AN ALTERNATIVE PROOF OF THE DIFFERENTIABILITY OF THE VOLUME WITH RESPECT TO THE L_p-SUM OF CONVEX BODIES

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One of the most useful facts when dealing with one-parameter functionals of the family of (p-)parallel bodies is the differentiability of the volume. In this paper, we provide an alternative proof for this differentiability at the origin in a restricted range of values of p.

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1. PRELIMINARIES

Let $\mathcal{K}^n$ be the set of all convex bodies, *i.e.*, non-empty compact convex sets in the Euclidean space $\mathbb{R}^n$ endowed with the standard scalar product $\langle\cdot,\cdot\rangle$, and let $\mathcal{K}^n_0$ be the subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin.

We will denote by $h(K,u) = \max\{\langle x,u\rangle : x \in K\}$ the *support function* of $K \in \mathcal{K}^n$ in the direction u of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. For a set $M \subseteq \mathbb{R}^n$, we write $\operatorname{int} M$ and $\operatorname{vol}(M)$ to denote, respectively, its interior and its volume, that is, its n-dimensional Lebesgue measure (if M is measurable).

The vectorial or *Minkowski addition* of two non-empty sets $A, B \subseteq \mathbb{R}^n$ is defined as

$$A + B = \{a + b : a \in A, b \in B\},$$

and we write $A + x := A + \{x\}$, for $x \in \mathbb{R}^n$. Moreover, $\lambda A = \{\lambda x : x \in A\}$, for $\lambda \geq 0$. We refer the reader to the books [7, 14] for a detailed study of this.

The so-called Minkowski difference can be regarded as the substraction counterpart of the Minkowski addition: for two non-empty sets $A, B \subseteq \mathbb{R}^n$, the *Minkowski difference* of A and B is defined by

$$A \sim B = \{x \in \mathbb{R}^n : B + x \subseteq A\},$$

that is, $A \sim B$ is the largest set such that $(A \sim B) + B \subseteq A$. Minkowski's difference gives rise to the notion of inner parallel bodies, a notion which has many applications in the geometry of convex bodies. We refer the reader to [14, Note 2 for Section 7.5] for further applications of inner parallel bodies.

In 1962 Firey introduced the following generalization of the classical Minkowski addition (see [5]). For $1 \le p < \infty$ and $K, E \in \mathcal{K}^n_0$ the *p-sum* (or L_p*-sum*) of K and E is the convex body $K +_p E \in \mathcal{K}^n_0$ whose support function is given by

$$h(K +_p E, u) = \big(h(K, u)^p + h(E, u)^p\big)^{1/p},$$

for all $u \in \mathbb{S}^{n-1}$. The p-sum of convex bodies was the starting point of the nowadays known as the L_p-Brunn-Minkowski (or Firey-Brunn-Minkowski) theory.

In [12] the following analogous to the Minkowski difference in the framework of Firey-Brunn-Minkowski theory was introduced: for $K, E \in \mathcal{K}^n_0$, $E \subseteq K$, and $1 \le p < \infty$, the *p-difference* of K and E is defined as

$$K \sim_p E = \big\{x \in \mathbb{R}^n : \langle x, u\rangle \le \big(h(K, u)^p - h(E, u)^p\big)^{1/p},\ u \in \mathbb{S}^{n-1}\big\}.$$

When $p = 1$, in both cases above the usual Minkowski sum and difference are obtained; *i.e.*, $+_1 = +$ and $\sim_1 = \sim$ are the Minkowski addition and difference, respectively.

In order to develop a structured and systematic study of the p-difference, it is useful to work with the following subfamily of convex sets where also the trivial cases are avoided (see [12] for further details):

$$\mathcal{K}^n_{00}(E) = \big\{K \in \mathcal{K}^n_0 : 0 \in K \sim \mathrm{r}(K; E)E\big\},$$

where $\mathrm{r}(K; E) = \max\big\{r \ge 0 : x + rE \subseteq K \text{ for some } x \in \mathbb{R}^n\big\}$ is the *relative inradius* of K with respect to E.

For convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ and real numbers $\lambda_1, \dots, \lambda_m \ge 0$, the volume of the linear combination $\lambda_1 K_1 + \dots + \lambda_m K_m$ is expressed as a polynomial of degree at most n in the variables $\lambda_1, \dots, \lambda_m$,

$$\mathrm{vol}\big(\lambda_1 K_1 + \dots + \lambda_m K_m\big) = \sum_{i_1,\dots,i_n=1}^{m} \mathrm{V}(K_{i_1}, \dots, K_{i_n})\lambda_{i_1} \cdots \lambda_{i_n},$$

whose coefficients $\mathrm{V}(K_{i_1}, \dots, K_{i_n})$ are the *mixed volumes* of $K_1, \dots, K_m$. Notice that such a polynomial expression is not possible for the sum $+_p$ when $p > 1$ (see *e.g.* [6]). Further, it is well-known that there exist finite Borel measures on $\mathbb{S}^{n-1}$, the *mixed area measures* $\mathrm{S}(K_2, \dots, K_n, \cdot)$, such that

$$\mathrm{V}(K_1, \dots, K_n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K_1, u)\, \mathrm{dS}(K_2, \dots, K_n, u).$$

We refer to [14, Chapter 5] for an extensive study of mixed volumes and mixed area measures. If only two convex bodies $K, E \in \mathcal{K}^n$ are involved in the above sum, the arising mixed volumes $\mathrm{V}\big(K[n-i], E[i]\big) =: \mathrm{W}_i(K;E)$ are called the *quermassintegrals* of K (relative to E); $[i]$ to the right of a convex body indicates that it appears i times. In particular, we have $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$, $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$ and $\mathrm{S}(K) := n\mathrm{W}_1(K;B_n)$ is the *surface area* of K. We notice that

$$\mathrm{W}_i(K;E) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(K,u)\,\mathrm{dS}\big(K[n-i-1], E[i], u\big).$$

Let $E \in \mathcal{K}_0^n$ and $K \in \mathcal{K}_{00}^n(E)$. The *full system of p-parallel bodies* of K relative to E, $1 \le p < \infty$, is defined as follows [12].

Definition 1.1. Let $E \in \mathcal{K}_0^n$ and $K \in \mathcal{K}_{00}^n(E)$. For $1 \le p < \infty$,

$$K_\lambda^p = \begin{cases} K \sim_p |\lambda| E & \text{if } -\mathrm{r}(K;E) \le \lambda \le 0, \\ K +_p \lambda E & \text{if } 0 \le \lambda < \infty. \end{cases}$$

K_λ^p is the *p-inner* (respectively, *p-outer*) parallel body of K at distance $|\lambda|$ relative to E.

Differentiability properties of functions that depend on one-parameter families of convex bodies play an important role in some proofs in Convex Geometry, see *e.g.* [14, Theorem 7.6.19 and Notes to Section 7.6]. In particular, for $E \in \mathcal{K}^n$ with interior points and $K \in \mathcal{K}^n$, the differentiability of functions depending on the full system of 1-parallel bodies was already addressed by Bol [1] and Hadwiger [8]. In this case ($p = 1$), the considered functions are the (relative) quermassintegrals $\mathrm{W}_i(K_\lambda^1;E)$, $i = 0, \dots, n-1$.

One of the most useful classical tools in this context is the differentiability of the function $\mathrm{vol}(K_\lambda^1)$ on $-\mathrm{r}(K;E) \le \lambda \le 0$. Further results and applications of the differentiability of quermassintegrals with respect to the one-parameter family of 1-parallel bodies can be found in [10] and the references therein.

In [9] Hernández Cifre, Martínez Fernández and Saorín Gómez proved, among other related results, the differentiability of the quermassintegrals $\mathrm{W}_i(K_\lambda^p;E)$, $i = 0, \dots, n-1$, on the range $(0, \infty)$. Moreover, as in the classical case ($p = 1$), the differentiability of the volume functional $\mathrm{vol}(K_\lambda^p)$ was also established, based on bounds of left and right derivatives of quermassintegrals.

