ENTROPY MAXIMIZING CURVES

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The paper defines the entropy of an oval curve as a function of its curvature and finds the ovals with maximum entropy. The problem of finding the entropy maximizing curves between two points is treated using the Lagrangian formalism and solved in closed form. The paper studies also the smooth isometric deformations of the type $\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = \sigma(t)\, \varphi_t(s)$ and proves that they are both area and entropy increasing, the oval with the maximum entropy being a circle.

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1. INTRODUCTION

The present paper continues the idea of [1], dealing with a problem related to elastica curves – *the maximum entropy curves.* The employed methods use tools of differential equations applied to differential geometry of curves, following the spirit of previous works [2–4].

The entropy of a curve is defined using its curvature. Section 2 shows that among all oval curves, *i.e.*, plane, closed, simple curves with positive curvature, the circle has the maximum entropy. The relation with the elastic potential is mentioned in Section 2. The maximum entropy curves between two given points and having prescribed directions at the endpoints is explored using Lagrangian formalism in Sections 4 and 5. The isometric deformations are introduced in Section 6 and the evolution of curvature, area and entropy along these deformations are explored in Sections 7, 8 and 9, respectively. It turns out that in all investigated cases the oval curves of maximum entropy are circles. Finally, Section 10 defines a statistical distance-like measure between two oval curves, called cross-entropy, and advances an open question.

2. SIMPLE CLOSED CURVES

An *oval curve* is a smooth, plane, closed, convex, and simple curve. Its curvature is smooth and positive everywhere. In the following, we shall introduce a probability density associated with each oval curve, and then provide a characterization theorem regarding maximum entropy ovals curves.

Let $c : [0, \tau] \to \mathbb{R}^2$ be an oval curve parameterized by the arc length s. Let $\kappa(s)$ denote the curvature of $c(s)$ and define $p(s) = \frac{1}{2\pi}\kappa(s)$. Since the convexity of the oval implies $p(s) \geq 0$, and by Fenchel's theorem, see Millman and Parker [11], we have $\int_0^\tau p(s)\,\mathrm{d}s = 1$, it follows that $p(s)$ can be regarded as a probability density function on $[0, \tau]$. By the Four Vertex Theorem (see for instance [11]) the curvature κ is either constant, or has two maxima and two minima. Consequently, the density $p(s)$ is either bimodal or uniform, see Fig. 1. We shall regard it as the *density function associated with the oval curve* $c(s)$.

The following definition is inspired by a similar concept from Thermodynamics.

Definition 1. The entropy of an oval curve $c(s)$ of length τ is the functional

$$H(c) = -\int_0^\tau p(s)\,\ln p(s)\,\mathrm{d}s = \ln(2\pi) - \frac{1}{2\pi}\int_0^\tau \kappa(s)\ln\kappa(s)\,\mathrm{d}s. \tag{2.1}$$

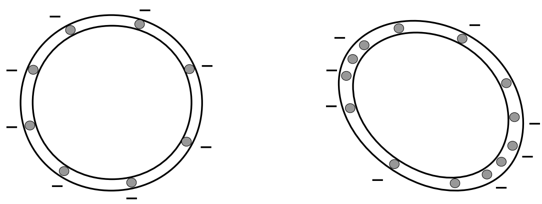
The negative sign in the definition is reminiscent from statistical mechanics where the entropy must be positive. However, in our case the entropy is not necessary positive. It is worth noting that some authors consider the entropy defined just by the more simple relation $\int_0^\tau \kappa(s)\ln\kappa(s)\,\mathrm{d}s$.

The physical interpretation is given in the following. Consider a closed metal thin wire of an oval shape and consider some charge on it. The charge is free to move through the wire and be distributed uniformly at an initial instance of time, see Fig. 1.a. However, after a while, it will prefer to concentrate in the regions of larger curvature, where the lattice structure of the metal is affected by bending, see Fig. 1.b. The equilibrium density of the charge will follow the density function associated with the oval curve, $p(s)$. The entropy of $c(s)$ is a measure of the uncertainty of the localization of the charge in the wire. The maximum entropy corresponds to the case when the localization of the charge is maximally uncertain, which corresponds to an uniform distribution. The next result deals with the proof of this statement.

THEOREM 1. *Let $\tau > 0$ be given. Among all oval curves of the same length τ, the curve with the maximum entropy is the circle. The maximum value of the entropy in this case is* $\ln\tau$.

Proof. For any convex function F, Jensen's inequality states that

$$F\Big(\frac{1}{\tau}\int_0^\tau f(s)\,\mathrm{d}s\Big) \leq \frac{1}{\tau}\int_0^\tau F\big(f(s)\big)\,\mathrm{d}s,$$



a **b**

Fig. 1 – a. Uniform distribution of charges on a circular wire; b. bimodal distribution of charge on an oval wire.

