume that all points where the boundary conditions change are in $\mathcal{V}$. We also fix the resulting polar coordinates $r \circ \phi_p$ and $\theta \circ \phi_p$ on U_p, for all $p \in \mathcal{V}$.

Definition 3.5. The coordinate charts $\phi_p : U_p \to B_j \times B_{2-j}$ of Definition 3.1 that were chosen such that $j = i_p$ are called the *defining* coordinate charts of the curvilinear polygonal domain Ω. (Recall that $j = i_p = 0$ if, and only if $p \in \mathcal{V}$.)

Remark 3.6. Artificial vertices are useful, for instance, in the case when we have a change in boundary conditions or if there are point singularities in the coefficients, see [35, 36] and the references therein. The right framework is, of course, that of a stratified space [10], with j_p denoting the dimension of the stratum to which p belongs, but we do not need this in the simple case at hand.

Remark 3.7. It follows from Definition 3.1 that if Ω is a curvilinear polygonal domain, then the set $\partial\Omega \smallsetminus \mathcal{V}$ is the union of finitely many smooth, open curves $e_j : (-1, 1) \to \partial\Omega$. The curves e_j have as image the *open edges* of $\partial\Omega$ and we shall sometimes identify e_j with its image. The curves e_j are disjoint and have no self-intersections. The closure of the image of e_j, for any j, is called a *closed edge*. Thus, the vertices are not contained in the open edges (but they are, of course, contained in the closed edges). Our assumption that all points where the boundary conditions change are in $\mathcal{V}$ implies that $\partial_D\Omega$ consists of a union of closed edges of Ω.

3.2. **Equivalent definitions of weighted spaces**

In this section, we discuss some equivalent definitions of weighted Sobolev spaces. We adapt to our setting the results in [3], to which we refer for more details.

We shall fix, from now on, a *finite* set of defining coordinate charts $\phi_k = \phi_{p_k}$, for some $p_k \in \overline{\Omega}$, $1 \le k \le N$, so that $U_k := U_{p_k}$, $1 \le k \le N$, defines a finite covering of $\overline{\Omega}$. Thus, for $p = p_k$ such that $j_p \neq 0$, the coordinates are $(x, y) \in \mathbb{R}^2$. Otherwise, these coordinates will be denoted by $(r, \theta) \in (0, 1) \times S^1$. We may relabel these points such that p_k is a vertex if, and only if, $1 \le k \le N_0$. We then have the following alternative definition of the weighted Sobolev spaces $\mathcal{K}^m_a(\Omega)$. We denote

$$X_k u := \partial_x u \quad \text{and} \quad Y_k u := \partial_y u\,, \quad \text{for } N_0 < k \le N\,, \tag{23}$$

in the coordinate system defined by $\phi_k = \phi_{p_k} = (x, y) \in \mathbb{R}^2$ that corresponds to one of the chosen points p_k (recall that, for $N_0 < k \leq N$, p_k is *not* a vertex). If, however, p_k is a vertex, then we let

$$X_k u := r\partial_r u \quad \text{and} \quad Y_k u := \partial_\theta u\,, \quad \text{for } 1 \leq k \leq N_0\,, \tag{24}$$

in the (polar) coordinate system defined by $\phi_k = (r, \theta) \in (0,1) \times S^1$. Note the appearance of r in front of ∂_r! Recall that r_Ω equals the distance to the vertices close to the vertices. Thus $r_\Omega = r$ close to the vertex of a straight angle.

Remark 3.8. Assuming that the coefficients of p_β are locally Lipschitz, we can express the differential operator $r_\Omega^2 p_\beta$ in any of the coordinate systems $\phi_k : U_k \to \mathbb{R}^2$. That means that, for each $1 \leq k \leq N$, we can find coefficients $c, c_1, c_2, c_{11}, c_{12}, c_{22}$ such that

$$p_\beta u = (c_{11} X_k^2 + c_{12} X_k Y_k + c_{22} Y_k^2 + c_1 X_k + c_2 Y_k + c) u \quad \text{on } U_k\,, \tag{25}$$

with the vector fields X_k and Y_k introduced in Equations (23) and (24).

For each open subset $U \subset \Omega$, let us denote

$$\|u\|^2_{\mathcal{K}_a^m(U)} := \sum_{|\alpha| \leq m} \|r_\Omega^{|\alpha| - a} \partial^\alpha u\|^2_{L^2(U)}\,. \tag{26}$$

Thus, if $U = \Omega$, $\|u\|_{\mathcal{K}_a^m(U)} = \|u\|_{\mathcal{K}_a^m(\Omega)}$ is simply the norm on $\mathcal{K}_a^m(\Omega)$. Note that the weight r_Ω is not intrinsic to the set U, but depends on Ω, which is nevertheless not indicated in the notation $\|u\|^2_{\mathcal{K}_a^m(U)}$, in order not to overburden it. We define the spaces $\mathcal{W}^{m,\infty}(U)$ similarly as in (5) with the same weight r_Ω. Let $U_k := U_{p_k}$.

PROPOSITION 3.9. *Let $u : \Omega \to \mathbb{C}$ be a measurable function and $U \subset \Omega$ be an open subset. We have that $u \in \mathcal{K}_a^m(U)$ if, and only if, $r_\Omega^{-a} X_k^i Y_k^j u \in L^2(U \cap U_k)$, for all $1 \leq k \leq N$ and all $i + j \leq m$ (recall that $U_k = U_{p_k}$). Moreover, the $\mathcal{K}_a^m(U)$–norm is equivalent to the norm*

$$|||u|||'_U := \sum_{k=1}^{N} \sum_{i+j \leq m} \|r_\Omega^{-a} X_k^i Y_k^j u\|_{L^2(U \cap U_k)}\,.$$

Proof. This follows right away from the definition of the $\mathcal{K}_a^m(U)$-norm. Indeed, away from the vertices, both the $|||\cdot|||'_U$-norm and the $\mathcal{K}_a^m$-norm coincide with the usual H^m-norm. On the other hand, near a vertex, or more generally on an angle $\Xi := \{(r, \theta) | \, \alpha < \theta < \beta\}$, both norms are given by $\sum_{i+j \leq m} \|r^{-a}(r\partial_r)^i \partial_\theta^j u\|_{L^2(\Xi)}$. For the $\mathcal{K}_a^m(U)$-norm this is seen by writing ∂_x and ∂_y in polar coordinates, more precisely, from

$$r\partial_x = (\cos\theta) r\partial_r - (\sin\theta)\partial_\theta \quad \text{and} \quad r\partial_y = (\sin\theta) r\partial_r + (\cos\theta)\partial_\theta\,. \tag{27}$$

See [4, 36] for more details. $\square$

We finally have the following corollary.

