THE INTRODUCTION OF NEW MODULUS $\zeta_X(\varepsilon)$, UNIFORM NON-SQUARENESS AND UNIFORM NORMAL STRUCTURE IN BANACH SPACES

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Let X be a Banach space, and $B(X)$ and $S(X)$ be the unit ball and unit sphere of X. In this paper, a new modulus $\zeta_X(\varepsilon)$ which related to the Modulus of U-Convexity, Arc Length is introduced. The properties of $\zeta_X(\varepsilon)$ are studied. The relationships between the modulus $\zeta_X(\varepsilon)$ and some other known ones are given. Some sufficient conditions for uniform non-squareness and uniform normal structure of a Banach space are provided. Some results about fixed points of non-expansive mapping are obtained.

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1. INTRODUCTION

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ be the unit ball, and the unit sphere of X, respectively. Let X^* be the dual space of X. We define $\nabla_x \subset S(X^*)$ to be the set of norm 1 supporting functionals of $S(X)$ at x, that is, $f_x \in \nabla_x \iff \langle x, f_x \rangle = 1$. For $x_1, x_2 \in B(X)$, we use $[x_1, x_2]$ to denote the line segment connecting x_1 and x_2 in X. Let X_2 be a 2-dimensional subspace of X, for $x_1, x_2 \in S(X_2)$, we use $\overset{\frown}{x_1, x_2}$ to denote the curve on $S(X_2)$ from x_1 to x_2 clockwise.

In 1948, Brodskiĭ and Mil'man [1] introduced the following geometric concepts:

Definition 1.1. A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} \leq c \cdot d(K)$.

For a reflexive Banach space, the normal structure and weak normal structure coincide.

Let D be a nonempty subset of a Banach space X. A mapping $T : D \to D$ is called to be non-expensive whenever $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A Banach space has fixed point property if for every bounded closed and convex subset D of X and for each non-expansive mapping $T : D \to D$, there is a point $x \in D$ such that $x = Tx$. (See [14]).

In 1965, Kirk [13] proved that if a Banach space X has weak normal structure then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

In [3], Clarkson introduced the following modulus of convexity: $\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X, \|x - y\| \geq \varepsilon\}$, where $0 \leq \varepsilon \leq 2$. It was proved that if there exists $\varepsilon > 0$ such that $\delta_X(1 + \varepsilon) > \frac{\varepsilon}{2}$, then X has uniform normal structure [9].

In [7], Gao introduced the modulus of U-convexity which is a generalization of $\delta_X(\varepsilon)$: $U_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X, \langle x - y, f_x\rangle \geq \varepsilon$ for some $f_x \in \nabla_x\}$, where $0 \leq \varepsilon \leq 2$. It was also proved that if there exists $\delta > 0$ such that $U_X(\frac{1}{2} - \delta) > 0$, then X has uniform normal structure. Mazcuñán-Navarro [14] proved that a Banach space X has fixed point property if there exists $\delta > 0$ such that $U_X(1 - \delta) > 0$. This was strengthened by Saejung [15]. In fact, it was proved that if a Banach space X is super-reflexive, then the moduli of U-convexity of the ultra-power $X_{\mathcal{U}}$ of X and of X itself coincide. By using ultra-power method he showed that a Banach space X and its dual X^* have uniform normal structure whenever $U_X(1) > 0$.

For $x_1 \in S(X)$ and $y \in X$, let $B(x_1, \varepsilon) = \{y : \|y - x_1\| \leq \varepsilon, 0 \leq \varepsilon \leq 2\}$ be the ball with center at x_1 and radius ε, and let $sl_X(x_1, \varepsilon) = \{y : \langle x_1 - y, f_{x_1}\rangle \leq \varepsilon, 0 \leq \varepsilon \leq 2\} \bigcap B(X)$ be the slice of $B(X)$ with vertex x_1.

The modulus of uniform convexity, $\delta_X(\varepsilon)$ and the modulus of U-convexity, $U_X(\varepsilon)$ is the uniform measure of depth of midpoint between $S(X) \bigcap B(x_1, \varepsilon)$ and $S(X) \bigcap sl_X(x_1, \varepsilon)$ to vertex x_1, respectively. The both modulus of uniform convexity $\delta_X(\varepsilon)$ and the modulus of U-convexity $U_X(\varepsilon)$ are used to study uniform non-squareness, uniform normal structure and other geometric properties of Banach space and each of them plays an important role.