Choosing $F(x) = x \ln x$ and $f(s) = \kappa(s)$, and using $\int_0^\tau \kappa(s)\,\mathrm{d}s = 2\pi$, the inequality becomes

$$\frac{2\pi}{\tau}\ln\frac{2\pi}{\tau} \leq \frac{1}{\tau}\int_0^\tau \kappa(s)\ln\kappa(s)\,\mathrm{d}s \Longleftrightarrow$$

$$\ln(2\pi) - \ln\tau \leq \frac{1}{2\pi}\int_0^\tau \kappa(s)\ln\kappa(s)\,\mathrm{d}s,$$

which can be written as $H(c) \leq \ln\tau$. The maximum value is reached when Jensen's inequality becomes identity, *i.e.*, when $\kappa(s) = \text{constant}$, which corresponds to a circle of radius $R = \dfrac{\tau}{2\pi}$. □

3. RELATION WITH THE ELASTIC POTENTIAL

When the metal wire is constrained to take an oval shape, some elastic force in the wire will occur. This has the tendency of straightening the wire. The more bent the wire is, the larger the elastic tendency. Integrating, we obtain the value of the total elastic potential along the curve as $E(c) = \int_0^\tau \frac{1}{2}\kappa(s)^2\,\mathrm{d}s$.

The physical interpretation follows. Assume there is a flow of particles flowing through the wire with constant speed, $|\dot{c}(s)| = 1$. Due to curvature, the particles will act on the walls of the wire with a force $F(s) = m\ddot{c}(s)$, where m is the mass of the particle. The elastic potential is the integral of the square of the magnitude of F along the curve, assuming $m = 1/2$. The picture becomes more clear if we replace the wire by a light hose through which we run water at pressure. The shape of the hose tends to minimize the elastic potential $E(c)$, taking the shape of an *elastic curve*.

The problem of elastic curves has been proposed for the first time by Daniel Bernoulli to Leonhard Euler in 1744. They defined them as curves which minimize the bending energy of a thin inextensible rod. The mathematical model of this problem is that of minimizing the elastic potential energy, which is the integral of the squared curvature for curves of a fixed length satisfying given first order boundary data. More precise, an *elastic curve* in the plane is a regular curve $c : [0, L] \to \mathbb{R}^2$ with prescribed endpoints $A = c(0)$, $B = c(\tau)$ and given length τ, which minimizes the elastic potential energy $\int_0^L \frac{1}{2}\kappa^2(s)\,\mathrm{d}s$, where $\kappa(s)$ denotes the curvature of $c(s)$.

Finding plane elastic curves of given length τ, which pass through two given points A and B, and having prescribed tangent lines at these points was a problem first solved by Euler, who found the complete classification of plane elastic curves into 9 distinct types, see Love [7]. An explicit calculation of the equation of elastic curves is done in [1], using elliptic trigonometric functions, see [8].

The tradeoff between the entropy and the elastic potential of an oval is captured by the next result.

Proposition 1. *We have*

$$H(c) + \frac{1}{\pi}\int_0^\tau \frac{1}{2}\kappa^2(s)\,\mathrm{d}s \geq 1 + \ln(2\pi). \tag{3.1}$$

Hence the entropy and elastic potential cannot be made simultaneously too small for the same oval curve c. The identity is reached for $\kappa = 1$, i.e. in the case of the unit circle.

Proof. By the inequality $\ln x \leq x - 1$, $x > 0$, and using Fenchel's theorem, we have

$$\frac{1}{2\pi}\int_0^\tau \kappa \ln \kappa \,\mathrm{d}s \leq \frac{1}{2\pi}\int_0^\tau \kappa^2\,\mathrm{d}s - 1.$$

The inequality (3.1) follows easily after using formula (2.1). The identity occurs for $\kappa = 1$. $\square$

Since ovals are closed curves, in this section we did not need to consider boundary conditions. In the next section, we consider curves that join two distinct points in the plane. Consequently, for the purpose of the next section, we shall drop the closeness condition of the curve.

4. VARIATIONALLY CONTROLLED SYSTEM

This section studies the entropy maximizing plane curves using the Lagrangian formalism with constraints. Consider the curves joining points A and B in the plane and having prescribed tangents $\alpha, \beta \in \mathbb{R}^2$ at the endpoints

$$\Omega = \{c : [0, \tau] \to \mathbb{R}^2;\ c(0) = A, c(\tau) = B, c'(0) = \alpha, c'(\tau) = \beta\}.$$

Denote by $\Omega_u = \{c \in \Omega; \|c'\| = 1\}$ the subspace of unit speed curves. We plan to maximize the entropy $H(c)$ over all curves in the space Ω_u. Then, the solution is given by

$$c^* = \arg\max_{c\in\Omega_u} H(c).$$

To solve the problem, we employ the Lagrangian formalism.

The following formula for the curvature, due to Euler, will be useful when setting up the Lagrangian. Consider a plane curve $c(s) = \big(x(s), y(s)\big)$ parameterized by the arc length s, with $s \in [0, \tau]$, τ being the length of the curve. Since $\dot{x}^2(s)+\dot{y}^2(s) = 1$, we can write $\dot{x}(s) = \cos\theta(s)$ and $\dot{y}(s) = \sin\theta(s)$, where the smooth function $\theta(s)$ is the angle made by the velocity $\dot{c}(s)$ with the x-axis. The square of the curvature can be written as

$$\kappa^2(s) = \ddot{x}^2(s) + \ddot{y}^2(s) = \dot{\theta}^2(s). \tag{4.1}$$

The use of unit speed curves $c(s)$ implies the use of the following constraints, see [6]

$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta. \tag{4.2}$$

The variational method involving non-holonomic constraints considers the unit speed curve $c(s) = \big(x(s), y(s)\big)$, and expresses its curvature by the formula (4.1) considering the velocity constraints (4.2).