COROLLARY 3.10. *The norm* $\|u\|_{\mathcal{K}_a^{m+1}(\Omega)}$ *is equivalent to the norm*

$$|||u||| := \|u\|_{\mathcal{K}_a^m(\Omega)} + \sum_{k=1}^{N} \left(\|X_k u\|_{\mathcal{K}_a^m(U_k)} + \|Y_k u\|_{\mathcal{K}_a^m(U_k)} \right).$$

Proof. In the definition of $|||u|||'$, Proposition 3.9, with m replaced by $m+1$, we collect all the terms with $i+j \leq m$ and notice that they give a norm equivalent to the norm for $\mathcal{K}_a^m$. The rest of the terms will contain at least one differential X_k or one differential Y_k and thus are of the form $\|r_\Omega^{-a} X_k^i Y_k^j Y_k u\|_{L^2(\Omega)}$ or $\|r_\Omega^{-a} X_k^i Y_k^j X_k u\|_{L^2(\Omega)}$, $i+j \leq m$, since the differential operators X_k and Y_k commute on U_k. $\square$

3.3. The differential operators

We include in this subsection the definition of our differential operators and three needed intermediate results (lemmas).

We introduce now our set of coefficients. Recall the norm $\|\beta\|_{Z_m}$ introduced in Equation (6) and let

$$Z_m := \{\, \beta = (a_{ij}, b_i, c), \ \|\beta\|_{Z_m} < \infty \,\}. \tag{28}$$

Note that for example, the Schrödinger operator $-\Delta + r^{-2}$ is an operator of the form P^β for suitable $\beta \in Z_m$.

Below, we shall often use inequalities of the form $A \leq CB$, where A and B are expressions involving u and β and $C \in \mathbb{R}$. We shall say that C is an *admissible bound* if it does not depend on u and β, and then we shall write $A \leq_c B$.

LEMMA 3.11. *Let* $\beta = (a_{ij}, b_i, c) \in Z_m$, $m \geq 1$, *and let us express* p_β *as in Remark 3.8. Then* $c, c_1, c_2, c_{11}, c_{12}, c_{22} \in \mathcal{W}^{m-1,\infty}(U_k)$. *Moreover,*

$$\|c\|_{\mathcal{W}^{m-1,\infty}(U_k)} + \|c_1\|_{\mathcal{W}^{m-1,\infty}(U_k)} + \ldots + \|c_{22}\|_{\mathcal{W}^{m-1,\infty}(U_k)} \ \leq_c \ \|\beta\|_{Z_m}.$$

If p_β *is moreover uniformly strongly elliptic, then* $|c_{22}^{-1}| \leq_c \gamma_{use}^{-1}(\beta)$ *on* U_k.

Proof. We first notice that since $m \geq 1$, we can convert our operator to a non-divergence form operator. Indeed, one can simply replace a term of the form $\partial_i a \partial_j u$ with $a\partial_i\partial_j u + (\partial_i a)\partial_j u$, where $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ and $r_\Omega \partial_i a \in \mathcal{W}^{m-1,\infty}(\Omega)$. We deal similarly with the terms of the form $\partial_i(b_i u)$. This accounts for the loss of one derivative in the regularity of the coefficients of $c, \ldots, c_{22}$.

We need to show that the coefficients $c, \dots, c_{22}$ are in $\mathcal{W}^{m-1,\infty}(\Omega)(U_k)$ and that they have the indicated bounds. To this end, we consider the two possible cases: when U_k contains no vertices of Ω (equivalently, if $k > N_0$) and the case when U_k is centered at a vertex (equivalently, if $k \leq N_0$).

If $k > N_0$, then the coefficients $c, \dots, c_{22}$ can be expressed using the coordinate chart $\phi_k = \phi_{p_k}$ of Definition 3.1 and its derivatives linearly in terms of the coefficients β on the closure of U_k. Since there is a finite number of such neighborhoods and ϕ_k and its derivatives are bounded on the closure of U_k, the bound for the coefficients $c, \dots, c_{22}$ in terms of $\|\beta\|_{Z_m}$ on U_k follows using a compactness argument. In particular, the bound $|c_{22}^{-1}| \leq_c \gamma_{use}^{-1}(\beta)$ follows from the uniform ellipticity of p_β on $\overline{U_k}$.

If, on the other hand, $k \leq N_0$ (that is, U_k is centered at a vertex). Let us concentrate on the highest order terms, for simplicity. We then have, up to lower order terms (denoted $l.o.t$)

$$\begin{aligned} r^2\partial_x^2 &= (\cos\theta)^2(r\partial_r)^2 - 2(\sin\theta\cos\theta)r\partial_r\partial_\theta + (\sin\theta)^2\partial_\theta^2 + l.o.t. \\ r^2\partial_x\partial_y &= (\sin\theta\cos\theta)(r\partial_r)^2 + (\cos^2\theta - \sin^2\theta)r\partial_r\partial_\theta + (\sin\theta\cos\theta)\partial_\theta^2 + l.o.t. \\ r^2\partial_y^2 &= (\sin\theta)^2(r\partial_r)^2 + 2(\sin\theta\cos\theta)r\partial_r\partial_\theta + (\cos\theta)^2\partial_\theta^2 + l.o.t. \end{aligned}$$

The bound on the coefficients $c, \dots, c_{22}$ follows since $\sin\theta$ and $\cos\theta$ are in $\mathcal{W}^{m,\infty}(U_k)$ for all m. This gives also that $c_{22} = a_{11}\cos^2\theta + 2a_{12}\cos\theta\sin\theta + a_{22}\sin^2\theta \geq \gamma_{use}(\beta)$ for the coefficient c_{22} of $Y_k^2 = \partial_\theta^2$. (Thus $|c_{22}^{-1}| \leq \gamma_{use}^{-1}(\beta)$ on U_k, for $k \leq N_0$.) $\square$

For instance, for the Laplacian in polar coordinates, we have

$$r_\Omega^2\Delta = (r\partial_r)^2 + \partial_\theta^2 = X_k^2 + Y_k^2$$

in the neighborhood U_k of the vertex p_k of a straight angle.

