ON GLOBAL AND DECAY SOLUTION FOR LARGE SIZE DATA OF NONLINEAR KIRCHHOFF MODEL IN SLOWLY INCREASING MOVING DOMAINS

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In this paper, we study the global and decay solution for large size data of nonlinear hyperbolic-parabolic equation of Kirchhoff type

$$u_{tt} + \mu u_t - \tilde{M}\Big(\int_{\Omega_t} |\nabla u|^2 \mathrm{d}x\Big)\Delta u = 0 \quad \text{in } \Omega_t$$

Where $\Omega_t = \{x \in \mathbb{R}^n : \ x = y\sigma(t), \quad y \in \Omega\}$ with Ω being a bounded open domain in $\mathbb{R}^n$, μ is a positive constant and $\sigma(t)$ is a given suitable increasing positive function unbounded from above. The real function $\tilde{M}$ is such that $\tilde{M}(\lambda) > 0$ and $\tilde{M}'(\lambda) \geqslant 0$ for every $\lambda \in [0, \infty[$.

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1. INTRODUCTION

Let Ω be an open bounded domain of $\mathbb{R}^n$ which, without loss of generality, can be assumed to contain the origin, with boundary Γ of class C^2 and $\sigma : [0, \infty[\longrightarrow \mathbb{R}$ a positive continuously differentiable increasing function, unbounded from above. Let us consider the family of bounded increasing subdomains $\{\Omega_t\}_{0 \leqslant t < \infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = h_t(\Omega), \quad \Omega_0 = h_0(\Omega), \quad h_t : y \in \Omega \longmapsto x = \sigma(t)y$$

whose boundaries are denoted by Γ_t, and $\hat{Q}$ the non-cylindrical domain of $\mathbb{R}^{n+1}$

$$\hat{Q} = \bigcup_{0 \leqslant t < \infty} \Omega_t \times \{t\},$$

with lateral boundary

$$\hat{\Gamma} = \bigcup_{0 \leqslant t < \infty} \Gamma_t \times \{t\}.$$

We consider the following mixed problem related to a nonlinear equation of Kirchhoff type

$$u_{tt} + \mu u_t - \tilde{M}\Big(\int_{\Omega_t} |\nabla u|^2 \mathrm{d}x\Big)\Delta u = 0 \quad \text{in} \quad \hat{Q}\,, \tag{1.1}$$

$$u|_{\hat{\Gamma}} = 0\,, \tag{1.2}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \tag{1.3}$$

where the given function $\tilde{M}$ satisfies the following conditions

$$\tilde{M} \in C^2([0,\infty[), \quad \tilde{M}(\lambda) \geqslant m_0 > 0, \quad \tilde{M}'(\lambda) \geqslant 0 \quad \forall \lambda \in [0,\infty[. \tag{1.4}$$

Here we want to solve the problem (1.1)–(1.3) globally in time regardless of size of the initial data $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$ provided the expansion of moving domains Ω_t is fairly slow.

In the literature, the equation (1.1) is called of hyperbolic-parabolic type. This class of equations has been studied by several authors, for instance Lar'Kin [26] and Bensoussan *et al.* [6]. Bisognin proved in [9] the existence of local solution of (1.1) in both bounded and unbounded domains of $\mathbb{R}^n$.

Whenever $\mu = 0$, there is a large number of papers involving the Kirchhoff-Carrier operator

$$Lu = u_{tt} - \Big(1 + \tilde{M}\Big(\int_{\Omega} |\nabla u|^2 \mathrm{d}x\Big)\Big)\Delta u.$$

We recall that in the case $n = 1$ with $M(\lambda) = a\lambda + b$ and a, $b > 0$, the equation $Lu = 0$ was proposed by Kirchhoff [25] in his book of Mathematical Physics in 1883, to describe the oscillations of an elastic stretched string. This equation was studied by some other authors like, Carrier [13], Bernstein [7], Dickey [17, 18], Menzala [31]. The result of local existence for $Lu = 0$ was obtained by some of the authors quoted above with initial data taken in usual Sobolev spaces and for both Dirichlet and periodic boundary conditions. The first result on global solvability for the Kirchhoff equation was established by Bernstein [7] in dimension $n = 1$ for analytic initial data. This result was extended later by Pohozaev [36], Arosio-Spagnolo [1], Kajatani-Yamaguti [24] in dimension $n \geqslant 2$. Throughout the years, these results on the global solvability for analytical initial data were extended and refined later by several authors (see for instance, Nishihara [34], Ghisi-Gobbino [21]),

The global solvability for large non-analytic initial data has been till now a deep open problem. Several results on the global solvability for small non-analytical (mainly of class $\mathcal{C}^\infty$ with compact support, Gevrey class, or Sobolev spaces) initial data are well established in the literature (see for instance, [10, 15, 16, 22, 30, 38–40]). We also mention that, for non analytical initial data, Pohozaev [37] and Menzala-Pereira [32] for instance, have obtained some global existence results, using non physical functions $M(r)$ behaving like $(\alpha r+\beta)^{-2}$, α and β being positive constants.

In order to obtain a global solution for $Lu = f$ several authors (see for instance Nishihara [35]), have introduced damping terms like $-\Delta u_t$ or $\Delta^2 u$ which allow to get strong estimates in order to control the nonlinear term proving in this way the global existence result. Another class of dissipative mechanisms was considered by Ikehata and Okazawa in [23], where the authors studied the following equation of a stretched string with "frictional" damping

$$u_{tt} - (1 + \tilde{M}(\int_\Omega |\nabla u(x,t)|^2 \mathrm{d}x))\Delta u + \mu u_t = 0$$

and showed the existence of global strong solutions, provided μ (a parameter depending on the initial datum) is large enough. Other authors have considered a model with a nonlinear damping term $g(u_t)$ replacing the term of μu_t.

The problem (1.1)–(1.3) was studied in [2,3] globally in time in dimension two provided the initial data are small and with non homogeneous Dirichlet boundary condition. In the literature, several works have been devoted to evolution problems in non-cylindrical domains (see [4, 5, 8, 11, 14, 20, 28]). For instance, the heat equation, the Navier-Stokes equation and the wave equation have been studied in non-cylindrical domains. The proof of the existence of both local and global solutions in most of those articles is based on suitable change of variables which allows to transform the problem in another problem in a cylindrical domain. Other methods have developed to solve evolution problems in non cylindrical domains. For instance, Cannarsa *et al.* developed in [11] a method which consists in transforming the problem into a non autonomous initial boundary problem in the Lebesgue space $L^2(\Omega)$, involving a family of unbounded operators with variable coefficients.

As it is well known, the result about local existence of solutions was proved in cylindrical and non-cylindrical domains by many authors cited in the reference. Our principal attention in this paper is devoted to the global existence of solutions and their asymptotic behaviour. We follow here the change of variable method described above. As announced above, this problem has already been studied in [2,3] in the the two-dimensional space case.

Our goal in this paper is to extend the results in the articles [2,3] in higher dimensional space and for opportunely large initial data. We succeeded to do so under the further assumption that the expansion of the domains Ω_t is slow and that the size of the initial domain Ω_0 is small.

