LQG HOMING PROBLEMS FOR PROCESSES USED IN FINANCIAL MATHEMATICS

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Stochastic control problems that consist in minimizing the time spent by one-dimensional diffusion processes in a given interval are considered for processes used in financial mathematics. Because the exact optimal solutions are difficult to obtain, the control variable is instead assumed to be of a certain form. The best suboptimal solutions are derived and compared with the optimal solutions computed numerically.

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1. INTRODUCTION

Let $\{X(t),\ t \geq 0\}$ be a time-homogeneous controlled one-dimensional diffusion process defined by the stochastic differential equation

$$\mathrm{d}X(t) = m[X(t)]\,\mathrm{d}t + h[X(t)]\,u[X(t)]\,\mathrm{d}t + \{v[X(t)]\}^{1/2}\,\mathrm{d}B(t), \tag{1}$$

in which $u(\cdot)$ is the control variable and $\{B(t), t \geq 0\}$ is a standard Brownian motion starting at 0. We define the first hitting time

$$T(x) = \inf\{t > 0 : X(t) = d_1 \text{ or } d_2 \mid X(0) = x \in (d_1, d_2)\}. \tag{2}$$

Whittle (1982) considered the problem of finding the control u^* that minimizes the expected value of the cost function

$$J(x) = \int_0^{T(x)} \left\{ \frac{1}{2} q[X(t)]\,u^2[X(t)] + \lambda \right\} \mathrm{d}t, \tag{3}$$

where $q(\cdot) > 0$ and $\lambda \neq 0$ is a constant. This type of problem is known as *LQG homing*. When the parameter λ is positive, the optimizer wants the controlled process to leave the interval (d_1, d_2) as soon as possible, while taking the quadratic control costs into account. The function $q(\cdot)$ can be chosen in such a way that it is preferable to keep $X(t)$ as small (or as large) as possible, for instance. In many cases, it is assumed to be a positive constant q_0.

LQG homing problems have been considered by the author in a series of papers (see Lefebvre and Zitouni (2012) and (2014), for instance), and also by Makasu (2009) and (2013).

To solve LQG homing problems, we can try to make use of dynamic programming. We define the value function

$$F(x) = \inf_{u[X(t)],\, 0 \le t \le T(x)} E[J(x)]. \tag{4}$$

PROPOSITION 1. *The value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:*

$$\inf_{u(x)} \left\{ \frac{1}{2} q(x) u^2(x) + \lambda + [m(x) + h(x) u(x)] F'(x) + \frac{1}{2} v(x) F''(x) \right\} = 0. \tag{5}$$

Proof. First, by making use of Bellman's principle of optimality, we can write (remembering that $X(0) = x$) that

$$\begin{aligned} F(x) &= \inf_{u[X(t)],\, 0 \le t \le \Delta t} E\bigg[\int_0^{\Delta t} \left\{ \frac{1}{2} q[X(t)] u^2[X(t)] + \lambda \right\} \mathrm{d}t \\ &\quad + F\big(x + m(x)\Delta t + h(x) u(x) \Delta t + v^{1/2}(x) B(\Delta t)\big) + o(\Delta t) \bigg] \\ &= \inf_{u[X(t)],\, 0 \le t \le \Delta t} \bigg\{ \frac{1}{2} q(x) u^2(x) \Delta t + \lambda \Delta t \\ &\quad + E\Big[F\big(x + m(x)\Delta t + h(x) u(x) \Delta t + v^{1/2}(x) B(\Delta t)\big) + o(\Delta t) \Big] \bigg\}. \end{aligned}$$

Next, since $B(0) = 0$, we have

$$E[B(\Delta t)] = 0 \quad \text{and} \quad E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t.$$

Hence, assuming that F is twice differentiable with respect to x, we deduce from Taylor's formula that

$$\begin{aligned} & E\Big[F\big(x + m(x)\Delta t + h(x) u(x) \Delta t + v^{1/2}(x) B(\Delta t)\big) + o(\Delta t) \Big] \\ &= F(x) + [m(x) + h(x) u(x)] \Delta t F'(x) + \frac{1}{2} v(x) \Delta t F''(x) + o(\Delta t). \end{aligned}$$