The following lemma will be used in the proof of Theorem 4.4 and explains some of the calculations there.

LEMMA 3.12. *For two functions b and c, we have*

(i) $\|bc\|_{\mathcal{K}_a^m(\Omega)} \leq_c \|b\|_{\mathcal{W}^{m,\infty}(\Omega)}\|c\|_{\mathcal{K}_a^m(\Omega)}$.

(ii) $\|bc\|_{\mathcal{W}^{m,\infty}(\Omega)} \leq_c \|b\|_{\mathcal{W}^{m,\infty}(\Omega)}\|c\|_{\mathcal{W}^{m,\infty}(\Omega)}$, *so $\mathcal{W}^{m,\infty}(\Omega)$ is an algebra.*

(iii) *If $b \in \mathcal{W}^{m,\infty}(\Omega)$ and $b^{-1} \in L^\infty(\Omega)$, then $b^{-1} \in \mathcal{W}^{m,\infty}(\Omega)$ and*

$$\|b^{-1}\|_{\mathcal{W}^{m,\infty}(\Omega)} \leq_c C\,\|b^{-1}\|_{L^\infty(\Omega)}^{m+1}\,\|b\|_{\mathcal{W}^{m,\infty}(\Omega)}^m .$$

The parameter C in $\leq_c$ depends only on m and Ω.

Proof. This is a direct calculation. Indeed, the first two relations are based on the rule $\partial^\alpha(bc) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta b\,\partial^{\alpha-\beta}c$. The last one is obtained from the relation $\partial^\alpha(b^{-1}) = b^{-1-|\alpha|}Q$, where $Q = Q(b, \partial_1 b, \partial_2 b, \dots, \partial^\alpha b)$ is a polynomial of degree $|\alpha|$ in all derivatives $\partial^\beta b$, with $0 \le \beta \le \alpha$, where each $\partial^\beta u$ is considered of degree 1. This relation is proved by induction on $|\alpha|$. $\square$

For further reference, we shall need the following version of "Nirenberg's trick," (see, for instance, [2,28]).

LEMMA 3.13. *Let $T : X \to Y$ be a continuous, bijective operator between two Banach spaces X and Y. Let $S_X(t)$ and $S_Y(t)$ be two c_0 semi-groups of operators on X, respectively Y, with generators denoted by A_X and, respectively, A_Y. We assume that for any $t > 0$, there exists $T_t \in \mathcal{L}(X;Y)$ such that $S_Y(t)T = T_t S_X(t)$. Assume that $t^{-1}(T_t - T)$ converges strongly as $t \to 0$ to a bounded operator B. Then T maps bijectively the domain of A_X to the domain of A_Y and we have that $A_X T^{-1}\xi = T^{-1}\big(A_Y\xi - BT^{-1}\xi\big)$, for all ξ in the domain of A_Y. Consequently,*

$$\|A_X T^{-1}\xi\|_X \ \le \ \|T^{-1}\| \left(\|A_Y\xi\|_Y + \|B\|\|T^{-1}\xi\|_X \right).$$

Proof. We have that $\xi \in X$ is in $\mathcal{D}(A_X)$, the domain of the generator A_X of S_X if, and only if, the limit $A_X\xi := \lim_{t\to 0} t^{-1}\big(S_X(t) - 1\big)\xi$ exists. The definition of T_t gives

$$t^{-1}\big(S_Y(t) - 1\big)T\xi \ = \ t^{-1}\big(T_t - T\big)S_X(t)\xi \ + \ t^{-1}T\big(S_X(t) - 1\big)\xi\,.$$

Since $t^{-1}(T_t - T)\zeta \to B\zeta$ for all vectors $\zeta \in X$ and $B : X \to Y$ is bounded, we obtain that the limit $\lim_{t\to 0} t^{-1}\big(S_Y(t) - 1\big)T\xi$ exists if, and only if, the limit $\lim_{t\to 0} t^{-1}\big(S_X(t) - 1\big)\xi$ exists. This shows that T maps bijectively the domain of A_X to the domain of A_Y and that $A_Y T = B + TA_X$ as unbounded operators with domain $\mathcal{D}(A_X)$. Multiplying by T^{-1} to the left and to the right gives the desired result. $\square$

One can use Lemma 3.13 as a regularity estimate.

4. HIGHER REGULARITY IN WEIGHTED SOBOLEV SPACES

In this section, we prove our main result, Theorem 4.4. Theorem 1.1 is then an immediate consequence of this theorem and of Remark 4.3. Recall that $r_\Omega : \Omega \to [0, \infty)$ denotes a continuous function, smooth and > 0 outside the vertices, such that r_Ω is the distance to the vertices, close to the vertices.

4.1. The higher regularity problem

We now come back to the study of our mixed problem, as formulated in Equation (9). We are interested in solutions with more regularity than the ones provided by the space V appearing in its weak formulation, Equation (13) or Equation (15). While for the weak formulation the classical Sobolev spaces suffice, the higher regularity is formulated in the framework of the weighted Sobolev spaces considered by Kondratiev [33] and others, see also [23, 24].