Therefore, we consider the functional with constraints $F : \Omega \to \mathbb{R}$, defined by

$$\begin{aligned} F(c) \;=\; & \int_0^\tau \Big(-\frac{1}{2\pi}\kappa(s)\ln\kappa(s)\Big)\,\mathrm{d}s + \lambda_1\int_0^\tau \big(\dot{x}(s) - \cos\theta(s)\big)\,\mathrm{d}s \\ & +\lambda_2\int_0^\tau \big(\dot{y}(s) - \sin\theta(s)\big)\Big)\,\mathrm{d}s, \end{aligned}$$

where $\theta(s)$ is the angle made by the tangent vector $\dot{c}$ with the x-axis, and λ_i are Lagrange multipliers.

In the following, we assume that the curve c is convex, *i.e.*, $\dot{\theta} > 0$. Then the entropy can be written in terms of angle θ as

$$\kappa(s)\ln\kappa(s) = \dot{\theta}(s)\ln\dot{\theta}(s),$$

and the resulting Lagrangian is

$$L(\theta, x, y, \dot{\theta}, \dot{x}, \dot{y}) = -\frac{1}{2\pi}\dot{\theta}\ln\dot{\theta} + \lambda_1(\dot{x} - \cos\theta) + \lambda_2(\dot{y} - \sin\theta), \tag{4.3}$$

with λ_1, λ_2 Lagrange multipliers. Then the previous functional becomes $F(c) = \int_0^\tau L\,\mathrm{d}s$. The associated Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial\dot{x}} = \frac{\partial L}{\partial x}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial\dot{y}} = \frac{\partial L}{\partial y}$$

can be written as

$$-\ddot{\theta}/\dot{\theta} = \mu_1 \sin\theta - \mu_2 \cos\theta \tag{4.4}$$
$$\dot{\mu}_1 = 0 \tag{4.5}$$
$$\dot{\mu}_2 = 0, \tag{4.6}$$

where $\mu_i = 2\pi\lambda_i$. It is remarkable that this system can be integrated explicitly as in the following.

The equation (4.4) can be written after integration as

$$\begin{aligned}\dot{\theta} &= \mu_2 \sin\theta + \mu_1 \cos\theta + K \\ &= |\mu| \sin(\theta + \varphi_0) + K,\end{aligned}$$

with K integration constant, $|\mu| = \sqrt{\mu_1^2 + \mu_2^2}$, and $\tan\varphi_0 = \mu_1/\mu_2$. The condition $K > |\mu|$ implies the convexity condition $\dot{\theta} > 0$.

Set $A = |\mu|$, $u = \theta + \varphi_0$. We thus obtain the following ODE in u

$$\dot{u} = A \sin u + K,$$

which can be integrated by the method of separation as follows

$$\int \frac{\mathrm{d}u}{A \sin u + K} = \int \mathrm{d}s \Longleftrightarrow$$

$$\frac{2}{\sqrt{K^2 - A^2}} \tan^{-1}\Big(\frac{A + K \tan\frac{u}{2}}{\sqrt{K^2 - A^2}}\Big) - \widetilde{C} = s.$$

Solving for u, we obtain

$$u = 2\tan^{-1}\Big(\sqrt{1 - \Big(\frac{A}{K}\Big)^2} \tan\Big(C + \frac{s}{2}\sqrt{K^2 - A^2}\Big) - \frac{A}{K}\Big),$$

with C integration constant. The solution of the Euler-Lagrange system is

$$\theta(s) = 2\tan^{-1}\Big(\sqrt{1 - \Big(\frac{A}{K}\Big)^2} \tan\Big(C + \frac{s}{2}\sqrt{K^2 - A^2}\Big) - \frac{A}{K}\Big) - \varphi_0. \tag{4.7}$$

Integrating in the constraints (4.2) provides the curve components

$$\begin{aligned}x(s) &= x(0) + \int_0^s \cos\theta(t)\mathrm{d}t \\ y(s) &= y(0) + \int_0^s \sin\theta(t)\mathrm{d}t.\end{aligned}$$

Even if $\cos\theta(t)$ and $\sin\theta(t)$ can be computed using trigonometric formulas and formula (4.7), an explicit computation of their integrals is hard to obtain.

5. BOUNDARY CONDITIONS

There are four end point conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad x(\tau) = x_1, \quad y(\tau) = y_1,$$

and two given tangent directions at endpoints

$$\theta(0) = \theta_0, \quad \theta(\tau) = \theta_1.$$

There are some simplifying assumptions that can be made. Performing a rigid motion, we can always assume, without loss of generality, that the curve starts from the origin and has the initial velocity tangent to the x-axis

$$x_0 = 0, \qquad y_0 = 0, \qquad \theta_0 = 0.$$

These boundary conditions can be written equivalently as

$$\begin{aligned} x_1 &= \int_0^\tau \cos\theta(t)\mathrm{d}t \\ y_1 &= \int_0^\tau \sin\theta(t)\mathrm{d}t \\ \varphi_0 &= 2\arctan\left(\sqrt{1-\left(\frac{A}{K}\right)^2}\tan C - \frac{A}{K}\right) \\ \theta_1 &= 2\arctan\left(\sqrt{1-\left(\frac{A}{K}\right)^2}\tan\left(C + \frac{\tau}{2}\sqrt{K^2-A^2}\right) - \frac{A}{K}\right) - \varphi_0. \end{aligned}$$

The four independent parameters μ_1, μ_2, C and K are determined by the aforementioned four equations (A and φ_0 depend on μ_1 and μ_2); the solution is not necessarily unique. A study of an exact number of solutions is missing at the moment.