Then,

$$\begin{aligned} 0 &= \inf_{u[X(t)],\, 0 \le t \le \Delta t} \bigg\{ \frac{1}{2} q(x) u^2(x) \Delta t + \lambda \Delta t + [m(x) + h(x) u(x)] \Delta t F'(x) \\ &\quad + \frac{1}{2} v(x) \Delta t F''(x) + o(\Delta t) \bigg\}. \end{aligned}$$

Finally, we divide both sides of the above equation by Δt, and we let Δt decrease to zero to obtain the Hamilton-Jacobi-Bellman (or dynamic programming) equation (5). $\square$

We deduce from Eq. (5) that the optimal control $u^*(x)$ can be expressed as

$$u^*(x) = -\frac{h(x)}{q(x)} F'(x). \tag{6}$$

Substituting this expression into the HJB equation, we obtain that the value function satisfies the second-order non-linear ordinary differential equation

$$\lambda + m(x) F'(x) - \frac{h^2(x)}{2q(x)} [F'(x)]^2 + \frac{1}{2} v(x) F''(x) = 0. \tag{7}$$

This equation is subject to the boundary conditions

$$F(d_1) = F(d_2) = 0. \tag{8}$$

Remark. Notice that to obtain the optimal control explicitly, we only need to calculate $G(x) := F'(x)$. Therefore, we can consider the first-order equation

$$\lambda + m(x) G(x) - \frac{h^2(x)}{2q(x)} G^2(x) + \frac{1}{2} v(x) G'(x) = 0. \tag{9}$$

This equation is a particular Riccati equation. However, we need a condition on the function $G(x)$ in order to find the unique solution to Eq. (9) that we are looking for. In some problems, using symmetry arguments, we can assert that $G(x_0) = 0$ for a known value of $x_0 \in (d_1, d_2)$. Then, the optimal control problem is greatly simplified. Unfortunately, in general it is not possible to find such a condition on $G(x)$.

In practice, it is quite difficult to obtain the exact analytical solution to the problem (7), (8). Even if we are able to find $G(x)$, we must integrate this function and use the boundary conditions (8) to determine the unique solution to our optimal control problem. Again, the integral needed is generally difficult to evaluate.

In this note, instead of looking for the optimal solution to the control problem, we will assume that the control variable is of a certain form and we will find the best solution of this form. We will consider two particular problems for which we are not able to determine $u^*(x)$ analytically, and we will compute suboptimal solutions instead. We will then compare the expected cost obtained with these suboptimal solutions with the value function evaluated numerically. The two diffusion processes in the problems presented in the next sections are important processes that are used frequently in financial mathematics.

2. SUBOPTIMAL CONTROL OF A BESSEL PROCESS

In the first problem that we consider, the diffusion process $\{X(t), t \geq 0\}$ is a particular controlled Bessel process. As mentioned in Jeanblanc *et al.* (2009), Bessel processes are intensively used in finance, in particular to model the dynamics of asset prices.

Assume that $m[X(t)] = 1/X(t)$, $h[X(t)] \equiv 1$ and $v[X(t)] \equiv 1$ in Eq. (1), so that

$$\mathrm{d}X(t) = \frac{1}{X(t)}\,\mathrm{d}t + u[X(t)]\,\mathrm{d}t + \mathrm{d}B(t). \tag{10}$$

The uncontrolled process is a Bessel process of *dimension* 3.

Next, we choose $q[X(t)] = X^2(t)$ and $\lambda = 1/8$ in the cost function defined in (3), and we take $d_1 = 1$ and $d_2 = 2$ in (2). Hence, our aim is to minimize the expected value of

$$J_1(x) = \int_0^{T_1(x)} \left\{ \frac{1}{2} X^2(t)\, u^2[X(t)] + \frac{1}{8} \right\} \mathrm{d}t, \tag{11}$$

with

$$T_1(x) = \inf\{t > 0 : X(t) = 1 \text{ or } 2 \mid X(0) = x \in (1,2)\}. \tag{12}$$

