TERNARY GENERALIZATIONS OF GRADED ALGEBRAS WITH SOME PHYSICAL APPLICATIONS

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We discuss cubic and ternary algebras which are a direct generalization of Grassmann and Clifford algebras, but with Z_3-grading replacing the usual Z_2-grading.

Elementary properties and structures of such algebras are discussed, with special interest in low-dimensional ones, with two or three generators.

Invariant antisymmetric quadratic and cubic forms on such algebras are introduced, and it is shown how the $SL(2,C)$ group arises naturally in the case of lowest dimension, with two generators only, as the symmetry group preserving these forms.

We also show how the calculus of differential forms can be extended to include also second differentials d^2x^i, and how the Z_3 grading naturally appears when we assume that $\mathrm{d}^3 = 0$ instead of $\mathrm{d}^2 = 0$.

Ternary analogue of the commutator is introduced, and its relation with usual Lie algebras investigated, as well as its invariance properties.

We shall also discuss certain physical applications In particular, Z_3-graded gauge theory is briefly presented, as well as ternary generalization of Pauli's exclusion principle and ternary Dirac equation for quarks.

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1. INTRODUCTION

Of all symmetry groups characterizing physical phenomena and their mathematical models, the discrete groups seem to be the most fundamental. Among those, the simplest discrete group Z_2 is omnipresent and plays a crucial role in fundamental interactions between elementary particles and fields. All theoretical models of elementary interactions are checked by their response to the three representations of the Z_2 group, called "C" (charge conjugation, reflecting the symmetry between particles and anti-particles), "P" (parity, consisting in space reflection) and "T" (time reversal).

Although in some situations parity or time reversal may be broken, all known phenomena are invariant under the simultaneous application of all these idempotents. This is often referred to as the "CPT"-theorem in elementary particle physics.

Another important manifestation of Z_2 symmetry in physics is the distinction between bosons and fermions, which in the language of quantum field theory corresponds to commutators (for bosons) or anti-commutators (for fermions) in the constitutive relations between the creation and annihilation operators:

$$a_i^\dagger a_k - a_k a_i^\dagger = \delta_{ik}, \quad a_i^\dagger a_k + a_k a_i^\dagger = \delta_{ik}. \tag{1}$$

for the Bose-Einstein or Fermi-Dirac statistics, respectively.

What we have here is an example of two distinct representations of the Z_2 symmetry group, the trivial one in the case of bosons, and the faithful one in the case of fermions. Let us analyze the structure of all possible representations of Z_2 in the complex plane.

All *bilinear* mappings of vector spaces into complex numbers can be divided into irreducible symmetry classes according to the representations of the Z_2 group, *e.g.* symmetric, anti-symmetric, hermitian, or anti-hermitian:

i) The trivial representation defines the symmetric 2-valenced tensors:

$$S_{\pi(AB)} = S_{BA} = S_{AB},$$

ii) The sign reversal defines the anti-symmetric tensors:

$$A_{\pi(CD)} = A_{DC} = -A_{CD},$$

iii) The complex conjugation defines the hermitian tensors:

$$H_{\pi(AB)} = H_{BA} = \bar{H}_{AB},$$

iv) $(-1)\times$ complex conjugation defines the anti-hermitian tensors.

$$T_{\pi(AB)} = T_{BA} = -\bar{T}_{AB},$$

Similarly, all tri-linear mappings can be distinguished by their symmetry properties with respect to the permutations belonging to the Z_3 symmetry group.

There are several different representations of the action of the Z_3 cyclic permutation group on tensors with three indices. Consequently, such tensors can be divided into irreducible subspaces which are conserved under the action of the cyclic group Z_3. Let us remind that also the action of the full S_3 permutation group containing *six* elements, is possible on any set of three items, but for the grading, which has to be additive, only its Z_3 subgroup is necessary.

There are three possibilities of an action of Z_3 being represented in the complex plane by multiplication by a complex number:

– the trivial one (multiplication by 1), and the two other representations,

$$\text{the multiplication by } j = e^{2\pi i/3}$$

$$\text{or by its complex conjugate } j^2 = \bar{j} = e^{4\pi i/3}.$$

Now we can introduce the following three irreducible subspaces of the linear space of 3-forms:

$$T \in \mathcal{T}: \quad T_{ABC} = T_{BCA} = T_{CAB}, \tag{2}$$

$$\Lambda \in \mathcal{L}: \quad \Lambda_{ABC} = j\,\Lambda_{BCA} = j^2\,\Lambda_{CAB}, \tag{3}$$

$$\bar{\Lambda} \in \bar{\mathcal{L}}: \quad \bar{\Lambda}_{ABC} = j^2\,\bar{\Lambda}_{BCA} = j\,\bar{\Lambda}_{CAB}, \tag{4}$$

which can be called, respectively, totally symmetric, j-skew-symmetric and j^2-skew-symmetric.

Thus the space of all tri-linear forms is the sum of three irreducible subspaces,

$$\Theta_3 = \mathcal{T} \oplus \mathcal{L} \oplus \bar{\mathcal{L}}$$

corresponding dimensions being, respectively, $(N^3 + 2N)/3$ for $\mathcal{T}$, $(N^3 - N)/3$ for $\mathcal{L}$ and for $\bar{\mathcal{L}}$.

Any three-form W^{α}_{ABC} mapping $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ into a vector space $\mathcal{X}$ of dimension k, $\alpha, \beta = 1, 2, ...k$, so that $X^{\alpha} = W^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C$ can be represented as a linear combination of forms with specific symmetry properties,

$$W^{\alpha}_{ABC} = T^{\alpha}_{ABC} + \Lambda^{\alpha}_{ABC} + \bar{\Lambda}^{\alpha}_{ABC},$$

with:

$$T^{\alpha}_{ABC} := \frac{1}{3}\,(W^{\alpha}_{ABC} + W^{\alpha}_{BCA} + W^{\alpha}_{CAB}), \tag{5}$$

$$\Lambda^{\alpha}_{ABC} := \frac{1}{3}\,(W^{\alpha}_{ABC} + j\,W^{\alpha}_{BCA} + j^2\,W^{\alpha}_{CAB}), \tag{6}$$

$$\bar{\Lambda}^{\alpha}_{ABC} := \frac{1}{3}\,(W^{\alpha}_{ABC} + j^2\,W^{\alpha}_{BCA} + j\,W^{\alpha}_{CAB}). \tag{7}$$

As in the Z_2 case, the three symmetries above define irreducible 3-forms.