The aim of this work is to provide a different proof, under the spirit of looking for a Matheron-type lemma, of the differentiability of the volume functional $\mathrm{vol}(K_\lambda^p)$ at $\lambda = 0$ for the range $1 < p \le n$. We think that our technique could be employed to obtain similar results in the Firey-Brunn-Minkowski theory.

2. DIFFERENTIABILITY OF $\mathrm{vol}(K_\lambda^p)$ AT THE ORIGIN (for $1 < p \le n$)

The aim of this section is to prove the differentiability of the function $\lambda \mapsto \mathrm{vol}(K_\lambda^p)$ at the origin for $1 < p \le n$. In order to do that, we need some previous results. In [13] Matheron proved the following *Convexity Lemma*:

Lemma 2.1 ([13, Convexity Lemma]). *Let $K, E \in \mathcal{K}^n$ with $E \subseteq K$. Then, for all $0 \le \varepsilon \le \mathrm{r}(K; E)$, it holds*

$$\mathrm{vol}(K) - \mathrm{vol}(K \sim \varepsilon E) \le \mathrm{vol}(K + \varepsilon E) - \mathrm{vol}(K).$$

Our first step is to show that the Convexity Lemma remains true for $1 \le p \le n$ if the convex bodies K and E are the same. Before doing that, we will need a technical inequality:

Lemma 2.2. *Let $1 \le p \le n$ and $0 \le \varepsilon \le 1$. Then,*

$$1 - (1-\varepsilon^p)^{n/p} \le (1+\varepsilon^p)^{n/p} - 1. \tag{2.1}$$

Proof. If $p = n$, then (2.1) holds trivially. Suppose that $1 \le p < n$, and let us consider the function $\varphi : [0,1) \to \mathbb{R}$ given by $\varphi(\varepsilon) := (1+\varepsilon^p)^{n/p} + (1-\varepsilon^p)^{n/p} - 2$. The function φ is differentiable on $(0,1)$, with derivative

$$\varphi'(\varepsilon) = n\varepsilon^{p-1}\left[(1+\varepsilon^p)^{(n-p)/p} - (1-\varepsilon^p)^{(n-p)/p}\right].$$

Since $1 \le p < n$, the function $t \mapsto t^{(n-p)/n}$ is strictly increasing in $(0,\infty)$, which implies that $\varphi'(\varepsilon) > 0$ for all $0 < \varepsilon < 1$. Then, $\varphi(\varepsilon) \ge \varphi(0) = 0$, for all $0 \le \varepsilon \le 1$, and (2.1) is proved. □

Lemma 2.3. *Let $Q \in \mathcal{K}^n$ with $0 \in \mathrm{int}\, Q$ and let $1 \le p \le n$. Then, for all $0 \le \varepsilon \le 1$, it holds*

$$\mathrm{vol}(Q) - \mathrm{vol}(Q \sim_p \varepsilon Q) \le \mathrm{vol}(Q +_p \varepsilon Q) - \mathrm{vol}(Q). \tag{2.2}$$

Proof. Firstly, we notice that for all $u \in \mathbb{S}^{n-1}$ we have that

$$h\big(Q +_p \varepsilon Q, u\big)^p = h(Q,u)^p + \varepsilon^p h(Q,u)^p = h\big((1+\varepsilon^p)^{1/p} Q, u\big)^p,$$

from where $Q +_p \varepsilon Q = (1+\varepsilon^p)^{1/p} Q$. On the other hand, it is easy to check that $Q \sim_p \varepsilon Q = (1-\varepsilon^p)^{1/p} Q$ (see [12, Lemma 2.2 (vi)]). Just by replacing these expressions, we immediately get that (2.2) is equivalent to

$$\mathrm{vol}(Q) - \mathrm{vol}\left((1-\varepsilon^p)^{1/p} Q\right) \le \mathrm{vol}\left((1+\varepsilon^p)^{1/p} Q\right) - \mathrm{vol}(Q). \tag{2.3}$$

Taking into consideration that the volume functional is homogeneous of degree n and that $\mathrm{vol}(Q) > 0$ (because Q has interior points), we deduce that (2.3) is equivalent to (2.1) and we finish the proof. □

Remark 2.1. Lemmas 2.2 and 2.3 show that a general Convexity Lemma does not exist for $p > n$, since (2.1) does not hold for $p > n$.