To this aim, we will first study our problem in the cylinder $Q = \Omega\times]0,\infty[$. The domains Q and $\hat{Q}$ are related by the diffeomorphism $\tau : \hat{Q} \longrightarrow Q$ defined by

$$\tau(x,t) := (y,t) = (\frac{x}{\sigma(t)}, t) \quad \text{for} \quad (x,t) \in \hat{Q}. \tag{1.5}$$

Whose inverse $\tau^{-1} : Q \longrightarrow \hat{Q}$ is given by

$$\tau^{-1}(y,t) := (x,t) = (y\sigma(t), t). \tag{1.6}$$

If we set

$$v(y,t) := u \circ \tau^{-1}(y,t) = u(y\sigma(t), t), \tag{1.7}$$

then the initial boundary value problem (1.1)–(1.3) becomes

$$v_{tt} + \mu v_t - \frac{1}{\sigma^2}\tilde{M}\Big(\int_\Omega |\sigma^{\frac{n-2}{2}}\nabla v|^2 \mathrm{d}y\Big)\Delta v = \tilde{F}(t,v)\,, \tag{1.8}$$

$$v|_{\partial\Omega} = 0, \qquad v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \tag{1.9}$$

where

$$\tilde{F}(t,v) := -\Big(\frac{\sigma'}{\sigma}\Big)^2 \sum_{i,j=1}^{n} \partial_{y_i}(y_i y_j \partial_{y_j} v) + a_1(t,y)\cdot\nabla v_t + a_2(t)\cdot\nabla v, \tag{1.10}$$

$$a_1(t,y) := 2\frac{\sigma'}{\sigma}y, \quad a_2(t,y) := \sigma^{-2}y(\sigma\sigma'' + \mu\sigma\sigma' + (n-1)\sigma'^2). \tag{1.11}$$

Remark 1.1. Note that the initial data (v_0, v_1) is determined by the given couple (1.3) (u_0, u_1) and depends of course on the initial position $\sigma(0)$ and the initial velocity $\sigma'(0)$, thus (see (1.25)) on σ_0 and σ_1. But considering subsequent assumption (see (2.2)) on σ_0 and σ_1, the only dependency of (v_0, v_1) in terms of σ_0 is meaningful. To emphasize this dependency, when required it will be noted $(v^0_{\sigma_0}, v^1_{\sigma_0})$ instead of (v_0, v_1).

Indeed, given (u_0, u_1), the couple of initial data (v_0, v_1) is determined using equations

$$x \in \Omega_0 = \sigma(0)\Omega, \quad u_0(x) = u(\sigma(0)y, 0) = v_0(y), \quad y \in \Omega \tag{1.12}$$

and (see (1.7) and (1.25))

$$u_1(x) = v_1(y) - \alpha\frac{\sigma_1}{\sigma_0}y\cdot\nabla v_0(y), \quad v_1(y) = v_t(y,t)|_{t=0}. \tag{1.13}$$

We set

$$M(s) := \tilde{M}(s) - \frac{m_0}{2}, \tag{1.14}$$

$$a_{ij}(t,y) := \frac{m_0}{2\sigma^2}\delta_{ij} - \Big(\frac{\sigma'}{\sigma}\Big)^2 y_i y_j \quad (i,j = 1,n)\,. \tag{1.15}$$

According to (1.14) and (1.4), it follows that

$$M(\lambda) \geqslant \frac{m_0}{2}, \quad M \in C^2([0,\infty[), \quad M'(\lambda) \geqslant 0, \qquad \forall\lambda \in [0,\infty[\,. \tag{1.16}$$

Given (1.14)–(1.15), the problem (1.8) and (1.9) is rewritten as

$$v_{tt} + \mu v_t - \frac{1}{\sigma^2}M\Big(\int_\Omega |\sigma^{\frac{n-2}{2}}\nabla v|^2 \mathrm{d}y\Big)\Delta v = F(t,v)\,, \tag{1.17}$$

$$v|_{\partial\Omega} = 0\,, \tag{1.18}$$

$$v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \tag{1.19}$$

with

$$F(t,v) = A(t)v + a_1(t,y)\cdot\nabla v_t + a_2(t,y)\cdot\nabla v, \tag{1.20}$$

where

$$A(t) = \sum_{i,j=1}^{n} \partial_{y_i}(a_{ij}(t,y)\partial_{y_j}v).$$

We set

$$a(t,u,v) = \sum_{i,j=1}^{n}\int_\Omega a_{ij}(t,y)(\partial_{y_i}u)(\partial_{y_j}v)\mathrm{d}y \tag{1.21}$$

$$a'(t,u,v) = \sum_{i,j=1}^{n}\int_\Omega a'_{ij}(t,y)(\partial_{y_i}u)(\partial_{y_j}v)\mathrm{d}y. \tag{1.22}$$

To study (1.17)–(1.19) we need some hypotheses on the function σ. Let us first recall that the function σ is positive, increasing and unbounded from above. Moreover, we assume that

$$\sigma\in C^3([0,\infty[), \quad \sigma(0)>0, \quad 0\leqslant\sigma'(t)\leqslant\frac{1}{d}\sqrt{\frac{m_0}{2}} \quad \forall t>0, \tag{1.23}$$

where $d=\mathrm{diam}(\Omega)$. The second condition (1.23) implies that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant 0 \quad \forall\xi\in\mathbb{R}^n\backslash\{0\}. \tag{1.24}$$

In order to avoid tedious abstract computations, we work throughout the paper with a typical family of functions σ which satisfy (1.23), that is

$$\sigma(t) = (\sigma_0+\sigma_1 t)^\alpha, \quad 0<\alpha<\frac{1}{2}. \tag{1.25}$$

where σ_0 and σ_1 are positive constants chosen so that (1.23) is satisfied. Note that this assumption means that $\hat{Q}$ is increasing in the sense that, if $t>t'$ then Ω_t contains $\Omega_{t'}$.

This paper is organised as follows. In Section 2, we present the result on the local existence for the problem (1.17)–(1.19) (and hence, for the problem (1.1)–(1.3)). The main difficulty in this paper as well as in [2, 3] lies in the derivation of the *a priori* estimates in Section 3, needed in order to extend the local solution and get the results of global existence for (1.17)–(1.19) and (1.1)–(1.3). The estimates in the Lemmas 3.1–3.6 are obtained by carefully choosing test functions for the equation (1.17), which are products of the unknown function

v (or some of its time derivatives) with suitable powers of the function σ describing the expansion of the domain (see (3), (3), (3), (3) and (3)). Section 4 is devoted to the existence of the global solution and its asymptotic behaviour with initial data opportunely large enough.

2. LOCAL SOLUTION

As mentioned in the introduction, the results about local existence of solutions were proved in cylindrical and non-cylindrical domains by many authors cited in the reference (see for exemple [4]). Through a process of approximation and compactness arguments, we can show that, for any initial data $(v_0, v_1) \in H^2(\Omega) \times H^1(\Omega)$, there exists $\bar{t} > 0$ such that the problem (1.17)–(1.19) has a unique local solution v such that

$$v \in L^\infty(0,\bar{t}; H_0^1(\Omega) \cap H^2(\Omega)), \quad v_t \in L^\infty(0,\bar{t}; H^1(\Omega)) \text{ and } v_{tt} \in L^\infty(0,\bar{t}; L^2(\Omega)).$$

So, it follows that $u = v \circ \tau$ (see (1.5) for the definition of τ) is the unique local solution of the problem (1.1)–(1.3) with

$$\begin{gathered} u \in L^\infty(0,\bar{t}; H_0^1(\Omega_t) \cap H^2(\Omega_t)), \\ u_t \in L^\infty(0,\bar{t}; H^1(\Omega_t)), \quad u_{tt} \in L^\infty(0,\bar{t}; L^2(\Omega_t)). \end{gathered} \tag{2.1}$$

The global existence and asymptotic behavior of the problem (1.1)–(1.3) with small initial data have been studied in [2] in dimensional $n = 2$. Here, we want to improve the result in [2] in the sense that $n \geqslant 3$ and size of initial data may be large enough. The global solution will be obtained by combining the result of local existence and suitable a priori estimates. These estimates which will be obtained in the following lemmas require a more elaborate treatment unlike the case σ bounded, because the assumption $\sigma(t) \to \infty$ for $t \to \infty$ makes the equation (1.17) degenerate at infinity. However, these estimates will be established under the assumptions (see (1.25))

$$0 < \varepsilon_1 \leqslant \sigma_0 \leqslant \varepsilon_0 < 1, \quad 0 \leqslant \sigma_1 \leqslant K\varepsilon_1^{1+\alpha}, \tag{2.2}$$

where ε_0, ε_1, K are positive constants. It should be noted (see (1.25)) that for all $j \geqslant 2$

$$\begin{gathered} |\sigma^{(j)}(t)| \leqslant K^{j-1}|(\alpha-1)(\alpha-2)\cdots(\alpha-(j-1))|\varepsilon_0^{\alpha(j-1)}\sigma'(t) \text{ for all } t \geqslant 0, \\ 0 < \sigma'(t) \leqslant \alpha K\varepsilon_0^{2\alpha}, \quad \frac{\sigma'}{\sigma} \leqslant \alpha K\varepsilon_0^{\alpha} \text{ for all } t \geqslant 0, \end{gathered} \tag{2.3}$$

which follows immediately from (2.2), (1.25) and the inequality

$$|\sigma^{(j)}(t)| \leqslant K\varepsilon_0^\alpha |\alpha - j + 1||\sigma^{(j-1)}(t)| \text{ for all } t \geqslant 0.$$