Consequently, two different cubic commutation relations can be imposed on an associative algebra, say Λ-type and $\bar{\Lambda}$-type: for any three elements a, b, c belonging to the algebra $\mathcal{A}_{\Lambda}$ we shall have

$$abc = j\;bca = j^2\;cab,$$

and for any three elements $\bar{a}, \bar{b}, \bar{c}$ belonging to algebra $\bar{\mathcal{A}}_{\bar{\Lambda}}$ we shall have

$$\bar{a}\bar{b}\bar{c} = j^2\;\bar{b}\bar{c}\bar{a} = j\;\bar{c}\bar{a}\bar{b}.$$

The Z_2-grading of ordinary (binary) algebras is well known and widely studied and applied (*e.g.* in the super-symmetric field theories in Physics [1, 2]).

2. TERNARY ALGEBRAS

The Grassmann and Clifford algebras are perhaps the oldest and the best known examples of a Z_2-graded structure. Other gradings are much less popular. The Z_3-grading was introduced and studied in early nineties [3]; the Z_N grading was discussed in papers by M. Dubois-Violette [4].

An approach to ternary Clifford algebra based on ternary triples and a successive process of ternary Galois extensions is proposed in [5]. More general case of N-algebras, in which only the product of N elements is defined, was studied in L. Vainerman, R. Kerner [6].

The usual definition of an algebra involves a linear space $\mathcal{A}$ (over real or complex numbers) endowed with a *binary* constitutive relations:

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}. \tag{8}$$

In a finite dimensional case, dim $\mathcal{A}$ = N, in a chosen basis $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N$, the constitutive relations (8) can be encoded in *structure constants* c^k_{ij} as follows:

$$\mathbf{e}_i \mathbf{e}_j = c^k_{ij}\, \mathbf{e}_k. \tag{9}$$

With the help of these structure constants all essential properties of a given algebra can be expressed, *e.g.* they will define a *Lie algebra* if they are antisymmetric and satisfy the Jacobi identity:

$$c^k_{ij} = -c^k_{ji}, \quad c^k_{im} c^m_{jl} + c^k_{jm} c^m_{li} + c^k_{lm} c^m_{ij} = 0, \tag{10}$$

whereas an abelian algebra will have its structure constants symmetric, $c^k_{ij} = c^k_{ji}$.

In what follows, we shall be concerned exclusively with *ternary* algebras, defined via triple product mapping $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ onto $\mathcal{A}$:

$$X, Y, Z \in \mathcal{A} \ \to \{X, Y, Z\} \in \mathcal{A}. \tag{11}$$

In a chosen basis of n linearly independent vectors $e_k \in \mathcal{A}$, $\ k, l = 1, 2, 3$, the 3-product is defined via *ternary structure constants* f^i_{klm}:

$$\{e_i, e_j, e_k\} = f^m_{ijk}\, e_m, \quad i, j, k, m = 1, 2, 3. \tag{12}$$

Obviously enough, given any classical associative algebra with *binary* multiplication law $X, Y \in \mathcal{A} \to X \cdot Y \in \mathcal{A}$, one can easily introduce ternary multiplication law by simple iteration:

$$\{X, Y, Z\} := X(Y \cdot Z) = (X \cdot Y)Z = XYZ. \tag{13}$$

In such a case, ternary structure constants can be expressed by means of usual (binary) structure constants of the associative algebra $\mathcal{A}$:

$$\{e_i, e_k, e_l\} = f^m_{ikl}\, e_m = (e_i e_k) e_l = c^j_{ik} e_j e_l = c^j_{ik} c^m_{jl} e_m, \tag{14}$$

from which we infer that $f^m_{ikl} = c^j_{ik} c^m_{jl}$.

On the other hand, due to the associativity of algebra $\mathcal{A}$, the same ternary product can be represented alternatively as

$$\{e_i, e_k, e_l\} = f^m_{ikl}\, e_m = e_i(e_k e_l) = e_i(c^j_{kl} e_j) = c^j_{kl}(e_i e_j) = c^j_{kl} c^m_{ij}\, e_m,$$

Therefore in the case when ternary multiplication law is naturally induced by an associative binary product, the resulting ternary structure constants must satisfy an obvious symmetry constraint:

$$f^m_{ikl} = c^j_{ik} c^m_{jl} = c^j_{kl} c^m_{ij}. \tag{15}$$

Usually, when we speak of algebras, we mean *binary algebras*, understanding that they are defined via *quadratic* constitutive relations (9). On such algebras the notion of Z_2-grading can be naturally introduced. An algebra $\mathcal{A}$ is called a Z_2-graded algebra if it is a direct sum of two parts, with symmetric (abelian) and anti-symmetric product respectively,

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1, \tag{16}$$

with *grade* of an element being 0 if it belongs to $\mathcal{A}_0$, and 1 if it belongs to $\mathcal{A}_1$.

Under the multiplication in a Z_2-graded algebra the grades add up reproducing the composition law of the Z_2 permutation group: if the grade of an element A is a, and that of the element B is b, then the grade of their product will be $a+b$ *modulo 2*:

$$\text{grade}(AB) = \text{grade}(A) + \text{grade}(B). \tag{17}$$

A Z_2-graded algebra is called a Z_2-graded commutative if for any two homogeneous elements A, B we have

$$AB = (-1)^{ab} BA. \tag{18}$$

It is worthwhile to note that the above relationship can be written in an alternative form, with all the expressions on the left side as follows:

$$AB - (-1)^{a\,b} BA = 0, \quad \text{or} \quad AB + (-1)^{(a\,b+1)} BA = 0 \tag{19}$$

The equivalence between these two alternative definitions of commutation (anticommutation) relations inside a Z_2-graded algebra is no more possible if by analogy we want to impose *cubic* relations on algebras with Z_3-symmetry properties, in which the cubic root of unity, $j = e^{\frac{2\pi i}{3}}$ plays the role similar to that of -1 in binary relations displaying a Z_2-symmetry [3].