For $K, L \in \mathcal{K}^n$ we write $\mathrm{R}_0(K; L) := \inf\{t > 0 : K \subseteq tL\}$ to denote the *relative circumradius at the origin* of K with respect to L.

LEMMA 2.4. *Let $E \in \mathcal{K}^n$ with $0 \in \operatorname{int} E$, $Q \in \mathcal{K}^n_{00}(E)$ and let $1 < p \leq n$. Then, for all $0 \leq \varepsilon \leq 1/\mathrm{R}_0(E; Q)$, we have that*

$$\mathrm{vol}(Q) - \mathrm{vol}(Q \sim_p \varepsilon E) \leq \mathrm{vol}(Q +_p \varepsilon \alpha_{QE} E) - \mathrm{vol}(Q),$$

with $\alpha_{QE} := \mathrm{R}_0(Q; E)\mathrm{R}_0(E; Q)$.

Proof. Since $E \subseteq \mathrm{R}_0(E; Q)Q$ we have that $Q \sim_p \varepsilon E \supseteq Q \sim_p \varepsilon \mathrm{R}_0(E; Q)Q$, and thus $\mathrm{vol}(Q \sim_p \varepsilon E) \geq \mathrm{vol}(Q \sim_p \varepsilon \mathrm{R}_0(E; Q)Q)$. On the other hand, $Q \subseteq \mathrm{R}_0(Q; E)E$, which yields $Q +_p \varepsilon \mathrm{R}_0(E; Q)Q \subseteq Q +_p \varepsilon \alpha_{QE} E$. Since $0 \leq \varepsilon \mathrm{R}_0(E; Q) \leq 1$, we have by Lemma 2.3 that

$$\begin{aligned} \mathrm{vol}(Q) - \mathrm{vol}(Q \sim_p \varepsilon E) &\leq \mathrm{vol}(Q) - \mathrm{vol}(Q \sim_p \varepsilon \mathrm{R}_0(E; Q)Q) \\ &\leq \mathrm{vol}(Q +_p \varepsilon \mathrm{R}_0(E; Q)Q) - \mathrm{vol}(Q) \\ &\leq \mathrm{vol}(Q +_p \varepsilon \alpha_{QE} E) - \mathrm{vol}(Q). \quad \square \end{aligned}$$

From Lemma 2.4 we deduce that, for all $1 < p \leq n$,

$$\mathrm{vol}(Q) - \mathrm{vol}(Q \sim_p \varepsilon E) \leq \mathrm{vol}(Q +_p \varepsilon E) - \mathrm{vol}(Q) + F(\varepsilon),$$

for all $0 \leq \varepsilon \leq 1/\mathrm{R}_0(E; Q)$, with

$$F(\varepsilon) := \mathrm{vol}\big(Q +_p \varepsilon \alpha_{QE} E\big) - \mathrm{vol}(Q +_p \varepsilon E) \geq 0, \tag{2.4}$$

because

$$\alpha_{QE} = \mathrm{R}_0(Q; E)\mathrm{R}_0(E; Q) \geq \mathrm{R}(Q; E)\mathrm{R}(E; Q) = \frac{\mathrm{R}(Q; E)}{\mathrm{r}(Q; E)} \geq 1, \tag{2.5}$$

where $\mathrm{R}(K; L) := \inf\{t > 0 : \text{ there exits } x \in \mathbb{R}^n \text{ with } x + tL \supseteq K\}$ is the *relative circumradius* of K with respect to L.