We can state that the optimal control is given by

$$u^*(x) = -\frac{1}{x^2}\, F'(x), \tag{13}$$

where the value function satisfies the following differential equation:

$$\frac{1}{8} + \frac{1}{x} F'(x) - \frac{1}{2x^2}\,[F'(x)]^2 + \frac{1}{2} F''(x) = 0. \tag{14}$$

The mathematical software *Maple* was able to express $F(x)$ as follows:

$$F(x) = \int \frac{x}{2} \left[3 + \sqrt{10}\tanh\left(\frac{\sqrt{10}}{2} \big(-\ln(x) + c_1 \big) \right) \right] \mathrm{d}x + c_2, \tag{15}$$

but it was unable to find explicitly the constants c_1 and c_2 for which the boundary conditions $F(1) = F(2) = 0$ are satisfied. However, we can find $F(x)$ for any $x \in (1,2)$ by using numerical methods. To be more precise, the *dsolve* command of *Maple* with the *numeric* option was used.

Now, suppose that instead of looking for the optimal solution, we assume that the control variable $u[X(t)]$ is of the form

$$u[X(t)] = \frac{c}{X(t)}, \tag{16}$$

where c is a constant. This choice for the control variable is based on the fact that $m[X(t)] = 1/X(t)$. Then, the diffusion process, which we shall denote by $\{X_c(t), t \geq 0\}$, satisfies the stochastic differential equation

$$\mathrm{d}X_c(t) = \frac{c+1}{X_c(t)}\,\mathrm{d}t + \mathrm{d}B(t). \tag{17}$$

Thus, $\{X_c(t), t \geq 0\}$ is a Bessel process of dimension $2c+3$.

Next, the cost function whose expected value we want to minimize is

$$J_c(x) := \int_0^{T_1(x)} \left\{ \frac{c^2}{2} + \frac{1}{8} \right\} \mathrm{d}t. \tag{18}$$

Hence, we are now looking for the constant c that minimizes

$$E[J_c(x)] = \left(\frac{c^2}{2} + \frac{1}{8} \right) E[T_1(x)]. \tag{19}$$

Let $m_1(x) := E[T_1(x)]$. The function $m_1(x)$ satisfies (see, for instance, Lefebvre 2007) the ordinary differential equation (ode)

$$\frac{1}{2} m_1''(x) + \frac{c+1}{x} m_1'(x) = -1, \tag{20}$$

subject to the boundary conditions $m_1(1) = m_1(2) = 0$. It is a simple matter to find that

$$m_1(x) = \frac{1 - x^2 + (4x^2 + 6x^{-2c-1} - 10)\,4^c + (16 - 12x^{-2c-1} - 4x^2)\,4^{2c}}{(2c+3)[1 - 2(4^c)]^2}. \tag{21}$$

The best possible choice for the control variable of the form given in (16) is obtained by finding the constant c^* that minimizes the function $m_1(x)$ above, multiplied by $\left(\frac{c^2}{2} + \frac{1}{8}\right)$.

Remark. The constant c^* depends on the initial value $X(0) = x$ of the controlled process. However, once it has been determined for a given x, it will remain the same until the final time $T_1(x)$.

Making use of a mathematical software, it is not difficult to estimate the constant c as precisely as we want. We found the approximate value of c^* for $x = 1.1, 1.3, \ldots, 1.9$. We then computed the expected cost $E[J_{c^*}(x)]$ obtained with this best suboptimal control. Moreover, we also computed the expected cost $E[J_0(x)]$ obtained when the optimizer chooses $u[X(t)] \equiv 0$. The results are presented in Table 1, as well as the value of $F(x)$ computed with the help of a mathematical software, by using a numerical method.

Looking at Table 1, we notice at once that, for this particular example, there is very little difference between $E[J_{c^*}(x)]$ and $E[J_0(x)]$. Furthermore, the difference between the expected cost using the suboptimal controls and

