ON SOME FUNCTIONAL EQUATIONS RELATED TO VARIOUS ENTROPIES

PREM NATH and D.K. SINGH*

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The general solutions of the functional equations

$$f(pq) = g(p)h(q) + qh(p)$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i)\sum_{j=1}^{m} h(q_j) + \sum_{i=1}^{n} h(p_i)$$

in which f, g, h are real-valued mappings each with domain I, the unit closed interval, and $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers have been obtained. Some of the solutions are related to the Shannon entropies and the entropies of degree α.

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1. INTRODUCTION

For $n = 1, 2, \ldots$; let $\Gamma_n = \left\{(p_1, \ldots, p_n) : p_i \geq 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} p_i = 1\right\}$ denote the set of all n-component finite discrete complete probability distributions with nonnegative elements. Throughout this paper, $\mathbb{R}$ will denote the set of all real numbers and $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$, the unit closed interval.

Consider the mappings $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ which satisfy the Pexider functional equation

$$f(pq) = g(p)h(q) \tag{1.1}$$

for all $p \in I$, $q \in I$. If we replace p by p_i; q by q_j, in (1.1) and sum the resulting equations with respect $i = 1$ to n and $j = 1$ to $m, n \geq 2$, $m \geq 2$ integers; we obtain the sum form functional equation

* Corresponding author.

(1.2) $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} h(q_j)$$

valid for all probability distributions $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Nath and Singh [11] obtained the general solutions of (1.2) by taking $n \geq 3$ and $m \geq 3$ fixed integers.

Every solution (f, g, h) of (1.1) satisfies (1.2) for all probability distributions $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers. However, the converse is not true ([11], Theorem 2).

Consider the functional equation

(FE1) $$f(pq) = g(p)h(q) + qh(p)$$

in which $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ are unknown mappings and $p \in I$, $q \in I$. The functional equation (1.1) cannot be regarded as a particular case of (FE1) as we cannot allow the possibility $qh(p) = 0$, $p \in I$, $q \in I$ in it. So, (FE1) may be regarded as an enlargement of the functional equation (1.1). Now we replace p by p_i; q by q_j in (FE1) and sum the resulting equation with respect $i = 1$ to n and $j = 1$ to $m; n \geq 2$ and $m \geq 2$ integers, we obtain the functional equation

(FE2) $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} h(q_j) + \sum_{i=1}^{n} h(p_i)$$

valid for all probability distributions $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$.

The authors [13] proposed the functional equation

(1.3) $$f_r(pq) = qf_r(p) + a_1 f_{r-1}(p) f_1(q) + \ldots + a_{r-1} f_1(p) f_{r-1}(q) + p\, f_r(q)$$

where $p \in I$, $q \in I$, $f_i : I \to \mathbb{R}$ are unknown mappings; $i = 1, \ldots, r$, $r \geq 3$ an integer; and $a_1, \ldots, a_{r-1}$ are given real constants. The general solutions of the functional (1.3) are not known to us. However, some special cases of it seem to be important. For instance, consider $r = 3$. The equation (1.3) reduces to

(1.4) $$f_3(pq) = qf_3(p) + a_1 f_2(p) f_1(q).$$

Let us write $f_3 = f$, $a_1 f_2 = g$ and $f_1 = h$. Then (1.4) reduces to

(1.5) $$f(pq) = g(p)h(q) + qf(p).$$

The functional equation (FE1) is a Pexider-type generalization of (1.5) containing three unknown mappings.

If $g(x) = x$ for all $x \in I$, then (FE2) reduces to

(1.6) $$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} h(p_i) + \sum_{j=1}^{m} h(q_j).$$

The general solutions of (1.6), for fixed integers $n \geq 3$ and $m \geq 3$, were obtained by the authors [10]. If $g(x) = x^\alpha$ for all $x \in I$; α being a fixed positive real power which satisfies the convention $0^\alpha := 0$, $1^\alpha := 1$; and $h(x) = f(x)$ for all $x \in I$, then (FE2) reduces to the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} f(q_j) \tag{1.7}$$

whose importance in information theory has already been discussed by Nath [9]. Hence it is desirable to find solutions of (FE2) and discuss their importance in information theory.

