A NOTE ON CLASSIFICATION
OF GENERALIZED POLARIZED MANIFOLDS
BY THE $c_r$-SECTIONAL HODGE NUMBER OF TYPE $(1,1)$
AND THE $c_r$-SECTIONAL BETTI NUMBER

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Let $(X, \mathcal{E})$ be a generalized polarized manifold of dimension $n \geq 3$ such that $\text{rank}(\mathcal{E}) = n - 2$ and $\mathcal{E}$ is generated by its global sections. In this short note, we classify $(X, \mathcal{E})$ with $h^{1,1}_{n,n-2}(X, \mathcal{E}) = 1$ (resp. $b_{n,n-2}(X, \mathcal{E}) = 1$).

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1. INTRODUCTION

Let $X$ be a projective variety of dimension $n$ defined over the field of complex numbers, and let $L$ be an ample line bundle on $X$. Then the pair $(X, L)$ is called a polarized variety. Moreover if $X$ is smooth, then $(X, L)$ is called a polarized manifold.

When we study polarized varieties, it is useful to use their invariants. The sectional genus $g(X, L)$ of $(X, L)$ is one of the well-known invariants of $(X, L)$. In [4] we defined the notion of the $i$th sectional geometric genus $g_i(X, L)$ of $(X, L)$ for every integer $i$ with $0 \leq i \leq n$. Here we explain the meaning of these invariants if $X$ is smooth, $L$ is base point free and $i$ is an integer with $1 \leq i \leq n - 1$. Let $H_1, \ldots, H_{n-i}$ be general members of $|L|$. We put $X_{n-i} := H_1 \cap \cdots \cap H_{n-i}$. Then $X_{n-i}$ is smooth with $\dim X_{n-i} = i$, and we can show that $g_i(X, L) = h^i(\mathcal{O}_{X_{n-i}})$.

These induce the notion of the $i$th sectional invariant of $(X, L)$ associated with an invariant.

Definition 1. Let $(X, L)$ be a polarized manifold of dimension $n$. Let $I(Y)$ (or $I$) be an invariant of a smooth projective variety $Y$ of dimension $i$, where $i$ is an integer with $0 \leq i \leq n$. Then an invariant $F_i(X, L)$ of $(X, L)$
is called the *ith sectional invariant of* $(X, L)$ associated with the invariant $I$ if $F_i(X, L) = I(X_{n-i})$ under the assumption that $Bs|L| = \emptyset$.

The *ith sectional geometric genus* is the *ith sectional invariant* of $(X, L)$ associated with the geometric genus. By the definition of the *ith sectional invariants*, the *ith sectional invariants* are expected to reflect properties of *i*-dimensional geometry. So we can expect that we are able to find interesting properties of $(X, L)$ by using its *ith sectional invariants*.

In [5], we defined other *ith sectional invariants*, that is, the *ith sectional Euler number* $e_i(X, L)$, the *ith sectional Betti number* $b_i(X, L)$, and the *ith sectional Hodge number* $h^{j,i-j}_i(X, L)$ of type $(j, i-j)$ of $(X, L)$ and we studied some properties of these. If $X$ is smooth, $L$ is base point free and $i$ is an integer with $1 \leq i \leq n-1$, then by using the above notation we see that $e_i(X, L) = e(X_{n-i})$, $b_i(X, L) = h^i(X_{n-i}, \mathbb{C})$ and $h^{j,i-j}_i(X, L) = h^{j,i-j}(X_{n-i})$, where $e(X_{n-i})$ denotes the Euler number of $X_{n-i}$.

Let $X$ be a smooth projective variety with $\dim X = n$ and let $E$ be an ample vector bundle on $X$ with rank $E = r$. We assume that $r \leq n$. In [6], we defined ample vector bundles’ version of the invariants above. Namely we defined the $c_r$-sectional geometric genus $g_{n,r}(X, E)^1$, the $c_r$-sectional Euler number $e_{n,r}(X, E)$, the $c_r$-sectional Betti number $b_{n,r}(X, E)$ and the $c_r$-sectional Hodge number $h^{j,n-r-j}_{n,r}(X, E)$ of type $(j, n-r-j)$ of $(X, E)$.