Table 1

Numerical values of the expected costs $E[J_{c^*}(x)]$ and $E[J_0(x)]$, and the value function when $\lambda = 1/8$, $d_1 = 1$ and $d_2 = 2$ in the case of the controlled Bessel process

x	c^*	$E[J_{c^*}(x)]$	$E[J_0(x)]$	$F(x)$
1.1	−0.023	0.01395	0.01398	0.01381
1.3	−0.007	0.02894	0.02894	0.02875
1.5	0.005	0.03125	0.03125	0.03109
1.7	0.014	0.02417	0.02419	0.02407
1.9	0.023	0.00965	0.00967	0.00962

the value function $F(x)$ is also almost negligible. This can, at least partly, be explained by the fact that the interval $(1, 2)$ is rather short, and the value of the parameter $\lambda = 1/8$ is not very large. For longer continuation regions and larger values of λ, the difference between the various expected costs will surely increase. Nevertheless, it is interesting to see that, in certain situations, it is probably not worth the effort of trying to find an exact and explicit expression for the optimal control (from the one for the value function). A suboptimal control like the one defined in (16) is easier to compute, and above all to implement, and yields very acceptable results. In fact, using no control at all here is perhaps the best thing to do.

Remark. Notice that the random variable $T_1(x)$ defined in (12) is a first exit time from a *finite* interval. If we replace the interval $(1, 2)$ by $(1, \infty)$, then $E[T_1(x)]$ could sometimes be infinite if $u[X(t)] \equiv 0$. Therefore, the optimizer would have to use some control in order to avoid receiving an infinite penalty for survival in the continuation region. Moreover, Eq. (20) is only valid if the function $m_1(x)$ exists (and is finite). Actually, in theory, $E[T_1(x)]$ could be infinite even in the case of a finite interval (d_1, d_2).

In the next section, another optimal control problem will be considered for an important diffusion process.

3. BEST CONSTANT CONTROL OF A CEV PROCESS

Let the controlled process $\{X(t),\ t \geq 0\}$ be defined by the stochastic differential equation

$$\mathrm{d}X(t) = u[X(t)]\,\mathrm{d}t + [X(t)]^{1/2}\mathrm{d}B(t). \tag{22}$$

Thus, this time we choose $m[X(t)] \equiv 0$, $h[X(t)] \equiv 1$ and $v[X(t)] = X(t)$ in Eq. (1). The uncontrolled process is a CEV (Constant Elasticity of Variance) process with zero drift. The CEV process is frequently used to model equities and commodities; see Cox and Ross (1976).

We set $q[X(t)] \equiv 1$ and $\lambda = 1/8$ in the cost function $J(x)$ defined in (3) and, as in the previous section, we take $d_1 = 1$ and $d_2 = 2$ in (2). It follows that the optimizer wants to minimize the expected value of

$$J_2(x) = \int_0^{T_2(x)} \left\{ \frac{1}{2} u^2[X(t)] + \frac{1}{8} \right\} \mathrm{d}t, \tag{23}$$

in which

$$T_2(x) = \inf\{t > 0 : X(t) = 1 \text{ or } 2 \mid X(0) = x \in (1, 2)\}. \tag{24}$$

From (6) and (7), we deduce that the optimal control is simply

$$u^*(x) = -F'(x), \tag{25}$$

where the function $F(x)$ satisfies the non-linear differential equation

$$\frac{1}{4} - [F'(x)]^2 + x F''(x) = 0, \tag{26}$$

subject to the same boundary conditions as above, namely $F(1) = F(2) = 0$.

Here, *Maple* was able to obtain a simple expression for $F(x)$:

$$F(x) = -2c_1\sqrt{x} - \frac{x}{2} - 2c_1^2 \ln\left(-\sqrt{x} + c_1\right) + c_2. \tag{27}$$

Unfortunately, it could not find explicitly the constants c_1 and c_2 such that $F(1) = F(2) = 0$. As in the previous section, we will compute $F(x)$ using the *dsolve* command of *Maple* with the *numeric* option, for $x = 1.1, 1.3, \ldots, 1.9$.

We will now try to determine the best constant control of the CEV process. This choice is motivated by the fact that if we set $u[X(t)] \equiv c$, then the controlled diffusion process $\{X_c(t), t \geq 0\}$ satisfies

$$\mathrm{d}X_c(t) = c\,\mathrm{d}t + [X_c(t)]^{1/2}\,\mathrm{d}B(t). \tag{28}$$

If we replace $[X_c(t)]^{1/2}$ by $2\,[X_c(t)]^{1/2}$, we obtain that $\{X_c(t), t \geq 0\}$ is a squared Bessel process of dimension c, which is important in financial mathematics; see Revuz and Yor (1999).