The object of this paper is to investigate the general solutions of (FE1) for all $p \in I$, $q \in I$; and of (FE2) when $n \geq 3$ and $m \geq 3$ are fixed integers.

2. SOME DEFINITIONS AND RESULTS

In this section, we mention some definitions and known results needed for the development of Sections 3 to 5.

A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive if $A(x+y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative if $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in \,]0,1]$, $q \in \,]0,1]$.

Result 2.1 ([8]). Let $f : I \to \mathbb{R}$ be a mapping which satisfies the equation $\sum_{i=1}^{n} f(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$; c being a given real constant and $n \geq 3$ a fixed integer. Then there exists an additive mapping $b : \mathbb{R} \to \mathbb{R}$ such that $f(p) = b(p) - \frac{1}{n} b(1) + \frac{c}{n}$ for all $p \in I$.

Result 2.2 ([12]). Let $G : I \to \mathbb{R}$, $H : I \to \mathbb{R}$ be mappings which satisfy the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} H(p_i q_j) = \sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} H(p_i) + n(m-1)H(0) \tag{2.1}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, any general solution (G, H) of (2.1) is of the form (for all $p \in I$):

$$\left.\begin{array}{ll} \text{(i)} & G(p) = \bar{b}_1(p) + G(0);\ \bar{b}_1(1) = -\,nG(0) \\ \text{(ii)} & H(p) = \bar{b}_2(p) + H(0);\ \bar{b}_2(1) = -\,mH(0) - c,\ c \neq 0 \end{array}\right\} \tag{2.2}$$

or

$$\text{(2.3)} \qquad \left.\begin{array}{ll} \text{(i)} & G \text{ an arbitrary real-valued mapping} \\ \text{(ii)} & H(p) = b_1(p) + H(0);\ b_1(1) = -\,mH(0) \end{array}\right\}$$

or

$$\text{(2.4)} \qquad \left.\begin{array}{ll} \text{(i)} & G(p) = b_2(p) + G(0);\ b_2(1) = 1 - nG(0) \\ \text{(ii)} & H(p) = -\,mH(0)p + a(p) + D(p,p) + H(0) \end{array}\right\}$$

or

(2.5)

$$\left.\begin{array}{l} G(p) = M(p) + b_4(p) + G(0);\ b_4(1) = -\,nG(0) \\ H(p) = -\,[b_3(1) + mH(0)]M(p) + b_3(p) + H(0);\ b_3(1)+mH(0) \neq 0 \end{array}\right\}$$

where $b_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3, 4)$; $\bar{b}_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2)$ are additive mappings; $M : I \to \mathbb{R}$ a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$; $a : \mathbb{R} \to \mathbb{R}$ is additive; $D : \mathbb{R} \times I$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ additive in both variables such that

$$\text{(2.6)} \qquad a(1) = E(1,1)$$

and

$$\text{(2.7)} \qquad D(pq,pq) = D(pq,p) + D(pq,q) + E(p,q)$$

for all $p \in I$, $q \in I$.

Note. From (2.6) and (2.7), it is easy to conclude that

$$\text{(2.8)} \qquad a(1) + D(1,1) = 0.$$

3. ON THE FUNCTIONAL EQUATION (FE2)

In this section, we prove the following:

THEOREM 3.1. *Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ be mappings which satisfy the functional equation (FE2) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, any general solution (f, g, h) of (FE2) is one of the form (for all $p \in I$):*

$$(\text{S}_1) \qquad \left.\begin{array}{ll} \text{(i)} & f(p) = A_2(p) + f(0) \\ \text{(ii)} & g(p) = A_1(p) + g(0) \\ \text{(iii)} & h(p) = A_3(p) + h(0) \end{array}\right\}$$

or

$$(\mathrm{S}_2)\qquad \left.\begin{array}{rl} \text{(i)} & f(p) = A_5(p) + f(0) \\ \text{(ii)} & g \textit{ an arbitrary real-valued mapping} \\ \text{(iii)} & h(p) = A_4(p) + h(0) \end{array}\right\}$$