We assume from now on that $V := \{u \in \mathcal{K}_1^1(\Omega),\ u = 0 \text{ on } \partial_D\Omega\}$ and that it has the induced norm. We then introduce

$$V_m(a) := \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \quad \text{for} \quad m \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \quad \text{and}$$
$$V_m^-(a) := \mathcal{K}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega) \quad \text{for} \quad m \in \mathbb{N} = \{1, 2, \ldots\}.$$

In particular, $V_m(a) = \mathcal{K}_{a+1}^{m+1}(\Omega) \cap r_\Omega^a V$. The spaces $\mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)$, $m \geq 1$, are the spaces of traces of functions in $\mathcal{K}_a^m(\Omega)$, in the sense that the restriction at the boundary defines a continuous, surjective map $\mathcal{K}_a^m(\Omega) \to \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega)$ [4]. The space $\mathcal{K}_a^m(\partial_N\Omega)$ can be defined directly for $m \in \mathbb{Z}_+$ in a manner completely analogous to the usual Kondratiev spaces. For non-integer regularity, they can be obtained by interpolation, [3, 4].

We recall that for all $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$, the differentiation defines continuous maps $\partial_j : \mathcal{K}_a^m(\Omega) \to \mathcal{K}_{a-1}^{m-1}(\Omega)$. In the same way, the combination of the normal derivative at the boundary $(\partial_\nu^\beta v) := \sum_{i=1}^d \nu_i \big(\sum_{j=1}^d a_{ij}\partial_j v + b_{d+i} v\big)$ and restriction at the boundary define a continuous, surjective map $\partial_\nu^\beta : \mathcal{K}_a^m(\Omega) \to \mathcal{K}_{a-3/2}^{m-3/2}(\partial_N\Omega)$, $m \geq 2$.

Lemma 4.1. *We have continuous maps*

$$P^\beta(m, a) := (p_\beta, \partial_\nu^\beta) : V_m(a) \to V_m^-(a), \quad m \geq 1,$$
$$P^\beta(m, a)(u) = \Big(\sum_{ij} \partial_i(a_{ij}\partial_j u) + \sum_i b_i \partial_i u + cu, \sum_{ij} \nu_i a_{ij} \partial_j u|_{\partial_N\Omega} \Big).$$

Therefore the operators $P^\beta(m, a)$, $m \in \mathbb{N}$, $a \in \mathbb{R}$, are given by the same formula (but have different domains and ranges).

Remark 4.2. Let us assume for this remark that $a = 0$ and discuss this case in more detail. If $\partial_N\Omega$ contains no adjacent edges, then the Hardy inequality [12, 34] shows that the natural inclusion

$$\mathcal{K}_1^1(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \to H_D^1(\Omega) := H^1(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \tag{29}$$

is an isomorphism (that is, it is continuous with continuous inverse). We thus consider $V := V_0(0)$ in general (for all $\partial_N\Omega$, that is, even if it contains adjacent Neumann edges). For symmetry, we also let $V_0^-(0) := V^*$ and

$$P^\beta(0, 0) := P^\beta : V_0(0) = V \to V_0^-(0) := V^*, \tag{30}$$

which is, of course, nothing but the operator studied before.

We then have

$$V_{m+1}(0) \subset V_m(0) \quad \text{and} \quad V_{m+1}^-(0) \subset V_m^-(0) \quad \text{for all } m \geq 0\,.$$

This is trivially true for $m > 0$. For $m = 0$, in which case we need to construct the natural inclusion $\Phi : V_m^-(0) \to V_0^-(0)$, $m \geq 1$. The map Φ associates to $(f, h) \in V_m^-(0) := \mathcal{K}_{-1}^{m-1}(\Omega) \oplus \mathcal{K}_{-1/2}^{m-1/2}(\partial_N \Omega)$ the linear functional $F := \Phi(f, h)$ on V, $F \in V^*$ defined by the formula

$$F(v) \;=\; \Phi(f,h)(v) \;:=\; \int_\Omega f\, v\, \mathrm{d}x + \int_{\partial_N\Omega} h\, v\, dS\,, \tag{31}$$

where $\mathrm{d}x$ is the volume element on Ω and dS is the surface element on $\partial\Omega$. With this definition of the inclusion $\Phi : V_m^-(0) \to V_0^-(0) := V^*$, we obtain that $P^\beta(m, 0)$ is the restriction of $P^\beta(0, 0)$ to $V_m(0)$. In other words, we have the commutative diagram

$$\begin{array}{ccc} V_m(0) & \xrightarrow{P^\beta(m,0)} & V_m^-(0) \\ \downarrow & & \downarrow \\ V_0(0) := V & \xrightarrow{P^\beta(0,0):=P^\beta} & V_0^-(0) \end{array} \tag{32}$$

with the operators P^β introduced in Lemma 4.1 and in Equation (30).

See also Remark 2.1. We now return to the general case $a \in \mathbb{R}$.

Remark 4.3. We then have

$$V_m(a) \;=\; r_\Omega^a V_m(0) \quad \text{for} \quad m \geq 0 \quad \text{and} \quad V_m^-(a) \;=\; r_\Omega^a V_m^-(0) \quad \text{for } m > 0\,.$$

We then let

$$V_0^-(a) \;:=\; r_\Omega^a V_0^-(0) \;=\; r_\Omega^a V^*\,.$$

By symmetry, we obtain

$$V_{m+1}(a) \subset V_m(a) \quad \text{and} \quad V_{m+1}^-(a) \subset V_m^-(a) \quad \text{for all } m \geq 0\,, \tag{33}$$

in general (for all a). In fact, the relation between the spaces above for different values of a allows us to reduce to the case $a = 0$ since, if $\beta \in Z_m$, then there exists $\beta(a) \in Z_m$ such that

$$P^\beta(m, a) \;=\; r_\Omega^a P^{\beta(a)}(m, 0) r_\Omega^{-a}\,, \quad m \geq 1\,. \tag{34}$$

This can be seen from $r^a \partial_j (r^{-a} u) = \partial_j u - r^{-1}(a x_j r^{-1})u$ and $r^{-1}x_j \in \mathcal{W}^{m,\infty}(\Omega)$ for all m, which then gives

$$\begin{aligned} r^a \partial_j \partial_k (r^{-a} u) \;&=\; \partial_j \partial_k u + r^{-1}\phi \partial_k u + \partial_j (r^{-1}\psi u) + r^{-2}\phi\psi u \\ &= \partial_j \partial_k u + r^{-1}\phi\partial_k u + r^{-1}\psi\partial_j + r^{-2}(r\partial_j\psi + r^{-1}x_j\psi + \phi\psi)u\,, \end{aligned} \tag{35}$$

where $\phi := -ar^{-1}x_j$ and $\psi := -ar^{-1}x_k$. In particular,

$$\beta(a) = \beta + a\beta_1 + a^2\beta_2 \tag{36}$$

with $\beta_1, \beta_2 \in Z_m$ depending linearly and continuously on $\beta \in Z_m$. (This explains why it is crucial to consider coefficients in weighted spaces of the form $\mathcal{W}^{m,\infty}(\Omega)$ as well as in terms of the form $\partial_i(b_i u)$ in the definition of p_β.) We use Equation (34) to define $P^\beta(0,a)$ for all a. Of course, $P^\beta(0,0) = P^\beta : V \to V^*$.