6. SMOOTH ISOMETRIC DEFORMATIONS

Any plane, simple, and closed smooth curve of length $\tau = 2\pi$ can be considered as the image of an isometric immersion φ of the unit circle $\mathbb{S}^1$ into the plane $\mathbb{R}^2$

$$\varphi(s) = \big(\varphi^1(s), \varphi^2(s)\big),$$

where $s \in [0, 2\pi]$ is the arc length. Consequently, the following periodicity conditions hold

$$\varphi(0) = \varphi(2\pi), \qquad \dot\varphi(0) = \dot\varphi(2\pi).$$

The immersion φ is deformed smoothly with respect to time t. This deformation is denoted by φ_t and is defined as follows:

(i) $\varphi_0 = \varphi$.

(ii) φ_t is an isometric immersion for $t \geq 0$, satisfying $\varphi_t(0) = \varphi_t(2\pi)$, $\dot{\varphi}_t(0) = \dot{\varphi}_t(2\pi)$.

(iii) The evolution of the deformation satisfies the heat-type equation

$$\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = F(t, \varphi_t(s)),$$

with the source function F to be specified later. In [1] we proved the following result:

THEOREM 2. *Let $\sigma(t)$ be a smooth function defined on $[0, \infty)$ such that*

$$\sigma(t) > 0, \qquad \sigma'(t) < 0, \qquad \lim_{t \to \infty} \sigma(t) = \sigma = \frac{1}{2}.$$

If the immersion φ_t satisfies the initial value problem

$$\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = \sigma(t)\, \varphi_t(s) \tag{6.1}$$

$$\varphi_{|t=0} = \varphi_0, \tag{6.2}$$

then the elastic potential $\frac{1}{2}\int \kappa_t^2$ is a decreasing function of t. The limit curve, obtained by taking $t \to \infty$, if exists, is a circle.

We shall use this deformation in the later sections and prove a similar result for the entropy.

7. THE EVOLUTION OF CURVATURE

We shall start with a few basic notions. Recall that $s \in [0, 2\pi]$ denotes the arc length. The tangent vector field to the unit speed curve $s \to \varphi_t(s)$ is given by $T_t(s) = \dfrac{\partial \varphi_t(s)}{\partial s}$. By Frenet's formula we have

$$\frac{\partial T_t(s)}{\partial s} = \kappa_t(s) N_t,$$

where N_t and κ_t stand for the unit inner normal vector field and the curvature of the curve $s \to \varphi_t(s)$, respectively. Therefore, the equation (6.1) becomes

$$\partial_t \varphi_t = \frac{1}{2}\kappa_t N_t + \sigma(t)\varphi_t. \tag{7.1}$$

Next we shall parameterize the problem in terms of the normal angle μ, which is the angle made by the normal N and the horizontal direction Ox. If θ is the angle made by the tangent direction T with the horizontal axis Ox, then $\mu = \theta - \frac{\pi}{2}$, see Fig. 2. We shall recall a few known facts regarding this parametrization.

If T and N are respectively the tangent vector and the inner unit normal vector to the unit speed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, then $T = \dot{\gamma} = (\cos\theta, \sin\theta)$ and

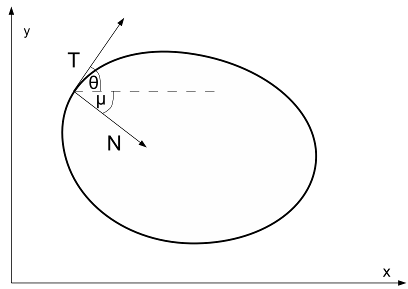


Fig. 2 – The normal angle, μ, is measured counter-clock wise, $\mu = \theta - \frac{\pi}{2}$.

$N = (\cos\mu, \sin\mu)$, see Fig. 2. The normal angle depends on the arc length, $\mu = \mu(s)$, and we may assume without loss of generality that $\mu(0) = 0$ and $\mu(2\pi) = 2\pi$, with $\mu(s)$ increasing. Then the curve γ can be parametrized by the normal angle μ. An important role in this approach is played by the *support function* of γ, which is defined by

$$S(\mu) = \langle \gamma(\mu), N(\mu) \rangle = \gamma^1(\mu)\cos\mu + \gamma^2(\mu)\sin\mu. \tag{7.2}$$

Differentiating with respect to μ, and using that $\gamma'(\mu)$ and $N(\mu)$ are perpendicular, we have

$$\partial_\mu S_\mu = -\gamma^1(\mu)\sin\mu + \gamma^2(\mu)\cos\mu \tag{7.3}$$

$$\partial_\mu^2 S(\mu) + S(\mu) = -\gamma^1_\mu(\mu)\sin\mu + \gamma^2_\mu(\mu)\cos\mu. \tag{7.4}$$