3. A PRIORI ESTIMATES

Let us bear in mind that, given the initial expansion $\Omega_0 = \sigma_0\Omega$ which will be assumed (see (2.2)) small enough and initial data $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$, our goal here is to show that the problem (1.1)–(1.3) admits a global solution u if the size

$$R(\Omega_0) = \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} \tag{3.1}$$

of initial data is large enough. Since $u = vo\tau$ (see (1.6)), it suffices to show that the problem (1.17) and (1.19) has a global solution v if the size of initial data $(v^0_{\sigma_0}, v^1_{\sigma_0}) \in H^2(\Omega) \times H^1(\Omega)$ is large enough. For that purpose, we set

$$R(\sigma_0) = \|v^0_{\sigma_0}\|^2_{H^2(\Omega)} + \|v^1_{\sigma_0}\|^2_{H^1(\Omega)}, \quad \lambda(\sigma_0) = \sigma_0^{\alpha(n-3+r)} R(\sigma_0), \qquad n \geqslant 3, \quad 0 < r < 1. \tag{3.2}$$

and we suppose

$$\lim_{\sigma_0 \to 0} \lambda(\sigma_0) = 0. \tag{3.3}$$

The assumption (3.3) specifies in what sense the size of our initial data $(v^0_{\sigma_0}, v^1_{\sigma_0})$ can be considered rather large if σ_0 is small enough. Moreover, if (3.3) is satisfied, then the size $R(\Omega_0)$ can be considered large enough. More precisely, we have

$$R(\Omega_0) \leqslant \frac{C_\Omega}{|\Omega_0|^{\frac{1+r}{n}}}. \tag{3.4}$$

In fact, recalling (1.12), (1.13) and (3.2), by easy computations, we can verify that

$$\begin{aligned} \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} &\leqslant C_\Omega \sigma_0^{\alpha(n-4)} (\|v^0_{\sigma_0}\|^2_{H^2(\Omega)} + \|v^1_{\sigma_0}\|^2_{H^1(\Omega)}) \\ &\leqslant C_\Omega \sigma_0^{-\alpha(1+r)} \lambda(\sigma_0) \end{aligned} \tag{3.5}$$

and considering (3.3), necessarily we have

$$\sigma_0^{\alpha(1+r)} \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} \leqslant C_\Omega \lambda(\sigma_0) \leqslant C_\Omega \tag{3.6}$$

whence (3.6) because $|\Omega_0| = \sigma_0^{\alpha n}|\Omega|$.

In statements following lemmas, we denote by C_i $(i = 0, \ldots, 4)$ the constants which depend on Ω, n , μ, m_0 and (see (3.8)) M_0 but (see (2.2)) neither on σ_0 nor σ_1. In addition, in the proof of each lemma, we will denote by $\tilde{C}_i$ $(i = 1, \cdots 8)$ the constants that depend only on Ω, n, μ and possibly on m_0 and M_0. As for the constants that depend only on Ω and n, they will be designated by C_Ω. Moreover, we denote $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^m}$ for the usual norms in the spaces $L^2 = L^2(\Omega)$ and $H^m = H^m(\Omega)$ respectively. In the sequel, fixed σ_0 small enough we consider the family of initial data $(v^0_{\sigma_0}, v^1_{\sigma_0}) \in H^2(\Omega) \cap H^1(\Omega)$ verifying (3.3) and we will derive estimates of the local solution of problem (1.17)–(1.19) that will allow us to extend this to a global solution. We begin by showing a crucial estimate which will be used essentially in the proof of Lemmas 3.5–3.7.

LEMMA 3.1. *Let σ_0 small enough and $(v^0_{\sigma_0}, v^1_{\sigma_0}) \in H^2(\Omega) \times H^1(\Omega)$. Given (3.3), (2.2) and (1.25), we have*

$$\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2} \leqslant C_0 \sigma_0^{\alpha(n-2)} (\|v^1_{\sigma_0}\|^2_{L^2} + \|v^0_{\sigma_0}\|^2_{H^1}), \quad n \geqslant 3 \tag{3.7}$$

$$\left|M^{(i)}(\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2})\right| \leqslant M_0, \quad i = 0, 1, 2 \tag{3.8}$$

where $M^{(i)}$ is the ith derivative of M and M_0 a positive constant independent of on σ_0 and σ_1.

Proof. To prove (3.7), we first multiply equation (1.17) with

$$\sigma^2 \sigma^{\frac{n-2}{2}} (\sigma^{\frac{n-2}{2}} v)_t.$$

By integrating over Ω, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} E(t) + \mu \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} = \sum_{k=1}^{5} I_k, \tag{3.9}$$

where

$$E(t) := \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} + \sigma^2 a(t, \sigma^{\frac{n-2}{2}} v, \sigma^{\frac{n-2}{2}} v) + \hat{M}(\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2}), \tag{3.10}$$

$$\hat{M}(\lambda) := \int_0^\lambda M(s) \mathrm{d}s. \tag{3.11}$$

and

$$\begin{aligned}
I_1 &:= (n-1) \frac{\sigma'}{\sigma} \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} - \frac{1}{2} \sigma^2 \int_\Omega (\nabla \cdot a_1) |(\sigma^{\frac{n-2}{2}} v)_t|^2 \mathrm{d}y, \\
I_2 &:= -\frac{n-2}{2} \frac{\sigma'}{\sigma} \sigma^2 \int_\Omega (a_1 . \nabla (\sigma^{\frac{n-2}{2}} v)(\sigma^{\frac{n-2}{2}} v)_t \mathrm{d}y, \\
I_3 &:= \sigma^2 \int_\Omega (a_2 . \nabla (\sigma^{\frac{n-2}{2}} v)(\sigma^{\frac{n-2}{2}} v)_t \mathrm{d}y, \\
I_4 &:= \frac{n-2}{2} [\frac{\sigma''}{\sigma} - \frac{n}{2} \left|\frac{\sigma'}{\sigma}\right|^2 + \mu \frac{\sigma'}{\sigma}] \sigma^2 \int_\Omega (\sigma^{\frac{n-2}{2}} v)(\sigma^{\frac{n-2}{2}} v)_t \mathrm{d}y, \\
I_5 &:= \frac{1}{2} \sigma^2 a'(t, \sigma^{\frac{n-2}{2}} v, \sigma^{\frac{n-2}{2}} v) + \frac{\sigma'}{\sigma} \sigma^2 a(t, \sigma^{\frac{n-2}{2}} v, \sigma^{\frac{n-2}{2}} v).
\end{aligned}$$

Recalling the expressions (1.11) of a_1 and a_2, it is easy to see that

$$I_1 = -\frac{\sigma'}{\sigma}\sigma^2\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} \leqslant 0, \tag{3.12}$$

$$|I_2+I_3+I_4| \leqslant \frac{\mu}{4}\sigma^2\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + C_\Omega\big(|\sigma'|^2+|\sigma''|^2+|\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}. \tag{3.13}$$

Furthermore, by recalling (1.21), (1.22) and (1.15)) we get

$$I_5 = -\sigma''\sigma'\sum_{i,j=1}^n\int_\Omega y_iy_j(\sigma^{\frac{n-2}{2}}\partial_{y_i}v)(\sigma^{\frac{n-2}{2}}\partial_{y_j}v)\mathrm{d}y \leqslant \mathcal{C}_\Omega(|\sigma''|^2+|\sigma'|^2)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}.$$