In this paper, a new parameter $\zeta_X(\varepsilon)$, the uniform measure of arc length from points on $S(X) \bigcap sl_X(x_1, \varepsilon)$ to vertex x_1, is introduced. The properties of

$\zeta_X(\varepsilon)$ are studied. The relationships between the modulus $\zeta_X(\varepsilon)$ and some other known ones are given. Some sufficient conditions for uniform non-squareness and uniform normal structure of a Banach space are provided. Finally, we study uniformly normal structure in Section 3. Some results about fixed points of non-expansive mapping are obtained.

2. PRELIMINARIES AND MAIN RESULTS

A curve in a Banach space X is a continuous mapping $x : [a, b] \to X$ and in this case it is denoted by $C := \{x(t) : a \le t \le b\}$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

For a normed linear space X, it is clear that $S(X_2)$, where X_2 is 2-dimensional subspace of X is a simple closed curve which is symmetric about the origin and unique up to orientation.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces. For a curve $C = \{x(t) : t \in [a, b]\}$ and a partition $P := \{t_0, t_1, t_2, \dots, t_n\} \subset [a, b]$ where

$$a = t_0 < t_1 < t_2 < ... < t_i < ... < t_n = b,$$

let

$$l(C, P) = \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|.$$

The length $l(C)$ of a curve C is defined as the least upper bound of $l(C, P)$ for all partitions P of $[a, b]$, that is,

$$l(C) = \sup\{l(C, P) : P \text{ is a partition of } [a, b]\}.$$

If $l(C)$ is finite, then the curve C is called rectifiable. (See [4, 8]).

Let $\|P\| = max_{1 \le i \le n}\{|t_i - t_{i-1}|\}$ for a partition P of $[a, b]$.

THEOREM 2.1 ([2, 17]). *If curve C is rectifiable, then for all $\epsilon > 0$, there exists $\delta > 0$, such that $\|P\| < \delta$ implies $l(C) - l(C, P) < \epsilon$. Furthermore if $\{P_k\}$ is a sequence of partitions of $[a, b]$ with $\|P_k\| \to 0$, then $lim_{k \to \infty} l(C, P_k) = l(C)$.*

Let $l_a^t(C)$ denote the length of curve $C = x(t)$ from a to t. For a rectifiable curve $C = x(t), a \le t \le b$, the arc length $l_a^t(C)$ is a continuous function of t.

Definition 2.2 ([2, 17]). Let $y(s)$ represent the point $x(t)$ on the curve C for which $l_a^t(C) = s$, then $C = y(s), 0 \le s \le l(C)$, is called standard form of the rectifiable curve C.

Theorem 2.3 ([2, 17]). *Let X_2 be a two dimensional Banach space and K_1, K_2 be closed convex subsets of X_2 with non-void interiors. If $K_1 \subseteq K_2$, then $l(\partial(K_1)) \le l(\partial(K_2))$, where $l(\partial(K_i))$ denote the lengths of the circumferences of $K_i, i = 1, 2$.*

Theorem 2.4 ([17]). *Let X_2 be a two dimensional Banach space. The following statements are true:*

(1) $6 \le l(S(X_2)) \le 8$;

(2) $l(S(X_2)) = 8$ *if and only if* $S(X_2)$ *is a parallelogram;*

(3) $l(S(X_2)) = 6$ *if and only if* $S(X_2)$ *is an affine regular hexagon.*

Lemma 2.5 ([11]). *If $x_1, x_2 \in B(X)$ and $0 < \epsilon < 1$ are such that $\frac{\|x_1+x_2\|}{2} > 1-\epsilon$, then for all $0 \le c \le 1$ and $z = cx_1 + (1-c)x_2 \in [x_1, x_2]$, the line segment connecting x_1 and x_2, it follows that $\|z\| > 1 - 2\epsilon$.*

Lemma 2.6. *If $x_1, x_2 \in S(X)$ and $0 < \epsilon < 1$ are such that $\frac{\|x_1+x_2\|}{2} > 1-\epsilon$, then there exists $0 \le c \le 1$, $z = cx_1 + (1-c)x_2 \in [x_1, x_2]$, such that for $y = \frac{z}{\|z\|} \in S(X)$ we have $1 - 2\varepsilon < \langle x_1, f_y\rangle = \langle x_2, f_y\rangle \le 1$, where $f_y \in \nabla_y$.*

Proof. We consider the 2-dimensional subspace X_2 of X spanned by x_1 and x_2, and x_1 and x_2 are clockwise located on $\widetilde{x_1, x_2} \subseteq S(X_2)$, then we can use Hahn-Banach theorem to extend the result to X.

