A NOTE ON CLASSIFICATION OF GENERALIZED POLARIZED MANIFOLDS BY THE c_r-SECTIONAL HODGE NUMBER OF TYPE $(1,1)$ AND THE c_r-SECTIONAL BETTI NUMBER

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Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension $n \geq 3$ such that $\operatorname{rank}(\mathcal{E}) = n-2$ and $\mathcal{E}$ is generated by its global sections. In this short note, we classify $(X,\mathcal{E})$ with $h^{1,1}_{n,n-2}(X,\mathcal{E}) = 1$ (resp. $b_{n,n-2}(X,\mathcal{E}) = 1$).

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1. INTRODUCTION

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then the pair (X,L) is called a *polarized variety*. Moreover if X is smooth, then (X,L) is called a polarized *manifold*.

When we study polarized varieties, it is useful to use their invariants. The sectional genus $g(X,L)$ of (X,L) is one of the well-known invariants of (X,L). In [4] we defined the notion of the *ith sectional geometric genus* $g_i(X,L)$ *of* (X,L) for every integer i with $0 \leq i \leq n$. Here we explain the meaning of these invariants if X is smooth, L is base point free and i is an integer with $1 \leq i \leq n-1$. Let $H_1,\ldots,H_{n-i}$ be general members of $|L|$. We put $X_{n-i} := H_1 \cap \cdots \cap H_{n-i}$. Then X_{n-i} is smooth with $\dim X_{n-i} = i$, and we can show that $g_i(X,L) = h^i(\mathcal{O}_{X_{n-i}})$.

These induce the notion of the *ith sectional invariant of* (X,L) *associated with an invariant.*

Definition 1. Let (X,L) be a polarized manifold of dimension n. Let $I(Y)$ (or I) be an invariant of a smooth projective variety Y of dimension i, where i is an integer with $0 \leq i \leq n$. Then an invariant $F_i(X,L)$ of (X,L)

is called the *ith sectional invariant of* (X,L) *associated with the invariant* I if $F_i(X,L)=I(X_{n-i})$ under the assumption that $\mathrm{Bs}|L|=\emptyset$.

The ith sectional geometric genus is the ith sectional invariant of (X,L) associated with the geometric genus. By the definition of the ith sectional invariants, the ith sectional invariants are expected to reflect properties of i-dimensional geometry. So we can expect that we are able to find interesting properties of (X,L) by using its ith sectional invariants.

In [5], we defined other ith sectional invariants, that is, the *ith sectional Euler number* $e_i(X,L)$, the *ith sectional Betti number* $b_i(X,L)$, and the *ith sectional Hodge number* $h_i^{j,i-j}(X,L)$ *of type* $(j,i-j)$ *of* (X,L) and we studied some properties of these. If X is smooth, L is base point free and i is an integer with $1\leq i\leq n-1$, then by using the above notation we see that $e_i(X,L)=e(X_{n-i})$, $b_i(X,L)=h^i(X_{n-i},\mathbb{C})$ and $h_i^{j,i-j}(X,L)=h^{j,i-j}(X_{n-i})$, where $e(X_{n-i})$ denotes the Euler number of X_{n-i}.

Let X be a smooth projective variety with $\dim X=n$ and let $\mathcal{E}$ be an ample vector bundle on X with rank $\mathcal{E}=r$. We assume that $r\leq n$. In [6], we defined ample vector bundles' version of the invariants above. Namely we defined the c_r-*sectional geometric genus* $g_{n,r}(X,\mathcal{E})$[1], the c_r*-sectional Euler number* $e_{n,r}(X,\mathcal{E})$, the c_r*-sectional Betti number* $b_{n,r}(X,\mathcal{E})$ and the c_r*-sectional Hodge number* $h_{n,r}^{j,n-r-j}(X,\mathcal{E})$ *of type* $(j,n-r-j)$ of $(X,\mathcal{E})$.