Therefore, on account of (3.13) and (3.12)

$$\sum_{k=1}^5 I_k \leqslant \frac{\mu}{2}\sigma^2\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + C_\Omega\big(|\sigma'|^2+|\sigma''|^2+|\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \tag{3.14}$$

and by adding (3.14) to (3.9) it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant C_\Omega\big(|\sigma'|^2+|\sigma''|^2+|\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}. \tag{3.15}$$

Since (see (3.10), (3.11) and (1.16))

$$E(t) \geqslant \hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})) \geqslant \frac{m_0}{2}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}, \tag{3.16}$$

and (see (2.3)) if ε_0 is small enough

$$|\sigma''|^2+|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2 \leqslant 2\big(|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2\big),$$

from (3.15) it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant \frac{C_\Omega}{m_0}\varphi(t)E(t), \quad \varphi(t) = |\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2. \tag{3.17}$$

Note that, given (2.2) and (1.25) it is easy to see that φ is, relative to σ_0 and σ_1, uniformly bounded in $L^1(0,\infty)$. By applying the Gronwall lemma, we get

$$E(t) \leqslant E(0)\exp(\frac{C_\Omega}{m_0}\|\varphi\|_{L^1}) \tag{3.18}$$

and thanks to (3.16), we have

$$\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \leqslant \frac{2}{m_0}E(0)\exp(\frac{C_\Omega}{m_0}\|\varphi\|_{L^1}). \tag{3.19}$$

From (3.10) (see also (1.25) and (1.21)), we have

$$E(0) := \sigma_0^{\alpha n}\big\|v^1_{\sigma_0}+\alpha\frac{n-2}{2}v^0_{\sigma_0}\big\|^2_{L^2}$$

$$+\ \sigma_0^{\alpha(n-2)} \sum_{i,j=1}^{n} \int_\Omega \big(\frac{m_0}{2}\delta_{ij} - \alpha^2\sigma_0^{2\alpha}\frac{\sigma_1^2}{\sigma_0^2} y_i y_j\big)(\partial_{y_i} v^0_{\sigma_0})(\partial_{y_j} v^0_{\sigma_0})\mathrm{d}y$$
$$+\ \hat{M}(\|\sigma_0^{\alpha(n-2)}\nabla v^0_{\sigma_0}\|^2_{L^2}).$$

According to (2.2), we have

$$E(0) \leqslant \tilde{C}_0\sigma_0^{\alpha(n-2)}(\|v^1_{\sigma_0}\|^2_{L^2} + \|v^0_{\sigma_0}\|^2_{H^1}) + \hat{M}(\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2}).$$

Furthermore, since (see (3.3) and (3.2)) $\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2} \leqslant \lambda(\sigma_0) \leqslant 1$, thanks to (1.16), we have

$$M\big(\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2}\big) \leqslant M(\lambda(\sigma_0)) \leqslant \sup_{0\leqslant\lambda\leqslant 1} M(\lambda) \tag{3.20}$$

given (3.11), we can see that

$$\hat{M}(\|\sigma_0^{\alpha(n-2)}\nabla v^0_{\sigma_0}\|^2_{L^2}) \leqslant \sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2} \sup_{0\leqslant\lambda\leqslant 1} M(\lambda) \tag{3.21}$$

and so

$$E(0) \leqslant \tilde{C}_0\sigma_0^{\alpha(n-2)}(\|v^1_{\sigma_0}\|^2_{L^2} + \|v^0_{\sigma_0}\|^2_{H^1}) \tag{3.22}$$

which, together with (3.19) gives us (3.7).

As to (3.8), since (see (1.16)) $M \in C^2([0,\infty[)$, we set

$$\sup_{0\leqslant\lambda\leqslant 1} |M^{(i)}(\lambda)| = N_i, \quad M_0 = \max(N_0, N_1, N_2). \tag{3.23}$$

Given (3.7) and (3.3), we have $\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \leqslant C_0\lambda(\sigma_0) \leqslant 1$ for σ_0 small enough and so

$$\big|M^{(i)}\big(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big)\big| \leqslant \sup_{0\leqslant\lambda\leqslant 1} |M^{(i)}(\lambda)| \leqslant M_0$$

from which follows (3.8). $\square$

LEMMA 3.2. *Let $0 < r < 1$. Under the same assumptions as the lemma 3.1, we have*

$$\Big[\frac{1}{\sigma^{\frac{1-r}{2}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\Big]^2 \leqslant C_0\sigma_0^{\alpha(1-r)}\lambda_0^2 \leqslant 1, \tag{3.24}$$
$$\frac{1}{\sigma^{\frac{1-r}{2}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2} \leqslant C_0\lambda_0^{\frac{1}{2}} \leqslant 1$$

$$\|\sigma^{\frac{n-2}{2}}\nabla v\|^4_{L^2} \leqslant C_0\sigma_0^{2\alpha(1-r)}\lambda_0^2 \leqslant 1, \tag{3.25}$$
$$\Big[\frac{1}{\sigma^{\frac{1-r}{3}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\Big]^{\frac{3}{2}} \leqslant C_0\sigma_0^{\alpha(1-r)}\lambda_0^{\frac{3}{2}} \leqslant 1$$

if (see (2.2)) σ_0 is small enough.

Proof. The proof follows immediately from (3.7) and (3.3) (see also (3.2) and (1.25)). □

LEMMA 3.3. *Let* $0 < r < 1$ *and*

$$L_1(t) := \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v) + \sigma^{1-r}\hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}). \tag{3.26}$$

Given (2.2) and (1.25), we have the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}L_1(t) + \mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} \leqslant C_1\varepsilon_0^\alpha\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + C_1\varphi(t)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}, \tag{3.27}$$

where φ *is given by (3.17).*

Proof. An easy computation of the scalar product in $L^2(\Omega)$ of the equation (1.17) with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v)_t$$

gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L_1(t) + \mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} = \sum_{k=1}^{5} I_k \tag{3.28}$$

with

$$\begin{aligned}
I_1 &:= \big(n-2+\frac{3-r}{2}\big)\frac{\sigma'}{\sigma}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} - \frac{1}{2}\sigma^{3-r}\int_\Omega(\nabla\cdot a_1)|(\sigma^{\frac{n-2}{2}}v)_t|^2\mathrm{d}y,\\
I_2 &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(a_1.\nabla(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\,,\\
I_3 &:= \sigma^{3-r}\int_\Omega(a_2\cdot\nabla(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\\
I_4 &:= \frac{n-2}{2}\Big[\frac{\sigma''}{\sigma} - \frac{n}{2}\Big|\frac{\sigma'}{\sigma}\Big|^2 + \mu\frac{\sigma'}{\sigma}\Big]\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\,,\\
I_5 &:= \frac{1}{2}\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v) + \frac{3-r}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v)\\
&\quad + \frac{1-r}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}\hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\,.
\end{aligned}$$

Recalling the expression (1.11) of a_1 and of a_2, it is easy to see that

$$I_1 = -\frac{1}{2}(1+r)\frac{\sigma'}{\sigma}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} \leqslant 0\,,$$

$$I_2 \leqslant \frac{\mu}{4}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + C_\Omega\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\,,$$

$$I_3+I_4 \leqslant \frac{\mu}{4}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + C_\Omega\big(|\sigma''|^2+|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2.$$

Furthermore, on account of (1.21), (1.22) and (1.15)) we get

$$I_5 = \frac{1-r}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}\big[\frac{m_0}{2}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\big]$$
$$- \ \big(\sigma''\sigma' - \frac{\sigma'}{\sigma}|\sigma'|^2\big)\sigma^{1-r}\sum_{i,j=1}^{n}\int_\Omega y_i y_j(\sigma^{\frac{n-2}{2}}\partial_{y_i}v)(\sigma^{\frac{n-2}{2}}\partial_{y_i}v)\mathrm{d}y.$$