The Z_3 cyclic group is an abelian subgroup of the S_3 symmetry group of permutations of three objects. The S_3 group contains *six* elements, including the group unit e (the identity permutation, leaving all objects in place: $(abc) \rightarrow (abc)$), the two cyclic permutations

$$(abc) \rightarrow (bca) \quad \text{and} \quad (abc) \rightarrow (cab),$$

and three odd permutations,

$$(abc) \to (cba), \quad (abc) \to (bac) \quad \text{and} \quad (abc) \to (acb).$$

There was a unique definition of *commutative* binary algebras given in two equivalent forms,

$$xy + (-1)yx = 0 \quad \text{or} \quad xy = yx. \tag{20}$$

In the case of cubic algebras we have the following four generalizations of the notion of *commutative* algebras:

a) Generalizing the first form of the commutativity relation (20), which amounts to replacing the -1 generator of Z_2 by j-generator of Z_3 and binary products by products of three elements, we get

$$S: \;\; x^\mu x^\nu x^\lambda + j\, x^\nu x^\lambda x^\mu + j^2\, x^\lambda x^\mu x^\nu = 0, \tag{21}$$

where $j = e^{\frac{2\pi i}{3}}$ is the primitive third root of unity.

b) Another primitive third root, $j^2 = e^{\frac{4\pi i}{3}}$ can be used in place of the former one; this will define the conjugate algebra $\bar{S}$, satisfying the following cubic constitutive relations:

$$\bar{S}: \;\; x^\mu x^\nu x^\lambda + j^2\, x^\nu x^\lambda x^\mu + j\, x^\lambda x^\mu x^\nu = 0. \tag{22}$$

Both algebras are infinitely-dimensional and have the same structure. Each of them is a possible generalization of infinitely-dimensional algebra of usual commuting variables with a finite number of generators. In the usual Z_2-graded case such algebras are just polynomials in variables $x^1, x^2, \ldots, x^N$; the algebras S and $\bar{S}$ defined above are also spanned by polynomials, but with different symmetry properties, and as a consequence, with different dimensions corresponding to a given power.

c) Then we can impose the following "weak" commutation, valid only for cyclic permutations of factors:

$$S_1: \;\; x^\mu x^\nu x^\lambda = x^\nu x^\lambda x^\mu \neq x^\nu x^\mu x^\lambda, \tag{23}$$

d) Finally, we can impose the following "strong" commutation, valid for arbitrary (even or odd) permutations of three factors:

$$S_0: \;\; x^\mu x^\nu x^\lambda = x^\nu x^\lambda x^\mu = x^\nu x^\mu x^\lambda \tag{24}$$

Let us turn now to the Z_3 generalization of anti-commuting generators, which in the usual homogeneous case with Z_2-grading define Grassmann algebras. Here, too, we have four different choices:

a) The "strong" cubic anti-commutation,

$$\mathcal{L}_0: \quad \Sigma_{\pi \in S_3}\, \theta^{\pi(A)} \theta^{\pi(B)} \theta^{\pi(C)} = 0, \tag{25}$$

i.e. the sum of *all* permutations of three factors, even and odd ones, must vanish.

b) The somewhat weaker "cyclic" anti-commutation relation,

$$\mathcal{L}_1 : \quad \theta^A \theta^B \theta^C + \theta^B \theta^C \theta^A + \theta^C \theta^A \theta^B = 0, \tag{26}$$

i.e. the sum of *cyclic* permutations of three elements must vanish. The same independent relation for the odd combination $\theta^C \theta^B \theta^A$ holds separately.

c) The j-skew-symmetric algebra:

$$\mathcal{L} : \theta^A \theta^B \theta^C = j \, \theta^B \theta^C \theta^A. \tag{27}$$

and its conjugate algebra $\bar{\mathcal{L}}$, isomorphic with $\mathcal{L}$, which we distinguish by putting a bar on the generators and using dotted indices:

d) The j^2-skew-symmetric algebra:

$$\bar{\mathcal{L}} : \quad \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \tag{28}$$

Both these algebras are finite dimensional. For j or j^2-skew-symmetric algebras with N generators the dimensions of their subspaces of given polynomial order are given by the following generating function:

$$H(t) = 1 + Nt + N^2 t^2 + \frac{N(N-1)(N+1)}{3} \, t^3, \tag{29}$$

where we include pure numbers (dimension 1), the N generators θ^A (or $\bar{\theta}^{\dot{B}}$), the N^2 independent quadratic combinations $\theta^A \theta^B$ and $N(N-1)(N+1)/3$ products of three generators $\theta^A \theta^B \theta^C$.

3. Z_3-GRADED GRASSMAN

Let us consider N generators spanning a linear space over complex numbers, satisfying the following cubic relations [3, 13]:

$$\theta^A \theta^B \theta^C = j \, \theta^B \theta^C \theta^A = j^2 \, \theta^C \theta^A \theta^B, \tag{30}$$

with $j = e^{2i\pi/3}$, the primitive root of 1. We have $1 + j + j^2 = 0$ and $\bar{j} = j^2$. It is worth mentioning that there are no relations between binary products $\theta^A \theta^B$, *i.e.* all these products are linearly independent. Let us denote the algebra spanned by the θ^A generators by $\mathcal{A}$.

We shall also introduce a similar set of *conjugate* generators, $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, ... = 1, 2, ..., N$, satisfying similar condition with j^2 replacing j:

$$\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \, \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} = j \, \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}, \tag{31}$$

Let us denote this algebra by $\bar{\mathcal{A}}$.

We shall endow the algebra $\mathcal{A} \oplus \bar{\mathcal{A}}$ with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2. The grades add up modulo 3, so that the products $\theta^A \theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A \theta^B \theta^C$ are of grade 0.

Similarly, all quadratic expressions in conjugate generators, $\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}$ are of grade $2 + 2 = 4 \pmod 3 = 1$, whereas their cubic products are again of grade 0, like the cubic products of θ^A's [7].

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the Z_3-graded generators.

As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta^A \theta^B \theta^C \theta^D = j\, \theta^B \theta^C \theta^A \theta^D = j^2\, \theta^B \theta^A \theta^D \theta^C =$$

$$= j^3\, \theta^A \theta^D \theta^B \theta^C = j^4\, \theta^A \theta^B \theta^C \theta^D, \tag{32}$$

and because $j^4 = j \neq 1$, the only solution is $\theta^A \theta^B \theta^C \theta^D = 0$.

Under associative multiplication the grade of the resulting element is the sum of the grades of two factors modulo 3. Let us form a vector represented by a column with entries ordered by their Z_3 grades:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Consider a 3×3 matrix acting on such a vector, with all entries of defined Z_3 grade.

In particular, we can form matrices which conserve the grades in the column vector, or raise the grade of each component by 1, or by 2. Such matrices can be called *grade raising operators*, of grades $0, 1$ or 2, respectively.