Let $y \in \widetilde{x_1, x_2}$ such that $\langle x_1 - x_2, f_y\rangle = 0$, therefore $\langle x_1, f_y\rangle = \langle x_2, f_y\rangle$ and let $z \in [x_1, x_2]$ such that $y = \frac{z}{\|z\|}$, then $\langle x_1, f_y\rangle = \langle x_2, f_y\rangle = \langle z, f_y\rangle = \|z\| > 1 - 2\epsilon$, from lemma 2.5. $\square$

Lemma 2.7 ([6]). *Let X be a Banach space without weak normal structure, then for any $0 < \epsilon < 1$, there exists a sequence $\{z_n\} \subseteq S(X)$ with $z_n \to^w 0$, and*

$$1 - \epsilon < \|z_{n+1} - z\| < 1 + \epsilon$$

for sufficiently large n, and any $z \in co\{z_k\}_{k=1}^n$.

Lemma 2.8 ([11]). *Let X be a Banach space without weak normal structure, then for any $0 < \epsilon < 1$, there exist x_1, x_2, x_3 in $S(X)$ satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a - 1| < \epsilon$;

(ii) $|||x_1 - x_2|| - 1|, |||x_3 - (-x_1)|| - 1| < \epsilon$; *and*

(iii) $||\frac{x_1+x_2}{2}||, ||\frac{x_3+(-x_1)}{2}|| > 1 - \epsilon$.

The lemma 2.8 can be extended to the following:

Lemma 2.9. *Let X be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$, there exist x_1, x_2, x_3 in S(X) satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a - 1| < \varepsilon$;

(ii) $|\|x_1 - x_2\| - 1|$, $|\|x_3 - (-x_1)\| - 1| < \varepsilon$;

(iii) $\|\frac{x_1+x_2}{2}\|$, $\|\frac{x_3+(-x_1)}{2}\| > 1-\varepsilon$; *and*

(iv) $\langle x_1, f_{x_2}\rangle \geq 1-\varepsilon$.

Proof. Let $\eta = \frac{\varepsilon}{4}$ and $\{z_n\}$ be chosen as in lemma 2.7 with ϵ replaced by η.

Let f_1 be the supporting functional of $z_1 \in S(X)$, *i.e.* $\|f_1\| = < z_1, f_1 > = 1$, and let z_{n_0} satisfies $| < z_{n_0}, f_1 > | < \eta$, and $w = \frac{z_1 - z_{n_0}}{||z_1 - z_{n_0}||} \in S(X)$.

Let $x_1 = w, x_2 = z_1$, and $x_3 = z_{n_0}$ respectively.

Then it is proved in [11] that x_1, x_2, and x_3 satisfy 3 conditions in lemma 2.8.

We prove (iv):

$$\langle x_1, f_{x_2}\rangle = \langle w, f_{z_1}\rangle = \langle \frac{z_1 - z_{n_0}}{||z_1 - z_{n_0}||}, f_{z_1}\rangle = \frac{1}{||z_1 - z_{n_0}||}\langle z_1 - z_{n_0}, f_{z_1}\rangle$$
$$= \frac{1}{||x_2 - x_3||}(1 - \langle z_{n_0}, f_{z_1}\rangle) \geq \frac{1}{1+\eta}(1-\eta) \geq 1-\varepsilon. \quad \square$$

In the following we assume $l(\widetilde{x_1, x_2}) \leq \frac{1}{2} l(S(X_2))$ where $x_1, x_2 \in S(X)$.

Definition 2.10. $\zeta_X(\varepsilon) = \sup\{l(\widetilde{x_1, x_2}) : x_1, x_2 \in S(X)$ with $\langle x_1 - x_2, f_{x_1}\rangle \leq \varepsilon$ for some $f_{x_1} \in \nabla_{x_1}\}$, where $0 \leq \varepsilon \leq 2$.

PROPOSITION 2.11. *$\zeta_X(\varepsilon)$ is an increasing function in $0 \leq \varepsilon \leq 2$.*

Proof. Let $\varepsilon_1 < \varepsilon_2$, then $\{x_1, x_2 \in S(X)$ with $\langle x_1 - x_2, f_{x_1}\rangle \leq \varepsilon_1\} \subseteq \{x_1, x_2 \in S(X)$ with $\langle x_1 - x_2, f_{x_1}\rangle \leq \varepsilon_2\}$.