LEMMA 2.5. *Let $E \in \mathcal{K}^n$ with $0 \in \operatorname{int} E$, $Q \in \mathcal{K}^n_{00}(E)$, $1 < p \leq n$, and let $F(\varepsilon)$ as in (2.4), with $0 \leq \varepsilon \leq 1/\mathrm{R}_0(E; Q)$. Then, there exists a constant $C > 0$ (which depends on Q and E) such that $F(\varepsilon) \leq C\varepsilon^p$, for all $0 < \varepsilon \leq 1/\alpha_{QE}$.*

Proof. We write, for brevity, $\alpha = \alpha_{QE}$. See Lemma 2.4 and (2.5). If $\alpha = 1$, then $F(\varepsilon) \equiv 0$ and the result becomes true. Suppose that $\alpha > 1$, and let us consider the function $k : [1, \alpha] \times \mathbb{S}^{n-1} \to (0, \infty)$ given by

$$k(t, u) := h(Q +_p t\varepsilon E, u).$$

By the continuity of support functions and the p-sum of convex bodies, the function k is continuous in each variable. The following claim is a technical step, which is proved with standard arguments. Nevertheless, we will include here a detailed proof for the sake of completeness.

Claim 2.1.
$$\lim_{s\to 0}\frac{k(t+s,u)-k(t,u)}{s}=\frac{\partial k(t,u)}{\partial t}=\varepsilon^p t^{p-1}h(E,u)^p h(Q+_p t\varepsilon E,u)^{1-p}$$
uniformly on $\mathbb{S}^{n-1}$*, for all* $t\in(1,\alpha)$.

Proof of Claim 2.1. Notice first that $0<\varepsilon\leq 1/\alpha$ is a fixed number. Let $t\in(1,\alpha)$ and $\eta>0$. We are going to prove that there exists some $\delta>0$ such that
$$|s|<\delta \implies \left|\frac{k(t+s,u)-k(t,u)}{s}-\frac{\partial k(t,u)}{\partial t}\right|<\eta, \qquad \text{for all } u\in\mathbb{S}^{n-1}.$$
As a consequence of the mean value theorem applied to the function $t^{1/p}$, $p\geq 1$, we have that for $\alpha,\beta\geq 0$, there exists some γ between α and β such that
$$\alpha^{1/p}-\beta^{1/p}=\frac{1}{p}(\alpha-\beta)\gamma^{(1-p)/p}, \tag{2.6}$$
and similarly
$$\alpha^{p-1}-\beta^{p-1}=(p-1)(\alpha-\beta)\gamma^{p-2}. \tag{2.7}$$
Taking $\alpha=h(Q,u)^p+(t+s)^p\varepsilon^p h(E,u)^p$ and $\beta=h(Q,u)^p+t^p\varepsilon^p h(E,u)^p$ in (2.6) we deduce that there exists some $t+\theta$ between $t+s$ and t such that
$$\begin{aligned}
&k(t+s,u)-k(t,u)-s\frac{\partial k(t,u)}{\partial t}=\\
&\quad=h\big(Q+_p(t+s)\varepsilon E,u\big)-h\big(Q+_p t\varepsilon E,u\big)-s\frac{\partial k(t,u)}{\partial t}\\
&\quad=\alpha^{1/p}-\beta^{1/p}-s\frac{\partial k(t,u)}{\partial t}\\
&\quad=\frac{1}{p}\varepsilon^p h(E,u)^p\big[(t+s)^p-t^p\big]h\big(Q+_p(t+\theta)\varepsilon E,u\big)^{1-p}-s\frac{\partial k(t,u)}{\partial t},
\end{aligned}$$
because $h(Q,u)^p+(t+\theta)^p\varepsilon^p h(E,u)^p=h\big(Q+_p(t+\theta)\varepsilon E,u\big)^p$. Again by the mean value theorem, we have that $(t+s)^p-t^p=sp(t+w)^{p-1}$, with $t+w$ between $t+s$ and t. Notice that $|\theta|,|w|\leq|s|$. Thus,
$$\begin{aligned}
&k(t+s,u)-k(t,u)-s\frac{\partial k(t,u)}{\partial t}=\\
&\quad=s\varepsilon^p h(E,u)^p\left[\left(\frac{t+w}{h\big(Q+_p(t+\theta)\varepsilon E,u\big)}\right)^{p-1}-\left(\frac{t}{h\big(Q+_p t\varepsilon E,u\big)}\right)^{p-1}\right].
\end{aligned}$$
In the following, we will use the *inradius of* Q *at the origin*, $\mathrm{r}_0(Q):=\max\{\delta>0:\delta B_n\subseteq Q\}>0$, and we will write $\mathrm{R}_0(K):=\mathrm{R}_0(K;B_n)$ to denote the *circumradius of* K *at the origin.*