Therefore, considering (1.25), (2.2) and (3.8), we have

$$I_5 \leqslant \frac{1-r}{2}\alpha K\varepsilon_0^\alpha\big(\frac{m_0}{2}+M_0\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2$$
$$+ C_\Omega(|\sigma''|^2+|\sigma'|^2+\frac{\sigma'}{\sigma}|\sigma'|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2$$

and so

$$I_5 \leqslant \tilde{C}_1\varepsilon_0^\alpha\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + C_\Omega(|\sigma''|^2+|\sigma'|^2+\big|\frac{\sigma'}{\sigma}\big|^2|\sigma'|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2.$$

Given estimates of terms I_i $(i=1,\dots,5)$, we obtain

$$\sum_{k=1}^{5} I_k \leqslant \frac{\mu}{2}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \tilde{C}_1\varepsilon_0^\alpha\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2$$
$$+ \tilde{C}_1\big(|\sigma''|^2+|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2. \tag{3.29}$$

Recalling the expression (3.17) of φ and taking account of (2.3), we have if ε_0 is small enough

$$|\sigma''|^2+|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2 \leqslant 2\varphi(t), \tag{3.30}$$

and given (3.29) and (3.28) we have (3.27) and therefore Lemma 3.3. □

Lemma 3.4. *Let* $0<r<1$ *and*

$$L_2(t) := \mu\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2 + 2\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\,. \tag{3.31}$$

Given (2.2) and (1.25), the following inequality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}L_2(t) + \frac{m_0}{2}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \leqslant \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2$$
$$C_2\varphi(t)(\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2). \tag{3.32}$$

Proof. Taking the scalar product in $L^2(\Omega)$ of equation (1.17) with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v),$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L_2(t)+\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v)=\sum_{k=1}^{4}I_k, \tag{3.33}$$

where

$$I_1:=\frac{1}{2}\Big[\mu(n+1-r)\frac{\sigma'}{\sigma}+(n-2)\frac{\sigma''}{\sigma}+\frac{1}{2}n(n-2)\Big|\frac{\sigma'}{\sigma}\Big|^2\Big]\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2,$$

$$I_2:=\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+(n+1-r)\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y,$$

$$I_3:=-\frac{1}{2}\sigma^{3-r}\int_\Omega(\nabla\cdot a_2)|\sigma^{\frac{n-2}{2}}v|^2\mathrm{d}y,$$

$$I_4:=-\sigma^{3-r}\int_\Omega(\nabla\cdot a_1)(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v_t)\mathrm{d}y-\sigma^{3-r}\int_\Omega(a_1\cdot\sigma^{\frac{n-2}{2}}\nabla v)(\sigma^{\frac{n-2}{2}}v_t)\mathrm{d}y.$$

We have

$$I_1\leqslant\frac{m_0}{16}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+\tilde{C}_2\big[|\sigma''|^2+|\sigma'|^2+\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2\big]\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2,$$

$$I_2\leqslant\frac{m_0}{16}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+(1+\tilde{C}_2|\sigma'|^2)\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2.$$

Moreover, given the expressions (1.11) of a_1 and a_2, we can estimate the last terms so that

$$I_3+I_4\leqslant\frac{m_0}{8}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \tilde{C}_2(|\sigma''|^2+|\sigma'|^2)(\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2).$$

By adding these estimates to (3.33) and taking into account (3.30), we obtain (3.32) and so the lemma 3.4. □

Lemma 3.5. *Let* $0<r<1$ *and*

$$\begin{aligned}L_3(t):=&\,\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2+\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t)\\&+\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\\&+\frac{1}{2}\sigma^{1-r}M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)[\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2]^2\\&+2\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v_t)\\&-4\frac{\sigma'}{\sigma}\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)\mathrm{d}y.\end{aligned}\tag{3.34}$$

Given (3.3), (2.2) and (1.25), the following inequality holds

$$
\begin{aligned}
(3.35)\qquad \frac{\mathrm{d}}{\mathrm{d}t}L_3(t)+\mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} &\leqslant C_3\varepsilon_0^{\alpha}(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\\
&\quad+\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\\
&+\ C_3\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}+\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\\
&+C_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}}+\ C_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}}.
\end{aligned}
$$

Proof. If we differentiate (1.17) with respect to t and we take the scalar product in $L^2(\Omega)$ with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v_t)_t,$$

we obtain

(3.36)

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L_3(t)+\mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}+\frac{3+r}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2},\\
=\sum_{k=1}^{7}I_k+\sum_{k=1}^{6}J_k,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= (n-2+\frac{3-r}{2})\frac{\sigma'}{\sigma}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}-\frac{1}{2}\sigma^{3-r}\int_\Omega|(\sigma^{\frac{n-2}{2}}v_t)_t|^2\nabla\cdot a_1\mathrm{d}y,\\
I_2 &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(a_1.\nabla(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y\,,\\
I_3 &:= \frac{n-2}{2}\Big[\frac{\sigma''}{\sigma}-(n-1)\big(\frac{\sigma'}{\sigma}\big)^2+\mu\frac{\sigma'}{\sigma}\Big]\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y\,,\\
I_4 &:= \sigma^{3-r}\int_\Omega a_2\cdot\nabla(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y\,,\\
I_5 &:= \sigma^{3-r}\int_\Omega a_2'.\nabla(\sigma^{\frac{n-2}{2}}v))(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y\\
I_6 &:= \sigma^{3-r}\int_\Omega a_1'\cdot\nabla(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y\,,\\
I_7 &:= \frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v)+\sigma^{3-r}a''(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v)\\
&\quad+\frac{3}{2}\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t)+\frac{3-r}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t)\,,
\end{aligned}
$$

and the nonlinear terms J_k they are given by

$$
\begin{aligned}
J_1 := -\big[2\frac{\sigma''}{\sigma}+(n+2-2r)\big(\frac{\sigma'}{\sigma}\big)^2\big]\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\\
\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)\mathrm{d}y\,,
\end{aligned}
$$

$$J_2 := \frac{n-6}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\frac{\mathrm{d}}{\mathrm{d}t}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}) \int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)\mathrm{d}y,$$
$$J_3 := \big((\frac{n-2}{2}+\frac{1-r}{4}\big)\frac{\sigma'}{\sigma}\sigma^{1-r}M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\big[\frac{\mathrm{d}}{\mathrm{d}t}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\big]^2,$$
$$J_4 := \frac{n-2}{2}(\frac{\sigma'}{\sigma})'\sigma^{1-r}M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\frac{\mathrm{d}}{\mathrm{d}t}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2},$$
$$J_5 := \frac{1}{4}\sigma^{1-r}M''(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\big[\frac{\mathrm{d}}{\mathrm{d}t}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\big]^3,$$
$$J_6 := \frac{3}{2}\sigma^{1-r}M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\frac{\mathrm{d}}{\mathrm{d}t}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2.$$

The terms I_k are similar to those in the identity (3.28) and therefore, can be (in particular the first and last term) estimated in the same way. We then obtain taking into account (2.3), (2.2) and (1.25)

$$\sum_{k=1}^{7} I_k \leqslant \frac{\mu}{2}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} + C_\Omega\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}, \tag{3.37}$$

where, let us recall the here

$$\varphi(t) := |\sigma'|^2 + \big|\frac{\sigma'}{\sigma}\big|^2. \tag{3.38}$$

As for the non-linear terms, given (3.8), (3.7), (3.3), (2.3), (2.2) and (1.25), one has

$$\sum_{k=1}^{4} J_k \leqslant \tilde{C}_3\varepsilon_0^\alpha(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2} + \sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}). \tag{3.39}$$

Regarding the last terms J_5 and J_6, given (3.8), we get

$$J_5 \leqslant \tilde{C}_3\big|\frac{\sigma'}{\sigma}\big|^3(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v\|^4_{L^2} + \tilde{C}_3\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}}\big(\frac{1}{\sigma^{\frac{1-r}{3}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big)^{\frac{3}{2}},$$

and in view of (3.25)

$$J_5 \leqslant \tilde{C}_3\big|\frac{\sigma'}{\sigma}\big|^3(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}) + \tilde{C}_3\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}}.$$