or

$$(\mathrm{S}_3)\qquad \left.\begin{array}{rl} \text{(i)} & f(p) = [g(1) + (n-1)g(0)]\{b_1(p) + [h(1) + (m-1)h(0)]p\} \\ & \qquad + A_6(p) + f(0) \\ \text{(ii)} & g \textit{ an arbitrary real-valued mapping} \\ \text{(iii)} & h(p) = b_1(p) + [h(1) + (m-1)h(0)]p + h(0) \end{array}\right\}$$

or

$$(\mathrm{S}_4)\qquad \left.\begin{array}{rl} \text{(i)} & f(p) = [g(1) + (n-1)g(0)]\,\{[h(1) - h(0)]p + a(p) + D(p,p)\} \\ & \qquad + A_6(p) + f(0) \\ \text{(ii)} & g(p) = [g(1) + (n-1)g(0)]b_2(p) + g(0),\ \ g(1) + (n-1)g(0) \neq 0 \\ \text{(iii)} & h(p) = [h(1) - h(0)]p + a(p) + D(p,p) + h(0) \end{array}\right\}$$

or

$$(\mathrm{S}_5)\qquad \left.\begin{array}{rl} \text{(i)} & f(p) = [g(1) + (n-1)g(0)]\{[h(1) + (m-1)h(0)]p \\ & \qquad - [b_3(1) + mh(0)]M(p) + b_3(p)\} + A_6(p) + f(0) \\ \text{(ii)} & g(p) = [g(1) + (n-1)g(0)][M(p) + b_4(p)] + g(0), \\ & \qquad g(1) + (n-1)g(0) \neq 0 \\ \text{(iii)} & h(p) = [h(1) + (m-1)h(0)]p - [b_3(1) + mh(0)]M(p) \\ & \qquad + b_3(p) + h(0),\ \ b_3(1) + mh(0) \neq 0. \end{array}\right\}$$

In (S_1) *to* (S_5), $A_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3, 4, 5, 6)$, $b_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2, 3, 4)$ *are additive mappings;* $a : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R} \times I$ *are mappings as described in Result 2.2;* $M : I \to \mathbb{R}$ *a multiplicative mapping which is not additive and* $M(0) = 0$, $M(1) = 1$. *Moreover,*

$$(\mathrm{S}_6)\qquad \begin{aligned} &[1 - g(1) - (n-1)g(0)][h(p) - h(0)] \\ &\qquad = A_7(p) - [h(1) + (m-1)h(0)][g(p) - g(0)] \end{aligned}$$

where $A_7 : \mathbb{R} \to \mathbb{R}$ *is an additive mapping and*

$$\text{(3.1)}\quad \left.\begin{aligned}
&\text{(i)} && A_1(1) = -ng(0)\\
&\text{(ii)} && A_2(1) = -nmf(0) + [h(1) + (n-1)h(0)]\\
&\text{(iii)} && A_3(1) = h(1) - h(0)\\
&\text{(iv)} && A_4(1) = -mh(0)\\
&\text{(v)} && A_5(1) = -nmf(0) + (n-m)h(0)\\
&\text{(vi)} && A_6(1) = -nmf(0) + m[g(1) + (n-1)g(0)]h(0)\\
& && \qquad + [h(1) + (n-1)h(0)]\\
&\text{(vii)} && A_7(1) = m[g(1) + (n-1)g(0)]h(0)\\
& && \qquad - n[h(1) + (m-1)h(0)]g(0) + [h(1) - h(0)]\\
&\text{(viii)} && b_1(1) = -mh(0)\\
&\text{(ix)} && b_2(1) = 1 - n[g(1) + (n-1)g(0)]^{-1}g(0)\\
&\text{(x)} && b_4(1) = -n[g(1) + (n-1)g(0)]^{-1}g(0).
\end{aligned}\right\}$$

Proof. We divide our discussion into three cases:

Case 1. $\sum_{i=1}^{n} g(p_i) = 0$ for all $(p_1, \dots, p_n) \in \Gamma_n$.