As mentioned in the introduction, a less common example for Theorems 1.1 and 4.4 is the Schrödinger operator $H := -\Delta + cr_\Omega^{-2}$, $c > 0$, on Ω with *pure Neumann* boundary conditions, hence $V := \mathcal{K}_1^1(\Omega)$ in this example. See also Theorem 5.4.

Our *higher regularity problem* is then to establish conditions for $P^\beta(m,a)$ to be an isomorphism, which is achieved in Theorem 4.4.

4.2. **Extension of Theorem 1.1 and its proof**

For its proof, it will be convenient to extend the differential operators X_k, Y_k from U_k to the whole domain Ω. We choose these extensions so that

(i) If p_k is a vertex, then all X_j, Y_j, $j \neq k$, vanish close to p_k.

(ii) For all k, X_k (regarded as a vector field) is tangent to all edges (if X_k vanishes at a point on an edge, it is considered to be tangent to the edge at that point).

Recall that $\rho(P) := \inf_{v \neq 0} \Re(Pv, v)/\|v\|_V$, for any linear map $P : V \to V^*$, that $\rho(\beta) := \rho(P^\beta)$, and that $\gamma_{use}^{-1}(\beta) \leq \rho(\beta)^{-1}$, by Corollary 2.10. Also, recall that $\beta(a)$ is given by Equation (34).

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear polygonal domain and $m \in \mathbb{Z}_+$. There exist $C_m > 0$ and $N_m \geq 0$ such that, if $\beta = (a_{ij}, b_i, c) \in Z_m$ and $P^\beta : V \to V^*$ is strongly coercive, then $P^\beta(m,0) : V_m(0) \to V_m^-(0)$ is invertible and*

$$\|P^\beta(m,0)^{-1}\|_{\mathcal{L}(V_m^-;V_m)} \leq C_m \rho(\beta)^{-N_m-1} \|\beta\|_{Z_m}^{N_m}.$$

Proof. Since the statement is for $a = 0$, we shall write $V_m(0) = V_m$ and $V_m^-(0) = V_m^-$. We shall also denote $\|(P^\beta)^{-1}\|_m := \|(P^\beta)^{-1}\|_{\mathcal{L}(V_m^-;V_m)}$.

For $m = 0$, we can just take $C_0 = 1$ and $N_0 = 0$ and then the result reduces to the Lax-Milgram Lemma (see Lemma 2.5). In general, we adapt to our setting the classical method based on finite differences (see for example [21, 28, 39]), which was used in similar settings in [12, 13, 18, 37, 44] and many other papers. We thus give a summary of the argument. For simplicity, we drop Ω from the notation of the norms. In this proof, as throughout the paper,

C is a parameter that is independent of β or F, and hence it depends only on Ω, $\partial_N\Omega$, m, and the choice of the vector fields X_k and Y_k (and of their initial domains U_k), but not on F, u, β. We shall usually write $A \leq_c B$ instead of $A \leq CB$, if C is such a bound.

Let us notice that $\|P^\beta(m,0)\|\|P^\beta(m,0)^{-1}\| \geq \|P^\beta(m,0)P^\beta(m,0)^{-1}\| \geq 1$. Since $\|P^\beta(m,0)\|_m \leq_c \|\beta\|_{Z_m}$, we have that

$$\|\beta\|_{Z_m}\|(P^\beta)^{-1}\|_m \geq 1/C > 0\,.$$

When $m = 0$, we also have $\rho(\beta)^{-1} \geq \|(P^\beta)^{-1}\| =: \|(P^\beta)^{-1}\|_0$, and hence

$$R(\beta) := \|\beta\|_{Z_m}\rho(\beta)^{-1} \geq \|\beta\|_{W^{0,\infty}}\|(P^\beta)^{-1}\|_0 \geq 1/C > 0\,. \tag{37}$$

To show that the operator $P^\beta(m,0) : V_m(0) \to V_m^-(0)$ is invertible and to obtain estimates on $\|(P^\beta)^{-1}\|_m := \|P^\beta(m,0)\|$, we proceed by induction on m. As we have explained above, for $m = 0$, this has already been proved. We thus assume that $P^\beta(m-1,0)$ is invertible and that it satisfies the required estimate, which we write as

$$\|(P^\beta)^{-1}\|_{m-1} := \|P^\beta(m-1,0)^{-1}\|_{\mathcal{L}(V_m^-;V_m)} \leq C_{m-1}\frac{R(\beta)^{N_{m-1}}}{\rho(\beta)}\,. \tag{38}$$

Let $F \in V_m^-$ be arbitrary but fixed. We know by the induction hypothesis that $u := (P^\beta)^{-1}F = P^\beta(m-1,0)^{-1}F \in V_{m-1}$, but we need to show that it is in fact in V_m and to estimate its norm in terms of $\|F\|_{V_m^-}$. Recall that $V := \{u \in \mathcal{K}_1^1(\Omega),\ u = 0 \text{ on } \partial_D\Omega\}$. Since $V_m := \mathcal{K}_1^{m+1}(\Omega) \cap V$, it is enough to show that $u \in \mathcal{K}_1^{m+1}(\Omega)$ and to estimate $\|u\|_{\mathcal{K}_1^{m+1}} = \|(P^\beta)^{-1}F\|_{\mathcal{K}_1^{m+1}}$ (recall that we drop Ω from the notation of our norms).

