ANALYSIS AND NUMERICAL APPROXIMATION OF A FRICTIONAL CONTACT PROBLEM WITH ADHESION

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Communicated by Dorin Ieşan

In this paper, we study a quasistatic contact problem for viscoelastic materials with long-term memory. The contact is modelled with a coupled system of a general nonlocal friction law and an ordinary differential equation which describes the adhesion effect. Under sufficient assumptions, we provide a weak formulation of the mechanical problem and establish the existence and uniqueness of a weak solution. The proof is based on arguments of elliptic variational inequalities, a version of Cauchy-Lipschitz theorem and the Banach fixed point theorem. We then introduce a fully discrete scheme based on the finite element method to the case of Tresca's friction law. The convergence of the scheme is established, error estimates are derived and under suitable regularity assumptions a linear convergence result is deduced.

AMS 2010 Subject Classification: 47J20, 49J40, 74M10, 74M15.

Key words: adhesion, quasistatic process, viscoelastic materials, long-term memory, general nonlocal friction law, bilateral, Tresca's friction law, Finite Element Method.

1. INTRODUCTION

Adhesive contact between bodies appears in many applications of solid mechanics where parts, usually nonmetallic, are glued together. For this reason, adhesive contact problems have recently received increased attention in the mathematical literature. An early attempt to study models of contact with adhesion, based on thermodynamical consideration, was done in [12, 13, 14]. Analysis and numerical simulations of adhesive contact with or without friction can be found in [4, 5, 9, 18, 20, 25] and references therein. The main new idea in these papers is the introduction of an internal variable β, the bonding field, defined on the contact surface, which has values between zero and one and which describes the fractional density of active bonds on the contact surface. When $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ partial adhesion takes place. The unilateral quasistatic contact problem with local friction and adhesion was studied in [9]; an existence result, for a friction coefficient small enough, was established. In [5] the dynamic frictionless adhesive contact problem, with normal compliance condition, was modelled and analyzed when the material is linearly viscoelastic. This paper is a continuation and an extension of [1] and [25]. There, the constitutive law was assumed to be nonlinear viscoelastic with short memory. In [1] the bilateral contact problem with a general nonlocal friction law was studied and numerical analysis of the particular problem with Tresca's friction law was included. In [25], the quasistatic adhesive contact problem with Tresca's friction law was investigated when the rate of the bonding field is assumed to be irreversible. The novelty of the present paper consists in dealing with a quasistatic viscoelastic problem modeling the adhesive frictional bilateral contact in which the evolution of the adhesion field is described by a general function which may change sign and allows for rebonding after debonding took place. Moreover, the viscoelastic constitutive law is assumed to be nonlinear with long-term memory and the friction is described by a nonlocal version of Coulomb's law of dry friction. We derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain regularity results for the solution. The proof is based on arguments of time-dependent elliptic variational inequalities, a version of Cauchy-Lipschitz theorem and the Banach fixed point theorem. We then propose a fully discrete numerical scheme for the model in the particular case of Tresca's friction law. We deduce error estimates and under suitable regularity hypothesis we derive a linear convergence result.

The rest of this paper is organized as follows. In Section 2, we present the notation and some preliminaries we shall use in our study. Section 3 is dedicated to describe the mechanical problem and derive its variational formulation. The main existence and uniqueness theorem is established in Section 4. Finally in Section 5, we perform numerical analysis of the particular problem with Tresca's friction law.

2. NOTATION AND PRELIMINARIES

Here we introduce the notation and some preliminaries we shall use later. Let $\Omega \subset \mathbb{R}^d$ (d=2,3) be a bounded domain with regular boundary Γ. We assume that Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. Here and below $\mathbb{R}^d = \{x = (x_1, ..., x_d),\ x_i \in \mathbb{R},\ 1 \leq i \leq d\}$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v = \sum_{i=1}^{d} u_i v_i,\ \ |u| = \sqrt[2]{u \cdot u}, \forall u,\ v\ \in \mathbb{R}^d;$$

$$\sigma \cdot \xi = \sum_{1 \le i,j \le d} \sigma_{ij}\xi_{ij},\ |\sigma| = \sqrt[2]{\sigma \cdot \sigma},\ \forall \sigma,\ \xi \in \mathbb{S}^d.$$

We introduce the spaces

$$H = L^2(\Omega, \mathbb{R}^d),\ \mathcal{Q} = L^2(\Omega, \mathbb{S}^d) = \{\sigma = (\sigma_{ij}) \mid \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{u \in H \mid \varepsilon(u) \in \mathcal{Q}\},$$
$$\mathcal{Q}_1 = \{\sigma \in \mathcal{Q} \mid Div\sigma \in H\}.$$

Here and below $\varepsilon : H_1 \to \mathcal{Q}$ and $Div : \mathcal{Q}_1 \to H$ are the linearized strain tensor and the divergence operator, respectively, defined by

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),\ \forall u \in H_1,$$

$$Div\sigma = \left((Div\sigma)_i\right)_{1 \le i \le d},\ \ (Div\sigma)_i = \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j},\ \forall \sigma \in \mathcal{Q}_1,$$

where ∇u is the gradient of u defined by

$$\nabla u = \left(\frac{\partial u_i}{\partial x_j}\right)_{1 \le i,j \le d},$$

$\dfrac{\partial u_i}{\partial x_j}$ represents the partial derivative of the function $u_i : \Omega \to \mathbb{R}$ with respect to the component x_j of the spatial variable and $(\nabla u)^T$ is the transpose of the matrix ∇u. Note that H, $\mathcal{Q}$, H_1, $\mathcal{Q}_1$ are Hilbert spaces equipped with the respective canonical inner products

$$(u, v)_H = \int_\Omega u \cdot v dx,\ (\sigma, \tau)_{\mathcal{Q}} = \int_\Omega \sigma \cdot \tau dx,$$
$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{Q}},\ \ (\sigma, \tau)_{\mathcal{Q}_1} = (Div\sigma, Div\tau)_H + (\sigma, \tau)_{\mathcal{Q}},$$

where the associated norms are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{Q}}$, $\|.\|_{\mathcal{Q}_1}$.

Let $H_\Gamma = H^{1/2}(\Gamma; \mathbb{R}^d)$ and let $\tilde{\gamma} : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$ we also use the notation v to denote the trace $\tilde{\gamma}(v)$ of v on Γ and for all $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ:

$$v_\nu = v \cdot \nu,\ v_\tau = v - v_\nu \nu \text{ on } \Gamma.$$

Here and below ν represents the unit outward normal vector to Γ. In a similar manner, the normal and tangential components of a regular (say C^1) tensor field σ are defined by

$$\sigma_\nu = \sigma\nu \cdot \nu,\ \sigma_\tau = \sigma\nu - \sigma_\nu \nu \text{ on } \Gamma.$$

Moreover, the following Green's formula holds:

$$(2.1)\qquad (\text{Div}\sigma, v)_H + (\sigma, \varepsilon(v))_{\mathcal{Q}} = \int_\Gamma \sigma\nu \cdot v\text{d}a,\ \forall\ v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1,\ v= 0 \text{ on } \Gamma_1,\ v_\nu= 0 \text{ on } \Gamma_3\}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$(2.2)\qquad C_K \left\| v \right\|_{H_1} \leq \left\| \varepsilon\left(v\right)\right\|_{\mathcal{Q}},\ \forall v \in V,$$

where C_K is a positive constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found, for instance, in [17, page 79]. Over the space V, we consider the inner product given by

$$(2.3)\qquad (u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{Q}},\ \forall u,\ v \in V,$$

and let $\|.\|_V$ be the associated norm. It follows from Korn's inequality (2.2) that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, (.,.)_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$(2.4)\qquad \|v\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq c_0\|v\|_V,\ \forall v \in V.$$

Also, we will use the space of fourth-order tensor fields,

$$\mathbf{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \mid\ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij};\ \mathcal{E}_{ijkl} \in L^\infty(\Omega),\ \forall i,\ j,\ k,\ l \in \{1, ..., d\}\},$$

which is a real Banach space with the norm,

$$(2.5)\qquad \|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1\leq i,j,k,l\leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Finally, for every real Banach space $(X, \|.\|_X)$ and $T > 0$, we denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X and we use the standard notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $p \in [1,\infty]$ and $k \geqslant 1$.

3. PROBLEM STATEMENT. ASSUMPTIONS. VARIATIONAL FORMULATION

The physical setting is as follows. A deformable body occupies the reference configuration $\Omega \subset \mathbb{R}^{d=2,3}$ which is a bounded domain with Lipschitz boundary Γ. The body is assumed to have a viscoelastic law with long-term memory and the process is quasistatic in the time interval of interest $[0,T]$. We assume that Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω and surface tractions of density f_2 act on Γ_2. The contact is supposed to be bilateral, adhesive and governed by a general nonlocal friction law. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 3.1. Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{G}(t-s)\,\varepsilon(u(s))\mathrm{d}s, \text{ in } \Omega \times (0,T), \tag{3.1}$$

$$\text{Div}\sigma + f_0 = 0, \text{ in } \Omega \times (0,T), \tag{3.2}$$

$$u = 0, \text{ on } \Gamma_1 \times (0,T), \tag{3.3}$$

$$\sigma\nu = f_2, \text{ on } \Gamma_2 \times (0,T), \tag{3.4}$$

$$u_\nu = 0, \text{ on } \Gamma_3 \times (0,T), \tag{3.5}$$

$$\begin{cases} |\sigma_\tau + p_\tau(\beta, u_\tau)| \leqslant p(|R\sigma_\nu|), \\ |\sigma_\tau + p_\tau(\beta, u_\tau)| < p(|R\sigma_\nu|) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau + p_\tau(\beta, u_\tau)| = p(|R\sigma_\nu|) \Rightarrow \exists\lambda \geq 0 \text{ such that:} \\ \sigma_\tau + p_\tau(\beta, u_\tau) = -\lambda\dot{u}_\tau \end{cases} \text{ on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$\dot{\beta} = H_{ad}(\beta, |R_\tau(u_\tau)|), \text{ on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$\beta(0) = \beta_0, \text{ on } \Gamma_3, \; u(0) = u_0, \text{ in } \Omega, \tag{3.8}$$