The three matrices acting as grade raising operators should have their entries graded as follows:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

If we restrain matrices to have elements of grade 0 exclusively, we get the following three types, which we shall denote symbolically by B, Q and $Q^\dagger$:

$$B \simeq \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad Q \simeq \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \quad Q^\dagger \simeq \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix},$$

Natural Z_3 grading can be attributed to the three types of matrices above: the diagonal ones (of B-type) are of Z_3-grade 0, the off-diagonal of Q-type are of Z_3-grade 1, and the off-diagonal ones of type $Q^\dagger$ are of Z_3-grade 2. It is easy to check that so attributed Z_3 grades add up modulo three under matrix multiplication. As a matter of fact, multiplying by any diagonal matirx of type B does not change the form of any Q or $Q^\dagger$ type matrices, thus keeping their grade unchanged, $0+1=1, \quad 0+2=2$. A product of two Q-type matrices produces a matrix of $Q^\dagger$ type, and the grades add up $1+1=2$; a product of two matrices of $Q^\dagger$ type produces a matrix of Q type, according to the grade addition $2+2=4$, but $4\,|_{mod 3}=1$. Finally, a product of a Q-matrix with a $Q^\dagger$-matrix is a diagonal 3×3 matrix of B-type, according to $1+2=3$, and $3\,|_{mod 3}=0$. Now we can proceed further and show how a special subset of the above 3×3 matrices spans a ternary generalization of Clifford algebras.

4. TERNARY CLIFFORD

Let us introduce the following three 3×3 matrices:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{33}$$

and their hermitian conjugates

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \quad Q_2^\dagger = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \quad Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{34}$$

These matrices can be endowed with natural Z_3-grading,

$$\text{grade}(Q_k) = 1, \quad \text{grade}(Q_k^\dagger) = 2, \tag{35}$$

The above matrices span a very interesting ternary algebra. Out of three independent Z_3-graded ternary combinations, only one leads to a non-vanishing result. One can check without much effort that both j and j^2 skew ternary commutators do vanish:

$$\{Q_1, Q_2, Q_3\}_j = Q_1Q_2Q_3 + jQ_2Q_3Q_1 + j^2Q_3Q_1Q_2 = 0,$$
$$\{Q_1, Q_2, Q_3\}_{j^2} = Q_1Q_2Q_3 + j^2Q_2Q_3Q_1 + jQ_3Q_1Q_2 = 0,$$

and similarly for the odd permutation, $Q_2Q_1Q_3$.

On the contrary, the totally symmetric combination does not vanish; it is proportional to the 3×3 identity matrix $\mathbf{1}$:

$$Q_aQ_bQ_c + Q_bQ_cQ_a + Q_cQ_aQ_b = 3\,\eta_{abc}\,\mathbf{1}, \quad a, b, ... = 1, 2, 3. \tag{36}$$

with η_{abc} given by the following non-zero components:

$$\eta_{111} = \eta_{222} = \eta_{333} = 1, \quad \eta_{123} = \eta_{231} = \eta_{312} = 1,$$

$$\eta_{213} = \eta_{321} = \eta_{132} = j^2. \tag{37}$$

all other components vanishing.

The above relation may serve as the definition of *ternary Clifford algebra*, as introduced in [5, 17].

Another set of three matrices formed by the hermitian conjugates of Q_a, which we shall endow with dotted indices $\dot{a}, \dot{b}, ... = 1, 2, 3$:

$$Q^{\dagger}_{\dot{a}} = \bar{Q}^T_a$$

satisfies the conjugate identities

$$Q^{\dagger}_{\dot{a}} Q^{\dagger}_{\dot{b}} Q^{\dagger}_{\dot{c}} + Q^{\dagger}_{\dot{b}} Q^{\dagger}_{\dot{c}} Q^{\dagger}_{\dot{a}} + Q^{\dagger}_{\dot{c}} Q^{\dagger}_{\dot{a}} Q^{\dagger}_{\dot{b}} = 3\, \eta_{\dot{a}\dot{b}\dot{c}}\, \mathbf{1}, \quad \dot{a}, \dot{b}, ... = 1, 2, 3. \tag{38}$$

with $\eta_{\dot{a}\dot{b}\dot{c}} = \bar{\eta}_{cba}$.

It is obvious that any similarity transformation of the generators Q_a will keep the ternary anti-commutator (36) invariant. As a matter of fact, if we define $\tilde{Q}_b = P^{-1} Q_b P$, with P a non-singular 3×3 matrix, the new set of generators will satisfy the same ternary relations, because

$$\tilde{Q}_a \tilde{Q}_b \tilde{Q}_c = P^{-1} Q_a P P^{-1} Q_b P P^{-1} Q_c P = P^{-1} (Q_a Q_b Q_c) P,$$

and on the right-hand side we have the unit matrix which commutes with all other matrices, so that $P^{-1}\, \mathbf{1}\, P = \mathbf{1}$.

It is also worthwhile to note that the six matrices displayed in (33), (34) together with two traceless diagonal matrices

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}$$

form the basis for certain representation of the SU(3), which was shown in the nineties by V. Kac in 1994 [16].

We shall endow the two diagonal matrices B and $B^{\dagger} = B^2$ with the Z_3 grade 0, the three matrices Q_a with grade 1, and their three hermitian conjugates $\bar{Q}_{\dot{b}}$ with Z_3 grade 2. Under matrix multiplication the grades add up modulo 3.

Let us introduce in the matrix algebra spanned by $B,\ B^{\dagger},\ Q_a$ and $\bar{Q}_{\dot{b}}$ the following Z_3-graded commutator:

$$[B,\ C]_{Z_3} = BC - j^{bc}\, CB, \ \text{ where } \ b = \mathrm{grad}(B), \ c = \mathrm{grad}(C). \tag{39}$$

Let us choose a matrix of grade 1 such that its cube is equal to the unit matrix, *e.g.*

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

and let us define *differentiation* of our matrix algebra as follows:

$$\mathrm{d}B = [\eta, B]_{Z_3} = \eta B - j^b \, B\eta. \tag{40}$$

One can easily show that $\mathrm{d}^3 = 0$ on any element $B \in \mathcal{A}$.

$$\mathrm{d}B = \eta B - j^b B\eta; \tag{41}$$

$$\begin{aligned} \mathrm{d}^2 B = \mathrm{d}(\eta B - j^b B\eta) = \eta(\eta B - j^b B\eta) - j^{b+1}(\eta B - j^b B\eta)\eta = \\ = \eta^2 - j^b \, \eta B\eta - j^{b+1} \, \eta B\eta + j^{2b+1} \, B\eta^2; \end{aligned} \tag{42}$$