We want to minimize with respect to c the expected value of the cost function

$$J_c(x) := \int_0^{T_2(x)} \left\{ \frac{c^2}{2} + \frac{1}{8} \right\} \mathrm{d}t. \tag{29}$$

That is, we are looking for the constant c for which

$$E[J_c(x)] = \left(\frac{c^2}{2} + \frac{1}{8}\right) E[T_2(x)] \tag{30}$$

is minimum.

As in the previous section, we define $m_2(x) = E[T_2(x)]$; we must solve the linear second-order ode

$$\frac{x}{2} m_2''(x) + c m_2'(x) = -1, \tag{31}$$

with $m_2(1) = m_2(2) = 0$. We easily find that

$$m_2(x) = \frac{x^{-2c+1}}{c[-1 + 2(4^{-c})]} - \frac{x}{c} + \frac{2(4^c - 1)}{c(4^c - 2)}. \tag{32}$$

Therefore, we want to obtain

$$E[J_{c^*}(x)] := \min_c \left\{ \left(\frac{c^2}{2} + \frac{1}{8}\right) \left[\frac{x^{-2c+1}}{c[-1 + 2(4^{-c})]} - \frac{x}{c} + \frac{2(4^c - 1)}{c(4^c - 2)}\right] \right\}. \tag{33}$$

The same calculations as in the previous section were carried out. The results obtained are shown in Table 2.

Table 2

Numerical values of the expected costs $E[J_{c^*}(x)]$ and $E[J_0(x)]$, and the value function when $\lambda = 1/8$, $d_1 = 1$ and $d_2 = 2$ in the case of the controlled CEV process

x	c^*	$E[J_{c^*}(x)]$	$E[J_0(x)]$	$F(x)$
1.1	−0.0283	0.00842	0.00845	0.00837
1.3	−0.0145	0.01869	0.01870	0.01859
1.5	−0.0025	0.02124	0.02124	0.02113
1.7	0.0082	0.01708	0.01708	0.01699
1.9	0.0181	0.00703	0.00704	0.00699

We see that the conclusions are the same as in the case of the controlled Bessel process. There is again very little difference between $E[J_{c^*}(x)]$ and $E[J_0(x)]$, and the value function is only about 0.5% smaller than $E[J_{c^*}(x)]$.

As mentioned above, the difference between the optimal and the suboptimal solutions should be larger when we choose a larger value of λ and a longer interval (d_1, d_2). To check this assertion, we calculated the same quantities as in Table 2, but with $\lambda = 1$ and $(d_1, d_2) = (1, 7)$; see Table 3. Although there is indeed a larger difference between the optimal and suboptimal expected costs, we can state that the suboptimal control $u[X(t)] \equiv c^*$, and even $u[X(t)] \equiv 0$, still yield acceptable results. Next, we chose the interval $(d_1, d_2) = (1, 11)$; see Table 4. This time, the control $u[X(t)] \equiv 0$ did rather poorly for small values of x. Finally, we took $\lambda = 8$ and $(d_1, d_2) = (1, 2)$; see Table 5. Again, we must conclude that using no control at all is not a good idea, especially near the endpoints of the interval. However, the suboptimal constant control still performs relatively well.

Table 3

Numerical values of the expected costs $E[J_{c^*}(x)]$ and $E[J_0(x)]$, and the value function when $\lambda = 1$, $d_1 = 1$ and $d_2 = 7$ in the case of the controlled CEV process

x	c^*	$E[J_{c^*}(x)]$	$E[J_0(x)]$	$F(x)$
2	−1.022	1.316	1.768	1.196
3	−0.606	2.244	2.489	1.905
4	−0.246	2.486	2.531	2.021
5	0.044	2.066	2.067	1.647
6	0.309	1.173	1.201	0.933

Table 4

Numerical values of the expected costs $E[J_{c^*}(x)]$ and $E[J_0(x)]$, and the value function when $\lambda = 1$, $d_1 = 1$ and $d_2 = 11$ in the case of the controlled CEV process

x	c^*	$E[J_{c^*}(x)]$	$E[J_0(x)]$	$F(x)$
2	−1.288	1.391	2.503	1.338
4	−0.902	3.782	4.736	3.259
6	−0.397	4.674	4.876	3.641
8	0.046	3.655	3.657	2.723
10	0.835	1.348	1.427	1.025