Equation (3.1) represents the viscoelastic constitutive law in which ε denotes the linearized strain tensor, $\mathcal{A}$ is the viscosity operator, $\mathcal{B}$ is the elasticity operator, $\mathcal{G}$ denotes the tensor of relaxation. Here and below the dot above a variable denotes its first derivative with respect to the time variable. We note that for $\mathcal{G} = 0$ the constitutive law (3.1) reduces to the well-known Kelvin-Voigt viscoelastic constitutive law

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}\varepsilon(u), \text{ in } \Omega \times (0,T).$$

Equation (3.2) represents the equilibrium equation. Equations (3.3)–(3.4) are the displacement-traction boundary conditions where ν stands for the unit outward normal vector to Γ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (3.5)–(3.6) represent the bilateral contact with nonlocal version of Coulomb's law in which adhesion is taken into account. Here p is a nonnegative function, $R : H^{-1/2}(\Gamma_3) \to L^2(\Gamma_3)$, represents a normal regularization operator, i.e. a linear continuous operator and satisfies

$$\|R\zeta_\nu\|_{L^2(\Gamma_3)} \leq L_R \|\zeta\|_{\mathcal{Q}_1}, \ \forall \zeta \in \mathcal{Q}_1 \tag{3.9}$$

with $L_R > 0$. The introduction of the nonlocal smoothing operator R is used for technical reasons, since the trace of the stress tensor on the boundary is too rough (see, *e.g.*, [11]). In the case where p is a constant function, (3.6) becomes the well known Tresca's friction law. In [10, 19], the friction bound was used with

$$p(|R\sigma_\nu|) = \mu |R\sigma_\nu|$$

where μ is the coefficient of friction. From thermodynamic consideration, a modified version for Coulomb's law of friction was obtained in [22, 23]. It consists of using the friction bound function

$$p(|R\sigma_\nu|) = (|R\sigma_\nu| (1 - \kappa |R\sigma_\nu|))_+$$

where κ is a small positive coefficient and $[r]+ = \max\{r, 0\}$. In (3.6), p_τ is a general prescribed function. In particular, we may consider the case

$$p_\tau(\beta, v) = \begin{cases} q_\tau(\beta)\, v \text{ if } 0 \leq |v| \leq L, \\ q_\tau(\beta)\, L \dfrac{v}{|v|} \text{ if } |v| > L, \end{cases}$$

where $L > 0$ is a limit bound constant (see, *e.g.*, [18]), and q_τ is nonnegative tangential stiffness function. Equation (3.7) represents the evolution of the bonding field described by a general function H_{ad} may change sign. This condition means that rebonding may take place after debonding. Here and below $R_\tau : \mathbb{R}^d \to \mathbb{R}^d$ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v \text{ if } 0 \leq |v| \leq L, \\ L \dfrac{v}{|v|} \text{ if } |v| > L, \end{cases} \tag{3.10}$$

The introduction of the operator R_τ is motivated by the mathematical arguments where $L > 0$ is a characteristic length of the bond, beyond which there is no any additional traction (see, *e.g.*, [18]). Clearly, R_τ satisfies

$$\begin{cases} |R_\tau(v)| \leq L, \ \forall v \in \mathbb{R}^n, \\ |(|R_\tau(w)| - |R_\tau(v)|)| \leqslant |w - v|, \ \forall w, \ v \in \mathbb{R}^n. \end{cases} \tag{3.11}$$

An example for the evolution rate function H_{ad} is

$$H_{ad}(\beta, r) = -\left(c_\tau \beta r^2 - \epsilon_a\right)_+, \ \textit{on } \Gamma_3 \times (0, T),$$

where c_τ, ϵ_a are given positive material parameters and since $\dot{\beta} \leq 0$, the process is irreversible, once debonding occurs bonding cannot be reestablished (see, *e.g.*, [4, 20, 25]). Finally, (3.8) are the initial conditions.

In the study of the mechanical Problem (3.1)–(3.8) we consider the following assumptions. We assume that the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l}\text{(i) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathcal{A}} \left|\varepsilon_1 - \varepsilon_2\right|^2, \\ \quad \forall \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d,\ a.e.\ x \in \Omega; \\ \text{(ii) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \left|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\right| \leqslant L_{\mathcal{A}} \left|\varepsilon_1 - \varepsilon_2\right|, \\ \quad \forall \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d,\ a.e.\ x \in \Omega; \\ \text{(iii) The mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \varepsilon \in \mathbb{S}^d; \\ \text{(iv) The mapping } x \mapsto \mathcal{A}(x,0_{\mathbb{S}^d}) \text{ belongs to } \mathcal{Q}. \end{array}\right. \tag{3.12}$$

We assume that the elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l}\text{(i) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad \left|\mathcal{B}(x,\varepsilon_1) - \mathcal{B}(x,\varepsilon_2)\right| \leqslant L_{\mathcal{B}} \left|\varepsilon_1 - \varepsilon_2\right|, \\ \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.e.\ x \in \Omega; \\ \text{(ii) The mapping } x \mapsto \mathcal{B}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \varepsilon \in \mathbb{S}^d; \\ \text{(iii) The mapping } x \mapsto \mathcal{B}(x,0_{\mathbb{S}^d}) \text{ belongs to } \mathcal{Q}. \end{array}\right. \tag{3.13}$$

The operator $\mathcal{G}$ satisfies

$$\mathcal{G} \in L^\infty(0,T;\mathbf{Q}_\infty). \tag{3.14}$$

We assume that the tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l}\text{(i) There exists } L_\tau > 0 \text{ such that} \\ \quad \left|p_\tau(x,\beta_1,r_1) - p_\tau(x,\beta_2,r_2)\right| \leqslant L_\tau \left(\left|\beta_1 - \beta_2\right| + \left|r_1 - r_2\right|\right), \\ \quad \forall \beta_1,\ \beta_2 \in \mathbb{R},\ \forall r_1,\ r_2 \in \mathbb{R}^d,\ a.e.\ x \in \Gamma_3; \\ \text{(ii) } r \cdot \nu(x) = 0 \Longrightarrow p_\tau(x,\beta,r) \cdot \nu(x) = 0, \\ \quad \forall \beta \in \mathbb{R},\ \forall r \in \mathbb{R}^d,\ a.e.\ x \in \Gamma_3; \\ \text{(iii) The mapping } x \mapsto p_\tau(x,\beta,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \text{for any } \beta \in \mathbb{R},\ \forall r \in \mathbb{R}^d; \\ \text{(iv) The mapping } x \mapsto p_\tau(x,0_{\mathbb{R}},0_{\mathbb{R}^d}) \text{ belongs to } L^2(\Gamma_3,\ \mathbb{R}^d). \end{array}\right. \tag{3.15}$$

The adhesion rate function $H_{ad}:\Gamma_3\times\mathbb{R}\times[0,L]\to\mathbb{R}$ satisfies

(3.16)
$$\begin{cases}\text{(i) There exists } L_{H_{ad}}>0 \text{ such that}\\ \quad |H_{ad}(x,\beta_1,r)-H_{ad}(x,\beta_2,r)|\leqslant L_{H_{ad}}|\beta_1-\beta_2|,\\ \quad \forall\beta_1,\ \beta_2\in\mathbb{R},\ \forall r\in[0,L],\ a.e.\ x\in\Gamma_3\,;\\ \text{(ii) } |H_{ad}(x,\beta_1,r_1)-H_{ad}(x,\beta_2,r_2)|\leqslant L_{H_{ad}}(|\beta_1-\beta_2|+|r_1-r_2|),\\ \quad \forall\beta_1,\ \beta_2\in[0,1],\ \forall r_1,\ r_2\in[0,L],\ a.e.\ x\in\Gamma_3\,;\\ \text{(iii) The mapping } x\mapsto H_{ad}(x,\beta,r) \text{ is Lebesgue measurable on } \Gamma_3,\\ \quad \forall\beta\in\mathbb{R},\ \forall r\in[0,L]\,;\\ \text{(iv) The mapping } (\beta,r)\mapsto H_{ad}(x,\beta,r) \text{ is continuous on } \mathbb{R}\times[0,L],\\ \quad a.e\ x\in\Gamma_3\,;\\ \text{(v) } H_{ad}(x,0,r)=0,\ \forall r\in[0,L],\ a.e.\ x\in\Gamma_3\,;\\ \text{(vi) } H_{ad}(x,\beta,r)\geqslant 0,\ \forall\beta\leqslant 0,\ \forall r\in[0,L],\ a.e.\ x\in\Gamma_3 \text{ and}\\ \qquad H_{ad}(x,\beta,r)\leqslant 0,\ \forall\beta\geqslant 1,\ \forall r\in[0,L],\ a.e\ x\in\Gamma_3\,.\end{cases}$$

We also assume that the friction bound function $p:\Gamma_3\times\mathbb{R}\to\mathbb{R}^+$ satisfies:

$$(3.17)\qquad \begin{cases}\text{(i) There exists } L_p>0 \text{ such that}\\ \quad |p(x,r_1)-p(x,r_2)|\leqslant L_p|r_1-r_2|,\\ \quad \forall r_1,\ r_2\in\mathbb{R}\ ,\ a.e.\ x\in\Gamma_3\,;\\ \text{(ii) The mapping } x\mapsto p(x,r) \text{ is Lebesgue measurable on } \Gamma_3\\ \quad \text{for any } r\in\mathbb{R};\\ \text{(iii) The mapping } x\mapsto p(x,0) \text{ belongs to } L^2(\Gamma_3).\end{cases}$$

The densities of forces satisfy

$$(3.18)\qquad \text{(i) } f_0\in C([0,T];H),\ \text{(ii) } f_2\in C([0,T];L^2(\Gamma_2,\mathbb{R}^d)).$$