It is natural to study a lower bound for these invariants. If $\mathcal{E}$ is generated by its global sections and $r\leq n-1$, then $g_{n,r}(X,\mathcal{E})$, $b_{n,r}(X,\mathcal{E})$ and $h_{n,r}^{j,n-r-j}(X,\mathcal{E})$ are non-negative (see Proposition 2.1 below). Under this setting, it is interesting to characterize $(X,\mathcal{E})$ whose invariants are small.

If $r=n$, then $g_{n,n}(X,\mathcal{E})=b_{n,n}(X,\mathcal{E})=h_{n,n}^{0,0}(X,\mathcal{E})=c_n(\mathcal{E})$ (see [6, Remarks 3.2.1 (i) and 3.3.1]), and $(X,\mathcal{E})$ with small $c_n(\mathcal{E})$ has been studied by several authors (for example [11, 12, 14, 16]).

If $r=n-1$, then $2g_{n,n-1}(X,\mathcal{E})=b_{n,n-1}(X,\mathcal{E})$ and $g_{n,n-1}(X,\mathcal{E})=h_{n,n-1}^{1,0}(X,\mathcal{E})=h_{n,n-1}^{0,1}(X,\mathcal{E})$ (see Theorem 2.1).

Here we note that $g_{n,n-1}(X,\mathcal{E})$ is the curve genus of $(X,\mathcal{E})$, and $(X,\mathcal{E})$ with small $g_{n,n-1}(X,\mathcal{E})$ has been classified (for example, see [10, 13]).

Assume that $r=n-2$. Then Lanteri [8, (3.4)Corollary] studied the classification of $(X,\mathcal{E})$ with $g_{n,n-2}(X,\mathcal{E})=0$ for $\mathcal{E}=L_1\oplus\cdots\oplus L_{n-2}$, where each L_i is ample and spanned line bundle on X. But in general, we don't know the case of $(X,\mathcal{E})$ such that $\mathcal{E}$ is generated by its global sections and $g_{n,n-2}(X,\mathcal{E})=0$.

In this short note, we will study the case where $r=n-2$. In particular, we will treat the c_r-sectional Hodge number $h_{n,n-2}^{1,1}(X,\mathcal{E})$ and the c_r-sectional

[1] This invariant $g_{n,r}(X,\mathcal{E})$ is equal to the invariant $g_{n-r}(X,\mathcal{E})$ in [3, Definition 2.1].

Betti number $b_{n,n-2}(X,\mathcal{E})$. In this case, we can prove $h^{1,1}_{n,n-2}(X,\mathcal{E}) \geq 1$ and $b_{n,n-2}(X,\mathcal{E}) \geq 1$ under the assumption that $\mathcal{E}$ is generated by its global sections (see [6, Proposition 4.1 (iv)] and Theorem 3.2). Moreover, it is natural to consider the classification of $(X,\mathcal{E})$ with $h^{1,1}_{n,n-2}(X,\mathcal{E}) = 1$ and the classification of $(X,\mathcal{E})$ with $b_{n,n-2}(X,\mathcal{E}) = 1$. In Theorem 3.1 (resp. Theorem 3.2) we will classify $(X,\mathcal{E})$ with $h^{1,1}_{n,n-2}(X,\mathcal{E}) = 1$ (resp. $b_{n,n-2}(X,\mathcal{E}) = 1$).

2. **PRELIMINARIES**

Notation:

(1) Let Y be a smooth projective variety of dimension $i \geq 1$, let $\mathcal{T}_Y$ be the tangent bundle of Y and let Ω_Y be the dual bundle of $\mathcal{T}_Y$. For every integer j with $0 \leq j \leq i$, we put

$$h_{i,j}(c_1(Y), \cdots, c_i(Y)) := \chi(\Omega_Y^j) = \int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y).$$

(Here $\mathrm{ch}(\Omega_Y^j)$ (resp. $\mathrm{Td}(\mathcal{T}_Y)$) denotes the Chern character of Ω_Y^j (resp. the Todd class of $\mathcal{T}_Y$). See [7, example 3.2.3 and example 3.2.4].)