This implies $\zeta_X(\varepsilon)$ is increasing. $\square$

Example 2.12. For Hilbert space H, $\zeta_H(\varepsilon) = 2 \cdot tan^{-1}\sqrt{\frac{\varepsilon}{2-\varepsilon}}$, for $0 \leq \varepsilon < 2$, and $\zeta_H(2) = \pi$.

Proof. Let $x_1, x_2 \in S(H)$ with $\langle x_2, f_{x_1}\rangle = 1-\varepsilon$ where $0 \leq \varepsilon \leq 2$.

Consider the two-dim Euclidean subspace H_2 spanned by x_1 and x_2 of H.

From $\langle x_2 - (1-\varepsilon)x_1, f_{x_1}\rangle = 0$, we have $x_2 - (1-\varepsilon)x_1$ perpendicular to x_1, therefore

$$\|x_2 - (1-\varepsilon)x_1\|^2 = \|x_1 - (1-\varepsilon)x_1\| \cdot \| - x_1 - (1-\varepsilon)x_1\| = \varepsilon(2-\varepsilon),$$
$$\|x_2 - (1-\varepsilon)x_1\| = \sqrt{\varepsilon(2-\varepsilon)}.$$

We have

$$\tan \frac{l(\widetilde{x_1, x_2})}{2} = \frac{\|x_1 - (1-\varepsilon)x_1\|}{\|x_2 - (1-\varepsilon)x_1\|} = \frac{\varepsilon}{\sqrt{\varepsilon(2-\varepsilon)}} = \sqrt{\frac{\varepsilon}{2-\varepsilon}}.$$

So,

$$\zeta_H(\varepsilon) = l(\widetilde{x_1, x_2}) = 2 \cdot tan^{-1}\sqrt{\frac{\varepsilon}{2-\varepsilon}}. \quad \square$$

Definition 2.13 ([12]). A Banach space X is called uniformly non-square if there exists $\delta > 0$ such that if $x, y \in S(X)$, then either $\frac{\|x+y\|}{2} \leq 1-\delta$ or $\frac{\|x-y\|}{2} \leq 1-\delta$.

Definition 2.14 ([4, 5]). Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$, $(1-\varepsilon)\|y\| \leq \|Ty\| \leq (1+\varepsilon)\|y\|$.

The Banach space X is called *super-reflexive* if any space Y which is finitely representable in X is reflexive.

Remark 2.15. It is well known that if X is uniformly non-square then X is supper-reflexive and therefore X is reflexive.

Theorem 2.16. *For a Banach space X, if $\zeta_X(\varepsilon) < 1+\varepsilon$ for $0 \leq \varepsilon \leq 1$, or $\zeta_X(\varepsilon) < 2\varepsilon$ for $1 \leq \varepsilon \leq 2$, then X is uniformly non-square.*

Proof. Suppose X is not uniformly non-square, then for any $\delta > 0$, let $x, y \in S(X)$ such that both $\frac{\|x+y\|}{2} \geq 1-\delta$ and $\frac{\|x-y\|}{2} \geq 1-\delta$.

Case 1. We first prove that if $\zeta_X(\varepsilon) < 1+\varepsilon$ for $0 \leq \varepsilon \leq 2$, then X is uniformly non-square.

Let $z \in \widetilde{x,y}$ such that $\langle x, f_z\rangle = \langle y, f_z\rangle$, then from lemma 2.6, $\langle x, f_z\rangle = \langle y, f_z\rangle > 1-2\delta$, where $f_z \in \nabla_z$.

Since $l(\widetilde{x,y}) \geq \|x-y\| \geq 2-2\delta$, without loss of generality, we may assume $l(\widetilde{z,y}) \geq \|z-y\| \geq 1-\delta$.