Finally, (we skip the intermediary calculations)

$$\begin{aligned} \mathrm{d}^3 B = \eta^3 B - (j^b + j^{b+1} + j^{b+2}) \, \eta^2 B + \\ (j^{2b+1} + j^{2b+2} + j^{2b+3}) \, \eta B\eta^2 - B\eta^3 = 0, \end{aligned} \tag{43}$$

because $\eta^3 = \mathbf{1}$ and commutes with B, and because

$$(j^b + j^{b+1} + j^{b+2}) = j^b(1 + j + j^2) = 0,$$

and

$$(j^{2b+1} + j^{2b+2} + j^{2b+3}) = j^{2b}(j + j^2 + j^3) = 0.$$

5. Z_3-GRADED DIFFERENTIALS

Instead of the usual exterior differential operator satisfying $\mathrm{d}^2 = 0$, let us postulate its Z_3-graded generalization satisfying

$$\mathrm{d}^2 \neq 0, \quad \mathrm{d}^3 f = 0 \tag{44}$$

The first differential of a smooth function $f(x^i)$ is as usual $\mathrm{d}f = \partial_i f \, \mathrm{d}x^i$, whereas the second differential is formally

$$\mathrm{d}^2 f = (\partial_k \partial_i f) \, \mathrm{d}x^k \mathrm{d}x^i + (\partial_i f) \, \mathrm{d}^2 x^i \tag{45}$$

We shall attribute the grade 1 to the 1-forms $\mathrm{d}x^i$, $(i, j, k = 1, 2, ...N)$, and grade 2 to the forms $\mathrm{d}^2 x^i$, $(i, j, k = 1, 2, ...N)$; under associative multiplication of these forms the grades add up *modulo* 3

$$\mathrm{grade}(\omega \, \theta) = [\mathrm{grade}(\omega) + \mathrm{grade}(\theta)] \; (\mathit{modulo} \, 3). \tag{46}$$

The Z_3-graded differential operator d has the following property, compatible with grading we have chosen:

$$\mathrm{d}(\omega \, \theta) = (\mathrm{d}\omega) \, \theta + j^{\mathrm{grade}_\omega} \, \omega \, \mathrm{d}\theta. \tag{47}$$

We have:

$$\mathrm{d}^2 f = (\partial_i \partial_k f) \mathrm{d}x^i \mathrm{d}x^k + (\partial_i f) \, \mathrm{d}^2 x^i, \tag{48}$$

$$\begin{aligned} \mathrm{d}^3 f &= (\partial_m \partial_i \partial_k f)\mathrm{d}x^m \mathrm{d}x^i \mathrm{d}x^k + (\partial_i \partial_k f)\mathrm{d}^2 x^i \mathrm{d}x^k \\ &+ j\, (\partial_i \partial_k f)\mathrm{d}x^i \mathrm{d}^2 x^k + (\partial_k \partial_i f)\mathrm{d}x^k \mathrm{d}^2 x^i + (\partial_i f)\, \mathrm{d}^3 x^i. \end{aligned} \tag{49}$$

equivalent with

$$\mathrm{d}^3 f = (\partial_m \partial_i \partial_k f)\mathrm{d}x^m \mathrm{d}x^i \mathrm{d}x^k + (\partial_i \partial_k f)[\mathrm{d}^2 x^k \mathrm{d}x^i - j^2\, \mathrm{d}x^i \mathrm{d}^2 x^k] + (\partial_i f). \tag{50}$$

(because of $\mathrm{d}^3 x^i = 0$).

Consequently, assuming that $\mathrm{d}^3 x^k = 0$ and $\mathrm{d}^3 f = 0$, to make the remaining terms vanish we must impose the following commutation relations on the products of forms:

$$\mathrm{d}x^i \mathrm{d}x^k \mathrm{d}x^m = j\, \mathrm{d}x^k \mathrm{d}x^m \mathrm{d}x^i, \qquad \mathrm{d}x^i \mathrm{d}^2 x^k = j\, \mathrm{d}^2 x^k \mathrm{d}x^i, \tag{51}$$

therefore

$$\mathrm{d}^2 x^k \mathrm{d}x^i = j^2\, \mathrm{d}x^i \mathrm{d}^2 x^k \tag{52}$$

As in the case of the abstract Z_3-graded Grassmann algebra, the fourth order expressions must vanish due to the associativity of the product:

$$\mathrm{d}x^i \mathrm{d}x^k \mathrm{d}x^l \mathrm{d}x^m = 0. \tag{53}$$

Consequently, we shall assume that also

$$\mathrm{d}^2 x^i \mathrm{d}^2 x^k = 0. \tag{54}$$

This completes the construction of algebra of Z_3-graded exterior forms (see, *e.g.* [4, 8, 9]), which were used in the construction of gauge fields satisfying higher order equations (see [10]).

Although the first differentials $\mathrm{d}x^i$ behave like tensors under coordinate-dependent transformations:

$$x^i \to y^{k'}(x^i), \quad \text{with } \det\left(\frac{\partial y^{k'}}{\partial x^i}\right) \neq 0,$$

produces new differentials which are linear combinations of previous ones:

$$\mathrm{d}y^{k'} = \left(\frac{\partial y^{k'}}{\partial x^i}\right)\, \mathrm{d}x^i,$$

The second differentials do not follow this rule. In fact, employing the Leibniz rule, we have

$$\mathrm{d}^2 y^{k'} = \left(\frac{\partial^2 y^{k'}}{\partial x^j \partial x^i}\right) dx^j \mathrm{d}x^i + \left(\frac{\partial y^{k'}}{\partial x^i}\right) \mathrm{d}^2 x^i.$$

We can ensure the tensorial transformation rule if we restrain the products $dx^j \mathrm{d}x^i$ to their anti-symmetric part,

$$dx^j \wedge \mathrm{d}x^i = \frac{1}{2}(dx^j \otimes \mathrm{d}x^i - \mathrm{d}x^i \otimes dx^j)$$

and impose the rule $\mathrm{d}^2 = 0$.

Let Φ be a module on which algebra $\mathcal{A}$ acts effectively and transitively.

A covariant differential can be introduced as follows. For $\varphi \in \Phi$,

$$\mathrm{D}\phi = \mathrm{d}\varphi + A\,\varphi$$

where A is a 1-form $A = A_i \mathrm{d}x^i$ with values in $\mathcal{A}$. Then one has

$$\begin{aligned}\mathrm{D}^2\varphi &= (\mathrm{d} + A)(\mathrm{d}\varphi + A\varphi) = \mathrm{d}^2\varphi + A\mathrm{d}\varphi + \mathrm{d}A\varphi + (-1)^1 A\mathrm{d}\varphi + AA\varphi \\ &= A\mathrm{d}\varphi - A\mathrm{d}\varphi + (\mathrm{d}A + AA)\varphi = (\mathrm{d}A + AA)\varphi = F\,\varphi.\end{aligned}$$

where the 2-form $F = \mathrm{d}A + AA$ is called the *curvature* of the *connection 1-form* A. The products are the wedge products of forms.