(2) Let X be a smooth projective variety of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$H_1(X;i,j) := \begin{cases} \displaystyle\sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega_X^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(X;i,j) := \begin{cases} \displaystyle\sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

In this paper, we consider the following case $(*)$:

$(*)$ Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X with $r \leq n$.

Definition 2.1. Let $(X,\mathcal{E})$, n and r be as in $(*)$. For every integer p with $0 \leq p \leq n-r$ we set

$$C_p^{n,r}(X,\mathcal{E}) := \sum_{k=0}^{p} c_k(X) s_{p-k}(\mathcal{E}^\vee).$$

Definition 2.2 ([6, Definition 3.1.1]). Let $(X,\mathcal{E})$, n and r be as in $(*)$. The *c_r-sectional H-arithmetic genus* $\chi^H_{n,r}(X,\mathcal{E})$ and the *c_r-sectional Euler number*

$e_{n,r}(X,\mathcal{E})$ of $(X,\mathcal{E})$ are defined by the following[2]:

$$\begin{aligned}\chi^H_{n,r}(X,\mathcal{E}) &:= \mathrm{td}_{n-r}\left(C^{n,r}_1(X,\mathcal{E}),\cdots,C^{n,r}_{n-r}(X,\mathcal{E})\right)c_r(\mathcal{E}).\\ e_{n,r}(X,\mathcal{E}) &:= C^{n,r}_{n-r}(X,\mathcal{E})c_r(\mathcal{E}).\end{aligned}$$

Definition 2.3 ([6, Definition 3.2.1]). Let $(X,\mathcal{E})$, n and r be as in $(*)$. The *c_r-sectional geometric genus* $g_{n,r}(X,\mathcal{E})$ and the *c_r-sectional Betti number* $b_{n,r}(X,\mathcal{E})$ *of* $(X,\mathcal{E})$ are defined by the following:

$$g_{n,r}(X,\mathcal{E}) := (-1)^{n-r}\chi^H_{n,r}(X,\mathcal{E}) + (-1)^{n-r+1}\chi(\mathcal{O}_X) + \sum_{k=0}^{r}(-1)^{r-k}h^{n-k}(\mathcal{O}_X).$$

$$b_{n,r}(X,\mathcal{E}) := \begin{cases} (-1)^{n-r}\left(e_{n,r}(X,\mathcal{E}) - \sum_{j=0}^{n-r-1}2(-1)^j h^j(X,\mathbb{C})\right), & \text{if } r<n,\\ e_{n,n}(X,\mathcal{E}), & \text{if } r=n.\end{cases}$$

Definition 2.4 ([6, Definition 3.3.1]). Let $(X,\mathcal{E})$, n and r be as in $(*)$. The *c_r-sectional Hodge number* $h^{j,n-r-j}_{n,r}(X,\mathcal{E})$ *of type* $(j,n-r-j)$ *of* $(X,\mathcal{E})$ is defined by the following:

$$h^{j,n-r-j}_{n,r}(X,\mathcal{E}) := (-1)^{n-r-j}\left\{w^j_{n,r}(X,\mathcal{E}) - H_1(X;n-r,j) - H_2(X;n-r,j)\right\}.$$

Here we set

$$w^j_{n,r}(X,\mathcal{E}) := \begin{cases} h_{n-r,j}(C^{n,r}_1(X,\mathcal{E}),\cdots,C^{n,r}_{n-r}(X,\mathcal{E}))c_r(\mathcal{E}), & \text{if } r<n,\\ c_n(\mathcal{E}), & \text{if } r=n.\end{cases}$$

for every integer j with $0\le j\le n-r$.

These invariants satisfiy the following properties.