From (7.2) and (7.3) one can retrieve the curve γ in terms of the support function S as

$$\gamma(\mu) = \begin{pmatrix} \cos\mu & -\sin\mu \\ \sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} S \\ \partial_\mu S \end{pmatrix}.$$

This represents the curve $\gamma(\mu)$ as a rotation applied to a vector depending on the support function S. The curvature κ of γ can be also represented in terms of the support function as in the following

$$\begin{aligned} \frac{1}{\kappa} &= \frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{\mathrm{d}s}{\mathrm{d}\mu} = \frac{\mathrm{d}s}{\mathrm{d}\mu}\|\dot\gamma\|^2 = \langle \frac{\mathrm{d}\gamma}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\mu}, \dot\gamma \rangle = \langle \gamma_\mu, (\cos\theta, \sin\theta) \rangle \\ &= \langle \gamma_\mu, (-\sin\mu, \cos\mu) \rangle = -\gamma^1_\mu \sin\mu + \gamma^2_\mu \cos\mu, \end{aligned}$$

which in the virtue of (7.4) implies

$$\kappa = \frac{1}{\partial_\mu^2 S + S}. \tag{7.5}$$

This formula can be also found in Zhu [10], p. 2.

Now, we shall assume that the curve is transformed isometrically by the immersion (6.1). Let $\varphi_t(s) = \varphi(s,t) = \varphi\big(s(\mu,t),t\big) = \gamma(\mu,t)$ and using (7.1) yields

$$\partial_t\gamma(\mu,t) \quad = \quad \frac{\partial\varphi}{\partial s}\frac{\partial s}{\partial t} + \partial_t\varphi_t = T\frac{\partial s}{\partial t} + \frac{1}{2}\kappa N + \sigma(t)\gamma.$$

Then $S = S(\mu,t) = \langle\gamma(\mu,t)$ has the total derivative with respect to t given by

$$\begin{aligned}\partial_t S &= \frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma,N\rangle = \langle\partial_t\gamma,N\rangle + \langle\gamma,\partial_t N\rangle \\ &= \langle T\frac{\partial s}{\partial t} + \frac{1}{2}\kappa N + \sigma(t)\gamma, N\rangle + \langle\gamma,\partial_t N\rangle \\ &= \frac{1}{2}\kappa + \sigma(t)S + \langle\gamma,\partial_t N\rangle.\end{aligned}$$

Using

$$\mu'(t) = \frac{\partial\mu}{\partial t} = \frac{\partial\mu}{\partial\theta}\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial t} = k\frac{\partial s}{\partial t} = 0,$$

since the arc s is invariant by isometry reasons, we have

$$\partial_t N = \partial_t(\cos\mu(t), \sin\mu(t)) = (-\sin\mu, \cos\mu)\mu'(t) = 0,$$

and hence

$$\partial_t S = \frac{1}{2}\kappa + \sigma(t)S, \tag{7.6}$$

where $\kappa = \kappa(\mu,t)$ is given by (7.5). Differentiating in (7.5) and using (7.6) yields

$$\begin{aligned}\partial_t\kappa &= -\frac{1}{(\partial_\mu^2 S + \partial_t S)^2}\big(\partial_t\partial_\mu^2 S + \partial_t S\big) \\ &= -\kappa^2\big(\frac{1}{2}\partial_\mu^2\kappa + \frac{1}{2}\kappa + \sigma(\partial_\mu^2 S + S)\big) \\ &= -\kappa^2\big(\frac{1}{2}\partial_\mu^2\kappa + \frac{1}{2}\kappa + \sigma\frac{1}{\kappa}\big).\end{aligned}$$

Hence, the curvature satisfies the following evolution equation

$$\partial_t\kappa = -\frac{1}{2}\kappa^2(\partial_\mu^2\kappa + \kappa) - \sigma(t)\kappa, \qquad \mu\in\mathbb{S}^1, t>0. \tag{7.7}$$

From the maximum principle of parabolic operators

$$\min_{\mu\in\mathbb{S}^1}\kappa(\mu,t) \geq \min_{\mu\in\mathbb{S}^1}\kappa(\mu,0) > 0, \qquad t\geq 0.$$

This shows that if the initial curve $\varphi_0(\mathbb{S}^1)$ is a convex curve and the isometric deformation φ_t satisfies the evolution equation (6.1), then the curve $\varphi_t(\mathbb{S}^1)$ is also convex, for any $t>0$.

8. THE EVOLUTION OF ENCLOSED AREA

In this section, we shall use the previous results to study how the area enclosed by a simple, plane curve evolves during a heat-type isometric deformation.