Finally, we assume that the initial data satisfy

$$(3.19)\qquad \beta_0\in L^\infty(\Gamma_3),\ 0\leqslant\beta_0\leqslant 1,\ a.e.\ x\in\ \Gamma_3.$$

$$(3.20)\qquad u_0\in V,$$

Now, using (3.18), we find that the function $f:[0,T]\to V$ defined by

$$(3.21)\qquad (f(t),w)_V=\int_\Omega f_0(t)\cdot w\mathrm{d}x+\int_{\Gamma_2}f_2(t)\cdot w\mathrm{d}a,\ \forall w\in V,\ \forall t\in[0,T],$$

has the regulariy

$$(3.22)\qquad f\in C([0,T];V).$$

In the sequel, we use the functionals $j_\tau:\mathcal{Q}_1\times V\longrightarrow\mathbb{R}$, $j_{ad}:L^2(\Gamma_3)\times V\times V\longrightarrow\mathbb{R}$ defined by

$$(3.23)\qquad j_\tau(\zeta,w)=\int_{\Gamma_3}p(|R\zeta_\nu|)|w_\tau|\,\mathrm{d}a,\ \forall(\zeta,w)\in\mathcal{Q}_1\times V.$$

$$j_{ad}(\beta, v, w) = \int_{\Gamma_3} p_\tau(\beta, v_\tau) \cdot w_\tau \mathrm{d}a, \ \forall(\beta, v, w) \in L^2(\Gamma_3) \times V \times V. \tag{3.24}$$

Thanks to (2.4) and (3.15), the functional j_{ad} satisfies

$$\begin{cases} \text{There exists } L_{ad} > 0 \text{ such that} \\ |j_{ad}(\beta_1, u_1, w) - j_{ad}(\beta_2, u_2, w)| \\ \leqslant L_{ad} \left(\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} + \|u_1 - u_2\|_V \right) \|w\|_V, \ \forall u_1, \ u_2, \ w \in V, \\ \forall \beta_1, \ \beta_2 \in L^2(\Gamma_3). \end{cases} \tag{3.25}$$

Also, using (2.4), (3.9) and (3.17), we have

$$\begin{cases} j_\tau(\zeta_1, v_2) - j_\tau(\zeta_1, v_1) + j_\tau(\zeta_2, v_1) - j_\tau(\zeta_2, v_2) \\ \leq L_p L_R c_0 \|\zeta_1 - \zeta_2\|_{\mathcal{Q}_1} \times \|v_1 - v_2\|_V, \ \forall \zeta_1, \ \zeta_2 \in \mathcal{Q}_1, \ \forall v_1, \ v_2 \in V. \end{cases} \tag{3.26}$$

Now, assume u, σ and β are smooth functions satisfying Problem (3.1)–(3.8) and use Green's formula (2.1) to obtain the following variational formulation for this mechanical problem.

Problem 3.2. Find a displacement field $u : [0, T] \to V$, a stress field $\sigma : [0, T] \to \mathcal{Q}_1$ and a bonding field $\beta : [0, T] \to L^\infty(\Gamma_3)$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}(\varepsilon(u(t))) + \int_0^t \mathcal{G}(t-s)\varepsilon(u(s))\mathrm{d}s, \tag{3.27}$$

$$\begin{cases} (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} + j_{ad}(\beta(t), u(t), w - \dot{u}(t)) \\ + j_\tau(\sigma(t), w) - j_\tau(\sigma(t), \dot{u}(t)) \geq (f(t), w - \dot{u}(t))_V, \\ \text{for all } w \in V, \ for \ a.e. \ t \in (0, T), \end{cases} \tag{3.28}$$

$$u(0) = u_0, \tag{3.29}$$

$$\dot{\beta}(t) = H_{ad}(\beta(t), |R_\tau(u_\tau(t))|), \text{ for } a.e. \ t \in (0, T), \tag{3.30}$$

$$\beta(0) = \beta_0. \tag{3.31}$$

4. **EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION**

Our main existence and uniqueness result in this section is the following.

THEOREM 4.1. *Assume that* (3.12)–(3.20) *are fulfilled. Then, there exists* $L_0 > 0$ *such that for* $L_p < L_0$, *Problem* (3.27)–(3.31) *has a unique solution* $\{u, \sigma, \beta\}$. *Moreover, the solution satisfies*

$$\text{(i)} \ u \in C^1([0, T]; V), \ (ii) \ \sigma \in C^1([0, T]; \mathcal{Q}_1). \tag{4.1}$$

$$\beta \in C^1([0, T]; L^\infty(\Gamma_3)), \ 0 \leqslant \beta(t) \leqslant 1, \ a.e. \ x \in \Gamma_3, \ \forall t \in [0, T]. \tag{4.2}$$

The proof of Theorem 4.1 will be carried out in several steps.

First step. Consider the following problem.

Problem 4.2. Let $\eta \in C([0,T];V)$, find a function $\beta_\eta : [0,T] \to L^\infty(\Gamma_3)$, such that

$$\dot{\beta}_\eta(t) = H_{ad}\left(\beta_\eta(t), |R_\tau(\eta_\tau(t))|\right), \ for\ a.e.\ t \in (0,T), \tag{4.3}$$

$$\beta_\eta(0) = \beta_0. \tag{4.4}$$

LEMMA 4.3. *Assume that* (3.16) *and* (3.19) *are fulfilled. Then, for each* $\eta \in C([0,T];V)$, *Problem* (4.3)–(4.4) *has a unique solution* β_η *which satisfies*

$$\begin{cases} (i)\ \ \beta_\eta \in C^1([0,T];L^\infty(\Gamma_3)), \\ (ii)\ \ 0 \le \beta_\eta(t) \le 1, \forall\ t \in [0,T],\ for\ a.e.\ x \in \Gamma_3. \end{cases} \tag{4.5}$$

Moreover, there exists a constant $c > 0$ *such that for all* η_1, $\eta_2 \in C([0,T];V)$,

$$\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \mathrm{d}s,\ \ \forall\ t \in [0,T]. \tag{4.6}$$

Proof. Let $\eta \in C([0,T];V)$, consider the following operator $\Psi : [0,T] \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$, such that

$$\Psi(t,\beta) = H_{ad}\left(\beta, |R_\tau(\eta_\tau(t))|\right), \forall\ t \in [0,T]. \tag{4.7}$$

Let β_1, $\beta_2 \in L^\infty(\Gamma_3)$. From (4.7) and (3.16), we obtain

$$\begin{aligned} |\Psi(t,\beta_1) - \Psi(t,\beta_2)| &\le |H_{ad}\left(\beta_1, |R_\tau(\eta_\tau(t))|\right) - H_{ad}\left(\beta_2, |R_\tau(\eta_\tau(t))|\right)| \\ &\le L_{H_{ad}}|\beta_1 - \beta_2|, \forall\ t \in [0,T], \end{aligned}$$

which implies that

$$\|\Psi(t,\beta_1) - \Psi(t,\beta_2)\|_{L^\infty(\Gamma_3)} \le L_{H_{ad}}\|\beta_1 - \beta_2\|_{L^\infty(\Gamma_3)}, \forall\ t \in [0,T].$$

Hence Ψ is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, the mapping $t \to \Psi(t,\beta)$ belongs to $L^\infty(0,T;L^\infty(\Gamma_3))$, $\forall \beta \in L^\infty(\Gamma_3)$. Thus, using a version of Cauchy-Lipschitz theorem (see *e.g.*, [24, page 60]), we obtain the existence of a unique solution $\beta_\eta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$ to system (4.3)–(4.4) and keeping in mind (3.16), (4.3) and using the fact that $\beta_\eta \in C([0,T];L^\infty(\Gamma_3))$, $\eta \in C([0,T];V)$ we get $\beta_\eta \in C^1([0,T];L^\infty(\Gamma_3))$. Now, from (4.3)–(4.4), (3.16), (3.19) and using arguments similar to those used in [5, Proposition 4.1], we deduce ((*ii*) 4.5). To continue, let β_{η_k} be two solutions for $\eta = \eta_k \in C([0,T];V)$, $k = 1,2$ and let $s \in [0,T]$. It follows from (4.3)–(4.4), (3.16) and (3.11) that

$$\|\beta_{\eta_1}(t)-\beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \leq c\int_0^t \|\beta_{\eta_1}(s)-\beta_{\eta_2}(s)\|_{L^2(\Gamma_3)}\mathrm{d}s + c\int_0^t \|\eta_1(s)-\eta_2(s)\|_{L^2(\Gamma_3,\mathbb{R}^d)}\mathrm{d}s.$$

Using Gronwall's lemma in the last inequality and keeping in mind (2.4) we obtain (4.6). □

Here and in the rest of this Section, the same letter c will be used to denote different positive constants independent of $t \in (0,T)$, and whose values may change from place to place.

Second step. To continue, for each $\eta \in C([0,T];V)$, let β_η be the unique solution of Problem (4.3)–(4.4) and let $f_\eta : [0,T] \to V$ be the function defined by

(4.8)
$$\begin{cases} (f_\eta(t),w)_V = (f(t),w)_V - \left(\mathcal{B}\left(\varepsilon(\eta\left(t\right))\right) + \int_0^t \mathcal{G}\left(t-s\right)\varepsilon(\eta\left(s\right))\mathrm{d}s, \varepsilon\left(w\right)\right)_{\mathcal{Q}} \\ \qquad -j_{ad}(\beta_\eta(t),\eta(t),w),\ \forall t\in[0,T],\ \forall\, w\in V, \end{cases}$$

which satisfies

$$f_\eta \in C([0,T];V). \tag{4.9}$$

We consider the following problem.

Problem 4.4 ($\mathcal{P}_{\eta\zeta}$). Let $(\eta,\zeta) \in C\left([0,T];V\right)\times C\left([0,T];\mathcal{Q}_1\right)$. Find a function $v_{\eta\zeta} \in C\left([0,T];V\right)$ such that

$$\begin{cases} \left(\mathcal{A}\left(\varepsilon(v_{\eta\zeta}\left(t\right)\right),\varepsilon\left(w-v_{\eta\zeta}\left(t\right)\right)\right)_{\mathcal{Q}} + \\ +j_\tau(\zeta\left(t\right),w) - j_\tau\left(\zeta\left(t\right),v_{\eta\zeta}\left(t\right)\right) \geq \left(f_\eta\left(t\right),w-v_{\eta\zeta}\left(t\right)\right)_V, \\ \text{for all } w\in V,\ \forall t\in[0,T]. \end{cases} \tag{4.10}$$

We have the following existence and uniqueness result for Problem $\mathcal{P}_{\eta\zeta}$.