It is natural to study a lower bound for these invariants. If $E$ is generated by its global sections and $r \leq n-1$, then $g_{n,r}(X, E)$, $b_{n,r}(X, E)$ and $h^{j,n-r-j}_{n,r}(X, E)$ are non-negative (see Proposition 2.1 below). Under this setting, it is interesting to characterize $(X, E)$ whose invariants are small.

If $r = n$, then $g_{n,n}(X, E) = b_{n,n}(X, E) = h^{0,0}_{n,n}(X, E) = c_n(E)$ (see [6, Remarks 3.2.1 (i) and 3.3.1]), and $(X, E)$ with small $c_n(E)$ has been studied by several authors (for example [11, 12, 14, 16]).

If $r = n-1$, then $2g_{n,n-1}(X, E) = b_{n,n-1}(X, E)$ and $g_{n,n-1}(X, E) = h^{1,0}_{n,n-1}(X, E) = h^{0,1}_{n,n-1}(X, E)$ (see Theorem 2.1).

Here we note that $g_{n,n-1}(X, E)$ is the curve genus of $(X, E)$, and $(X, E)$ with small $g_{n,n-1}(X, E)$ has been classified (for example, see [10, 13]).

Assume that $r = n-2$. Then Lanteri [8, (3.4)Corollary] studied the classification of $(X, E)$ with $g_{n,n-2}(X, E) = 0$ for $E = L_1 \oplus \cdots \oplus L_{n-2}$, where each $L_i$ is ample and spanned line bundle on $X$. But in general, we don't know the case of $(X, E)$ such that $E$ is generated by its global sections and $g_{n,n-2}(X, E) = 0$.

In this short note, we will study the case where $r = n-2$. In particular, we will treat the $c_r$-sectional Hodge number $h^{1,1}_{n,n-2}(X, E)$ and the $c_r$-sectional.

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1This invariant $g_{n,r}(X, E)$ is equal to the invariant $g_{n-r}(X, E)$ in [3, Definition 2.1].
Betti number $b_{n,n-2}(X, \mathcal{E})$. In this case, we can prove $h_{n,n-2}^{1,1}(X, \mathcal{E}) \geq 1$ and $b_{n,n-2}(X, \mathcal{E}) \geq 1$ under the assumption that $\mathcal{E}$ is generated by its global sections (see [6, Proposition 4.1 (iv)] and Theorem 3.2). Moreover, it is natural to consider the classification of $(X, \mathcal{E})$ with $h_{n,n-2}^{1,1}(X, \mathcal{E}) = 1$ and the classification of $(X, \mathcal{E})$ with $b_{n,n-2}(X, \mathcal{E}) = 1$. In Theorem 3.1 (resp. Theorem 3.2) we will classify $(X, \mathcal{E})$ with $h_{n,n-2}^{1,1}(X, \mathcal{E}) = 1$ (resp. $b_{n,n-2}(X, \mathcal{E}) = 1$).

2. PRELIMINARIES

Notation:
(1) Let $Y$ be a smooth projective variety of dimension $i \geq 1$, let $T_Y$ be the tangent bundle of $Y$ and let $\Omega_Y$ be the dual bundle of $T_Y$. For every integer $j$ with $0 \leq j \leq i$, we put

$$h_{i,j}(c_1(Y), \cdots, c_i(Y)) := \chi(\Omega^j_Y) = \int_Y \text{ch}(\Omega^j_Y) \text{Td}(T_Y).$$

(Here $\text{ch}(\Omega^j_Y)$ (resp. $\text{Td}(T_Y)$) denotes the Chern character of $\Omega^j_Y$ (resp. the Todd class of $T_Y$). See [7, example 3.2.3 and example 3.2.4].)