As for the last term J_6, we have (see (3.8))

$$J_6 \leqslant \tilde{C}_3\big[\frac{1}{\sigma^{\frac{1-r}{2}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}\big]\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}} + \tilde{C}_3\frac{\sigma'}{\sigma}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}.$$

Since

$$\begin{aligned}&\frac{\sigma'}{\sigma}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\\ &= \frac{\sigma'}{\sigma}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{4}}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{4}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\\ &\leqslant \frac{|\sigma'|^2}{2\sigma^2}\big(\frac{1}{\sigma^{1-r}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big)\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big)\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}}\\ &\qquad + \frac{1}{2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}},\end{aligned}$$

then thanks to (3.24) (see also (2.3))

$$J_6 \leqslant \tilde{C}_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}} + \tilde{C}_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}}$$

and consequently, we have

(3.40)
$$J_5+J_6 \leqslant \tilde{C}_3\Big|\frac{\sigma'}{\sigma}\Big|^3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}+\tilde{C}_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}} + \tilde{C}_3\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}}.$$

Finally, putting together (3.40), (3.39) and (3.37), from (3.36) it follows the inequality (3.35) and thus this achieves the proof of lemma 3.5. □

Lemma 3.6. *Let $0 < r < 1$ and*

$$L_4(t) := 2\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)_t(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y. \tag{3.41}$$

Given (3.3), (2.2) and (1.25), the following inequality holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_4(t) + \frac{m_0}{2}\sigma^{1-r}\|(\sigma^{\frac{n-2}{2}}\nabla v_t)\|^2_{L^2} &\leqslant C_4\big(\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2}\\ &+ \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}\big) + C_4\varphi(t)\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\\ &+ \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}\big).\end{aligned} \tag{3.42}$$

Proof. As in the proof of lemma 3.5, if we differentiate (1.17) with respect to t and we take the scalar product in $L^2(\Omega)$ of the new equation with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v)_t\,,$$

we obtain

(3.43)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L_4(t)+\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2+\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t)$$
$$+\sigma^{1-r}M'(\sigma^{\frac{n-2}{2}}\|\nabla v\|_{L^2}^2)\Big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\Big]^2=\sum_{i=1}^{8}I_i$$

where

$$\begin{aligned}
I_1 &:= \frac{3n-2r}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)_t(\sigma^{\frac{n-2}{2}}v_t)_t\mathrm{d}y+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2\,,\\
I_2 &:= [\frac{n-2}{2}(\frac{\sigma''}{\sigma}+\frac{n}{2}|\frac{\sigma'}{\sigma}|^2)+\mu(\frac{3-r}{2}+n-2)\frac{\sigma'}{\sigma}]\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2\,,\\
I_3 &:= \frac{n-2}{2}(\frac{\sigma''}{\sigma}-\frac{n}{2}|\frac{\sigma'}{\sigma}|^2)[\frac{n-2}{2}\frac{\sigma'}{\sigma}+\mu]\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\,,\\
I_4 &:= \sigma^{3-r}\int_\Omega[(a_2+a_1')\cdot(\sigma^{\frac{n-2}{2}}\nabla v_t+a_2'\cdot(\sigma^{\frac{n-2}{2}}\nabla v)](\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\,,\\
I_5 &:= -\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v,(\sigma^{\frac{n-2}{2}}v)_t)\,,\\
I_6 &:= \sigma^{3-r}\int_\Omega(a_1\cdot(\sigma^{\frac{n-2}{2}}\nabla v_{tt}))(\sigma^{\frac{n-2}{2}}v)_t\mathrm{d}y\,,\\
I_7 &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)\mathrm{d}y\,,\\
I_8 &:= (n-2)\big|\frac{\sigma'}{\sigma}\big|^2\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2\,.
\end{aligned}$$

Recalling the expression (1.11) of a_1 and a_2 (see also (1.22), (1.21) and (1.15)) and taking account of (2.3) and (2.2), we can estimate the first six terms so that

$$\sum_{i=1}^{5}I_k\leqslant\frac{m_0}{12}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2$$
$$+C_\Omega(\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2)$$
$$+\tilde{C}_4\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2).$$

Furthermore, by integrating by parts, the term I_6 can be estimated ensure that

$$I_6\leqslant\frac{m_0}{12}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2$$
$$+\tilde{C}_4\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2).$$

As for the latter terms, considering (3.8), (3.7), (2.3) and

$$I_7+I_8\leqslant \frac{m_0}{12}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2+\tilde{C}_4\varphi(t)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2.$$

The proof of the lemma is completed by adding the above estimates of the terms I_i to (3.43). $\square$

LEMMA 3.7. *Let $0<r<1$. We set*

$$D(t):=\sigma^{3-r}\big[\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2\big]+\sigma^{1-r}\big[\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\big], \tag{3.44}$$

$$L(t)=\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2+D(t), \tag{3.45}$$

$$\mathcal{L}(t):=k_1L_1(t)+k_2L_2(t)+k_3L_3(t)+L_4(t), \tag{3.46}$$

where $L_i(t)$ $(i=1,\dots 4)$ are given by (3.26), (3.31), (3.34) and (3.41) as for k_1, k_2 and k_3 are positive constants. Then the following inequalities hold

$$\mathcal{L}(t)\geqslant b_0L(t)\,, \tag{3.47}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t)+\frac{b_1}{4}D(t)\leqslant b_2D(t)\mathcal{L}(t)\,, \tag{3.48}$$

with positive constants $b_0,\dots,b_2$ independent of σ_0.

Proof. Given (3.26), (3.31), (3.34) and (3.41), it is easy to see that

$$\begin{aligned}
&L_1(t)\geqslant \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\frac{m_0}{2}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2,\\
&L_2(t)\geqslant \frac{\mu}{2}\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2-\frac{2}{\mu}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2\\
&L_3(t)\geqslant \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2+\frac{m_0}{4}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2-C_\Omega\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2,\\
&L_4(t)\geqslant -\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2-\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2\,.
\end{aligned}$$

From which (see also (3.45)) we get

$$\begin{aligned}
\mathcal{L}(t)\geqslant{}&\lambda_1\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\lambda_2\sigma^{1+r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+\frac{k_2\mu}{2}\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2\\
&+\ \lambda_3\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2+\ \frac{m_0}{4}k_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2,
\end{aligned} \tag{3.49}$$

where

$$\lambda_1:=k_1-\frac{2k_2}{\mu}-1,\quad \lambda_2:=\frac{k_1m_0}{2}-C_\Omega k_3,\quad \lambda_3:=k_3-1\,. \tag{3.50}$$

On the order hand, if we multiply inequality (3.27), (3.32), (3.35) and (3.42) by k_1, k_2, k_3 and $k_4=1$ respectively and summing, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t)+\lambda_4\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\lambda_5\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \tag{3.51}$$

$$+\lambda_6\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2+\lambda_7\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2$$
$$\leqslant(C_1k_1+C_2k_2+C_3k_3+C_4)\varphi(t)\big[\sigma^{3-r}(\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2+\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2)$$
$$+\sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2+\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2)\big]+k_3C_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\big]^{\frac{3}{2}}$$
$$+k_3C_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\big]^{\frac{1}{2}}.$$

where

$$\lambda_4:=k_1\mu-k_2-C_4,\quad \lambda_5:=\frac{k_2m_0}{2}-k_1C_1\varepsilon_0^\alpha-k_3C_3\varepsilon_0^\alpha,\tag{3.52}$$
$$\lambda_6:=k_3\mu-C_4,\quad \lambda_7:=\frac{m_0}{2}-k_3C_3\varepsilon_0^\alpha.$$

If ε_0 is small enough, it is easy to see that we can choose k_1, k_2 and k_3 such as (see (3.50) and (3.52))

$$\lambda_i>0\quad i=1,\dots,7.\tag{3.53}$$

Indeed, we first choose

$$k_3=2\max\big(1,\frac{C_4}{\mu},\frac{2C_4m_0}{3C_\Omega\mu},\frac{2m_0}{3C_\Omega\mu}\big),\quad k_1=\frac{3C_\Omega k_3}{m_0},\quad 0<\varepsilon_0^\alpha<\min\big(1,\frac{m_0}{4C_3k_3}\big)\tag{3.54}$$

so that

$$\lambda_2=\frac{1}{2}C_\Omega k_3,\quad \lambda_3\geqslant 1,\quad \lambda_6\geqslant C_4,\quad \lambda_7\geqslant\frac{m_0}{4}.\tag{3.55}$$