LEMMA 4.5. *Assume that* (3.12)–(3.19) *are fulfilled. Then, for each* $(\eta,\zeta)\in C\left([0,T];V\right)\times C\left([0,T];\mathcal{Q}_1\right)$*, Problem* $\mathcal{P}_{\eta\zeta}$ *has a unique solution* $v_{\eta\zeta}$*, which satisfies*

$$v_{\eta\zeta} \in C\left([0,T];V\right). \tag{4.11}$$

Proof. Let $A: V\to V$ be the operator defined by

$$(Aw,z)_V = \left(\mathcal{A}\left(\varepsilon(w)\right),\varepsilon\left(z\right)\right)_{\mathcal{Q}},\ \forall\, w,\, z\in V.$$

It follows from assumption (3.12) that A is a strongly monotone and Lipschitz continuous operator. On the other hand the functional $j_\tau(\zeta\left(t\right),.)$:

$V \to \mathbb{R}$ is a continuous semi-norm on V, $\forall t \in [0,T]$. Then by a classical argument of elliptic variational inequalities (see *e.g.*, [3]), we deduce that $\mathcal{P}_{\eta\zeta}$ has a unique solution, $v_{\eta\zeta}$ satisfies (4.11). □

Third step. For each $(\eta,\zeta) \in C([0,T];V) \times C([0,T];\mathcal{Q}_1)$, let $v_{\eta\zeta}$ be the unique solution of Problem (4.10), let $\sigma_{\eta\zeta} : [0,T] \to \mathcal{Q}$ be the function defined by

$$\sigma_{\eta\zeta}(t) = \mathcal{A}(\varepsilon(v_{\eta\zeta}(t)) + \mathcal{B}(\varepsilon(\eta(t))) + \int_0^t \mathcal{G}(t-s)\,\varepsilon(\eta(s))\mathrm{d}s,\ \forall t \in [0,T], \tag{4.12}$$

and choose $w = v_{\eta\zeta}(t) \pm \varphi$ where $\varphi \in [D(\Omega)]^d$ in (4.10), we deduce that

$$\mathrm{Div}\sigma_{\eta\zeta}(t) = -f_0(t) \ \text{ in } \Omega,\ \forall t \in [0,T]. \tag{4.13}$$

Therefore, using (3.12)–(3.14), ((*i*) 3.18) and (4.11)–(4.13) we obtain

$$\sigma_{\eta\zeta} \in C([0,T];\mathcal{Q}_1) \tag{4.14}$$

Now, for each $\eta \in C([0,T];V)$, let $\Lambda_\eta : C([0,T];\mathcal{Q}_1) \to C([0,T];\mathcal{Q}_1)$ be the operator defined by

$$\Lambda_\eta\zeta = \sigma_{\eta\zeta},\ \forall\, \zeta \in C([0,T];\mathcal{Q}_1). \tag{4.15}$$

We have the following result.

Lemma 4.6. *The operator Λ_η has a unique fixed point $\zeta_\eta^* \in C([0,T];\mathcal{Q}_1)$.*

Proof. Let $\eta \in C([0,T];V)$, let ζ_1, $\zeta_2 \in C([0,T];\mathcal{Q}_1)$, let $t \in [0,T]$, then by (4.10) and (3.12), we get

$$\begin{aligned} m_{\mathcal{A}}\|v_{\eta\zeta_1}(t) - v_{\eta\zeta_2}(t)\|_V^2 \le j_\tau(\zeta_1(t), v_{\eta\zeta_2}(t)) - j_\tau(\zeta_1(t), v_{\eta\zeta_1}(t)) \\ + j_\tau(\zeta_2(t), v_{\eta\zeta_1}(t)) - j_\tau(\zeta_2(t), v_{\eta\zeta_2}(t)), \end{aligned}$$

and keeping in mind (3.26), we find

$$\|v_{\eta\zeta_1}(t) - v_{\eta\zeta_2}(t)\|_V \leqslant \frac{L_p L_R c_0}{m_{\mathcal{A}}}\|\zeta_1(t) - \zeta_2(t)\|_{\mathcal{Q}_1}.$$

Now, using (4.15), (4.12)–(4.13) and (3.12), we obtain

$$\begin{aligned} \|\Lambda_\eta\zeta_1(t) - \Lambda_\eta\zeta_2(t)\|_{\mathcal{Q}_1} &= \|\sigma_{\eta\zeta_1}(t) - \sigma_{\eta\zeta_2}(t)\|_{\mathcal{Q}_1} \\ &= \|\sigma_{\eta\zeta_1}(t) - \sigma_{\eta\zeta_2}(t)\|_{\mathcal{Q}} \\ &\leqslant L_{\mathcal{A}}\|v_{\eta\zeta_1}(t) - v_{\eta\zeta_2}(t)\|_V. \end{aligned}$$

Therefore, we find that

$$\|\Lambda_\eta\zeta_1(t) - \Lambda_\eta\zeta_2(t)\|_{\mathcal{Q}_1} \leqslant \frac{L_p L_{\mathcal{A}} L_R c_0}{m_{\mathcal{A}}}\|\zeta_1(t) - \zeta_2(t)\|_{\mathcal{Q}_1},\ \forall\, t \in [0,T].$$

This inequality implies that for $L_p < L_0 = \dfrac{m_{\mathcal{A}}}{L_{\mathcal{A}} L_R c_0}$, Λ_η has a unique fixed point $\zeta_\eta^* \in C([0,T];\mathcal{Q}_1)$. □

Fourth step. For each $\eta \in C([0,T];V)$, let $\sigma_{\eta\zeta_\eta^*} = \zeta_\eta^*$ be the fixed point of Λ_η, we denote by $v_\eta = v_{\eta\zeta_\eta^*}$ the unique solution of Problem (4.10) and by β_η the unique solution of Problem (4.3)-(4.4). Also, keeping in mind (3.20), and let $\Lambda : C([0,T];V) \to C([0,T];V)$ be the operator defined by

$$\Lambda\eta(t) = \int_0^t v_\eta(s)\,\mathrm{d}s + u_0,\ \forall\ \eta \in C([0,T];V),\ \forall\ t \in [0,T]. \tag{4.16}$$

We have the following result.

Lemma 4.7. *The operator Λ has a unique fixed point $\eta^* \in C([0,T];V)$.*

Proof. Let η_1, $\eta_2 \in C([0,T];V)$, let $t \in (0,T)$, let $s \in (0,t)$, we denote by $\sigma_i = \sigma_{\eta_i\zeta_{\eta_i}^*} = \zeta_{\eta_i}^*$, $v_i = v_{\eta_i}$, $\beta_i = \beta_{\eta_i}$ for i=1, 2, then by (4.12)–(4.13), we get

$$\begin{aligned} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{Q}_1} &= \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{Q}} \\ &\leqslant L_{\mathcal{A}} \|v_1(s) - v_2(s)\|_V + L_{\mathcal{B}} \|\eta_1(s) - \eta_2(s)\|_V \\ &\quad + c\int_0^s \|\eta_1(r) - \eta_2(r)\|_V \,\mathrm{d}r. \end{aligned} \tag{4.17}$$

On the other hand, using (4.10) and (4.8), one has

$$\begin{cases} (\mathcal{A}(\varepsilon(v_1(s))) - \mathcal{A}(\varepsilon(v_2(s))), \varepsilon(v_1(s) - v_2(s)))_{\mathcal{Q}} \\ \leqslant j_\tau(\sigma_1(s), v_2(s)) - j_\tau(\sigma_1(s), v_1(s)) + j_\tau(\sigma_2(s), v_1(s)) - j_\tau(\sigma_2(s), v_2(s)) \\ + j_{ad}(\beta_2(s), \eta_2(s), v_1(s) - v_2(s)) - j_{ad}(\beta_1(s), \eta_1(s), v_1(s) - v_2(s)) \\ + (\mathcal{B}(\varepsilon(\eta_2(s))) - \mathcal{B}(\varepsilon(\eta_1(s))), \varepsilon(v_1(s) - v_2(s)))_{\mathcal{Q}} + \\ + \left(\displaystyle\int_0^s \mathcal{G}(s-r)\varepsilon(\eta_2(r) - \eta_1(r))\mathrm{d}r, \varepsilon(v_1(s) - v_2(s))\right)_{\mathcal{Q}}, \end{cases}$$

which, together with (3.12)–(3.14) and (3.25)–(3.26), gives

$$\begin{cases} m_{\mathcal{A}} \|v_1(s) - v_2(s)\|_V^2 \\ \leqslant L_p L_R c_0 \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{Q}_1} \|v_1(s) - v_2(s)\|_V + \\ + c\left(\|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} + \|\eta_1(s) - \eta_2(s)\|_V\right) \|v_1(s) - v_2(s)\|_V \\ + L_{\mathcal{B}} \|\eta_1(s) - \eta_2(s)\|_V \|v_1(s) - v_2(s)\|_V \\ + c\displaystyle\int_0^s \|\eta_1(r) - \eta_2(r)\|_V \,\mathrm{d}r \times \|v_1(s) - v_2(s)\|_V, \end{cases}$$

and using (4.17), we obtain

$$\begin{cases} m_{\mathcal{A}} \|v_1(s) - v_2(s)\|_V \leqslant c \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} + c \|\eta_1(s) - \eta_2(s)\|_V \\ +c \int_0^s \|\eta_1(r) - \eta_2(r)\|_V \, \mathrm{d}r + L_{\mathcal{A}} L_p L_R c_0 \|v_1(s) - v_2(s)\|_V . \end{cases}$$

Therefore, if $L_p < L_0 = \dfrac{m_{\mathcal{A}}}{L_{\mathcal{A}} L_R c_0}$, and keeping in mind (4.6), we deduce that

$$\|v_1(s) - v_2(s)\|_V \leqslant c \|\eta_1(s) - \eta_2(s)\|_V + c \int_0^s \|\eta_1(r) - \eta_2(r)\|_V \, \mathrm{d}r,$$

which gives

$$\|v_1(s) - v_2(s)\|_V^2 \leqslant c \|\eta_1(s) - \eta_2(s)\|_V^2 + c \int_0^s \|\eta_1(r) - \eta_2(r)\|_V^2 \, \mathrm{d}r,$$

integrating both sides of the previous inequality on $(0, t)$, we get

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2 \, \mathrm{d}s \leqslant c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 \, \mathrm{d}s,$$

which, together with (4.16), implies that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V^2 \leqslant c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 \, \mathrm{d}s.$$