(2) Let $X$ be a smooth projective variety of dimension $n$. For every integers $i$ and $j$ with $0 \leq j \leq i \leq n$, we put

$$H_1(X; i, j) := \begin{cases} \sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega^j_X) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(X; i, j) := \begin{cases} \sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega^{i-j}_X) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

In this paper, we consider the following case ($\ast$):

($\ast$) Let $X$ be a smooth projective variety of dimension $n$ and let $\mathcal{E}$ be an ample vector bundle of rank $r$ on $X$ with $r \leq n$.

**Definition 2.1.** Let $(X, \mathcal{E})$, $n$ and $r$ be as in ($\ast$). For every integer $p$ with $0 \leq p \leq n - r$ we set

$$C_{n,r}^p(X, \mathcal{E}) := \sum_{k=0}^{p} c_k(X) s_{p-k}(\mathcal{E}^\vee).$$

**Definition 2.2** ([6, Definition 3.1.1]). Let $(X, \mathcal{E})$, $n$ and $r$ be as in ($\ast$). The $c_r$-sectional $H$-arithmetic genus $\chi_{n,r}^H(X, \mathcal{E})$ and the $c_r$-sectional Euler number
$e_{n,r}(X, \mathcal{E})$ of $(X, \mathcal{E})$ are defined by the following$^2$:
\[
\chi_{n,r}^H(X, \mathcal{E}) := \text{td}_{n-r} \left( C_1^{n,r}(X, \mathcal{E}), \ldots, C_{n-r}^{n,r}(X, \mathcal{E}) \right) c_r(\mathcal{E}).
\]
\[
e_{n,r}(X, \mathcal{E}) := C_{n-r}^{n,r}(X, \mathcal{E}) c_r(\mathcal{E}).
\]

**Definition 2.3** ([6, Definition 3.2.1]). Let $(X, \mathcal{E})$, $n$ and $r$ be as in ($\ast$). The $c_r$-sectional geometric genus $g_{n,r}(X, \mathcal{E})$ and the $c_r$-sectional Betti number $b_{n,r}(X, \mathcal{E})$ of $(X, \mathcal{E})$ are defined by the following:
\[
g_{n,r}(X, \mathcal{E}) := (-1)^{n-r} \chi_{n,r}^H(X, \mathcal{E}) + (-1)^{n-r+1} \chi(\mathcal{O}_X) + \sum_{k=0}^{r} (-1)^{r-k} h^{n-k}(\mathcal{O}_X).
\]
\[
b_{n,r}(X, \mathcal{E}) := \begin{cases} 
(-1)^{n-r} \left( e_{n,r}(X, \mathcal{E}) - \sum_{j=0}^{n-r-1} 2(-1)^j h^j(X, \mathbb{C}) \right), & \text{if } r < n, \\
es_{n,n}(X, \mathcal{E}), & \text{if } r = n.
\end{cases}
\]

**Definition 2.4** ([6, Definition 3.3.1]). Let $(X, \mathcal{E})$, $n$ and $r$ be as in ($\ast$). The $c_r$-sectional Hodge number $h_{n,r}^{j,n-r-j}(X, \mathcal{E})$ of type $(j, n-r-j)$ of $(X, \mathcal{E})$ is defined by the following:
\[
h_{n,r}^{j,n-r-j}(X, \mathcal{E}) := (-1)^{n-r-j} \left\{ w_{n,r}^j(X, \mathcal{E}) - H_1(X; n-r, j) - H_2(X; n-r, j) \right\}.
\]
Here we set
\[
w_{n,r}^j(X, \mathcal{E}) := \begin{cases} 
h_{n-r,j} C_1^{n,r}(X, \mathcal{E}), \cdots, C_{n-r}^{n,r}(X, \mathcal{E}) c_r(\mathcal{E}), & \text{if } r < n, \\
C_n(\mathcal{E}), & \text{if } r = n.
\end{cases}
\]
for every integer $j$ with $0 \leq j \leq n - r$.

These invariants satisfy the following properties.