In this case, by Result 2.1, there exists an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ such that (S_1)(ii) holds with $A_1(1)$ given by (3.1)(i). Now, from (FE2), it follows that

$$\text{(3.2)}\qquad \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} h(p_i)$$

valid for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$. Now, let us put $p_1 = 1, p_2 = \dots = p_n = 0$ in (3.2). We obtain $\sum_{j=1}^{m} f(q_j) = [h(1) + (n-1)h(0)] - m(n-1)f(0)$. By Result 2.1, there exists an additive mapping $A_2 : \mathbb{R} \to \mathbb{R}$ such that (S_1)(i) holds with $A_2(1)$ given by (3.1)(ii). Now, from (3.2), we obtain the equation $\sum_{i=1}^{n} h(p_i) = h(1) + (n-1)h(0)$. By Result 2.1, there exists an additive mapping $A_3 : \mathbb{R} \to \mathbb{R}$ such that (S_1)(iii) holds with $A_3(1)$ given by (3.1)(iii). Equations (S_1)(i), (S_1)(ii), (S_1)(iii), together with (3.1)(i), (3.1)(ii) and (3.1)(iii), constitute the solution (S_1) of (FE2).

Case 2. $\sum_{j=1}^{m} h(q_j) = 0$ for all $(q_1, \dots, q_m) \in \Gamma_m$.

In this case, by Result 2.1, there exists an additive mapping $A_4 : \mathbb{R} \to \mathbb{R}$ such that (S_2)(iii) holds with $A_4(1)$ given by (3.1)(iv). Also, since $\sum_{j=1}^{m} h(q_j) = 0$

for all $(q_1, \dots, q_m) \in \Gamma_m$, it follows, from (FE2), that g is an arbitrary real-valued mapping. Thus, S_2(ii) holds. From (S_2)(iii) and (FE2), it also follows that $\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = (n-m)h(0)$. Putting $p_1 = 1, p_2 = \ldots = p_n = 0$ in this equation, we obtain the equation $\sum_{j=1}^{m} f(q_j) = (n-m)h(0) - m(n-1)f(0)$. By Result 2.1, there exists an additive mapping $A_5 : \mathbb{R} \to \mathbb{R}$ such that (S_2)(i) follows with $A_5(1)$ given by (3.1)(v). Equations (S_2)(i), (S_2)(ii), (S_2)(iii), together with (3.1)(iv) and (3.1)(v), constitute the solution (S_2) of (FE2).

Case 3. Neither $\sum_{i=1}^{n} g(p_i)$ vanishes identically on Γ_n nor $\sum_{j=1}^{m} h(q_j)$ vanishes identically on Γ_m.

In this case, there exist a probability distribution $(p_1^*, \dots, p_n^*) \in \Gamma_n$ and a probability distribution $(q_1^*, \dots, q_m^*) \in \Gamma_m$ such that

$$\text{(3.3)} \qquad \text{(i)} \quad \sum_{i=1}^{n} g(p_i^*) \neq 0 \qquad \text{(ii)} \quad \sum_{j=1}^{m} h(q_j^*) \neq 0.$$

Now, let us put $p_1 = 1, p_2 = \ldots = p_n = 0$ in (FE2). We obtain

$$\sum_{j=1}^{m} \{f(q_j) - [g(1)+(n-1)g(0)]h(q_j)\} = -m(n-1)f(0)+[h(1)+(n-1)h(0)]$$

valid for all $(q_1, \dots, q_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $A_6 : \mathbb{R} \to \mathbb{R}$ such that

$$\text{(3.4)} \qquad f(p) = [g(1) + (n-1)g(0)][h(p) - h(0)] + A_6(p) + f(0)$$

for all $p \in I$ with $A_6(1)$ given by (3.1)(vi). From (FE2), (3.4), (3.1)(vi) and the additivity of A_6, we obtain the equation

$$\begin{aligned} &[g(1) + (n-1)g(0)] \sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i q_j) \\ &= \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} h(q_j) + \sum_{i=1}^{n} h(p_i) + m(n-1)[g(1) + (n-1)g(0)]h(0) \\ \text{(3.5)} \qquad &- [h(1) + (n-1)h(0)] \end{aligned}$$

valid for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$. Putting $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (3.5), we obtain