Reiterating the last inequality n times, we infer that

$$\|\Lambda^n \eta_1 - \Lambda^n \eta_2\|_{C([0,T];V)}^2 \leqslant \frac{(cT)^n}{n!} \|\eta_1 - \eta_2\|_{C([0,T];V)}^2 ,$$

which implies that, for n sufficiently large, a power Λ^n of Λ is a contraction in the Banach space $C([0,T];V)$. Therefore, we deduce that Λ has a unique fixed point $\eta^* \in C([0,T];V)$. □

Now, we have all the ingredients to prove Theorem 4.1. Let η^* be the fixed point of Λ defined by (4.16), let β_{η^*} be the unique solution of Problem (4.3)–(4.4) for $\eta = \eta^*$ and let $\sigma_{\eta^*} = \sigma_{\eta^* \zeta^*_{\eta^*}}$ be the function defined by (4.12), where $\zeta^*_{\eta^*}$ is the fixed point of Λ_{η^*} defined by (4.15). Also, keeping in mind (3.20) and let $u_{\eta^*} : [0,T] \to V$ be the displacement field defined by

$$u_{\eta^*}(t) = \eta^*(t) = \int_0^t v_{\eta^*}(s) \, \mathrm{d}s + u_0, \ \forall\, t \in [0,T], \tag{4.18}$$

where $v_{\eta^*} = v_{\eta^* \zeta^*_{\eta^*}}$ is the unique solution of Problem (4.10) for $(\eta, \zeta) = \left(\eta^*, \zeta^*_{\eta^*}\right)$. Therefore, if $L_p < L_0 = \dfrac{m_{\mathcal{A}}}{L_{\mathcal{A}} L_R c_0}$, we conclude by (4.3)–(4.4), (4.10), (4.12) and (4.18) that $\{u_{\eta^*}, \sigma_{\eta^*}, \beta_{\eta^*}\}$ is a solution of Problem (3.27)–(3.31), and keeping in mind (4.11), (4.14) and (4.5), it follows that $\{u_{\eta^*}, \sigma_{\eta^*}, \beta_{\eta^*}\}$ has the regularity (4.1)–(4.2). The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ_η, the uniqueness of the fixed point of the operator Λ and of the uniqueness of the solution of Problem (4.3)–(4.4) and Problem (4.10).

5. NUMERICAL APPROXIMATION

In this section, we assume that the conditions (3.12)–(3.16) and (3.18)–(3.20) are satisfied and we suppose that the friction bound function p satisfies for all $r \in \mathbb{R}$,

(5.1) $$p\,(.,r) = g \in L^\infty(\Gamma_3), \text{ such that } g \geqslant 0 \ a.e \text{ on } \Gamma_3.$$

Here g represents the friction bound, i.e. the magnitude of the frictional force at which slip begins. In this case, (3.6) becomes the well known Tresca's friction law in which adhesion is taken into account. In the sequel, we use the notation j_τ to denote the functional $j_\tau : V \longrightarrow \mathbb{R}$ defined by

(5.2) $$j_\tau(w) = \int_{\Gamma_3} g\,|w_\tau|\,\mathrm{d}a, \ \forall w \in V.$$

Eliminating the stress field from (3.27)–(3.31), we obtain the following variational formulation of the Problem (3.1)–(3.8) in terms of displacement and adhesion fields only.

Problem 5.1. Find a displacement field $u : [0,T] \to V$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

(5.3) $$\begin{cases} (\mathcal{A}\,(\varepsilon(\dot{u}\,(t))) + \mathcal{B}\,(\varepsilon(u(t)))\,, \varepsilon\,(w - \dot{u}(t)))_{\mathcal{Q}} \\ + \left(\displaystyle\int_0^t \mathcal{G}\,(t-s)\,\varepsilon(u\,(s))\mathrm{d}s, \varepsilon\,(w - \dot{u}(t))\right)_{\mathcal{Q}} + j_{ad}(\beta(t), u(t), w - \dot{u}(t)) \\ + j_\tau(w) - j_\tau\,(\dot{u}(t)) \geq (f(t), w - \dot{u}(t))_V\,, \text{ for all } w \in V, \ \forall\, t \in [0,T]\,, \end{cases}$$

(5.4) $$\dot{\beta}\,(t) = H_{ad}\,(\beta(t), |R_\tau\,(u_\tau\,(t))|)\,, \ \forall t \in [0,T]\,,$$

(5.5) $$\text{(i) } u(0) = u_0, \ \text{(ii) } \beta(0) = \beta_0.$$

Clearly, in the view of Theorem 4.1, we have the following result.

COROLLARY 5.2. *Assume that* (3.12)–(3.16), (3.18)–(3.20) *and* (5.1) *are fulfilled. Then Problem* (5.3)–(5.5) *has a unique solution* $\{u, \beta\}$ *which satisfies*

(5.6) $$u \in C^1\,([0,T]\,; V)\,.$$

(5.7) $$\beta \in C^1\,([0,T]\,; L^\infty(\Gamma_3))\,, \ 0 \leqslant \beta\,(t) \leqslant 1, \ a.e. x \in \Gamma_3, \ \forall t \in [0,T]\,.$$

In the sequel, we consider a fully discrete approximation scheme for Problem (5.3)–(5.5). To this end, we introduce two finite-dimensional spaces $V^h \subset V$ and $B^h \subset L^2(\Gamma_3)$, approximating the spaces V and $L^2(\Gamma_3)$, respectively. Here $h > 0$ is a discretization parameter. Let $\mathcal{P}_{B^h} : L^2(\Gamma_3) \to B^h$ be the orthogonal projection operator defined by:

$$\left(\mathcal{P}_{B^h}\gamma, \gamma^h\right)_{L^2(\Gamma_3)} = \left(\gamma, \gamma^h\right)_{L^2(\Gamma_3)}, \ \forall \ \left(\gamma, \gamma^h\right) \in L^2(\Gamma_3) \times B^h,$$

which satisfies

$$\|\mathcal{P}_{B^h}\gamma\|_{L^2(\Gamma_3)} \leq \|\gamma\|_{L^2(\Gamma_3)}, \ \ \forall \gamma \in L^2(\Gamma_3). \tag{5.8}$$

Let $N \in \mathbb{N}^*$, let $k = \dfrac{T}{N}$ be the stepsize of a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < ... < t_N = T$, $t_n = nk$, $1 \leqslant n \leqslant N$. For a continuous function $w \in C([0,T]; X)$ with values in a normed space X, we use the notation $w_n = w(t_n)$, and for a sequence $\{w_n\}_{n=0}^N$, we denote $\delta w_n = \dfrac{w_n - w_{n-1}}{k}$, $1 \leqslant n \leqslant N$. Next, we make the following additional regularities

$$(i) \ u \in W^{2,1}(0,T;V), \ \ (ii) \ \mathcal{G} \in W^{1,\infty}(0,T;\mathbf{Q}_\infty). \tag{5.9}$$

Now, a fully discrete scheme for Problem (5.3)–(5.5) is the following.

Problem 5.3 (P_V^{hk}). Find a displacement field $u^{hk} = \left\{u_n^{hk}\right\}_{n=0}^N \subset V^h$, a velocity field $v^{hk} = \left\{v_n^{hk}\right\}_{n=1}^N \subset V^h$ and an adhesion field $\beta^{hk} = \left\{\beta_n^{hk}\right\}_{n=0}^N \subset B^h$ with

$$\text{(i)} \ u_0^{hk} = u_0^h, \ \text{(ii)} \ \beta_0^{hk} = \beta_0^h. \tag{5.10}$$

such that for all $n = 1, 2, ..., N$,

$$\begin{cases} \left(\mathcal{A}\left(\varepsilon(v_n^{hk})\right), \varepsilon\left(w^h - v_n^{hk}\right)\right)_{\mathcal{Q}} + \left(\mathcal{B}\left(\varepsilon(u_{n-1}^{hk})\right), \varepsilon(w^h - v_n^{hk})\right)_{\mathcal{Q}} \\ + \left(k\displaystyle\sum_{j=0}^{n-1}\mathcal{G}\left(t_n - t_j\right)\varepsilon(u_j^{hk}), \varepsilon\left(w^h - v_n^{hk}\right)\right) \\ + j_{ad}(\beta_{n-1}^{hk}, u_{n-1}^{hk}, w^h - v_n^{hk}) + j_\tau(w^h) - j_\tau\left(v_n^{hk}\right) \\ \geq \left(f_n, w^h - v_n^{hk}\right)_V, \text{ for all } w^h \in V^h, \end{cases} \tag{5.11}$$

$$\delta\beta_n^{hk} = \mathcal{P}_{B^h} H_{ad}\left(\beta_{n-1}^{hk}, \left|R_\tau\left(\left(u_{n-1}^{hk}\right)_\tau\right)\right|\right), \ \text{ on } \Gamma_3, \tag{5.12}$$

$$u_n^{hk} = u_0^h + k\sum_{j=1}^{n} v_j^{hk}. \tag{5.13}$$

Here, u_0^h and β_0^h are suitable approximations of the initial values u_0 and β_0, respectively. We assume that

$$\text{(5.14)}\qquad \text{(i)}\ u_0^h \in V^h,\ \ \text{(ii)}\ \beta_0^h \in B^h.$$

Under the conditions (3.12)–(3.13), (3.15)–(3.16), (3.18), (5.1), ((*ii*) 5.9) and (5.14), using classical results on variational inequalities, it is easy to verify that Problem P_V^{hk} has a unique solution. We turn now to estimate the numerical errors

$$\text{(5.15)}\qquad \text{(i)}\ a_n = \beta_n - \beta_n^{hk},\ \ \text{(ii)}\ e_n = v_n - v_n^{hk},\ 1 \leqslant n \leqslant N.$$

In the rest of this paper, we denote by

$$\text{(5.16)}\qquad v = \dot{u},$$

then, we get

$$\text{(5.17)}\qquad u(t) = \int_0^t v(s)\,\mathrm{d}s + u_0,\ \ \forall t \in [0,T].$$