In local coordinates we have $A = A_i \mathrm{d}x^i$,

$$\mathrm{d}A = \partial_k A_i\ \mathrm{d}x^k \wedge \mathrm{d}x^i = \frac{1}{2}(\partial_k A_i - \partial_i A_k)\ \mathrm{d}x^k \wedge \mathrm{d}x^i, \tag{55}$$

because $\mathrm{d}^2x^i = 0$. Therefore the curvature tensor has the form

$$F = \frac{1}{2}F_{ik}\ \mathrm{d}x^i \wedge \mathrm{d}x^k = \frac{1}{2}\left[(\partial_i A_k - \partial_k A_i) + [A_i, A_k]\right]\ \mathrm{d}x^i \wedge \mathrm{d}x^k, \tag{56}$$

so that $F_{ik} = (\partial_i A_k - \partial_k A_i) + [A_i, A_k]$.

In the case of an abelian gauge group the coefficients of the connection form A_i are commutative, therefore only the part $F_{ik} = (\partial_i A_k - \partial_k A_i)$ does not vanish, which is the case of the usual electromagnetic field.

Let us now show how a similar covariant derivative shall behave if we replace the Z_2-graded differential calculus by its Z_3-graded conterpart, with $\mathrm{d}^3 = 0$, but $\mathrm{d}^2 \neq= 0$.

The Z_3-graded Leibniz rule for differential forms becomes now:

$$\mathrm{d}(\omega\theta) = \mathrm{d}\omega\theta + j^{|\omega|}\ \omega\mathrm{d}\theta \tag{57}$$

Therefore the exterior differential of the 1-form $A = A_i \mathrm{d}x^i$ is

$$\mathrm{d}A = \mathrm{d}(A_i \mathrm{d}x^i) = (\partial_k A_i)\mathrm{d}x^k \mathrm{d}x^i + A_i \mathrm{d}^2 x^i. \tag{58}$$

Continuing to apply the exterior differential operator we get:

$$\begin{gathered}\mathrm{d}^2 A = \mathrm{d}\left[(\partial_k A_i)\mathrm{d}x^k \mathrm{d}x^i + A_i \mathrm{d}^2 x^i\right] = \\ (\partial_m \partial_k A_i)\mathrm{d}x^m \mathrm{d}x^k \mathrm{d}x^i + \partial_k A_i \mathrm{d}^2 x^k \mathrm{d}x^i \\ +j\ \partial_k A_i \mathrm{d}x^k \mathrm{d}^2 x^i + \partial_k A_i \mathrm{d}x^k \mathrm{d}^2 x^i + A_i \mathrm{d}^3 x^i.\end{gathered}$$

The last term vanishes by virtue of $\mathrm{d}^3 = 0$ in the Z_3-graded exterior calculus. As $1 + j = -j^2$, and $\mathrm{d}x^k \mathrm{d}^2 x^i = j\ \mathrm{d}^2 x^i \mathrm{d}x^k$, after swapping mute indices i and k in last two terms, we get

$$\mathrm{d}^2 A = (\partial_m \partial_k A_i)\mathrm{d}x^m \mathrm{d}x^k \mathrm{d}x^i + (\partial_k A_i - \partial_i A_k)\mathrm{d}^2 x^k \mathrm{d}x^i. \tag{59}$$

Let us consider first the abelian case. Both terms are gauge invariant: the second one is the well known antisymmetric Maxwell tensor. If $\tilde{A}_i = A_i + \partial_i f$, then

$$\tilde{F}_{ik} = \partial_i \tilde{A}_k - \partial_k \tilde{A}_i = \partial_i A_k - \partial_k A_i = F_{ik},$$

because $\partial_i \partial_k f - \partial_k \partial_i f = 0$.

But also the first term is gauge invariant: it vanishes if $A_i = \partial_i f$:

$$(\partial_i \partial_k \partial_m f) \mathrm{d}x^i \mathrm{d}x^k \mathrm{d}x^m = 0$$

because $\mathrm{d}x^i \mathrm{d}x^k \mathrm{d}x^m = j\ \mathrm{d}x^k \mathrm{d}x^m \mathrm{d}x^i = j^2\ \mathrm{d}x^m \mathrm{d}x^i \mathrm{d}x^k$, and $1 + j + j^2 = 0$.

It is not difficult to recognize (still in the abelian case) the following invariant form of the expression

$$\mathrm{d}^2 A = (\partial_m \partial_k A_i) \mathrm{d}x^m \mathrm{d}x^k \mathrm{d}x^i + (\partial_k A_i - \partial_i A_k) \mathrm{d}^2 x^k \mathrm{d}x^i. \tag{60}$$

which can be also written as

$$-\frac{1}{6} \left[\partial_k F_{mi} + \partial_i F_{mk} \right] + \frac{i\sqrt{3}}{2} \left[\partial_k F_{mi} - \partial_i F_{mk} \right].$$

Without explicit proof (a bit lengthy), we extend this formula to the non-abelian case:

$$D^2 A = -\frac{1}{6} \left[D_k F_{mi} + D_i F_{mk} \right] + \frac{i\sqrt{3}}{2} \left[D_k F_{mi} - D_i F_{mk} \right].$$

Similar construction in the case of linear connection over a metric differential manifold leads to analogous expressions involving the Christoffel coefficients and the Riemann tensor.

Let us assume that differentials $\mathrm{d}x^i$ and $\mathrm{d}^2 x^k$ form the Z_3-graded ternary Grassmann algebra, with constitutive relations

$$\mathrm{d}x^i \mathrm{d}x^k \mathrm{d}x^m = j\ \mathrm{d}x^k \mathrm{d}x^m \mathrm{d}x^i, \quad \mathrm{d}x^i \mathrm{d}^2 x^k = j\ \mathrm{d}^2 x^k \mathrm{d}x^i, \quad \text{while } \mathrm{d}^3 x^3 = 0. \tag{61}$$

Suppose now that a linear connection is defined in a chosen coordinate system, given by its coefficients Γ^i_{km}, so that we can define *covariant* differentials as follows:

$$\mathrm{D}x^i = \mathrm{d}x^i, \quad \mathrm{D}^2 x^i = \mathrm{d}^2 x^i + \Gamma^i_{km} \mathrm{D}x^k \mathrm{D}x^m. \tag{62}$$

Due to the transformation properties of connection coefficients, under a change of local coordinates the second differentials behave now as tensors: if $y^{i'} = y^{i'}(x^k)$, then we have

$$\mathrm{D}y^{i'} = \frac{\partial y^{i'}}{\partial x^k} \mathrm{D}x^k, \quad \text{and} \quad \mathrm{D}^2 y^{i'} = \frac{\partial y^{i'}}{\partial x^k} \mathrm{D}^2 x^k, \tag{63}$$