First of all, by Corollary 3.10, it is enough to estimate $\|X_k u\|_{\mathcal{K}_1^m}$ and $\|Y_k u\|_{\mathcal{K}_1^m}$. Indeed,

$$\|u\|_{\mathcal{K}_1^{m+1}} \leq_c \|u\|_{\mathcal{K}_1^m} + \sum_{k=1}^N \|X_k u\|_{\mathcal{K}_1^m(U_k)} + \sum_{k=1}^N \|Y_k u\|_{\mathcal{K}_1^m(U_k)}\,, \tag{39}$$

and the first term on the right hand side is estimated by induction on m by

$$\|u\|_{\mathcal{K}_1^m} \leq \frac{C_{m-1}R(\beta)^{N_{m-1}}}{\rho(\beta)}\|F\|_{V_{m-1}^-} \leq \frac{C_{m-1}R(\beta)^{N_{m-1}}}{\rho(\beta)}\|F\|_{V_m^-}\,. \tag{40}$$

Let us estimate now the remaining terms in the sum appearing on the right hand side of the inequality (39). Note that these terms are norms that are computed on smaller subsets $U_k \subset \Omega$. First, since X_k is tangent to all edges of Ω, it integrates to a one parameter family of diffeomorphisms of Ω, and hence to strongly continuous one-parameter groups of continuous operators on $X := V_{m-1}$ and $Y := V_{m-1}^-$, due to the particular form of boundary conditions used to define these spaces. Let us denote by $S_X(t) : X \to X$ and $S_Y(t) : Y \to Y$, $t \in \mathbb{R}$, the operators defining these one-parameter groups of operators. We have that

$$B := X_k P^\beta - P^\beta X_k = \lim_{t\to 0} t^{-1}(S_X(t) P^\beta S_Y(-t) - P^\beta) = P^{\beta'} ,$$

and hence $\beta' \in Z_{m-1}$ by 3.11. Therefore $B : X := V_{m-1} \to Y := V_{m-1}^-$ is bounded by Lemma 4.1. The assumptions of Lemma 3.13 are therefore satisfied. Moreover, $\|B\| \leq_c \|\beta'\|_{Z_{m-1}} \leq_c \|\beta\|_{Z_m}$, which allows us to conclude that

$$\|X_k u\|_{\mathcal{K}_1^m} \leq_c \|(P^\beta)^{-1}\|_{m-1}\big(\|X_k F\|_{V_{m-1}^-} + \|\beta\|_{Z_m}\|(P^\beta)^{-1}\|_{m-1}\|F\|_{V_{m-1}^-}\big) ,$$

which gives

$$\|X_k u\|_{\mathcal{K}_1^m} \leq_c \|(P^\beta)^{-1}\|_{m-1}\big(1 + \|(P^\beta)^{-1}\|_{m-1}\|\beta\|_{Z_m}\big)\|F\|_{V_m^-} .$$

Using the definition of $R(\beta)$, the induction estimate of Equation (38), and the relation $\|\beta\|_{Z_m}\|P^\beta(m-1,0)^{-1}\| \geq 1/C$ of Equation (37), we obtain

$$\|X_k u\|_{\mathcal{K}_1^m} \leq_c \frac{R(\beta)^{2N_{m-1}+1}}{\rho(\beta)}\|F\|_{V_m^-}. \tag{41}$$

We now turn to the study of the terms $\|Y_k u\|_{\mathcal{K}_1^m}$, for which we need to use the strong ellipticity of P^β (as in the classical methods [28, 39]) together with Lemmas 3.11 and 3.12. First of all, Lemma 3.11 provides us with the decomposition $c_k Y_k^2 u = r_\Omega^2 P^\beta u - Q_k u$, where $c_k \in \mathcal{W}^{m-1,\infty}(U_k)$ and Q_k is a sum of differential operators of the form $Y_k X_k$ and X_k^2 and lower order differential operators generated by X_k and Y_k with coefficients in $\mathcal{W}^{m-1,\infty}(U_k)$. This gives using first the general form of the $\|\cdot\|_{\mathcal{K}_1^m(U_k)}$-norm (recall that X_k and Y_k commute on U_k)

$$\begin{aligned} \|Y_k u\|_{\mathcal{K}_1^m(U_k)} &\leq_c \|Y_k u\|_{\mathcal{K}_1^{m-1}} + \|X_k Y_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k^2 u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\leq_c \|u\|_{\mathcal{K}_1^m} + \|Y_k X_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k^2 u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\leq_c \|u\|_{\mathcal{K}_1^m} + \|X_k u\|_{\mathcal{K}_1^m} + \|c_k^{-1}(r_\Omega^2 p_\beta - Q_k)u\|_{\mathcal{K}_1^{m-1}(U_k)} . \end{aligned} \tag{42}$$

The first term in the last line of Equation (42) is estimated by the induction hypothesis in Equation (40). The second one is estimated in Equation (41). To estimate the third term, we obtain directly from Lemma 3.11 the following

(1) each $c_k \in \mathcal{W}^{m-1,\infty}(U_k)$ is bounded in terms of $\|\beta\|_{Z_m}$,
(2) the coefficients of X_k^2, $X_k Y_k$, X_k, and Y_k and the free term of Q_k (which is no longer in divergence form) are in $\mathcal{W}^{m-1,\infty}(U_k)$ and are also bounded in terms of $\|\beta\|_{Z_m}$,
(3) $\|c_k^{-1}\|_{L^\infty} \leq_c \gamma_{use}^{-1}(\beta) \leq_c \rho(\beta)^{-1}$, by the uniform strong ellipticity of p_β.

Hence

$$\|c_k^{-1}\|_{\mathcal{W}^{m-1,\infty}(U_k)} \le_c \|c_k^{-1}\|^m_{L^\infty(U_k)}\|c_k\|^{m-1}_{\mathcal{W}^{m-1,\infty}(U_k)} \le_c \rho(\beta)^{-m}\|\beta\|^{m-1}_{\mathcal{W}^{m,\infty}} = \rho(\beta)^{-1}R(\beta)^{m-1}, \tag{43}$$

where the first inequality is by Lemma 3.12(iii).