Let $u = \frac{y-t(x+y)}{\|y-t(x+y)\|}, 0 \leq t \leq 1$. From lemma 2.5, we have

$$\begin{aligned}
1 \geq \langle u, f_z\rangle &= \langle \frac{y-t(x+y)}{\|y-t(x+y)\|}, f_z\rangle \\
&\geq \langle y-t(x+y), f_z\rangle - (\|(y-t(x+y)) - \frac{y-t(x+y)}{\|y-t(x+y)\|}\|) \\
&\geq \langle y-t(x+y), f_z\rangle - 2\delta = \langle -tx+(1-t)y, f_z\rangle - 2\delta \\
&\geq -t+(1-t)(1-2\delta) - 2\delta \geq 1-2t-4\delta.
\end{aligned}$$

So,

$$\langle z-u, f_z\rangle \leq 2t+4\delta.$$

From theorem 2.3 and lemma 2.5,
$l(\widetilde{z,u}) \geq \|z-y\| + \|y-u\| \geq 1-\delta + \|y-(y-t(x+y))\| - \|(y-t(x+y))-u\| \geq 1-\delta+2t(1-\delta)-2\delta \geq 1+2t-5\delta$.

Since δ can be arbitrarily small, we have $\zeta_X(2t) \geq 1+2t$ for $0 \leq t \leq 1$.

This is equivalent to $\zeta_X(\varepsilon) \geq 1+\varepsilon$, for any $0 \leq \varepsilon \leq 2$.

Case 2. We then prove that if $\zeta_X(1+\varepsilon) < 2+2\varepsilon$ for $0 \leq \varepsilon \leq 2$, then X is uniformly non-square.

Let $z \in \widetilde{-y, y}$ such that $\langle -y, f_z\rangle = \langle y, f_z\rangle$, then we have, $\langle -y, f_z\rangle = \langle y, f_z\rangle = 0$, where $f_z \in \nabla_z$.

Since $l(\widetilde{-y, y}) \geq 4-4\delta$, without loss of generality, we may assume $l(\widetilde{z, y}) \geq \|z-y\| \geq \|x-y\| \geq 2-2\delta$.

Let $u = \frac{y-t(x+y)}{\|y-t(x+y)\|}, 0 \leq t \leq 1$. From lemma 2.5, we have

$$
\begin{aligned}
1 \geq \langle u, f_z\rangle &= \langle \frac{y-t(x+y)}{\|y-t(x+y)\|}, f_z\rangle \\
&\geq \langle y-t(x+y), f_z\rangle - (\|(y-t(x+y)) - \frac{y-t(x+y)}{\|y-t(x+y)\|}\|) \\
&\geq \langle y-t(x+y), f_z\rangle - 2\delta = \langle -tx+(1-t)y, f_z\rangle - 2\delta \\
&= -t\langle x, f_z\rangle - 2\delta \geq -t-2\delta.
\end{aligned}
$$

So,

$$\langle z-u, f_z\rangle \leq 1+t+4\delta.$$

From theorem 2.3 and lemma 2.5,

$$
\begin{aligned}
l(\widetilde{z, u}) &\geq \|z-y\| + \|y-u\| \\
&\geq 2-2\delta + \|y-(y-t(x+y))\| - \|(y-t(x+y))-u\| \\
&\geq 2-2\delta+2t(1-\delta)-2\delta \geq 2+2t-6\delta.
\end{aligned}
$$

Since δ can be arbitrarily small, we have $\zeta_X(1+t) \geq 2+2t$ for $0 \leq t \leq 1$. This is equivalent to $\zeta_X(\varepsilon) \geq 2\varepsilon$, for any $1 \leq \varepsilon \leq 2$.

Combine Case 1 and Case 2, we have

$$\zeta_X(\varepsilon) < 1+\varepsilon \ for\ 0 \leq \varepsilon \leq 1,\ or\ \zeta_X(\varepsilon) < 2\varepsilon\ for\ 1 \leq \varepsilon \leq 2$$

implies X is uniformly non-square. $\square$

THEOREM 2.17. *For a Banach space X, if $\zeta_X(\varepsilon) < 1+\varepsilon$ for any $0 \leq \varepsilon \leq 1$, then X has normal structure.*

Proof. $\zeta_X(\varepsilon) < 1+\varepsilon$ for any $0 \leq \varepsilon \leq 1$ implies X is uniformly non-square, therefore reflexive. So the normal structure and weak normal structure coincide.

If X fails weak normal structure, and x_1, x_2, and x_3 be chosen to satisfy 4 conditions in lemma 2.9.