Let D be the domain enclosed by the simple curve γ, and denote its area by $\mathcal{A}$. First, we shall represent the area in terms of support function, S, and curvature, κ. From Green's formula, we have

$$\begin{aligned}\mathcal{A} &= \iint_D dA = \frac{1}{2}\int_\gamma x\mathrm{d}y - y\mathrm{d}x = \frac{1}{2}\int_0^{2\pi}(xy' - yx')\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^{2\pi}\langle \gamma, N\rangle\,\mathrm{d}s = \frac{1}{2}\int_0^{2\pi}\frac{S}{\kappa}\,\mathrm{d}\mu, \qquad (8.1)\end{aligned}$$

where we used the change of variables

$$\mathrm{d}s = \frac{\mathrm{d}s}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\mu}\mathrm{d}\mu = \frac{1}{\kappa}\,\mathrm{d}\mu.$$

The area $\mathcal{A}(t)$ enclosed by the curve $\varphi_t(\mathbb{S}^1)$ is the same as the one enclosed by the curve $\gamma(\cdot,t)$. Differentiating in (8.1) and using the evolution equations (7.6) and (7.7) as well as formula (7.5), we have

$$\begin{aligned}\mathcal{A}'(t) &= \frac{1}{2}\int_0^{2\pi}\frac{\partial}{\partial t}\Big(\frac{S}{\kappa}\Big)\,\mathrm{d}\mu = \frac{1}{2}\int_0^{2\pi}\frac{\partial_t S\,\kappa - S\,\partial_t\kappa}{\kappa^2}\,\mathrm{d}\mu \\ &= \frac{1}{2}\int_0^{2\pi}\Big(\frac{1}{2} + \sigma\frac{S}{\kappa}\Big)\,\mathrm{d}\mu + \frac{1}{2}\int_0^{2\pi}\Big[\frac{1}{2}S(\partial_\mu^2\kappa + \kappa) + \sigma\frac{S}{\kappa}\Big]\,\mathrm{d}\mu \\ &= \frac{\pi}{2} + 2\sigma\mathcal{A}(t) + \frac{1}{2}\int_0^{2\pi}\Big[\frac{1}{2}\partial_\mu^2 S\,\kappa + \frac{1}{2}S\kappa\Big]\,\mathrm{d}\mu \\ &= \frac{\pi}{2} + 2\sigma\mathcal{A}(t) + \frac{1}{2}\int_0^{2\pi}\Big[\frac{1}{2}(\partial_\mu^2 S + S)\kappa\Big]\,\mathrm{d}\mu \\ &= \frac{\pi}{2} + 2\sigma\mathcal{A}(t) + \frac{1}{2}\int_0^{2\pi}\frac{1}{2}\,\mathrm{d}\mu \\ &= \pi + 2\sigma\mathcal{A}(t).\end{aligned}$$

Therefore, $\mathcal{A}(t)$ verifies the following initial value linear differential equation

$$\begin{aligned} y' - 2\sigma(t)y &= \pi \\ y(0) &= \mathcal{A}(0).\end{aligned}$$

The solution is unique and is given by

$$\mathcal{A}(t) = e^{2\rho(t)}\Big(\mathcal{A}(0) + \pi\int_0^t e^{-2\rho(u)}\,\mathrm{d}u\Big), \qquad (8.2)$$

with $\rho(t) = \int_0^t \sigma > 0$.

Since the curve is isometric with the circle $\mathbb{S}^1$, from the isoperimetric inequality, we have $\mathcal{A}(t) \leq \pi$, with identity reached for the circle. We are interested in finding a time $T \geq 0$ (finite or infinite) such that $\mathcal{A}(T) = \pi$, case in which the curve becomes a circle. Using (8.2) the time T should satisfy the equation

$$e^{-2\rho(T)} = \int_0^T e^{-2\rho(u)}\,du + \frac{\mathcal{A}(0)}{\pi}. \tag{8.3}$$

We are interested in the existence of solutions T for the equation (8.3). The equation can be written equivalently as $f(T) = g(T)$, where

$$\begin{aligned} f(T) &= e^{-2\rho(T)} \\ g(T) &= \int_0^T e^{-2\rho(u)}\,du + \frac{\mathcal{A}(0)}{\pi}. \end{aligned}$$

Since $\sigma(t) > \frac{1}{2}$ we have $2\rho(T) > T$. Then $f(T)$ is decreasing with

$$0 < f(T) \leq e^{-T},$$

and with initial value $f(0) = 1$ and $\lim_{T\to\infty} f(T) = 0$. On the other side, the function $g(T)$ is increasing, with $g(0) = \frac{\mathcal{A}(0)}{\pi}$. It follows that as long as $\mathcal{A}(0) \leq \pi$ the equation $f(T) = g(T)$ has a unique finite solution $T^* \geq 0$.

In the particular case when $\mathcal{A}(0) = \pi$, the solution is $T^* = 0$, case in which the initial curve is already a circle.

If $\mathcal{A}(0) < \pi$, then $\varphi_{T^*}(\mathbb{S}^1)$ is a circle, with $T^* < \infty$.

There are no solutions in the case $\mathcal{A}(0) > \pi$, since this inequality contradicts the isoperimetric inequality.

We conclude this section with the following result:

THEOREM 3. *The deformation* (6.1) *satisfying the conditions of Theorem* 2 *increases the area enclosed by the curve to the maximum value* π *in finite time. In this case the curve becomes a circle.*

9. THE ENTROPY FLOW

We have seen in Theorem 1 that circles have maximum entropy. Theorem 3 shows that the isometric deformation transforms the curve into one with maximum entropy in finite time, increasing its area at all time instances, while keeping its perimeter constant. The question is whether this is done by increasing also the entropy of the curve at all instances. The present section deals with this problem.