Now it is easy to prove that due to the constitutive relations between the differentials (61) and the definition of $\mathrm{D}x^i$ and $\mathrm{D}^2 x^k$, we also have

(64)

$$\mathrm{D}x^i \mathrm{D}x^k \mathrm{D}x^m = j\ \mathrm{D}x^k \mathrm{D}x^m \mathrm{D}x^i, \quad \mathrm{D}x^i \mathrm{D}^2 x^k = j\ \mathrm{D}^2 x^k \mathrm{D}x^i, \quad \text{while } \mathrm{d}^3 x^3 \neq 0.$$

The third covariant differential $\mathrm{D}^3 x^i$ does not vanish automatically; it can be expressed by means of the lower order covariant differentials as follows (taking into account that now $\mathrm{d}^3 x^i = 0$):

$$\mathrm{D}^3 x^i = \tilde{B}^i_{km} \mathrm{D}x^k \mathrm{D}^2 x^m + \tilde{C}^i_{klm} \mathrm{D}x^k \mathrm{D}x^l \mathrm{D}x^m, \tag{65}$$

with quite complicated tensorial coefficients involving torsion tensor $S^i_{km} = \Gamma^i_{km} - \Gamma^i_{mk}$ in the expression of $\tilde{B}^i_{km}$, and combinations of the Riemann tensor R^i_{klm} in the expression of $\tilde{C}^i_{klm}$. For more details, see [9].

6. Z_3 ANALOG OF LIE ALGEBRAS

A most straightforward Z_3 generalization of Lie algebras is introduced by means of a Z_3-skew ternary product defined in any associative (ordinary) algebra A. For any three elements $X, Y, Z \in \mathcal{A}$, let us define

$$[X, Y, Z]_j = XYZ + jYZX + j^2 ZXY. \tag{66}$$

We obviously have

$$[X, Y, Z]_j = j\ [Y, Z, X]_j = j^2\ [Z, X, Y]_j\,, \tag{67}$$

from which it follows that $[X, X, X]_j = 0$.

We have not found yet a "Ternary Jacobi identity" similar to the usual one in the Z_2-graded case. $\mathcal{A}$, with unit element $\mathbf{1}$, we have:

$$[X, \mathbf{1}, Y]_j = X\mathbf{1}Y + j\ \mathbf{1}YX + j^2\ YX\mathbf{1} =$$

$$= XY + (j + j^2)\ YX = XY - YX = [X, Y], \tag{68}$$

the usual commutator defining a classical Lie algebra.

The simplest realization of ternary Lie algebra with Z_3-skew product is given by Pauli's matrices.

$$[\sigma_i, \sigma_k, \sigma_l] = \sigma_i \sigma_k \sigma_l + j\ \sigma_k \sigma_l \sigma_i + j^2\ \sigma_l \sigma_i \sigma_k, \tag{69}$$

with the obvious Z_3 symmetry:

$$[\sigma_i, \sigma_k, \sigma_l] = j\ [\sigma_k, \sigma_l, \sigma_i] = j^2\ [\sigma_l, \sigma_i, \sigma_k] \tag{70}$$

From the Z_3-skew symmetry property we infer that out of $3^3 = 27$ different combinations of three indices i, k, l the three ones with three identical values (111), (222) and (333) identically vanish. Out of the 24 remaining combinations it is enough to determine one out of three related by cyclic Z_3 permutations. This leaves only eight combinations to be computed effectively.

We find quite easily that in the case when all three indices are different, one has

$$[\sigma_1, \sigma_2, \sigma_3] = 0 \quad \text{and} \quad [\sigma_3, \sigma_2, \sigma_1] = 0. \tag{71}$$

The remaining six combinations display two identical indices and the third one different. They can be separated in three couples: $\{(121), (212), \quad \{(232), (323)\}$ and $\{(313), (131)\}$ which form three independent subalgebras. Their ternary commutation relations are all of the same form:

$$[\sigma_1, \sigma_2, \sigma_1] = -2\ \sigma_2, \quad [\sigma_2, \sigma_1, \sigma_2] = -2\ \sigma_1,$$
$$[\sigma_2, \sigma_3, \sigma_2] = -2\ \sigma_3, \quad [\sigma_3, \sigma_2, \sigma_3] = -2\ \sigma_2,$$
$$[\sigma_3, \sigma_1, \sigma_3] = -2\ \sigma_1, \quad [\sigma_1, \sigma_3, \sigma_1] = -2\ \sigma_3, \tag{72}$$

We give here the proof for one particular choice, the rest is computed in the same manner, using the well known properties of Pauli's matrices:

$$\sigma_k^2 = \mathbf{1}, \ \text{ and } \ \sigma_k \sigma_l = -\sigma_l \sigma_k \ \text{ if } \ k \neq l.$$

We have:

$$[\sigma_1, \sigma_2, \sigma_1] = \sigma_1\sigma_2\sigma_1 + j\ \sigma_2\sigma_1^2 + j^2\ \sigma_2^2\sigma_1 =$$
$$= -\sigma_2\sigma_1^2 + (j + j^2)\sigma_2 = -2\ \sigma_2$$

because $j + j^2 = -1$.

The calculus is exactly the same for all other independent combinations displayed in (72) above.

By the way, the results will be identical for quaternions, whose commutation relations are the same as for σ-matrices multiplied by imaginary unit i, to ensure that quaternion's squares be equal to -1.

Now all ternary structure constants are well determined. Defining

$$[\sigma_i, \sigma_k, \sigma_l] = f^m_{ikl}\ \sigma_m, \tag{73}$$

we find easily that

$$f^m_{iii} = 0; \quad f^m_{ijk} = 0 \ \text{ for } \ i \neq j \neq k; \quad f^m_{iki} = -2\ \delta^m_k.$$

and of course,

$$f^m_{kii} = -2j\ \delta^m_k, \quad f^m_{iik} = -2j^2\ \delta^m_k.$$

Similar ternary algebra can be defined with four generators taken to be the Dirac 4×4-matrices γ^μ, $\mu = 9, 1, 2, 3$. Their ternary commutation relations can be represented in the following fully covariant manner:

$$\left[\gamma^\mu, \gamma_\nu, \gamma^\lambda\right] = f^\rho_{\mu\nu\lambda}\ \gamma_\rho = 2\delta^\rho_\nu\, g_{\mu\lambda}\gamma_\rho \tag{74}$$

where $g_{\mu\lambda}$ is the Minkowskian metric tensor $g_{\mu\lambda} = \mathrm{diag}(+1, -1, -1, -1)$.