We have, successively

$$\|r_\Omega^2 p_\beta u\|_{\mathcal{K}_1^{m-1}(U_k)} \le_c \|p_\beta u\|_{\mathcal{K}_{-1}^{m-1}(U_k)} \le_c \|p_\beta u\|_{\mathcal{K}_{-1}^{m-1}} \le_c \|F\|_{V_{m-1}^-}. \tag{44}$$

Similarly, let n_Q be the $\mathcal{W}^{m-1,\infty}(U_k)$ norm of the coefficients of Q_k, then $n_Q \le_c \|\beta\|_{\mathcal{W}^{m,\infty}}$ and hence

$$\begin{aligned}\|Q_k u\|_{\mathcal{K}_1^{m-1}(U_k)} &\le_c n_Q \Big(\|X_k^2 u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k X_k u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\quad + \|X_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|u\|_{\mathcal{K}_1^{m-1}(U_k)}\Big) \\ &\le_c \|\beta\|_{Z_m}\Big(\|X_k u\|_{\mathcal{K}_1^m} + \|u\|_{\mathcal{K}_1^m}\Big) \\ &\le_c (R(\beta)^{2N_{m-1}+2} + R(\beta)^{N_{m-1}+1})\|F\|_{V_{m-1}^-} \le_c R(\beta)^{2N_{m-1}+2}\|F\|_{V_{m-1}^-},\end{aligned} \tag{45}$$

where we have used also Equations (40) and (41). Consequently,

$$\begin{aligned}\|c_k^{-1}(r_\Omega^2 p_\beta - Q_k)u\|_{\mathcal{K}_1^{m-1}(U_k)} &\le_c \|c_k^{-1}\|_{\mathcal{W}^{m-1,\infty}}\|r_\Omega^2 p_\beta u - Q_k u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\le_c \frac{R(\beta)^{m-1}}{\rho(\beta)}(1 + R(\beta)^{2N_{m-1}+2})\|F\|_{V_{m-1}^-} \le_c \frac{R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_{m-1}^-}.\end{aligned} \tag{46}$$

Substituting back into Equation (42) the estimates of Equations (40), (41), and (46) for the respective terms, and then using Equation (37), we obtain

$$\|Y_k u\|_{\mathcal{K}_1^m(U_k)} \le_c \frac{R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_m^-}. \tag{47}$$

In a completely analogous manner, substituting back into Equation (39) the estimates of Equations (40), (41), and (47), we obtain

$$\|u\|_{\mathcal{K}_1^{m+1}} \le_c \frac{R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_m^-}. \tag{48}$$

In all the statements above, saying $\|v\|_Z < \infty$ for some Banach space Z means, implicitly, that $v \in Z$. We thus have that $u \in \mathcal{K}_1^{m+1}$ and that it satisfies the required estimate with $N_m = 2N_{m-1} + m + 1$. The proof is complete. □

We now record the obvious modification needed to deal with the additional parameter a.

Corollary 4.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear polygonal domain, $a \in \mathbb{R}$, $m \in \mathbb{Z}_+$. There exist $C_m > 0$ and $N_m \geq 0$ such that, if $\beta = (a_{ij}, b_i, c) \in Z_m$ and $P^{\beta(a)} : V \to V^*$ is strongly coercive, then $P^\beta(m,a) : V_m(a) \to V_m^-(a)$ is invertible and*

$$\|P^\beta(m,a)^{-1}\|_{\mathcal{L}(V_m^-;V_m)} \ \leq \ C_m \rho(\beta(a))^{-N_m-1} \, \|\beta(a)\|_{Z_m}^{N_m} .$$

Proof. Because $\beta(a)$ depends analytically on β, in view of Remark 4.3 and of the relation in Equation (34), we can just substitute $\beta(a)$ for β in Theorem 4.4. □

Remark 4.6. Remark 4.3 gives that there exists γ_1 such that $\rho(\beta(a)) \geq \rho(\beta) - \gamma_1 |a| \|\beta\|_{Z_0}$, for a in a bounded interval. Moreover, an induction argument gives that $N_m = 2^{m+2} - m - 3 \geq 0$ in two dimensions. We ignore if this is true in higher dimensions as well.

See also [5, 20, 29, 31, 32, 38] for extensions and related results.

5. EXTENSIONS AND APPLICATIONS

We conclude with a few corollaries and extensions of our previous results. For simplicity, we formulate them only in the case $a = 0$, since Remark 4.3 allows us to reduce to the case $a = 0$. Throughout this section, we continue to assume that $\beta = (a_{ij}, b_i, c) \in Z_m$ and that Ω is a bounded, curvilinear polygonal domain with $\partial_D \Omega$ nonempty.

5.1. Corollaries of Theorem 4.4

Recall that $\mathcal{L}(V;V^*)_c$ denotes the set of strongly coercive operators and that we regard $Z \subset \mathcal{L}(V;V^*)$ with the induced topology. In particular, $\mathcal{L}(V;V^*)_c \cap Z$ denotes the set of coefficients that yield a strongly coercive operator.

Corollary 5.1. *Let $U := \mathcal{L}(V;V^*)_c \cap Z_m$. Then U is an open subset of Z_m and the map $U \times V_m^- \ni (\beta, F) \to (P^\beta)^{-1} F \in V_m$ is analytic. Moreover, there exist $C_m > 0$ and $N_m \geq 0$ such that*

$$\|(P^\beta)^{-1} F\|_{V_m} \ \leq \ C_m \frac{\|\beta\|_{Z_m}^{N_m}}{\rho(\beta)^{N_m+1}} \|F\|_{V_m^-} , \quad (\forall) \beta \in U, \ F \in V_m^- .$$