THEOREM 4. *Assume ϕ_0 is a curve of finite entropy. Then the deformation* (6.1)*, satisfying the conditions of Theorem* 2*, increases the entropy to the maximum value* $\ln(2\pi)$*, provided the entropy stays finite during the deformation.*

Proof. Using the change of variables $\mathrm{d}\theta = \kappa(s)\,\mathrm{d}s$, the entropy of a curve c can be written in terms of the angle θ as

$$H(c) = \ln(2\pi) - \frac{1}{2\pi}\int_0^{2\pi} \ln\kappa(\theta)\,\mathrm{d}\theta.$$

Given the linear relation between the normal angle, μ, and the tangent angle, θ, we have $\partial_\theta^2 = \partial_\mu^2$. Let H_t denote the entropy of the curve $\varphi_t(\mathbb{S}^1)$, with $t \in [0, T^*)$, where T^* is the value of t at which the curve $\varphi_{T}^{*}(\mathbb{S}^1)$ becomes a circle. Differentiating and using (7.7) yields

$$\begin{aligned}
\frac{\mathrm{d}H_t}{\mathrm{d}t} &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial_t\kappa_t(\theta)}{\kappa_t(\theta)}\,\mathrm{d}\theta = -\frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{1}{2}\kappa_t(\partial_\mu^2\kappa_t + \kappa_t) - \sigma(t)\Big]\,\mathrm{d}\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\kappa_t\big(\partial_\theta^2\kappa_t + \kappa_t\big)\,\mathrm{d}\theta + \sigma(t).
\end{aligned}$$

Differentiating again, yields

$$\begin{aligned}
\frac{\mathrm{d}^2H_t}{\mathrm{d}t^2} &= \sigma'(t) + \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{1}{2}\partial_t\kappa_t\Big(\partial_\theta^2\kappa_t + \kappa_t\Big) + \frac{1}{2}\kappa_t(\partial_t\partial_\theta^2\kappa_t + \partial_t\kappa_t)\Big]\,\mathrm{d}\theta \\
&= \sigma'(t) + \frac{1}{2\pi}\int_0^{2\pi}\Big(\partial_t\kappa_t\,\kappa_t + \partial_\theta^2\kappa_t\partial_t\kappa\Big)\,\mathrm{d}\theta \\
&= \sigma'(t) + \frac{1}{2\pi}\int_0^{2\pi}\partial_t\kappa_t\Big(\kappa_t + \partial_\theta^2\kappa_t\Big)\,\mathrm{d}\theta \\
&= \sigma'(t) - \frac{1}{2\pi}\int_0^{2\pi} 2\partial_t\kappa_t\frac{\partial_t\kappa_t + \sigma(t)\kappa_t}{\kappa_t^2}\,\mathrm{d}\theta \\
&= \sigma'(t) - \frac{1}{\pi}\int_0^{2\pi}\Big(\frac{\partial_t\kappa_t}{\kappa_t}\Big)^2\,\mathrm{d}\theta - \frac{\sigma(t)}{\pi}\int_0^{2\pi}\frac{\partial_t\kappa_t}{\kappa_t}\,\mathrm{d}\theta \\
&\le \sigma'(t) - 2\Big(\frac{\mathrm{d}H_t}{\mathrm{d}t}\Big)^2 + 2\sigma(t)\frac{\mathrm{d}H_t}{\mathrm{d}t}.
\end{aligned}$$

We note that in the last inequality we have used Jensen's integral inequality $\Big(\frac{1}{b-a}\int_a^b f\,\mathrm{d}\theta\Big)^2 \le \frac{1}{b-a}\int_a^b f^2\,\mathrm{d}\theta$. Then the variation of the entropy, $y(t) = \dfrac{\mathrm{d}H_t}{\mathrm{d}t}$, satisfies the inequality

$$y' \le \sigma'(t) - 2y^2 + 2\sigma(t)y,$$

with $\sigma'(t) < 0$ and $\sigma(t) > \frac{1}{2}$. In particular, we have the strict inequality

$$y' - 2\sigma(t)y < -2y^2.$$

Assume the function y is negative at some time $y(t_0) = -\frac{1}{a^2} < 0$. If $z = y^{-1}$, the previous inequality becomes $z' + 2\sigma z > 2$, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{2\rho(t)} z(t)\Big) > 2e^t$$

where we used that $2\rho(t) = 2\int_0^t \sigma > t$. Integrating between t_0 and t yields

$$z(t) > e^{-2\rho(t)}\Big[2e^t - (2e^{t_0} + a^2 e^{2\rho(t_0)})\Big]. \tag{9.1}$$

Since $z(t_0) = -a^2 < 0$ and the right side of (9.1) is positive for

$$t > t_1 = t_0 + \ln\Big[1 + \frac{a^2}{2} e^{2\rho(t_0) - t_0}\Big] > t_0,$$

it follows that there is a $t_2 \in (t_0, t_1]$ such that $\lim\limits_{t \to t_2-} z(t) = 0^-$. This implies that $\lim\limits_{t \to t_2-} y(t) = -\infty$. Hence, H_t is singular at $t = t_2$, contradiction. It follows that $y(t)$ must be positive all the time, *i.e.* the entropy H_t is increasing for any $t \in [0, T)$. The maximum value is reached at $t = T$ and it is $\ln(2\pi)$, see Theorem 1. $\square$

COROLLARY 1. *Let H_0 denote the entropy of φ_0. Then for any constant c, with $H_0 < c < 2\pi$, there is a unique $t > 0$ such that $H_t = c$.*

10. CROSS ENTROPY OF OVALS

Divergence functions, called also contrast functions, are distance-like quantities which measure the asymmetric difference of two probability density functions on a statistical manifold, see [5]. The following definition is an analog of the Kullback-Leibler divergence in the case of oval curves. Two ovals can be distinguished using the Kullback-Leibler divergence of the associated density functions.