Now, using (5.13), (5.17), ((ii) 5.15) and ((i) 5.9), we obtain

$$\begin{cases} \left\|u_n - u_n^{hk}\right\|_V \leq \left\|u_0 - u_0^h\right\|_V + k\sum_{j=1}^{n}\left\|v_j - v_j^{hk}\right\|_V \\ \qquad + \left\|\int_0^{t_n} v(s)\,\mathrm{d}s - k\sum_{j=1}^{n} v_j\right\|_V \\ \leq \left\|u_0 - u_0^h\right\|_V + k\sum_{j=1}^{n}\|e_j\|_V + ck\|\dot{v}\|_{L^1(0,T;V)}, \end{cases}$$

which gives

$$\text{(5.18)}\qquad \left\|u_n - u_n^{hk}\right\|_V \leq \left\|u_0 - u_0^h\right\|_V + k\sum_{j=1}^{n}\|e_j\|_V + ck,\ 1 \leqslant n \leqslant N.$$

Denote

$$\text{(5.19)}\quad H_{1,n} = \int_0^{t_n} H_{ad}\left(\beta(s), |R_\tau((u_\tau(s)))|\right)\mathrm{d}s - k\sum_{j=0}^{n-1} H_{ad}\left(\beta_j, \left|R_\tau\left((u_j)_\tau\right)\right|\right).$$

Using (5.6), we obtain

$$\text{(5.20)}\qquad \|u(s) - u_{j-1}\|_V \leq k\|v\|_{C([0,T];V)},\ \ \forall s \in [t_{j-1}, t_j],\ 1 \leqslant j \leqslant N.$$

Also, using (5.7), we get

$$\text{(5.21)}\quad \|\beta(s) - \beta_{j-1}\|_{L^2(\Gamma_3)} \leq k\left\|\dot{\beta}\right\|_{C([0,T];L^2(\Gamma_3))},\ \ \forall s \in [t_{j-1}, t_j],\ 1 \leqslant j \leqslant N.$$

From (5.19), (3.16) and (2.4), it is easy to see that

$$\|H_{1,n}\|_{L^2(\Gamma_3)} \leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|\beta(s)-\beta_j\|_{L^2(\Gamma_3)}\,\mathrm{d}s + c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|u(s)-u_j\|_V\,\mathrm{d}s,$$

which, together with (5.20)–(5.21), gives

$$\|H_{1,n}\|_{L^2(\Gamma_3)} \leq ck,\ 1 \leqslant n \leqslant N. \tag{5.22}$$

On the other hand, using (5.12), we obtain

$$\beta_n^{hk} = \beta_0^h + k\sum_{j=0}^{n-1}\mathcal{P}_{B^h}H_{ad}\left(\beta_j^{hk}, \left|R_\tau\left(\left(u_j^{hk}\right)_\tau\right)\right|\right), \tag{5.23}$$

and keeping in mind (5.4) and ((*ii*) 5.5), we get

$$\beta_n - \beta_n^{hk} = \beta_0 + \int_0^{t_n} H_{ad}(\beta(s), |R_\tau((u_\tau(s)))|)\,\mathrm{d}s - \beta_0^h$$
$$- k\sum_{j=0}^{n-1}\mathcal{P}_{B^h}H_{ad}\left(\beta_j^{hk}, \left|R_\tau\left(\left(u_j^{hk}\right)_\tau\right)\right|\right),$$

which gives

$$\beta_n - \beta_n^{hk} = \beta_0 - \beta_0^h + H_{1,n} + H_{2,n} + k\sum_{j=0}^{n-1}(I-\mathcal{P}_{B^h})\,H_{ad}\left(\beta_j, \left|R_\tau\left((u_j)_\tau\right)\right|\right), \tag{5.24}$$

where

$$H_{2,n} = k\sum_{j=0}^{n-1}\left[\mathcal{P}_{B^h}H_{ad}\left(\beta_j, \left|R_\tau\left((u_j)_\tau\right)\right|\right) - \mathcal{P}_{B^h}H_{ad}\left(\beta_j^{hk}, \left|R_\tau\left(\left(u_j^{hk}\right)_\tau\right)\right|\right)\right].$$

Taking into account (5.8), (5.22), (5.24) and (3.16), one has

$$\left\|\beta_n - \beta_n^{hk}\right\|_{L^2(\Gamma_3)} \leq \left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)} + ck\sum_{j=0}^{n-1}\left\|\beta_j - \beta_j^{hk}\right\|_{L^2(\Gamma_3)}$$
$$+ck\sum_{j=0}^{n-1}\left\|u_j - u_j^{hk}\right\|_V + R_N\left(\dot\beta\right) + ck, \tag{5.25}$$

where

$$R_N\left(\dot\beta\right) = k\sum_{j=0}^{N}\left\|(I-\mathcal{P}_{B^h})\,H_{ad}\left(\beta_j, \left|R_\tau\left((u_j)_\tau\right)\right|\right)\right\|_{L^2(\Gamma_3)}. \tag{5.26}$$

Thus, we have

$$\|a_1\|_{L^2(\Gamma_3)} \leq c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)} + c\left\|u_0 - u_0^h\right\|_V + R_N\left(\dot\beta\right) + ck. \tag{5.27}$$

Moreover, keeping in mind (5.18) and using a discrete version of Gronwall's lemma in (5.25) we get

$$
\left\{
\begin{aligned}
&\|a_n\|_{L^2(\Gamma_3)} \leq c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)} + c\left\|u_0 - u_0^h\right\|_V \\
&+ck\sum_{j=1}^{n-1}\|e_j\|_V + cR_N\left(\dot{\beta}\right) + ck,\ 2 \leqslant n \leqslant N.
\end{aligned}
\right. \tag{5.28}
$$

Now, let us consider the variational inequality (5.3) at time $t = t_n$ with $w = v_n^{hk}$ and the variational inequality (5.11) with $w^h = w_n^h$, ($w_n^h \in V^h$, $1 \leq n \leq N$, are arbitrary), then adding (5.3) and (5.11), it leads to the following inequality

$$
\left\{
\begin{aligned}
&\left(\mathcal{A}\left(\varepsilon(v_n)\right) - \mathcal{A}\left(\varepsilon(v_n^{hk})\right), \varepsilon\left(v_n - v_n^{hk}\right)\right)_{\mathcal{Q}} \\
&\leq I_n + \left(\mathcal{A}\left(\varepsilon(v_n^{hk})\right), \varepsilon\left(w_n^h - v_n\right)\right)_{\mathcal{Q}} + \left(\mathcal{B}\left(\varepsilon(u_n)\right), \varepsilon(v_n^{hk} - v_n)\right)_{\mathcal{Q}} + \\
&\left(\mathcal{B}\left(\varepsilon(u_{n-1}^{hk})\right), \varepsilon(w_n^h - v_n^{hk})\right)_{\mathcal{Q}} + j_{ad}(\beta_n, u_n, v_n^{hk} - v_n) \\
&+j_{ad}(\beta_{n-1}^{hk}, u_{n-1}^{hk}, w_n^h - v_n^{hk}) + j_\tau(w_n^h) - j_\tau\left(v_n\right) \\
&-\left(f_n, w_n^h - v_n\right)_V,
\end{aligned}
\right. \tag{5.29}
$$

where

$$
\left\{
\begin{aligned}
&I_n = \left(\int_0^{t_n} \mathcal{G}\left(t_n - s\right)\varepsilon(u\left(s\right))\mathrm{d}s, \varepsilon\left(v_n^{hk} - v_n\right)\right)_{\mathcal{Q}} + \\
&+\left(k\sum_{j=0}^{n-1}\mathcal{G}\left(t_n - t_j\right)\varepsilon(u_j^{hk}), \varepsilon\left(w_n^h - v_n^{hk}\right)\right)_{\mathcal{Q}}.
\end{aligned}
\right. \tag{5.30}
$$

Using (5.6), ((ii) 5.9) and (5.20), we get

$$
\begin{aligned}
&\left\|\int_0^{t_n} \mathcal{G}\left(t_n - s\right)\varepsilon(u\left(s\right))\mathrm{d}s - k\sum_{j=0}^{n-1}\mathcal{G}\left(t_n - t_j\right)\varepsilon(u_j^{hk})\right\|_{\mathcal{Q}} \\
&\quad\leq \left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(\mathcal{G}\left(t_n - s\right)\left(\varepsilon(u\left(s\right)) - \varepsilon(u_j)\right)\right)\mathrm{d}s\right\|_{\mathcal{Q}} \\
&\quad+\left\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(\mathcal{G}\left(t_n - s\right) - \mathcal{G}\left(t_n - t_j\right)\right)\varepsilon(u_j)\mathrm{d}s\right\|_{\mathcal{Q}}
\end{aligned}
$$

$$+\left\| k\sum_{j=0}^{n-1}\mathcal{G}\left(t_n-t_j\right)\left(\varepsilon(u_j)-\varepsilon(u_j^{hk})\right)\right\|_{\mathcal{Q}}$$

$$\leq ck+ck\sum_{j=0}^{n-1}\left\|u_j-u_j^{hk}\right\|_V,\ 1\leqslant n\leqslant N,$$

and writing

$$\begin{cases} I_n=\left(\int_0^{t_n}\mathcal{G}\left(t_n-s\right)\varepsilon(u\left(s\right))\mathrm{d}s-k\sum_{j=0}^{n-1}\mathcal{G}\left(t_n-t_j\right)\varepsilon(u_j^{hk}),\varepsilon\left(v_n^{hk}-v_n\right)\right)_{\mathcal{Q}} \\ +\left(\int_0^{t_n}\mathcal{G}\left(t_n-s\right)\varepsilon(u\left(s\right))\mathrm{d}s-k\sum_{j=0}^{n-1}\mathcal{G}\left(t_n-t_j\right)\varepsilon(u_j^{hk}),\varepsilon\left(v_n-w_n^{h}\right)\right)_{\mathcal{Q}} \\ +\left(\int_0^{t_n}\mathcal{G}\left(t_n-s\right)\varepsilon(u\left(s\right))\mathrm{d}s,\varepsilon\left(w_n^{h}-v_n\right)\right)_{\mathcal{Q}}, \end{cases}$$

we obtain

$$\begin{cases} I_n\leq\left(ck+ck\sum_{j=0}^{n-1}\left\|u_j-u_j^{hk}\right\|_V\right)\left(\left\|v_n^{hk}-v_n\right\|_V+\left\|w_n^{h}-v_n\right\|_V\right) \\ +\left(\int_0^{t_n}\mathcal{G}\left(t_n-s\right)\varepsilon(u\left(s\right))\mathrm{d}s,\varepsilon\left(w_n^{h}-v_n\right)\right),\ 1\leqslant n\leqslant N. \end{cases} \tag{5.31}$$