$$\sum_{i=1}^{n} h(p_i) = -[h(1) + (m-1)h(0)] \sum_{i=1}^{n} g(p_i) + [g(1) + (n-1)g(0)] \sum_{i=1}^{n} h(p_i)$$

$$+\,(m-n)[g(1)+(n-1)g(0)]h(0)+[h(1)+(n-1)h(0)]. \tag{3.6}$$

Applying Result 2.1 on equation (3.6), there exists an additive mapping $A_7:\mathbb{R}\to\mathbb{R}$ such that (S_6) holds with $A_7(1)$ given by (3.1)(vii). Now, from (3.5) and (3.6), it follows that

$$\begin{aligned}[g(1)+(n-1)g(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)\\ =\sum_{i=1}^{n}g(p_i)\left\{\sum_{j=1}^{m}h(q_j)-[h(1)+(m-1)h(0)]\right\}\\ +[g(1)+(n-1)g(0)]\sum_{i=1}^{n}h(p_i)+n(m-1)[g(1)+(n-1)g(0)]h(0).\end{aligned} \tag{3.7}$$

Case 3.1. $g(1)+(n-1)g(0)=0.$

In this case, (3.7) reduces to the equation

$$\sum_{i=1}^{n}g(p_i)\left\{\sum_{j=1}^{m}h(q_j)-[h(1)+(m-1)h(0)]\right\}=0 \tag{3.8}$$

valid for all $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$. If we substitute $p_1=p_1^*,\ldots,p_n=p_n^*$ in (3.8) and use (3.3)(i), we obtain $\sum\limits_{j=1}^{m}h(q_j)=[h(1)+(m-1)h(0)]$ valid for all $(q_1,\ldots,q_m)\in\Gamma_m$. By Result 2.1, there exists an additive mapping $A_3:\mathbb{R}\to\mathbb{R}$ such that $h(p)=A_3(p)+h(0)$ with $A_3(1)$ given by (3.1)(iii). Since $g(1)+(n-1)g(0)=0$, (3.4) gives $f(p)=A_6(p)+f(0)$ with $A_6(1)=-nmf(0)+[h(1)+(n-1)h(0)]$. Now, from (3.8), we observe that g is an arbitrary real-valued mapping with $g(1)+(n-1)g(0)=0$. Thus, the solution, obtained in this case, is included in (S_3) when $A_3(p)=b_1(p)+[h(1)+(m-1)h(0)]p$ and $b_1(1)$ given by (3.1)(viii).

Case 3.2. $g(1)+(n-1)g(0)\neq 0.$

In this case, (3.7) reduces to

(3.9)

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)=\sum_{i=1}^{n}\left[\frac{g(p_i)}{g(1)+(n-1)g(0)}\right]\left\{\sum_{j=1}^{m}h(q_j)-[h(1)+(m-1)h(0)]\right\}\\ +\sum_{i=1}^{n}h(p_i)+n(m-1)h(0).\end{aligned}$$

Define the mappings $G : I \to \mathbb{R}$ and $H : I \to \mathbb{R}$ as

$$G(p) = [g(1) + (n-1)g(0)]^{-1} g(p) \tag{3.10}$$

and

$$H(p) = h(p) - [h(1) + (m-1)h(0)]p \tag{3.11}$$

for all $p \in I$. Now (3.9) reduces to the functional equation (2.1). From (3.10) and (3.11), it can be easily verified that

$$G(1) + (n-1)G(0) = 1, \tag{3.12}$$

$$H(0) = h(0), \tag{3.13}$$

$$H(1) + (m-1)H(0) = 0. \tag{3.14}$$

In Result 2.2, we need to consider only those solutions of (2.1) which satisfy (3.12) and (3.14). So, we consider only (2.3), (2.4) and (2.5).

Solution (S_3) (with (3.1)(viii)) of (FE2), follows from (2.3), (3.10), (3.11), (3.13), (3.4).

Solution (with (3.1)(ix)) of (FE2), follows from (2.4), (2.6), (2.7), (2.8), (3.10), (3.11), (3.13), (3.4).

Solution (S_5) (with (3.1)(x)) of (FE2), follows from (2.5), (3.10), (3.11), (3.13), (3.4).