Let $y = \frac{x_1-t(x_3+x_1)}{\|x_1-t(x_3+x_1)\|}, 0 \leq t \leq 1$. From lemma 2.5, we have

$$
\begin{aligned}
\langle y, f_{x_2}\rangle &= \langle \frac{x_1-t(x_3+x_1)}{\|x_1-t(x_3+x_1)\|}, f_{x_2}\rangle \\
&\geq \langle x_1-t(x_3+x_1), f_{x_2}\rangle - \|(x_1-t(x_3+x_1)) - \frac{x_1-t(x_3+x_1)}{\|x_1-t(x_3+x_1)\|}\| \\
&\geq \langle x_1-t(x_3+x_1), f_{x_2}\rangle - 2\varepsilon = \langle x_1-t(x_2-ax_1+x_1), f_{x_2}\rangle - 2\varepsilon
\end{aligned}
$$

$$\geq 1-\varepsilon-t(1+|1-a|)-2\varepsilon \geq 1-\varepsilon-t-3\varepsilon = 1-t-4\varepsilon.$$

So,

$$\langle x_2 - y, f_{x_2}\rangle \leq t + 4\varepsilon.$$

From theorem 2.3, lemma 2.5 and lemma 2.9,

$$l(\widetilde{x_2, y}) \geq \|x_2 - x_1\| + \|x_1 - y\|$$
$$\geq \|x_2 - x_1\| + \|x_1 - (x_1 - t(x_3 + x_1))\| - \|(x_1 - t(x_3 + x_1)) - y\|$$
$$\geq (1-\varepsilon) + t - \varepsilon - 2\varepsilon = 1 + t - 4\varepsilon.$$

Since ε can be arbitrarily small, we have $\zeta_X(t) \geq 1+t$, for any $0 \leq t \leq 1$. This is equivalent to $\zeta_X(\varepsilon) \geq 1+\varepsilon$, for any $0 \leq \varepsilon \leq 1$. $\square$

3. UNIFORM NORMAL STRUCTURE

Let F be a filter on an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to converge to x with respect to F, denote by $\lim_F x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in F$.

A filter U on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion.

An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$.

Remark 3.1. We will use the fact that if U is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq U$ or $I \setminus A \subseteq U$;

(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_U x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 3.2 ([16]). Let U be an ultrafilter on I and let

$$N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_U ||x_i|| = 0\}.$$

The ultra-product of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_U$ equipped with the quotient norm.

We will use $(x_i)_U$ to denote the element of the ultra-product. It follows from (ii) of above remark 3.1, and the definition of quotient norm that

$$\|(x_i)_U\| = \lim_U \|x_i\| \tag{3.1}$$

In the following we will restrict our index set I to be $\mathbf{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbf{N}$ for some Banach space X. For an ultrafilter U on $\mathbf{N}$, we use X_U to denote the ultra-product.

Lemma 3.3 ([16]). *Suppose U is an ultrafilter on $\mathbf{N}$ and X is a Banach space. Then*

(i) $(X^*)_U = (X_U)^*$ *if and only if X is super-reflexive; and in this case,*

(ii) *the mapping J defined by $\langle (x_i)_U, J((f_i)_U)\rangle = \lim_U \langle x_i, f_i\rangle$ for all $(x_i)_U \in X_U$, is the canonical isometric isomorphism from $(X^*)_U$ onto $(X_U)^*$.*

Theorem 3.4. *For any Banach space X with $\zeta_X(\varepsilon) < 1+\varepsilon$ and $0 \le \varepsilon \le 2$, and for any nontrivial ultrafilter U on N, $\zeta_{X_U}(\varepsilon) = \zeta_X(\varepsilon)$.*

Proof. X with $\zeta_X(\varepsilon) < 1+\varepsilon$ and $0 \le \varepsilon \le 2$ implies X is uniformly non-square, so X is super-reflexive. We can use lemma 3.3.

Since X can be isometrically embedded onto X_U, we have $\zeta_X(\varepsilon) \le \zeta_{X_U}(\varepsilon)$.

We prove the reverse inequality. For any $\eta > 0$, from definition of $\zeta_X(\varepsilon)$ we can choose

$$(x_i^1)_U \in S(X_U),\ (x_i^2)_U \in S(X_U)$$

and an

$$f = (f_i^1)_U \in \nabla_{(x_i^1)_U} \in S((X_U)^*) = S((X^*)_U),$$

such that

$$\langle (x_i)_U - (y_i)_U, (f_i^1)_U\rangle \le \varepsilon, \text{ but } l(\widetilde{(x_i)_U, (y_i)_U}) > \zeta_{X_U}(\varepsilon) - \eta.$$

Without loss of generality, we may assume $\|x_i^1\| = \|x_i^2\| = \|f_{x_i^1}\| = 1$ for all $i \in \mathbf{N}$.