On the other hand, by choosing (see (3.54))

$$k_2=\frac{3C_\Omega k_3}{m_0}\mu-\max(2C_4,\mu)>0,\quad 0<\varepsilon_0^\alpha\leqslant\min\big(1,\frac{k_2m_0}{4(k_1C_1+C_3k_3)},\frac{m_0}{4C_3k_3}\big)\tag{3.56}$$

we have

$$\lambda_4=k_1\mu-k_2-C_4\geqslant C_4,\quad \lambda_1:=k_1-\frac{2k_2}{\mu}-1\geqslant 1,\tag{3.57}$$
$$\lambda_5=\frac{k_2m_0}{2}-k_1C_1\varepsilon_0^\alpha-C_3\varepsilon_0^\alpha k_3\geqslant\frac{k_2m_0}{4}.$$

So, considering (3.56)–(3.57), from (3.45) it follows

$$\mathcal{L}(t)\geqslant b_0L(t),\quad b_0=\min\big(1,\frac{k_3C_\Omega}{2},\frac{k_2\mu}{2},\frac{k_3m_0}{4}\big)\tag{3.58}$$

i.e. (3.47). Furthermore from (3.51) (see also (3.44) it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t)+b_1D(t)\tag{3.59}$$
$$\leqslant\tilde{C}_5\varphi(t)\big[\sigma^{3-r}(\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2+\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2+\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2)$$

$$+\ \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}+\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\big]+k_3C_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}}$$
$$+\ +\ k_3C_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{1}{2}},$$

where (see (3.56)–(3.57)).
(3.60)

$$b_1=\min(\lambda_j, j=4,\cdots 7)\geqslant \min(C_4,\frac{m_0}{4},\frac{m_0k_2}{4}),\quad \tilde{C}_5=C_1k_1+C_2k_2+C_3k_3+C_4.$$

Given (3.44), by recalling the expression (3.38) of φ (see also (2.3) and (2.2)), we have

$$\tilde{C}_5\varphi(t)\big[\sigma^{3-r}(\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2}+\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2}+\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}) \tag{3.61}$$
$$+\ \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}+\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\big]=\tilde{C}_5\varphi(t)(\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2}+D(t))$$
$$\leqslant \tilde{C}_5C_\Omega\varphi(t)\sigma^2\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}+\tilde{C}_5\varphi(t)D(t))\leqslant \tilde{C}_6\varepsilon_0^\alpha D(t)$$

and (see (3.58), (3.44) and (3.45))

(3.62)

$$k_3C_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}}+k_3C_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{1}{2}}$$
$$\leqslant \frac{k_3C_3}{\sqrt{b_0}}\mathcal{L}^{\frac{1}{2}}(t)(t)D(t)\leqslant \frac{b_1}{2}D(t)+b_2D(t)\mathcal{L}(t)$$

where

$$b_2=\frac{k_3^2C_3^2}{2b_1b_0}. \tag{3.63}$$

By adding (3.62) and (3.61) to (3.51) and choosing (see (3.56)) ε_0 so that

$$\varepsilon_0^\alpha\leqslant\frac{b_1}{4\tilde{C}_6}$$

we obtain (3.48). □

Lemma 3.8. *Let be* $(v^0_{\sigma_0}, v^1_{\sigma_0})\in H^2(\Omega)\cap H^1(\Omega)$. *We set*

$$\tilde{R}(\sigma_0)=\sigma_0^{-2\alpha r}\|v^0_{\sigma_0}\|^2_{H^2}+\|v^1_{\sigma_0}\|^2_{H^1},\quad \tilde{\lambda}(\sigma_0)=\sigma_0^{\alpha(n-3+r)}\tilde{R}(\sigma_0), \tag{3.64}$$

and we suppose

$$\lim_{\sigma_0\to 0}\tilde{\lambda}(\sigma_0)=0. \tag{3.65}$$

Then, if (see (2.2)) σ_0 *is small enough, we have*

$$0\leqslant \mathcal{L}(0)<\frac{b_1}{8b_2} \tag{3.66}$$

(see (3.63), (3.60) and (3.58)) for b_1 *and* b_2.

Proof. Let us first note that (3.65) implies the hypothesis (3.3) under which Lemmas 3.2–3.6 and therefore lemma 3.7 are established. That said, by recalling the expressions (see (3.34), (3.41) and (3.46)) of $L_3(t)$ and $L_4(t)$, we can see that $\mathcal{L}(0)$ contains the L^2-norm of the term $v_{tt}|_{t=0}$. This term is defined (see (1.11), (1.15) and (1.17)) by

$$v_{tt}|_{t=0} = -\mu v^1_{\sigma_0} + \frac{1}{\sigma_0^{2\alpha}}\tilde{A}v^0_{\sigma_0} + a_1(0,y).\nabla v^1_{\sigma_0} + a_2(0,y).\nabla v^0_{\sigma_0}\,,$$

where

$$\tilde{A}v^0_{\sigma_0} = \sum_{i,j}^{n} \partial_{y_i}\big((\tilde{M}(\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2})\delta_{ij} - \alpha^2|\frac{\sigma_1}{\sigma_0}|^2\sigma_0^{2\alpha}y_iy_j\big)\partial_{y_j}v^0_{\sigma_0}).$$

Therefore, considering (3.23), (1.14) and (2.2), we get

$$\|v_{tt}|_{t=0}\|_{L^2} \leqslant C_\Omega(\|v^1_{\sigma_0}\|_{H^1} + \|v^0_{\sigma_0}\|_{H^1}) + C_\Omega\sigma_0^{-2\alpha}\|v^0_{\sigma_0}\|_{H^2}. \tag{3.67}$$

Recalling (3.41), (3.34) and (3.67), given (2.2) and (3.23) the easy computations give us

$$L_3(0) + L_4(0) \leqslant \tilde{C}_7\sigma_0^{\alpha(n-3+r)}(\sigma_0^{-2\alpha r}\|v^0_{\sigma_0}\|^2_{H^2} + \|v^1_{\sigma_0}\|^2_{H^1}). \tag{3.68}$$

Moreover, one can easily see that

$$L_1(0) + L_2(0) \leqslant \tilde{C}_7\sigma_0^{\alpha(n-3+r)}(\|v^0_{\sigma_0}\|^2_{H^1} + \|v^1_{\sigma_0}\|^2_{L^2}). \tag{3.69}$$

So, from (3.69), (3.68) and (3.46) (see also (3.64)) it follows $\mathcal{L}(0) \leqslant \tilde{C}_8\tilde{\lambda}(\sigma_0)$ and from (3.65) it follows (3.66). $\square$

4. GLOBAL SOLUTION AND ITS ASYMPTOTIC BEHAVIOUR

Lemmas 3.7–3.8 being established, now we are in position to prove our main result on the existence and asymptotic behaviour of global solution of the initial boundary value problem (1.1)–(1.3). More precisely, fixed the initial expansion Ω_0, we give initial data $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$ verifying (3.4), we suppose Ω_0 small enough and we ask the question of the existence of global solution u of the initial boundary value problem (1.1)–(1.3). Here, we insist on the fact the initial data (u_0, u_1) can be large enough. In fact, recalling (3.4) (see also (3.1)) and fixed R_0 large enough, it can be seen that if

$$0 < |\Omega_0| \leqslant \frac{C_\Omega}{R_0^{\frac{n}{1+r}}} \quad n \geqslant 3, \quad 0 < r < 1$$

then

$$R(\Omega_0) = \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} \leqslant R_0.$$

Our main result enunciated above is a non trivial generalization in higher dimension of our previous papers [3] and [4], where the results are obtained in dimension one and two with an unbounded expansion of the domain and with initial data sufficiently small.