PROPOSITION 2.1. *Let $(X,\mathcal{E})$, n and r be as in $(*)$. Assume that $r\le n-1$ and there exists a smooth projective variety Z such that $\dim Z = n-r$ and Z is the zero locus of an element of $H^0(\mathcal{E})$. Then*

$$\chi^H_{n,r}(X,\mathcal{E}) = \chi(\mathcal{O}_Z),\qquad g_{n,r}(X,\mathcal{E}) = h^{n-r}(\mathcal{O}_Z),\qquad e_{n,r}(X,\mathcal{E}) = e(Z),$$

$$b_{n,r}(X,\mathcal{E}) = h^{n-r}(Z,\mathbb{C}),\qquad h^{j,n-r-j}_{n,r}(X,\mathcal{E}) = h^{j,n-r-j}(Z).$$

In particular, $g_{n,r}(X,\mathcal{E})$, $b_{n,r}(X,\mathcal{E})$ and $h^{j,n-r-j}_{n,r}(X,\mathcal{E})$ are non-negative if $\mathcal{E}$ is generated by its global sections with $r\le n-1$.

Proof. See [6, Propositions 3.1.1, 3.2.2 and 3.3.1]. $\square$

[2] Here td_{n-r} means the Todd polynomial of weight $n-r$ (see [3, Definition 1.4 (1)]).

THEOREM 2.1. *Let $(X,\mathcal{E})$, n and r be as in $(*)$. Assume that $r \le n-1$. For every integer j with $0 \le j \le n-r$, we get the following.*

(i) $b_{n,r}(X,\mathcal{E}) = \sum_{k=0}^{n-r} h_{n,r}^{k,n-r-k}(X,\mathcal{E})$.

(ii) $h_{n,r}^{j,n-r-j}(X,\mathcal{E}) = h_{n,r}^{n-r-j,j}(X,\mathcal{E})$.

(iii) $h_{n,r}^{n-r,0}(X,\mathcal{E}) = h_{n,r}^{0,n-r}(X,\mathcal{E}) = g_{n,r}(X,\mathcal{E})$.

(iv) *If $n-r$ is odd, then $b_{n,r}(X,\mathcal{E})$ is even.*

Proof. See [6, Theorem 4.1]. $\square$

3. MAIN RESULT

THEOREM 3.1. *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension $n \ge 3$ such that* $\operatorname{rank}(\mathcal{E}) = n-2$ *and $\mathcal{E}$ is generated by its global sections. In this setting, we have $h_{n,n-2}^{1,1}(X,\mathcal{E}) \ge 1$ by [6, Proposition 4.1 (iv)]. If $h_{n,n-2}^{1,1}(X,\mathcal{E}) = 1$, then $(X,\mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2})$.*

Proof. We note that by [6, Theorem 6.1] the following equality holds.

$$12\chi_{n,n-2}^{H}(X,\mathcal{E}) = (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) + e_{n,n-2}(X,\mathcal{E}). \tag{1}$$

(This is an analogue of Noether's equality [1, p.26 (4)].) Hence by (1), Theorem 2.1 and Definitions 2.2, 2.3 and 2.4 we have

$$h_{n,n-2}^{1,1}(X,\mathcal{E}) = 10\chi_{n,n-2}^{H}(X,\mathcal{E}) - (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) + 2h^1(\mathcal{O}_X). \tag{2}$$

Let Z be the zero locus of a general member s of $H^0(X,\mathcal{E})$. Then Z is a smooth projective surface and the following hold.

$$\chi_{n,n-2}^{H}(X,\mathcal{E}) = \chi(\mathcal{O}_Z), \tag{3}$$

$$(K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) = K_Z^2, \tag{4}$$

$$h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Z). \tag{5}$$

(i) Assume that $\kappa(Z) \ge 0$. Then $\chi(\mathcal{O}_Z) \ge 0$ and $9\chi(\mathcal{O}_Z) \ge K_Z^2$ hold. Hence by (3) and (4) we have $\chi_{n,n-2}^{H}(X,\mathcal{E}) \ge 0$ and $9\chi_{n,n-2}^{H}(X,\mathcal{E}) \ge (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E})$. Therefore by (2) we get

$$h_{n.n-2}^{1,1}(X,\mathcal{E}) \ge \chi_{n,n-2}^{H}(X,\mathcal{E}) + 2h^1(\mathcal{O}_X). \tag{6}$$