Table 5

Numerical values of the expected costs $E[J_{c^*}(x)]$ and $E[J_0(x)]$, and the value function when $\lambda = 8$, $d_1 = 1$ and $d_2 = 2$ in the case of the controlled CEV process

x	c^*	$E[J_{c^*}(x)]$	$E[J_0(x)]$	$F(x)$
1.1	−3.250	0.385	0.541	0.351
1.3	−2.144	1.062	1.197	0.892
1.5	−0.388	1.354	1.359	1.068
1.7	1.448	1.046	1.093	0.836
1.9	2.748	0.377	0.450	0.320

4. CONCLUDING REMARKS

LQG homing problems are usually difficult to solve analytically, even in one dimension. In Sections 2 and 3, we presented two such problems that *Maple* was unable to solve explicitly. For this reason, it is interesting to derive suboptimal solutions that are both satisfactory and easy to implement.

In Section 2, we saw that assuming that the control variable is of the form $u[X(t)] = c/X(t)$, in the case of a particular Bessel process, enabled us to find an adequate alternative to the optimal solution. Similarly, in Section 3, a very simple solution in the form of a constant control was found to be quite good. Actually, for a short interval and a relatively small value of the parameter λ in the cost function, even using no control at all yielded an expected cost that was close to the value function in both problems. However, we found that $u[X(t)] \equiv 0$ was not satisfactory for a longer interval and/or a larger value of λ.

Since the two suboptimal controls yielded acceptable results in all the examples that were presented, one could wonder whether it is necessary to make all the efforts needed to obtain the exact optimal solution.

In deriving the suboptimal solutions, we had to use a mathematical software to determine (approximately) the value of the constant c that yielded the smallest expected cost. This was done quite easily, and with sufficient accuracy for practical purposes.

To find the optimal solution, we had to use a numerical method, with the help of the same software, to compute the value function. Although it is of course interesting to obtain the best possible solution, a numerical method does not provide us with a mathematical expression for this optimal solution. We could try to find such an expression for the value function, and hence for the optimal control, by making use of curve fitting or regression methods. However, then the function obtained, although suitable, would not be the exact optimal solution, at any rate.

In conclusion, we saw, through two examples, that deriving suboptimal solutions to LQG homing problems can be a worthwhile alternative to trying to obtain the exact optimal solution. For problems in two or more dimensions, computing these suboptimal solutions is surely a more realistic goal to achieve. Indeed, so far the LQG homing problems that were solved explicitly were almost always in one dimension, unless symmetry could be used to reduce an n-dimensional to a one-dimensional problem. Finding the best constant controls, for instance, should be easier.

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REFERENCES

[1] J.C. Cox and S.A. Ross, *The valuation of options for alternative stochastic processes.* J. Financ. Econ. **3** (1976), 145–166.

[2] M. Jeanblanc, M. Yor and M. Chesney, *Mathematical Methods for Financial Markets.* Springer, London, 2009.

[3] M. Lefebvre, *Applied Stochastic Processes.* Springer, New York, 2007.

[4] M. Lefebvre and F. Zitouni, *General LQG homing problems in one dimension.* Int. J. Stoch. Anal. Article ID 803724, 2012. doi:10.1155/2012/803724.

[5] M. Lefebvre and F. Zitouni, *Analytical solutions to LQG homing problems in one dimension.* Systems Science and Control Engineering: An Open Access Journal **2** (2014), 41–47.

[6] C. Makasu, *Risk-sensitive control for a class of homing problems.* Automatica J. IFAC **45** (2009), 2454–2455.

[7] C. Makasu, *Explicit solution for a vector-valued LQG homing problem.* Optim. Lett. **7** (2013), 607–612.

[8] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 3rd Ed. Springer Verlag, Heidelberg, Berlin, New York, 1999.

[9] P. Whittle, *Optimization over Time*, Vol. I. Wiley, Chichester, 1982.

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