By applying (2.7) with $\alpha = (t+w)h\big(Q +_p t\varepsilon E, u\big)$ and $\beta = th\big(Q +_p (t+\theta)\varepsilon E, u\big)$ we have that there exists some $\Gamma_{u,w,\theta} > 0$ between α and β (this number is bounded so that $\Gamma_{u,w,\theta}^{p-2} \leq C'$ for all $u \in \mathbb{S}^{n-1}$) such that

$$
\begin{aligned}
&\left|\frac{k(t+s,u)-k(t,u)}{s} - \frac{\partial k(t,u)}{\partial t}\right| = \\
&= \frac{\varepsilon^p h(E,u)^p}{\big[h\big(Q +_p \varepsilon E, u\big) h\big(Q +_p (t+\theta)\varepsilon E, u\big)\big]^{p-1}} \times \\
&\quad \times \left|\big[(t+w)h\big(Q +_p t\varepsilon E, u\big)\big]^{p-1} - \big[th\big(Q +_p (t+\theta)\varepsilon E, u\big)\big]^{p-1}\right| \\
&\leq \frac{\big(\varepsilon \mathrm{R}_0(E)\big)^p}{\mathrm{r}_0(Q)^{2(p-1)}}(p-1)\Gamma_{u,w,\theta}^{p-2}\left|(t+w)h\big(Q +_p t\varepsilon E, u\big) - th\big(Q +_p (t+\theta)\varepsilon E, u\big)\right| \\
&\leq \widetilde{C}\left(t\left|h\big(Q +_p (t+\theta)\varepsilon E, u\big) - h\big(Q +_p t\varepsilon E, u\big)\right| + |w|\mathrm{R}_0(Q +_p E)\right),
\end{aligned}
\tag{2.8}
$$

where we have used that $t\varepsilon \leq \alpha\varepsilon \leq 1$ implies $h\big(Q +_p t\varepsilon E, u\big) \leq h(Q +_p E, u) \leq \mathrm{R}_0(Q +_p E)$ and we have denoted

$$
\widetilde{C} := (p-1)C' \frac{\big(\varepsilon \mathrm{R}_0(E)\big)^p}{\mathrm{r}_0(Q)^{2(p-1)}}.
$$

Again by the mean value theorem we have that there exists some ξ between t and $t+\theta$ such that

$$
(t+\theta)^p - t^p = \theta p \xi^{p-1}. \tag{2.9}
$$

It is important to observe that if $|s|$ (and so $|\theta|$) is small enough, then we will have that $|\xi| \leq \frac{3t}{2}$. Now (2.9) together with (2.6) with $\alpha = h(Q,u)^p + (t+\theta)^p\varepsilon^p h(E,u)^p$ and $\beta = h(Q,u)^p + t^p\varepsilon^p h(E,u)^p$ allows to deduce the existence of some $t+\Delta$ between $t+\theta$ and t (with $|\Delta| \leq |\theta| \leq |s|$) such that

$$
\begin{aligned}
h\big(Q +_p (t+\theta)\varepsilon E, u\big) - h\big(Q +_p t\varepsilon E, u\big) &= \alpha^{1/p} - \beta^{1/p} \\
&= \left[(t+\theta)^p - t^p\right]\varepsilon^p h(E,u)^p \frac{1}{p}\left(h(Q,u)^p + (t+\Delta)^p\varepsilon^p h(E,u)^p\right)^{\frac{1-p}{p}} \\
&= \theta\xi^{p-1}\varepsilon^p h(E,u)^p h\big(Q +_p (t+\Delta)\varepsilon E, u\big)^{1-p}.
\end{aligned}
$$