From remark (i) and (ii) of ultrafilter, equation (1) and the paragraphs above, the sets:

$$P = \{i \in \mathbf{N} : \langle x_i^1 - x_i^2, f_{x_i^1}\rangle \le \varepsilon\},$$

and

$$Q = \{i \in \mathbf{N} : l(\widetilde{(x_i)_U, (y_i)_U}) > \zeta_{X_U}(\varepsilon) - \eta\}$$

are all in U.

So the intersection $P \bigcap Q$ is in U too, and is hence not empty.

Let $i \in P \bigcap Q$ and $(X_i)_2$ be a two dimensional subspace of X spanned by x_i^1 and x_i^2, we have

$$\langle x_i^1 - x_i^2, f_{x_i^1}\rangle \le \varepsilon,$$

and

$$l(\widetilde{(x_i^1)_U, (x_i^2)_U}) > \zeta_{X_U}(\varepsilon) - \eta.$$

Hence $\zeta_X(\varepsilon) \ge \zeta_{X_U}(\varepsilon) - \eta$.

Since η can be arbitrarily small, we have $\zeta_X(\varepsilon) \ge \zeta_{X_U}(\varepsilon)$. $\square$

THEOREM 3.5. *If X is a Banach space with $\zeta_X(\varepsilon) < 1+\varepsilon$ and $0 \le \varepsilon \le 2$, then X has uniform normal structure.*

Proof. The idea of the proof is same as the proof of theorem 4.4 in [11]. Suppose $\zeta_X(\varepsilon) < 1+\varepsilon$ and $0 \le \varepsilon \le 2$, and X does not have uniform normal structure, we find a sequence $\{C_n\}$ of bounded closed convex subset of X such that for each n,

$$0 \in C_n, \quad d(C_n) = 1,$$

and

$$rad(C_n) = \inf_{x \in C_n} \sup_{y \in C_n} \|x-y\| > 1 - \frac{1}{n}.$$

Let U be any nontrivial ultrafilter on $\mathbf{N}$, and let

$$C = \{(x_n)_U : x_n \in C_n, n \in \mathbf{N}\},$$

then C is a nonempty bounded closed convex subset of X_U.

It follows from the properties of C_n above that $d(C) = rad(C) = 1$, so X_U does not have normal structure.

On the other hand, from theorem 3.4, $\zeta_{X_U}(\varepsilon) < 1+\varepsilon$ and $0 \le \varepsilon \le 2$.

This contradicts theorem 2.17, and X must have uniform normal structure. $\square$

We provide some relationships between the modulus $\zeta_X(\varepsilon)$ and some other known ones in the following:

In [10], Gao introduced a parameter $Q(X) = \sup\{l(S(X_2)) : X_2 \subseteq X\}$, where X_2 denotes two dimensional subspace of X, and proved that a Banach space X with $Q(X) < 6 + \frac{2\delta_X(1)}{1-\delta_X(1)}$ has uniform normal structure. For any Banach space X, $0 \le \delta_X(1) \le \frac{1}{2}$, so $6 \le 6 + \frac{2\delta_X(1)}{1-\delta_X(1)} \le 8$.

From $Q(X) = 2\zeta_X(2)$, we have

THEOREM 3.6. *A Banach space X with $\zeta_X(2) < 3 + \frac{\delta_X(1)}{1-\delta_X(1)}$ has uniform normal structure.*

We use Hilbert space as an example.

For the Hilbert space H, $\zeta_H(2) = \pi$ and $\delta_H(\varepsilon) = 1 - \frac{\sqrt{4-\varepsilon^2}}{2}$ for $0 \le \varepsilon \le 2$. We have $\delta_H(1) = 1 - \frac{\sqrt{3}}{2}$ and

$$3 + \frac{\delta_H(1)}{1-\delta_H(1)} = 3 + \frac{1-\frac{\sqrt{3}}{2}}{1-(1-\frac{\sqrt{3}}{2})} = 3 + \frac{2-\sqrt{3}}{\sqrt{3}} = 3.1547\cdots\cdots.$$

This shows $\zeta_H(2) < 3 + \frac{\delta_H(1)}{1-\delta_H(1)}$ for the Hilbert space.

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