**Proposition 2.1.** Let $(X, \mathcal{E})$, $n$ and $r$ be as in ($\ast$). Assume that $r \leq n-1$ and there exists a smooth projective variety $Z$ such that $\dim Z = n-r$ and $Z$ is the zero locus of an element of $H^0(\mathcal{E})$. Then
\[
\chi_{n,r}^H(X, \mathcal{E}) = \chi(\mathcal{O}_Z), \quad g_{n,r}(X, \mathcal{E}) = h^{n-r}(\mathcal{O}_Z), \quad e_{n,r}(X, \mathcal{E}) = e(Z),
\]
\[
b_{n,r}(X, \mathcal{E}) = h^{n-r}(Z, \mathbb{C}), \quad h_{n,r}^{j,n-r-j}(X, \mathcal{E}) = h_{n,r}^{j,n-r-j}(Z).
\]
In particular, $g_{n,r}(X, \mathcal{E})$, $b_{n,r}(X, \mathcal{E})$ and $h_{n,r}^{j,n-r-j}(X, \mathcal{E})$ are non-negative if $\mathcal{E}$ is generated by its global sections with $r \leq n-1$.

**Proof.** See [6, Propositions 3.1.1, 3.2.2 and 3.3.1]. □

$^2$Here $\text{td}_{n-r}$ means the Todd polynomial of weight $n-r$ (see [3, Definition 1.4 (1)]).
THEOREM 2.1. Let \((X, \mathcal{E})\), \(n\) and \(r\) be as in \((\ast)\). Assume that \(r \leq n - 1\). For every integer \(j\) with \(0 \leq j \leq n - r\), we get the following.

(i) \(b_{n,r}(X, \mathcal{E}) = \sum_{k=0}^{n-r} h_{n,r}^{k,n-r-k}(X, \mathcal{E})\).

(ii) \(h_{n,r}^{j,n-r-j}(X, \mathcal{E}) = h_{n,r}^{n-r-j}(X, \mathcal{E})\).

(iii) \(h_{n,r}^{n-r,0}(X, \mathcal{E}) = g_{n,r}(X, \mathcal{E})\).

(iv) If \(n - r\) is odd, then \(b_{n,r}(X, \mathcal{E})\) is even.

Proof. See [6, Theorem 4.1]. □

3. MAIN RESULT

THEOREM 3.1. Let \((X, \mathcal{E})\) be a generalized polarized manifold of dimension \(n \geq 3\) such that \(\text{rank}(\mathcal{E}) = n - 2\) and \(\mathcal{E}\) is generated by its global sections. In this setting, we have \(h_{n,n-2}^{1,1}(X, \mathcal{E}) \geq 1\) by [6, Proposition 4.1 (iv)]. If \(h_{n,n-2}^{1,1}(X, \mathcal{E}) = 1\), then \((X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2})\).

Proof. We note that by [6, Theorem 6.1] the following equality holds.

\[
(1) 12\chi_{n,n-2}^{H}(X, \mathcal{E}) = (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) + e_{n,n-2}(X, \mathcal{E}).
\]

(This is an analogue of Noether’s equality [1, p.26 (4)].) Hence by (1), Theorem 2.1 and Definitions 2.2, 2.3 and 2.4 we have

\[
(2) h_{n,n-2}^{1,1}(X, \mathcal{E}) = 10\chi_{n,n-2}^{H}(X, \mathcal{E}) - (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) + 2h^1(\mathcal{O}_X).
\]

Let \(Z\) be the zero locus of a general member \(s\) of \(H^0(X, \mathcal{E})\). Then \(Z\) is a smooth projective surface and the following hold.

\[
(3) \chi_{n,n-2}^{H}(X, \mathcal{E}) = \chi(\mathcal{O}_Z);
\]

\[
(4) (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) = K_Z^2;
\]

\[
(5) h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Z).
\]

(i) Assume that \(\kappa(Z) \geq 0\). Then \(\chi(\mathcal{O}_Z) \geq 0\) and \(9\chi(\mathcal{O}_Z) \geq K_Z^2\) hold. Hence by (3) and (4) we have \(\chi_{n,n-2}^{H}(X, \mathcal{E}) \geq 0\) and \(9\chi_{n,n-2}^{H}(X, \mathcal{E}) \geq (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E})\). Therefore by (2) we get

\[
(6) h_{n,n-2}^{1,1}(X, \mathcal{E}) \geq \chi_{n,n-2}^{H}(X, \mathcal{E}) + 2h^1(\mathcal{O}_X).
\]