This completes the proof of the theorem.

Note. It is easy to verify that (S_1) and (S_2), together with (3.1)((i) to (v)), satisfy (FE2). However, it is worth noticing that (S_6) with (3.1)(vii) is needed to verify the solutions (S_3), (S_4) and (S_5) together with (3.1)((viii),(ix),(x)) of (FE2).

4. ON THE FUNCTIONAL EQUATION (FE1)

In this section, we prove the following:

THEOREM 4.1. *Suppose $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are mappings which satisfy the functional equation (FE1) for all $p \in I$, $q \in I$. Then, any general solution (f, g, h) of (FE1) is one of the forms (for all $p \in I$):*

$$f(p) = h(1)[p\ell(p) + 2p], \quad g(p) = p, \quad h(p) = h(1)[p\ell(p) + p] \tag{4.1}$$

with $h(1) \neq 0$ or

$$f(p) = p\ell(p), \quad g(p) = p, \quad h(p) = p\ell(p) \tag{4.2}$$

or

$$f(p) = 0, \quad g \text{ not an identity mapping}, \quad h(p) = 0 \tag{4.3}$$

or

(4.4)
$$f(p) = \frac{[h(1)+\lambda]^2}{\lambda}M(p)-\lambda p,\ g(p) = \frac{[h(1)+\lambda]}{\lambda}M(p),\ h(p) = [h(1)+\lambda]M(p)-\lambda p$$

where $\lambda \neq 0$, $[h(1)+\lambda] \neq 0$ *are arbitrary real constants;* $\ell : I \to \mathbb{R}$ *is a logarithmic mapping and* $M : I \to \mathbb{R}$ *is a multiplicative mapping.*

Proof. The left hand side of (FE1) is symmetric in p and q. So, should also be its right hand side. This fact gives

(4.5) $$[g(p) - p]h(q) = [g(q) - q]h(p)$$

valid for all $p \in I$, $q \in I$. Now we divide our discussion into two cases:

Case 1. $q \mapsto g(q) - q$ vanishes identically on I.

In this case, we have $g(q) = q$ for all $q \in I$. Making use of this form of g in (FE1), we obtain the equation

(4.6) $$f(pq) = ph(q) + qh(p)$$

valid for all $p \in I$, $q \in I$. The substitution $q = 1$ in (4.6) gives

(4.7) $$f(p) = h(1)p + h(p)$$

for all $p \in I$. From (4.6) and (4.7), we obtain the equation

(4.8) $$h(pq) = ph(q) + qh(p) - h(1)pq$$

valid for all $p \in I$, $q \in I$. The substitutions $p = 0$, $q = 0$, in (4.8) gives $h(0) = 0$.

If $h(1) = 0$, then (4.8) reduces to the equation $h(pq) = ph(q)+qh(p)$ valid for all $p \in I$, $q \in I$. So, $h(p) = p\ell(p)$ for all $p \in I$; $\ell : I \to \mathbb{R}$ being a logarithmic mapping. Using this form of h in (4.7) and the fact that $h(1) = 0$, we obtain $f(p) = p\ell(p)$ for all $p \in I$. Thus, solution (4.2), of (FE1), has been obtained.

If $h(1) \neq 0$, then since $h(0) = 0$, it is enough to restrict our discussion to $p \in \,]0,1]$ and $q \in \,]0,1]$ where $]0,1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$. Since $h(1)pq \neq 0$, (4.8) can be written in the form (for $p \in \,]0,1]$, $q \in \,]0,1]$)

(4.9) $$\frac{h(pq)}{h(1)pq} - 1 = \left[\frac{h(p)}{h(1)p} - 1\right] + \left[\frac{h(q)}{h(1)q} - 1\right].$$

Define a mapping $\ell : I \to \mathbb{R}$ as

(4.10) $$\ell(p) = \begin{cases} \frac{h(p)}{h(1)p} - 1 & \text{if } 0 < p \leq 1 \\ 0 & \text{if } p = 0. \end{cases}$$

Form (4.9), (4.10) and the fact that $\ell(0) = 0$, it follows that $\ell : I \to \mathbb{R}$ is a logarithmic mapping and $h(p) = h(1)[p\ell(p) + p]$ for all $p \in I$ with $h(1) \neq 0$.