To continue, using (3.12), we get

$$\begin{cases} \left(\mathcal{A}\left(\varepsilon(v_n^{hk})\right),\varepsilon\left(w_n^{h}-v_n\right)\right)_{\mathcal{Q}}=\left(\mathcal{A}\left(\varepsilon(v_n^{hk})\right)-\mathcal{A}\left(\varepsilon(v_n)\right),\varepsilon\left(w_n^{h}-v_n\right)\right)_{\mathcal{Q}} \\ +\left(\mathcal{A}\left(\varepsilon(v_n)\right),\varepsilon\left(w_n^{h}-v_n\right)\right)_{\mathcal{Q}} \\ \leq c\left\|e_n\right\|_V\left\|w_n^{h}-v_n\right\|_V+\left(\mathcal{A}\left(\varepsilon(v_n)\right),\varepsilon\left(w_n^{h}-v_n\right)\right)_{\mathcal{Q}}. \end{cases} \tag{5.32}$$

Also, using (3.13), we have

$$\begin{cases} \left(\mathcal{B}\left(\varepsilon(u_n)\right),\varepsilon(v_n^{hk}-v_n)\right)_{\mathcal{Q}}+\left(\mathcal{B}\left(\varepsilon(u_{n-1}^{hk})\right),\varepsilon(w_n^{h}-v_n^{hk})\right)_{\mathcal{Q}}= \\ \left(\mathcal{B}\left(\varepsilon(u_n)\right)-\mathcal{B}\left(\varepsilon(u_{n-1}^{hk})\right),\varepsilon(v_n^{hk}-v_n)\right)_{\mathcal{Q}} \\ +\left(\mathcal{B}\left(\varepsilon(u_{n-1}^{hk})\right)-\mathcal{B}\left(\varepsilon(u_n)\right),\varepsilon(w_n^{h}-v_n)\right)_{\mathcal{Q}}+\left(\mathcal{B}\left(\varepsilon(u_n)\right),\varepsilon(w_n^{h}-v_n)\right)_{\mathcal{Q}} \\ \leq c\left\|u_n-u_{n-1}^{hk}\right\|_V\left(\left\|v_n^{hk}-v_n\right\|_V+\left\|w_n^{h}-v_n\right\|_V\right) \\ +\left(\mathcal{B}\left(\varepsilon(u_n)\right),\varepsilon(w_n^{h}-v_n)\right)_{\mathcal{Q}}, \end{cases} \tag{5.33}$$

and using (3.25), we obtain

(5.34)
$$\begin{cases} j_{ad}(\beta_n, u_n, v_n^{hk} - v_n) + j_{ad}(\beta_{n-1}^{hk}, u_{n-1}^{hk}, w_n^h - v_n^{hk}) \\ = j_{ad}(\beta_n, u_n, v_n^{hk} - v_n) - j_{ad}(\beta_{n-1}^{hk}, u_{n-1}^{hk}, v_n^{hk} - v_n) + \\ j_{ad}(\beta_{n-1}^{hk}, u_{n-1}^{hk}, w_n^h - v_n) - j_{ad}(\beta_n, u_n, w_n^h - v_n) \\ + j_{ad}(\beta_n, u_n, w_n^h - v_n) \\ \leq c\left(\left\|\beta_n - \beta_{n-1}^{hk}\right\|_{L^2(\Gamma_3)} + \left\|u_n - u_{n-1}^{hk}\right\|_V\right)\left(\left\|v_n^{hk} - v_n\right\|_V + \left\|w_n^h - v_n\right\|_V\right) \\ + j_{ad}(\beta_n, u_n, w_n^h - v_n). \end{cases}$$

Denote

(5.35)
$$\begin{cases} r_N\left(t_n, w_n^h\right) = \left(\mathcal{A}\left(\varepsilon(v_n)\right) + \mathcal{B}\left(\varepsilon(u_n)\right), \varepsilon\left(w_n^h - v_n\right)\right)_{\mathcal{Q}} \\ + \left(\int_0^{t_n} \mathcal{G}\left(t_n - s\right)\varepsilon(u\left(s\right))\mathrm{d}s, \varepsilon\left(w_n^h - v_n\right)\right)_{\mathcal{Q}} + j_{ad}(\beta_n, u_n, w_n^h - v_n) \\ + j_\tau(w_n^h) - j_\tau\left(v_n\right) - \left(f_n, w_n^h - v_n\right)_V, \ 1 \leqslant n \leqslant N, \end{cases}$$

and keeping in mind (3.12), (5.18), (5.29), (5.31)–(5.34) and using the inequality

$$\lambda\alpha \leq \frac{\lambda^2}{4\theta} + \theta\alpha^2, \ \forall\theta, \ \lambda, \ \alpha \in \mathbb{R}, \ \theta > 0,$$

we obtain

$$(5.36) \quad \begin{cases} \|e_n\|_V^2 \leq c\left\|u_0 - u_0^h\right\|_V^2 + c\left\|\beta_n - \beta_{n-1}\right\|_{L^2(\Gamma_3)}^2 + c\left\|a_{n-1}\right\|_{L^2(\Gamma_3)}^2 \\ + c\left\|u_n - u_{n-1}\right\|_V^2 + c\left\|u_{n-1} - u_{n-1}^{hk}\right\|_V^2 + ck\sum_{j=1}^{n-1}\|e_j\|_V^2 \\ + c\left\|w_n^h - v_n\right\|_V^2 + c\left|r_N\left(t_n, w_n^h\right)\right| + ck^2, \ 2 \leqslant n \leqslant N, \end{cases}$$

and for $n = 1$, we have

$$(5.37) \quad \begin{cases} \|e_1\|_V^2 \leq c\left\|u_0 - u_0^h\right\|_V^2 + c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)}^2 \\ + c\left\|w_1^h - v_1\right\|_V^2 + c\left|r_N\left(t_1, w_1^h\right)\right| + ck^2. \end{cases}$$

Using (5.36), (5.20)–(5.21) and (5.27)–(5.28), we obtain

$$\begin{cases} \|e_n\|_V^2 \leq c\left\|u_0 - u_0^h\right\|_V^2 + c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)}^2 + cR_N^2\left(\dot{\beta}\right) \\ + ck\sum_{j=1}^{n-1}\|e_j\|_V^2 + c\left\|w_n^h - v_n\right\|_V^2 \\ + c\left|r_N\left(t_n, w_n^h\right)\right| + ck^2, \ 2 \leqslant n \leqslant N. \end{cases}$$

which, together with (5.28), gives

$$\begin{cases} \|e_n\|_V^2 + \|a_n\|_{L^2(\Gamma_3)}^2 \le c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)}^2 + c\left\|u_0 - u_0^h\right\|_V^2 \\ +cR_N^2\left(\dot{\beta}\right) + ck\sum_{j=1}^{n-1}\|e_j\|_V^2 + ck^2 \\ +c\max_{1\le n\le N}\left(\left\|w_n^h - v_n\right\|_V^2 + \left|r_N\left(t_n, w_n^h\right)\right|\right),\ 2 \leqslant n \leqslant N. \end{cases}$$

Thus, using a discrete version of Gronwall's lemma in the last inequality, we deduce the following

$$\begin{cases} \|e_n\|_V^2 + \|a_n\|_{L^2(\Gamma_3)}^2 \le c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)}^2 + c\left\|u_0 - u_0^h\right\|_V^2 \\ +cR_N^2\left(\dot{\beta}\right) + \\ c\max_{1\le n\le N}\left(\left\|w_n^h - v_n\right\|_V^2 + \left|r_N\left(t_n, w_n^h\right)\right|\right) + ck^2,\ 2 \leqslant n \leqslant N \end{cases}$$

which together with (5.27) and (5.37) implies that there exists a positive constant c such that

$$\text{(5.38)}\qquad \begin{cases} \max_{1\le n\le N}\|e_n\|_V + \max_{1\le n\le N}\|a_n\|_{L^2(\Gamma_3)} \le c\left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)} \\ +c\left\|u_0 - u_0^h\right\|_V + cR_N\left(\dot{\beta}\right) \\ +c\max_{1\le n\le N}\left(\left\|w_n^h - v_n\right\|_V + \left|r_N\left(t_n, w_n^h\right)\right|^{1/2}\right) + ck, \end{cases}$$

where $w_n^h \in V^h$, $1 \le n \le N$, are arbitrary. Here and below, c is a positive constant independent of $t_n \in (0,T)$, k, h or of w_n^h and whose value may change from line to line. The main result of this section is the following.

Theorem 5.4. *Assume that* (3.12)–(3.13), (3.15)–(3.16), (3.18)–(3.20), (5.1), (5.9) *and* (5.14) *are fulfilled. Then, we have the error estimate* (5.38).