Since $h_{n,n-2}^{1,1}(X,\mathcal{E}) = 1$, we have $0 \le \chi_{n,n-2}^{H}(X,\mathcal{E}) \le 1$ and $h^1(\mathcal{O}_X) = 0$. Here we note that

$$\chi_{n,n-2}^{H}(X,\mathcal{E}) = 1 - h^1(\mathcal{O}_X) + g_{n,n-2}(X,\mathcal{E})$$

holds, and $g_{n,n-2}(X,\mathcal{E}) = h^2(\mathcal{O}_Z) \ge 0$ by (3) and (5). Therefore

$$\chi_{n,n-2}^{H}(X,\mathcal{E}) \ge 1. \tag{7}$$

Hence by (6) and (7) we have $9\chi^H_{n,n-2}(X,\mathcal{E}) = (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E})$, that is, $9\chi(\mathcal{O}_Z) = K_Z^2$. Thus Z is a ball quotient and hence Z is an Eilenberg-MacLane space of type $(\Pi, 1)$ (see [2, Remark 5.1.7]). On the other hand, by [15, Corollary 22] we have $\pi_1(X,Z) = \{0\}$ and $\pi_2(X,Z) = \{0\}$. Moreover we note that the restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(Z,\mathbb{Z})$ is injective by the Lefschetz-type theorem for ample vector bundles (see [9, Theorem 1.3]). Hence by the proof of [2, Theorem 5.1.5 and Corollary 5.1.6], this is impossible because we assume that $n = \dim X \geq 3$.

(ii) Assume that $\kappa(Z) = -\infty$. If $Z \not\cong \mathbb{P}^2$, then we have $8\chi(\mathcal{O}_Z) \geq K_Z^2$ holds, that is, we have

$$8\chi^H_{n,n-2}(X,\mathcal{E}) \geq (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}).$$

Hence

$$\begin{aligned} h^{1,1}_{n,n-2}(X,\mathcal{E}) &\geq 2\chi^H_{n,n-2}(X,\mathcal{E}) + 2h^1(\mathcal{O}_X) \\ &= 2 - 2h^1(\mathcal{O}_X) + 2h^1(\mathcal{O}_X) \\ &= 2 \end{aligned}$$

and the case is impossible. So we have $Z \cong \mathbb{P}^2$. By [9, Theorem A] we see that $(X,\mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2})$ and we get the assertion. □

THEOREM 3.2. *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension n such that* $\operatorname{rank}(\mathcal{E}) = n-2$ *and $\mathcal{E}$ is generated by its global sections. Then $b_{n,n-2}(X,\mathcal{E}) \geq 1$ holds. Moreover if $b_{n,n-2}(X,\mathcal{E}) = 1$, then $(X,\mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2})$.*

Proof. Here we note that $b_{n,n-2}(X,\mathcal{E}) = h^{2,0}_{n,n-2}(X,\mathcal{E}) + h^{1,1}_{n,n-2}(X,\mathcal{E}) + h^{0,2}_{n,n-2}(X,\mathcal{E})$ by Theorem 2.1 (i). Since $h^{2,0}_{n,n-2}(X,\mathcal{E}) = h^{0,2}_{n,n-2}(X,\mathcal{E}) = g_{n,n-2}(X,\mathcal{E}) \geq 0$ by Theorem 2.1 (iii) and assumption, we have $b_{n,n-2}(X,\mathcal{E}) \geq h^{1,1}_{n,n-2}(X,\mathcal{E})$. We also note that $h^{1,1}_{n,n-2}(X,\mathcal{E}) \geq 1$ (see [6, Proposition 4.1 (iv)]). Hence $b_{n,n-2}(X,\mathcal{E}) \geq 1$, and $b_{n,n-2}(X,\mathcal{E}) = 1$ implies that $h^{1,1}_{n,n-2}(X,\mathcal{E}) = 1$ and we get the assertion by Theorem 3.1. □

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