Using this form of h in (4.7), it follows that $f(p) = h(1)[p\ell(p)+2p]$ for all $p \in I$. Thus, solution (4.1), of (FE1), has been obtained.

Case 2. $q \mapsto g(q) - q$ does not vanish identically on I.

In this case, there exists an element $q_0 \in I$ such that $[g(q_0) - q_0] \neq 0$. Putting $q = q_0$ in (4.5), we obtain

$$h(p) = \lambda[g(p) - p] \tag{4.11}$$

for all $p \in I$ with $\lambda = [g(q_0) - q_0]^{-1}h(q_0)$. If $\lambda = 0$, then (4.11) gives $h(p) = 0$ for all $p \in I$. Now, from (FE1), it is easy to conclude that $f(p) = 0$ for all $p \in I$. Thus, solution (4.3), of (FE1), has been obtained. Now consider the case when $\lambda \neq 0$. Putting $q = 1$ in (FE1) and using (4.11), it follows that

$$f(p) = [h(1) + \lambda]g(p) - \lambda p \tag{4.12}$$

for all $p \in I$. From (FE1), (4.11) and (4.12), it follows that

$$[h(1) + \lambda]g(pq) = \lambda g(p)g(q) \tag{4.13}$$

in which $p \in I$, $q \in I$ and $\lambda \neq 0$.

If $h(1) + \lambda = 0$, then (4.13) gives $g(p) = 0$ for all $p \in I$ as $\lambda \neq 0$. Now (4.12) gives $f(p) = -\lambda p$ for all $p \in I$ with $\lambda \neq 0$. Also, from (FE1), $h(p) = f(p)$ for all $p \in I$ as $g(p) = 0$ for all $p \in I$. This solution is included in (4.4) when $M(p) \equiv 0$ on I.

If $h(1) + \lambda \neq 0$, then (4.13) reduces to

$$g(pq) = [h(1) + \lambda]^{-1}\lambda g(p)g(q). \tag{4.14}$$

Define a mapping $M : I \to \mathbb{R}$ as

$$M(x) = [h(1) + \lambda]^{-1}\lambda g(x) \tag{4.15}$$

for all $x \in I$. Now, from (4.14) and (4.15), it follows that $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$. Thus M, defined by (4.15), is a multiplicative mapping. From (4.15), it follows that

$$g(p) = \frac{[h(1) + \lambda]}{\lambda}M(p) \tag{4.16}$$

for all $p \in I$. From (4.12) and (4.16), we have $f(p) = \frac{[h(1)+\lambda]^2}{\lambda}M(p) - \lambda p$ for all $p \in I$. Also, from (4.11) and (4.16), it follows that $h(p) = [h(1) + \lambda]M(p) - \lambda p$ for all $p \in I$. Thus, solution (4.4), of (FE1), has been obtained.

Note. The solutions (S_1), (S_2), (S_3), (S_4) and (S_5) of (FE2) do not satisfy the (FE1). However, the solutions (4.1) and (4.2) are included in (S_4); solution (4.3) is included in (S_3) and solution (4.4) is included in (S_5).

5. APPLICATIONS

For any probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, the Shannon [16] entropies $H_n : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$ are defined as

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{5.1}$$

where $0 \log_2 0 := 0$. The Shannon entropies are useful in Ecology [14]; Biology [4,6]; Psychology [5] and Economics [17] etc. Let $\ell : I \to \mathbb{R}$ be the logarithmic mapping defined as

$$\ell(p) = \begin{cases} -\log_2 p & \text{if } 0 < p \leq 1 \\ 0 & \text{if } p = 0. \end{cases} \tag{5.2}$$