Going over (2.8) again and using the above inequalities we finally get

$$
\begin{aligned}
&\left|\frac{k(t+s,u)-k(t,u)}{s} - \frac{\partial k(t,u)}{\partial t}\right| = \\
&\leq \widetilde{C}\left(t|\theta|\left(\frac{3t}{2}\right)^{p-1}\big(\varepsilon \mathrm{R}_0(E)\big)^p \frac{1}{\mathrm{r}_0(Q)^{p-1}} + |w|\mathrm{R}_0(Q +_p E)\right) \\
&\leq \widehat{C}|s| < \eta, \qquad \text{for all } u \in \mathbb{S}^{n-1}
\end{aligned}
$$

whenever $|s| < \delta := \min\left\{\frac{\eta}{C^*}, \frac{t}{2}\right\}$, where

$$C^* := \widetilde{C}\left(\left(\varepsilon t \mathrm{R}_0(E)\right)^p \left(\frac{3}{2\mathrm{r}_0(Q)}\right)^{p-1} + \mathrm{R}_0(Q +_p E)\right).$$

We have proved thus that

$$\lim_{s\to 0} \frac{k(t+s,u) - k(t,u)}{s} = \frac{\partial k(t,u)}{\partial t} \tag{2.10}$$

uniformly on $\mathbb{S}^{n-1}$. It remains to see that the right-hand side of (2.10) equals to $\varepsilon^p t^{p-1} h(E,u)^p h\big(Q +_p t\varepsilon E, u\big)^{1-p}$. But this is a straightforward verification. In fact, since $k(t,u) = (h(Q,u)^p + t^p\varepsilon^p h(E,u)^p)^{1/p}$ we get by the chain rule that

$$\begin{aligned}\frac{\partial k(t,u)}{\partial t} &= \frac{1}{p}\left(h(Q,u)^p + t^p\varepsilon^p h(E,u)^p\right)^{\frac{1}{p}-1} \cdot p t^{p-1}\varepsilon^p h(E,u)^p \\ &= \varepsilon^p t^{p-1} h(E,u)^p h\big(Q +_p t\varepsilon E, u\big)^{1-p},\end{aligned}$$

and we finish the proof of Claim 2.1. □

Now we need a result proved by Böröczky, Lutwak, Yang and Zang:

Lemma 2.6 ([3, Lemma 2.1]). *Let $k : I \times \mathbb{S}^{n-1} \to (0,\infty)$ be a continuous function, where I is an open interval of $\mathbb{R}$. Suppose that*

$$\lim_{s\to 0} \frac{k(t+s,u) - k(t,u)}{s} = \frac{\partial k(t,u)}{\partial t}$$

uniformly on $\mathbb{S}^{n-1}$. If $\{K_t\}_{t\in I}$ is the family of Wulff-shapes *associated with k_t (i.e., $K_t = \bigcap_{u\in\mathbb{S}^{n-1}}\big\{x \in \mathbb{R}^n : \langle x,u\rangle \le k_t(u)\big\}$), then*

$$\frac{\mathrm{dvol}(K_t)}{\mathrm{d}t} = \int_{\mathbb{S}^{n-1}} \frac{\partial k(t,u)}{\partial t} \mathrm{dS}_{K_t}(u),$$

where $\mathrm{S}_{K_t}(u) := \mathrm{S}(K_t[n-1],u)$.

Claim 2.1 together with Lemma 2.6 yields then

$$\frac{\mathrm{dvol}(Q_t)}{\mathrm{d}t} = \int_{\mathbb{S}^{n-1}} \frac{\partial k(t,u)}{\partial t}\, \mathrm{dS}_{Q_t}(u),$$

where $Q_t := Q +_p t\varepsilon E$.