Definition 2. Let $c, \gamma : [0, \tau] \to \mathbb{R}^2$ be two oval curves. The relative entropy of c with respect to γ is

$$H(c, \gamma) = \int_0^\tau p_c(s) \ln \frac{p_c(s)}{p_\gamma(s)}\, \mathrm{d}s = \frac{1}{2\pi}\int_0^\tau \kappa_c(s) \ln \frac{\kappa_c(s)}{\kappa_\gamma(s)}\, \mathrm{d}s,$$

where κ_c, κ_γ and p_c, p_γ are the curvatures and the density functions associated with the oval curves $c(s)$ and $\gamma(s)$.

LEMMA 1. *Let c and γ be two oval curves.*
(i) *We have $H(c, \gamma) \geq 0$.*
(ii) *$H(c, \gamma) = 0$ if and only if the ovals c and γ are the same curve, up to a rigid motion.*

Proof. (i) This part follows from the inequality $\ln x \le x - 1$ and Fenchel's theorem as follows:

$$\begin{aligned} H(c,\gamma) &= -\frac{1}{2\pi}\int_0^{\tau} \kappa_c \ln\frac{\kappa_\gamma}{\kappa_c} \ge -\frac{1}{2\pi}\int_0^{\tau} \kappa_c\Big(\frac{\kappa_\gamma}{\kappa_c} - 1\Big) \\ &= -\frac{1}{2\pi}\Big(\int_0^{\tau} \kappa_\gamma - \int_0^{\tau} \kappa_c\Big) = 0. \end{aligned}$$

(ii) $H(c,\gamma) = 0$ when $\kappa_c(s) = \kappa_\gamma(s)$. From the fundamental theorem of differential geometry, see [11], the curves c and γ must be the same, up to a rigid motion. $\square$

Consider a deformation φ_t, $t \in [0,T]$ and consider the division $0 < t_1 < \cdots < t_n = T$, with $t_k = k\Delta t$, $\Delta t = T/n$. The relative entropy of the curve $\varphi_{t_i+\Delta t}$ with respect to the neighboring curve, φ_{t_i}, is given by $H(\varphi_{t_i+\Delta t}, \varphi_{t_i})$, and the amount of information changed during the time interval Δt is

$$H(\varphi_{t_i+\Delta t}, \varphi_{t_i})\Delta t.$$

The *total amount of information* associated with the deformation φ_t and the division $0 < t_1 < \cdots < t_n = T$ is the sum of all partial deformations

$$I(\varphi_t; t_0, t_1, \cdots, t_n) = \sum_{i=1}^{n} H(\varphi_{t_{i+1}}, \varphi_{t_i})\Delta t.$$

The information associated with the deformation is given by the following limit

$$I(\varphi_t) = \lim_{n\to\infty} \sum_{i=1}^{n} H(\varphi_{t_i+\Delta t}, \varphi_{t_i})\Delta t.$$

We end this paper with the following open problem with applications to computer graphics:

For which deformation φ_t is the information $I(\varphi_t)$ minimum?

REFERENCES

[1] F. Al-Azemi and O. Calin, *Isometric elastic deformations.* BGS Proc. **21** (2014), 1–8.

[2] O. Calin and D.C. Chang, *Sub-Riemannian Geometry, General Theory and Examples.* Encyclopedia of Mathematics and Its Applications **126**, Cambridge Univ. Press, Cambridge-New York, 2009.

[3] O. Calin and D.C. Chang, *Geometric Mechanics on Riemannian Manifolds: Applications to Partial Differential Equations.* Applied and Numerical Analysis. Birhäuser, Boston, 2004.

[4] O. Calin, D.C. Chang, K. Furutani and C. Iwasaki, *Heat Kernels for Elliptic and Sub-elliptic Operators, Methods and Techniques.* Applied and Numerical Analysis. Birhäuser, Boston, 2010.

[5] O. Calin and C. Udriste, *Geometric Modeling in Probability and Statistics.* Springer, New York, 2014.

[6] M. Giaquinta and S. Hildebrand, *Calculus of Variations,* Vol. I. A Series of Comprehensive Studies in Mathematics **310**, Springer, New York.

[7] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity.* Dover, New York, 1944.

[8] D.F. Lawden, *Elliptic Functions and Applications.* Applied Mathematical Sciences **80**, Springer-Verlag, New-York, 1989.

[9] W.H. Miller, *The Elements of Hydrostatics and Hydrodynamics.* H. & J.J. Deighton, 1931.

[10] X.P. Zhu, *Lectures on Mean Curvature Flows.* St. Adv. Math., AMS/IP, 2003.

[11] R.S. Millman and G.D. Parker, *Elements of Differential Geometry.* Prentice Hall, Inc., 1977.

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