From (4.2), (5.1) and (5.2), it is easy to conclude that $\sum_{i=1}^{n} f(p_i) = \sum_{i=1}^{n} h(p_i) = H_n(p_1, \ldots, p_n)$. On the other hand, from (4.1), (5.1) and (5.2),

$$\sum_{i=1}^{n} f(p_i) = AH_n(p_1, \ldots, p_n) + 2A \qquad \text{and} \qquad \sum_{i=1}^{n} h(p_i) = AH_n(p_1, \ldots, p_n) + A$$

where $A = h(1)$ is an arbitrary nonzero real constant. Thus, in this case, the summands $\sum_{i=1}^{n} f(p_i)$ and $\sum_{i=1}^{n} h(p_i)$ represent the Shannon entropies upto additive and multiplicative constants. The solution (4.3), of (FE1), is of no importance from information-theoretic point of view as the summands $\sum_{i=1}^{n} f(p_i)$ and $\sum_{i=1}^{n} h(p_i)$ are independent of the probabilities $p_1, \ldots, p_n$. Regarding the solution (4.4), of (FE1), it seems useful to choose the mapping $M : I \to \mathbb{R}$ defined as $M(p) = p^\alpha$, $p \in I$, $\alpha > 0$, $\alpha \neq 1$, $\alpha \in \mathbb{R}$, $0^\alpha := 0$, $1^\alpha := 1$. Then (4.4) gives

$$\sum_{i=1}^{n} f(p_i) = \frac{[h(1) + \lambda]^2}{\lambda} \{1 - (1 - 2^{1-\alpha}) H_n^\alpha(p_1, \ldots, p_n)\} - \lambda,$$

$$\sum_{i=1}^{n} g(p_i) = \frac{[h(1) + \lambda]}{\lambda} \{1 - (1 - 2^{1-\alpha}) H_n^\alpha(p_1, \ldots, p_n)\},$$

$$\sum_{i=1}^{n} h(p_i) = [h(1) + \lambda] \{1 - (1 - 2^{1-\alpha}) H_n^\alpha(p_1, \ldots, p_n)\} - \lambda$$

where $H_n^\alpha : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$ are the entropies

$$H_n^{\alpha}(p_1, \ldots, p_n) = (1 - 2^{1-\alpha})^{-1}\left(1 - \sum_{i=1}^{n} p_i^{\alpha}\right), \quad \alpha \neq 1 \tag{5.3}$$

given by Havrda and Charvat [3], also known as the entropies of degree α.

If $\alpha = 2$, then (5.3) reduces to

$$H_n^2(p_1, \ldots, p_n) = 2(1 - \sum_{i=1}^{n} p_i^2). \tag{5.4}$$

The Gini-Simpson [15] index of the probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, written as $(GS)_n(p_1, \ldots, p_n)$, is defined as

$$(GS)_n(p_1, \ldots, p_n) = 1 - \sum_{i=1}^{n} p_i^2. \tag{5.5}$$

Clearly $H_n^2(p_1, \ldots, p_n) = 2(GS)_n(p_1, \ldots, p_n)$. The Gini-Simpson index $(GS)_n$
$(p_1, \ldots, p_n)$ has been used in Sociology [1, 7]; and in Linguistics [2]. From the solution (4.4) (with $M(p) = p^2$) and (5.5), it can be easily derived that

$$\sum_{i=1}^{n} f(p_i) = \frac{[h(1)+\lambda]^2}{\lambda}\{1 - (GS)_n(p_1, \ldots, p_n)\} - \lambda,$$

$$\sum_{i=1}^{n} g(p_i) = \frac{[h(1)+\lambda]}{\lambda}\{1 - (GS)_n(p_1, \ldots, p_n)\},$$

$$\sum_{i=1}^{n} h(p_i) = [h(1)+\lambda]\{1 - (GS)_n(p_1, \ldots, p_n)\} - \lambda.$$

In this sense, the solution (4.4) is also related to Gini-Simpson index $(GS)_n(p_1, \ldots, p_n)$.

Proceeding as above, we can derive the relations of some of the solutions of (FE2) with the Shannon entropies; the entropies of degree α and Gini-Simpson index.

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University of Delhi,
Department of Mathematics,
Delhi 110007, India
pnathmaths@gmail.com

University of Delhi,
Zakir Husain Delhi College,
Department of Mathematics,
Jawaharlal Nehru Marg,
Delhi 110002, India
dhiraj426@rediffmail.com,
dksingh@zh.du.ac.in