Proof. Recall that $\mathcal{L}(V;V^*)_c$ is open in $\mathcal{L}(V;V^*)$ and that the map $Z_m \to \mathcal{L}(V;V^*)_c$ is continuous. Therefore $U := \mathcal{L}(V;V^*)_c \cap Z_m$ is open in Z_m. Next we proceed as in Lemma 2.5 using that the map $\Phi : U \times V_m^- \to V_m$, $\Phi(\beta, F) := (P^\beta)^{-1}F$ is the composition of the maps

$$U \times V_m^- \ni (\beta, F) \to (P^\beta, F) \in \mathcal{L}_{inv}(V_m; V_m^-) \times V_m^- ,$$
$$\mathcal{L}_{inv}(V_m; V_m^-) \times V_m^- \ni (P, F) \to (P^{-1}, F) \in \mathcal{L}(V_m^-; V_m) \times V_m^- , \quad \text{and}$$
$$\mathcal{L}(V_m^-; V_m) \times V_m^- \ni (Q, F) \to QF \in V_m .$$

The first of these three maps is well defined by Theorem 4.4. Since it is linear, it is also analytic. The other two maps are analytic by Lemma 2.2. Since the composition of analytic functions is analytic, the result follows. □

The following result is useful in approximating solutions of parametric problems when one has uniform measures. (Note however that the estimates in Theorem 4.4 provide errors that are integrable with respect to lognormal measures.) We keep the notation in the last corollary.

COROLLARY 5.2. *Let Y be a Banach space and let $U \subset Y$ be an open subset. Let $F : U \to V_m^-$ and $\beta : U \to \mathcal{L}(V; V^*)_c \cap \mathcal{W}^{m,\infty}(\Omega)$ be analytic functions. Then $U \ni y \to (P^{\beta(y)})^{-1}F(y) \in V_m$ is analytic and*

$$\|(P^{\beta(y)})^{-1}F(y)\|_{V_m} \;\le\; C_m \, \frac{\|\beta(y)\|_{Z_m}^{N_m}}{\rho(\beta(y))^{N_m+1}} \, \|F(y)\|_{V_m^-} .$$

In particular, if the functions $\|\beta(y)\|_{\mathcal{W}^{m,\infty}(\Omega)}$ and $\|F(y)\|_{V_m^-}$ are bounded and *there exists $c > 0$ such that $\rho(\beta(y)) > c$, then $(P^{\beta(y)})^{-1}F(y)$ is a bounded analytic function.*

Proof. The composition of two analytic functions is analytic. The first part is therefore an immediate consequence of the first part of Corollary 5.1. The second part follows also from Corollary 5.1. □

The method used to obtain analytic dependence of the solution in terms of coefficients can be extended to other settings.

Remark 5.3. Let us assume the following:

(i) We are given continuously embedded Banach spaces $W_D^{m+1} \subset V \subset H^1(\Omega)$, $\breve{W}^{m-1} \subset V^*$, and $\mathcal{Z} \subset Z_m$ satisfying the following properties:

(ii) For any $\beta \in \mathcal{Z}$, the operator P^β defines continuous maps $V \to V^*$ and $W_D^{m+1} \to \breve{W}^{m-1}$.

(iii) $\|P^\beta\|_{\mathcal{L}(W_D^{m+1}; \breve{W}^{m-1})} \le_c \|\beta\|_{\mathcal{Z}}$ and $\|P^\beta\|_{\mathcal{L}(V; V^*)} \le_c \|\beta\|_{\mathcal{Z}}$.

(iv) If $\beta \in \mathcal{Z}$ and $P^\beta : V \to V^*$ is strongly coercive, then the map $(P^\beta)^{-1} : V^* \to V$ maps $\breve{W}^{m-1}$ to W_D^{m+1} continuously and there exists a continuous, increasing function $\mathfrak{N}_m : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

$$\|(P^\beta)^{-1}\|_{\mathcal{L}(\breve{W}^{m-1}; W_D^{m+1})} \;\le\; \mathfrak{N}_m\big(\rho(\beta)^{-1} ,\, \|\beta\|_{Z}\big) .$$

Then our previous results (in particular, Corollaries 5.1 and 5.2) extend to the new setting by replacing $\mathcal{W}^{m,\infty}(\Omega)$ with Z, V_m with W_D^{m+1}, V_m^- with $\breve{W}^{m-1}$, and by using $\mathfrak{N}_m$ in the bounds for the norm. We thank Markus Hansen and Christoph Schwab for their input related to this remark.

5.2. General domains with conical points

The same argument as in the proof of Theorem 4.4 gives a proof of a similar result on general domains with conical points. In the neighborhood of a conical point, the domain is of the form $\Omega = \{rx', 0 < r < 1, x' \in \omega\}$, where $\omega \subset S^{n-1}$ is a smooth domain on the unit sphere S^{n-1}. The main difference is that we will need to additionally straighten the boundary of ω.

5.3. Dirichlet and Neumann boundary conditions

We conclude this paper by an application of Theorem 4.4 to estimates for Schrödinger operators. We note that the following result applies to arbitrary mixed boundary conditions (including pure Neumann).

THEOREM 5.4. *Let $P^\beta u = -\sum_{ij=1}^d \partial_i(a_{ij}\partial_j u) + \frac{c}{r_\Omega^2}u$, $c \geq 0$, be a strongly elliptic operator (so $b_i = 0$). In case $p \in \mathcal{V} \subset \partial\Omega$ is a vertex that belongs to two adjacent Neumann edges, we assume that $c(p) > 0$. Then P^β is strongly coercive on $V := \{u \in \mathcal{K}_1^1(\Omega),\, u = 0 \text{ on } \partial_D\Omega\}$. Moreover,*

$$P^\beta : \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \to \mathcal{K}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N\Omega) \tag{49}$$

is an isomorphism and its inverse has norm

$$\|(P^\beta)^{-1}\| \;\leq\; C_m\,\rho(\beta)^{-N_m-1}\|\beta\|^{N_m}\,, \quad |a| \leq 1\,,$$

with C_m and N_m as in Theorem 4.4 (hence independent of β and F).

Acknowledgments. We thank Martin Costabel, Monique Dauge, Markus Hansen, Serge Nicaise, and Christoph Schwab for useful discussions. We dedicate this paper to Monique Dauge for her many contributions to Scientific Computing and Analysis on singular domains.

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Received 22 May 2016

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