THEOREM 4.1. *Let σ_0 small enough, $\Omega_0 = \sigma_0^\alpha \Omega$ and $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$ such that (3.4) is satisfied then the initial boundary value problem (1.1)–(1.3) has a unique global solution*

$$u \in L^\infty(0,\infty; H_0^1(\Omega_t) \cap H^2(\Omega_t)) \tag{4.1}$$

$$u_t \in L^\infty(0,\infty; H^1(\Omega_t)), \quad u_{tt} \in L^\infty(0,\infty; L^2(\Omega_t)).$$

Moreover,

$$\|u_{tt}\|^2_{L^2(\Omega_t)} + \|u_t\|^2_{L^2(\Omega_t)} + \|u\|^2_{H^1(\Omega_t)} \leqslant \frac{C_\Omega}{|\Omega_t|^{\frac{1-r}{n}}}, \quad 0 < r < 1\,. \tag{4.2}$$

Remark 4.1. Theorem 4.1 results from the existence of a global solution v of problem (1.17) and (1.19) under the assumption (3.65) on the initial data $(v^0_{\sigma_0}, v^1_{\sigma_0})$.

Indeed, if under hypothesis (3.65) (see also (3.64)) such a solution v exists, one can easily verify that $u = vo\tau$ (see (1.6)) is a global solution of the initial boundary value problem (1.1)–(1.3) with the initial data (u_0, u_1) large enough. Thus the proof of theorem 4.1 is reduced to that of the existence of a global solution v of problem (1.17) and (1.19) under the assumption (3.65) on the initial data $(v^0_{\sigma_0}, v^1_{\sigma_0})$. The latter follows from combination of its local solution and some of these a priori estimates allowing to get the uniform boundedness with respect to $t \in [0,\infty)$ of the weighted norm $L(t)$ (see (3.45)). In fact, if this norm is bounded for all t by the same constant, as will be seen, we can then step by step extend the local solution v to the whole interval $[0,\infty)$.

4.1. **The Proof of Theorem 4.1.**

From (3.48) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) + \frac{b_1}{8}D(t) + b_2 D(t)\big[\frac{b_1}{8b_2} - \mathcal{L}(t)\big] \leqslant 0\,, \tag{4.3}$$

We set

$$\mathcal{E}(t) = \frac{b_1}{8b_2} - \mathcal{L}(t).$$

Considering (3.66), we have $\mathcal{E}(0) > 0$ and by continuity there exists τ_0 small enough such that $\mathcal{E}(t) > \mathcal{E}(0) - \frac{1}{2}\mathcal{E}(0) > 0$ for all $t \in]0, \tau_0[$. By integrating (4.3) on $[0, \tau_0[$, we obtain

$$\mathcal{L}(t) \leqslant \mathcal{L}(0), \quad \text{for any } \; t \in [0, \tau_0] \tag{4.4}$$

particularly

$$\mathcal{L}(\tau_0) \leqslant \mathcal{L}(0)$$

so $\mathcal{E}(\tau_0) \geqslant \mathcal{E}(0) > 0$ and by continuity we have

$$\mathcal{E}(t) > \mathcal{E}(\tau_0) - \frac{1}{2}\mathcal{E}(0) \geqslant \mathcal{E}(0) - \frac{1}{2}\mathcal{E}(0) > 0 \quad \text{for all} \quad t \in [\tau_0, 2\tau_0].$$

By integrating (4.3) on $[\tau_0, 2\tau_0[$, we obtain

$$\mathcal{L}(t) \leqslant \mathcal{L}(\tau_0) \leqslant \mathcal{L}(0), \quad \text{for any} \quad t \in [\tau_0, 2\tau_0] \tag{4.5}$$

from (4.5) and (4.4) it follows $\mathcal{L}(t) \leqslant \mathcal{L}(0)$ for all $t \in [0, 2\tau_0[$ and repetition of this process, give us $\mathcal{L}(t) \leqslant \mathcal{L}(0)$ for all $t \in [0, \infty[$. This last inequality gives us (see (3.47)) $L \in L^\infty(0,\infty)$ and from (4.3) is follows that $D \in L^1(0,\infty)$. By reminding (see (3.44) and (3.45)) expressions of L and D, we obtain

$$\begin{gathered}
\sigma^{\frac{3-r}{2}}(\sigma^{\frac{n-2}{2}} v) \in L^\infty(0,\infty; L^2), \quad \sigma^{\frac{1-r}{2}}(\sigma^{\frac{n-2}{2}} v) \in L^\infty \cap L^2(0,\infty; H^1), \\
\sigma^{\frac{3-r}{2}}(\sigma^{\frac{n-2}{2}} v_t) \in L^\infty \cap L^2(0,\infty; L^2), \quad \sigma^{\frac{1-r}{2}}(\sigma^{\frac{n-2}{2}} v_t) \in L^\infty \cap L^2(0,\infty; H^1), \\
\sigma^{\frac{3-r}{2}}(\sigma^{\frac{n-2}{2}} v_{tt}) \in L^\infty \cap L^2(0,\infty; L^2).
\end{gathered} \tag{4.6}$$

Now, we rewrite the equation (1.17) in the following form

$$-\sum_{i,j=1}^{n} \partial_{y_i}(\tilde{a}_{ij}\partial_{y_j} v) = \tilde{F},$$

where

$$\tilde{a}_{ij} = \tilde{M}(\sigma^{\frac{n-2}{2}}\|\nabla v\|^2_{L^2})\delta_{ij} - |\sigma'|^2 y_i y_j, \quad \tilde{F} = \sigma^2(-v_{tt} - \mu v_t + a_1.\nabla v_t + a_2.\nabla v).$$

From (2.3), (1.4) and (1.14) it is easy to see that

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\xi_j\xi_i \geqslant \frac{m_0}{2}|\xi|^2.$$

So, by standard regularity arguments of elliptic equations we have

$$\|\sigma^{\frac{n-2}{2}} v\|_{H^2} \leqslant C_\Omega \|\sigma^{\frac{n-2}{2}} \tilde{F}\|_{L^2} \leqslant \tag{4.7}$$

$$C_\Omega \frac{\sigma^4}{\sigma^{3-r}}\big[\sigma^{3-r}(\|(\sigma^{\frac{n-2}{2}} v_t)_t\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}} v_t\|^2_{L^2}) + \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\big].$$

The last inequality follows from the above expression of $\tilde{F}$ (see also (2.3), (1.25) and (1.11)). From (4.7) and (4.6) it follows that

$$\sigma^{\frac{1+r}{2}}(\sigma^{\frac{n-2}{2}} v) \in L^\infty \cap L^2(0,\infty, H^2), \quad 0 < r < 1\,. \tag{4.8}$$

Now, if we use $u = v \circ \tau$ (see (1.6) for definition of τ), given (4.8) and (4.6), by easy computations we can see that

$$u \in L^\infty(0,\infty; H^1_0(\Omega_t) \cap H^2(\Omega_t)),\ u_t \in L^\infty(0,\infty; H^1(\Omega_t)),\ u_{tt} \in L^2(0,\infty; L^2(\Omega_t)).$$

In order to complete the proof of Theorem 4.1, it remains to prove the asymptotic behaviour of global solution. Indeed, given (2.2) and (1.25), easy computations gives us

$$\|u\|^2_{H^1(\Omega_t)} + \|u_t\|^2_{H^1(\Omega_t)} + \|u_{tt}\|^2_{L^2(\Omega_t)} \leqslant$$

$$\frac{C_\Omega}{\sigma^{1-r}}\left[\sigma^{3-r}(\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}+\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2})+\sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}+\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\right]$$

and from (4.6) follows easily

$$\|u\|^2_{H^1(\Omega_t)} + \|u_t\|^2_{H^1(\Omega_t)} + \|u_{tt}\|^2_{L^2(\Omega_t)} \leqslant \frac{C_\Omega}{\sigma^{1-r}(t)},$$

that is to say (4.2) because $|\Omega_t| = \sigma^n(t)|\Omega|$. This concludes the proof of Theorem 4.1. $\square$

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