Since $p > 1$, we have that $h(Q_t,u)^{1-p} \le \mathrm{r}_0(Q)^{1-p}$ for all $u \in \mathbb{S}^{n-1}$. On the other hand, $h(E,u)^p \le \mathrm{R}_0(E)^p$ for all $u \in \mathbb{S}^{n-1}$. Moreover, since $0 \le \varepsilon \le 1/\alpha$, we have that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \mathrm{dS}_{Q_t}(u) &= \int_{\mathbb{S}^{n-1}} h(B_n,u)\, \mathrm{dS}(Q_t[n-1],u) \\ &= n\mathrm{V}(B_n, Q_t[n-1]) = \mathrm{S}(Q_t) \\ &\le \mathrm{S}(Q_\alpha) = \mathrm{S}(Q +_p \varepsilon\alpha E) \\ &\le \mathrm{S}(Q +_p E).\end{aligned}$$

Then,

$$
\begin{aligned}
F(\varepsilon) = \mathrm{vol}(Q_\alpha) - \mathrm{vol}(Q_1) &= \int_1^\alpha \left(\int_{\mathbb{S}^{n-1}} \frac{\partial k(t,u)}{\partial t} \, \mathrm{dS}_{Q_t}(u) \right) \mathrm{d}t \\
&= \varepsilon^p \int_1^\alpha t^{p-1} \left(\int_{\mathbb{S}^{n-1}} h(Q_t,u)^{1-p} h(E,u)^p \, \mathrm{dS}_{Q_t}(u) \right) \mathrm{d}t \\
&\leq \frac{\mathrm{R}_0(E)^p}{\mathrm{r}_0(Q)^{p-1}} \varepsilon^p \int_1^\alpha t^{p-1} \left(\int_{\mathbb{S}^{n-1}} \mathrm{dS}_{Q_t}(u) \right) \mathrm{d}t \\
&\leq C\varepsilon^p,
\end{aligned}
$$

where

$$
C := \frac{\mathrm{R}_0(E)^p}{\mathrm{r}_0(Q)^{p-1}} \mathrm{S}(Q +_p E) \frac{\alpha^p - 1}{p} > 0. \quad \square
$$

THEOREM 2.1. *Let* $E \in \mathcal{K}^n$ *with* $0 \in \operatorname{int} E$, $K \in \mathcal{K}^n_{00}(E)$ *and let* $1 < p \leq n$. *Then, the function* $\lambda \mapsto \mathrm{vol}(K^p_\lambda)$ *is differentiable at the origin, with*

$$
\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{vol}(K^p_\lambda) = 0.
$$

Proof. For $\varepsilon > 0$ small enough we have that $K^p_0 = K$ and

$$
K^p_{0-\varepsilon} = K^p_{-\varepsilon} = K \sim_p \varepsilon E, \quad K^p_{0+\varepsilon} = K^p_\varepsilon = K +_p \varepsilon E.
$$

Then, by Lemmas 2.4 and 2.5, we obtain that

$$
\begin{aligned}
\left.\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{vol}(K^p_\lambda) &= \lim_{\varepsilon\to 0^+} \frac{\mathrm{vol}(K) - \mathrm{vol}(K \sim_p \varepsilon E)}{\varepsilon} \\
&\leq \lim_{\varepsilon\to 0^+} \frac{\mathrm{vol}(K +_p \varepsilon E) - \mathrm{vol}(K)}{\varepsilon} + C \lim_{\varepsilon\to 0^+} \varepsilon^{p-1} \\
&= \left.\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{vol}(K^p_\lambda).
\end{aligned}
$$

The reverse inequality follows from [9, Proposition 2]. Finally, from [9, Theorem 3] we conclude that there exists

$$
\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{vol}(K^p_\lambda) = \left.\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\right|_{\lambda=0} \mathrm{vol}(K^p_\lambda) = 0. \quad \square
$$

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