Since \(h_{n,n-2}^{1,1}(X, \mathcal{E}) = 1\), we have \(0 \leq \chi_{n,n-2}^{H}(X, \mathcal{E}) \leq 1\) and \(h^1(\mathcal{O}_X) = 0\). Here we note that

\[
\chi_{n,n-2}^{H}(X, \mathcal{E}) = 1 - h^1(\mathcal{O}_X) + g_{n,n-2}(X, \mathcal{E})
\]

holds, and \(g_{n,n-2}(X, \mathcal{E}) = h^2(\mathcal{O}_Z) \geq 0\) by (3) and (5). Therefore

\[
(7) \chi_{n,n-2}^{H}(X, \mathcal{E}) \geq 1.
\]
Hence by (6) and (7) we have $9\chi^{H}_{n,n-2}(X, E) = (K_X + c_1(E))^2c_{n-2}(E)$, that is, $9\chi(O_Z) = K^2_Z$. Thus $Z$ is a ball quotient and hence $Z$ is an Eilenberg-MacLane space of type $(\Pi, 1)$ (see [2, Remark 5.1.7]). On the other hand, by [15, Corollary 22] we have $\pi_1(X, Z) = \{0\}$ and $\pi_2(X, Z) = \{0\}$. Moreover we note that the restriction homomorphism $H^2(X, \mathbb{Z}) \to H^2(Z, \mathbb{Z})$ is injective by the Lefschetz-type theorem for ample vector bundles (see [9, Theorem 1.3]). Hence by the proof of [2, Theorem 5.1.5 and Corollary 5.1.6], this is impossible because we assume that $n = \dim X \geq 3$.

(ii) Assume that $\kappa(Z) = -\infty$. If $Z \not\cong \mathbb{P}^2$, then we have $8\chi(O_Z) \geq K^2_Z$ holds, that is, we have

$$8\chi^{H}_{n,n-2}(X, E) \geq (K_X + c_1(E))^2c_{n-2}(E).$$

Hence

$$h^{1,1}_{n,n-2}(X, E) \geq 2\chi^{H}_{n,n-2}(X, E) + 2h^1(O_X) = 2 - 2h^1(O_X) + 2h^1(O_X) = 2$$

and the case is impossible. So we have $Z \cong \mathbb{P}^2$. By [9, Theorem A] we see that $(X, E) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(1)\oplus^{n-2})$ and we get the assertion. □

**Theorem 3.2.** Let $(X, E)$ be a generalized polarized manifold of dimension $n$ such that $\text{rank}(E) = n - 2$ and $E$ is generated by its global sections. Then $b_{n,n-2}(X, E) \geq 1$ holds. Moreover if $b_{n,n-2}(X, E) = 1$, then $(X, E) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(1)\oplus^{n-2})$.

*Proof.* Here we note that $b_{n,n-2}(X, E) = h^{2,0}_{n,n-2}(X, E) + h^{1,1}_{n,n-2}(X, E) + h^{0,2}_{n,n-2}(X, E)$ by Theorem 2.1 (i). Since $h^{2,0}_{n,n-2}(X, E) = h^{0,2}_{n,n-2}(X, E) = g_{n,n-2}(X, E) \geq 0$ by Theorem 2.1 (iii) and assumption, we have $b_{n,n-2}(X, E) \geq h^{1,1}_{n,n-2}(X, E)$. We also note that $h^{1,1}_{n,n-2}(X, E) \geq 1$ (see [6, Proposition 4.1 (iv)]). Hence $b_{n,n-2}(X, E) \geq 1$, and $b_{n,n-2}(X, E) = 1$ implies that $h^{1,1}_{n,n-2}(X, E) = 1$ and we get the assertion by Theorem 3.1. □

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