The estimate (5.38) is a basis for deriving error estimates for finite element solutions, and to that end, we briefly describe how to construct the spaces V^h and B^h. Details can be found in [8, 21]. In the rest of this paper, we assume that Ω is a polygonal/polyhedral domain and its boundary Γ is split into three relatively closed subsets Γ_1, Γ_2 and Γ_3, with mutually disjoint interiors. For $1 \le j \le 3$, we write $\Gamma_j = \bigcup_{i=1}^{i_j}\Gamma_j^i$ such that on each Γ_j^i, the unit outward normal vector is constant. Let $\mathcal{T}^h$ be a regular family of finite element partitions of $\bar{\Omega}$ assumed to be compatible with the decomposition of Γ. Let $h \in]0,1[$ be the maximal diameter of the elements. Let $V^h \subset V$ be the finite element space consisting of continuous piecewise linear functions corresponding to the partition $\mathcal{T}^h$. Denote by $\mathcal{T}^h_{\Gamma_3}$ the partition of Γ_3 induced by the triangulation $\mathcal{T}^h$. Then the space $L^2(\Gamma_3)$ is approximated by

$$B^h = \left\{ \gamma^h \in L^2(\Gamma_3) : \gamma^h_{|\Sigma} \in \mathbb{R},\ \forall \Sigma \in \mathcal{T}^h_{\Gamma_3} \right\}. \tag{5.39}$$

Let $\mathcal{P}_{B^h} : L^2(\Gamma_3) \to B^h$ be the orthogonal projection operator. Then $\mathcal{P}_{B^h}$ satisfies (5.8) and we have the following error estimate

$$\|\gamma - \mathcal{P}_{B^h}\gamma\|_{L^2(\Gamma_3)} \le ch \|\gamma\|_{H^1(\Gamma_3)},\ \forall \gamma \in H^1(\Gamma_3). \tag{5.40}$$

Let $\Pi^h : V \to V^h$ be the finite element interpolation operator (see [8]). Also we use the same symbol Π^h for the interpolation on Γ_3. Now, for the initial values, we assume

$$\text{(i) } u_0 \in H^2(\Omega),\ \text{(ii) } \beta_0 |_{\Gamma_3^i} \in H^1\left(\Gamma_3^i\right),\ 1 \leqslant i \leqslant i_3, \tag{5.41}$$

and take

$$\text{(i) } u_0^h = \Pi^h u_0,\ \text{(ii) } \beta_0^h = \mathcal{P}_{B^h}\beta_0. \tag{5.42}$$

We assume that the restriction of β to Γ_3^i satisfies

$$\beta |_{\Gamma_3^i} \in C^1\left([0,T]; H^1\left(\Gamma_3^i\right)\right),\ 1 \leqslant i \leqslant i_3. \tag{5.43}$$

Also, we assume

$$\text{(i) } v \in C\left([0,T]; H^2\left(\Omega, \mathbb{R}^d\right)\right),\ \text{(ii) } \sigma\nu \in C\left([0,T]; L^2\left(\Gamma_3, \mathbb{R}^d\right)\right), \tag{5.44}$$

where

$$\sigma(t) = \mathcal{A}(\varepsilon(v(t))) + \mathcal{B}(\varepsilon(u(t))) + \int_0^t \mathcal{G}(t-s)\,\varepsilon(u(s))\mathrm{d}s,\ \forall t \in [0,T]. \tag{5.45}$$

We have the following error estimate of the fully discrete solution of Problem (5.3)–(5.5).

THEOREM 5.5. *Assume that the hypothesis of Theorem* 5.4 *and the assumptions* (5.41)–(5.43) *are fulfilled. Then:*

(1) *Under the assumption* ((*i*) 5.44), *we have*

$$\max_{1\le n\le N} \|e_n\|_V + \max_{1\le n\le N} \|a_n\|_{L^2(\Gamma_3)} \le c\left(h^{1/2} + k\right). \tag{5.46}$$

(2) *Under the assumptions* ((*i*) 5.44) *and* ((*ii*) 5.44), *we have*

$$\max_{1\le n\le N} \|e_n\|_V + \max_{1\le n\le N} \|a_n\|_{L^2(\Gamma_3)} \le c\left(h^{3/4} + k\right). \tag{5.47}$$

(3) *Under the assumptions* ((*i*) 5.44), ((*ii*) 5.44) *and the following additional regularity assumption*

$$v_\tau \mid_{\Gamma_3^i} \in C\left([0,T]\,; H^2\left(\Gamma_3^i, \mathbb{R}^d\right)\right), \quad 1 \le i \le i_3, \tag{5.48}$$

we have the optimal order error estimate

$$\max_{1\le n\le N} \|e_n\|_V + \max_{1\le n\le N} \|a_n\|_{L^2(\Gamma_3)} \le c\,(h+k)\,. \tag{5.49}$$

Proof. Using (5.41)–(5.43), (5.4), (5.26) and keeping in mind (5.40), we have

$$\left\|u_0 - u_0^h\right\|_V + \left\|\beta_0 - \beta_0^h\right\|_{L^2(\Gamma_3)} + R_N\left(\dot{\beta}\right) \le ch. \tag{5.50}$$

From (3.12)–(3.15), (3.18), (5.1)–(5.2), (5.6)–(5.7) and (5.35), we obtain

$$\left|r_N\left(t_n, \Pi^h v_n\right)\right| \le c\left\|\Pi^h v_n - v_n\right\|_V, \; 1 \le n \le N. \tag{5.51}$$

Moreover, under the additional regularity condition ((*i*) 5.44), (see [8]), we get

$$\left\|v_n - \Pi^h v_n\right\|_V \le ch\,\|v_n\|_{H^2\left(\Omega,\mathbb{R}^d\right)}, \; 1 \le n \le N, \tag{5.52}$$

and choosing $w_n^h = \Pi^h v_n$ in (5.38), for $n = 1, 2, ..., N$, we get (5.46). To continue, using (5.35) and ((ii) 5.44), we obtain

$$\begin{aligned} r_N\left(t_n, \Pi^h v_n\right) &= \left(\sigma_n, \varepsilon\left(\Pi^h v_n - v_n\right)\right)_{\mathcal{Q}} + j_{ad}(\beta_n, u_n, \Pi^h v_n - v_n) \\ &\quad + j_\tau(\Pi^h v_n) - j_\tau\left(v_n\right) - \left(f_n, \Pi^h v_n - v_n\right)_V \\ &= \int_{\Gamma_3} \left(\sigma_n\right)_\tau \cdot \left(\left(\Pi^h v_n\right)_\tau - \left(v_n\right)_\tau\right) + j_{ad}(\beta_n, u_n, \Pi^h v_n - v_n) \\ &\quad + j_\tau(\Pi^h v_n) - j_\tau\left(v_n\right), \end{aligned}$$

which gives

$$\left|r_N\left(t_n, \Pi^h v_n\right)\right| \le c\left\|\left(v_n\right)_\tau - \left(\Pi^h v_n\right)_\tau\right\|_{L^2\left(\Gamma_3,\mathbb{R}^d\right)}.$$

Therefore, taking into account (5.50) and (5.52) the estimate (5.38) reduces to

$$\left\{\begin{aligned} \max_{1\le n\le N} \|e_n\|_V + \max_{1\le n\le N} \|a_n\|_{L^2(\Gamma_3)} \le c \max_{1\le n\le N} \left\|\left(v_n\right)_\tau - \left(\Pi^h v_n\right)_\tau\right\|^{1/2}_{L^2\left(\Gamma_3,\mathbb{R}^d\right)} \\ +ch + ck. \end{aligned}\right. \tag{5.53}$$

Now, since the regularity assumption ((i) 5.44), implies that

$$v_\tau \mid_{\Gamma_3^i} \in C\left([0,T]\,; H^{3/2}\left(\Gamma_3^i, \mathbb{R}^d\right)\right), \quad 1 \le i \le i_3,$$

we obtain the following interpolation error estimate (see [8])

$$\left\| (v_n)_\tau - \left(\Pi^h (v_n)\right)_\tau \right\|_{L^2(\Gamma_3,\mathbb{R}^d)} \leq c h^{3/2} \sum_{i=1}^{i_3} \|(v_n)_\tau\|_{H^{3/2}(\Gamma_3^i,\mathbb{R}^d)} . \tag{5.54}$$

Thus, the estimate (5.47) is now a direct consequence of (5.53) and (5.54). Finally, under the additional regularity (5.48) condition (5.54), see [8], becomes

$$\left\| (v_n)_\tau - \left(\Pi^h (v_n)\right)_\tau \right\|_{L^2(\Gamma_3,\mathbb{R}^d)} \leq c h^{2} \sum_{i=1}^{i_3} \|(v_n)_\tau\|_{H^{2}(\Gamma_3^i,\mathbb{R}^d)} ,$$

which, together with (5.53), gives (5.49). □

Remark 5.6. We notice that, under the same assumptions of Theorem 4.1 and Theorem 5.5, if we take in (3.6)

$$p_\tau(\beta, u_\tau) = c_\tau \beta^2 R_\tau(u_\tau),$$

and replace condition (3.15) by

$$c_\tau \in L^\infty(\Gamma_3),\ c_\tau \geq 0 \text{ a.e } x \in \Gamma_3,$$

where c_τ is material parameter, and define the functional j_{ad} in (3.24) by

$$j_{ad}(\beta, u, w) = \int_{\Gamma_3} c_\tau \beta^2 R_\tau(u_\tau) \cdot w_\tau \mathrm{d}a,\ \forall (\beta, u, w) \in L^\infty(\Gamma_3) \times V \times V.$$

Then, Theorem 4.1 and Theorem 5.5 hold true.

Indeed, in this case, the functional j_{ad} satisfies

$$\begin{cases} \text{There exists } L_{ad} > 0 \text{ such that} \\ |j_{ad}(\beta_1, u_1, w) - j_{ad}(\beta_2, u_2, w)| \\ \leqslant L_{ad}\left(\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} + \|u_1 - u_2\|_V\right)\|w\|_V,\ \forall u_1, u_2, w \in V, \\ \forall \beta_1, \beta_2 \in L^\infty(\Gamma_3),\ 0 \leqslant \beta_1 \leqslant 1,\ 0 \leqslant \beta_2 \leqslant 1,\ a.e. x \in \Gamma_3 , \end{cases}$$

which implies that Lemma 4.7 is still valid. Moreover, since B^h is defined by (5.39), then the orthogonal projection operator $\mathcal{P}_{B^h} : L^2(\Gamma_3) \to B^h$ satisfies (see [21])

$$\|\mathcal{P}_{B^h}\gamma\|_{L^\infty(\Gamma_3)} \leq \|\gamma\|_{L^\infty(\Gamma_3)},\ \forall \gamma \in L^\infty(\Gamma_3).$$

Now, using a discrete version of Gronwall's lemma in (5.23), we can show that $\left\{\|\beta_n^{hk}\|_{L^\infty(\Gamma_3)}\right\}_{0 \leqslant n \leqslant N}$ is uniformly bounded. Consequently, we get

$$\left| j_{ad}(\beta_{n-1}^{hk}, u_{n-1}^{hk}, w) - j_{ad}(\beta_n, u_n, w) \right|$$

$$\leqslant c\left(\left\|\beta_n-\beta_{n-1}^{hk}\right\|_{L^2(\Gamma_3)}+\left\|u_n-u_{n-1}^{hk}\right\|_V\right)\|w\|_V\,,\ \forall w\in V,$$

which implies that (5.34) is straightforward. Thus, using similar arguments we can prove that the results in Theorem 4.1 and Theorem 5.5 are still true.

Remark 5.7. We note that if the regularity conditions are different or the finite element space is changed, then we will have another different error estimate which can be again derived from (5.38).

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Received 25 July 2016

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