It can be shown that the invariant group of these constitutive relations is the Lorentz group.

What is not very surprising here is the conformity of ternary j-commutator with respect to the Lorentz transformations.

This ternary Z_3 analogue of Lie algebra is not restricted to a realization with a Z_3-commutator imposed on a classical associative algebra (*e.g.* matrices). One can imagine ternary algebras (usually non-associative) realized in a different way. For example, take the linear space of 3-forms on N-dimensional linear space. Given in a chosen linear basis, they will read as $F_{ABC}, \;\; A, B, ... = 1, 2, ...N$.

Let us define the following ternary product:

$$(F * G * H)_{ABC} = \Sigma_{J,K,L=1}^{N} F_{JAK} G_{KBL} H_{LCJ}.$$

In fact, the summation supposes the possibility of raising and lowering indices, by means of a symmetric metric g^{AB} or an anti-symmetric 2-form ϵ^{AB}.

Such an algebra becames interesting if we impose a Z_3 symmetry on our 3-forms.

For example, let us impose a Z_3-symmetry on 3-forms, requiring that

$$F_{ABC} = j^2 F_{BCA} = j F_{CAB}.$$

To guarantee that ternary Z_3-skew product of three such forms yields a 3-form having the same symmetry properties, we must introduce the following Z_3-skew product:

$$\{F, G, H\}_{ABC} = (F * G * H)_{ABC} + j(F * G * H)_{BCA} + j^2(F * G * H)_{CAB},$$

equivalent with

$$\{F, G, H\}_{ABC} = (F * G * H)_{ABC} + j(G * H * F)_{ABC} + j^2(H * F * G)_{ABC}.$$

Ternary algebras of this type were considered and investigated in [9, 11, 18].

7. INVARIANT THREE-FORMS

Let us consider multilinear forms defined on the algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$. Because only cubic relations are imposed on products in $\mathcal{A}$ and in $\bar{\mathcal{A}}$, and the binary relations on the products of ordinary and conjugate elements, we shall fix our attention on tri-linear and bi-linear forms.

Consider a tri-linear form ρ^{α}_{ABC}. We shall call this form Z_3-invariant if we can write, by virtue of (30).

$$\rho^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C + \rho^{\alpha}_{BCA}\, \theta^B \theta^C \theta^A + \rho^{\alpha}_{CAB}\, \theta^C \theta^A \theta^B\right] =$$

$$= \frac{1}{3}\left[\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\,(j^2\,\theta^A\theta^B\theta^C) + \rho^{\alpha}_{CAB}\,j\,(\theta^A\theta^B\theta^C)\right].$$

From this it follows that we should have

$$\rho^{\alpha}_{ABC}\;\theta^A\theta^B\theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC} + j^2\,\rho^{\alpha}_{BCA} + j\,\rho^{\alpha}_{CAB}\right]\theta^A\theta^B\theta^C, \tag{75}$$

from which we get the following properties of the ρ-cubic matrices:

$$\rho^{\alpha}_{ABC} = j^2\,\rho^{\alpha}_{BCA} = j\,\rho^{\alpha}_{CAB}. \tag{76}$$

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, *i.e.* j^2, so that they compensate each other.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{B}\dot{A}}$ satisfying the relations similar to (76) with j replaced be its conjugate, j^2:

$$\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}} = j\,\bar{\rho}^{\dot{\alpha}}_{\dot{B}\dot{C}\dot{A}} = j^2\,\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{A}\dot{B}} \tag{77}$$

In the simplest case of two generators, the j-skew-invariant forms have only two independent components:

$$\rho^1_{121} = j\,\rho^1_{211} = j^2\,\rho^1_{112},$$

$$\rho^2_{212} = j\,\rho^2_{122} = j^2\,\rho^2_{221},$$

and we can set

$$\rho^1_{121} = 1,\;\; \rho^1_{211} = j^2,\;\; \rho^1_{112} = j,$$
$$\rho^2_{212} = 1,\;\; \rho^2_{122} = j^2,\;\; \rho^2_{221} = j.$$

8. THE INVARIANCE GROUP OF CUBIC MATRICES

The constitutive cubic relations between the generators of the Z_3-graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, *i.e.* if they are independent of the choice of the basis. Let $U^{A'}_A$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_B\,\theta^B$.

We are looking for the solution of the covariance condition for the ρ-matrices:

$$S^{\alpha'}_{\beta}\;\rho^{\beta}_{ABC} = U^{A'}_A\,U^{B'}_B\,U^{C'}_C\;\rho^{\alpha'}_{A'B'C'}. \tag{78}$$

Now, $\rho^1_{121} = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-matrix on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

which leads to the following equation:

$$S^{1'}_1 = U^{1'}_1\, U^{2'}_2\, U^{1'}_1 + j^2\, U^{2'}_1\, U^{1'}_2\, U^{1'}_1 + j\, U^{1'}_1\, U^{1'}_2\, U^{2'}_1 = U^{1'}_1\, (U^{2'}_2\, U^{1'}_1 - U^{2'}_1\, U^{1'}_2), \tag{79}$$

because $j^2 + j = -1$.

For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$S^{2'}_1 = U^{2'}_1\, U^{1'}_2\, U^{2'}_1 + j^2\, U^{1'}_1\, U^{2'}_2\, U^{2'}_1 + j\, U^{2'}_1\, U^{2'}_2\, U^{1'}_1 = U^{2'}_1\, (U^{1'}_2\, U^{2'}_1 - U^{1'}_1\, U^{2'}_2), \tag{80}$$

The remaining two equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ-matrix on the left hand side must be ρ^2 whose component ρ^2_{212} is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$S^{1'}_2 = U^{1'}_2\, U^{2'}_1\, U^{1'}_2 + j^2\, U^{2'}_2\, U^{1'}_1\, U^{1'}_2 + j\, U^{1'}_2\, U^{1'}_1\, U^{2'}_2 = U^{1'}_2\, (U^{1'}_2\, U^{2'}_1 - U^{1'}_1\, U^{2'}_2), \tag{81}$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$S^{2'}_2 = U^{2'}_2\, U^{1'}_1\, U^{2'}_2 + j^2\, U^{1'}_2\, U^{2'}_1\, U^{2'}_2 + j\, U^{2'}_2\, U^{2'}_1\, U^{1'}_2 = U^{2'}_2\, (U^{1'}_1\, U^{2'}_2 - U^{2'}_1\, U^{1'}_2). \tag{82}$$

The determinant of the 2×2 complex matrix $U^{A'}_B$ appears everywhere on the right-hand side.

$$S^{2'}_1 = -U^{2'}_1\, [\det(U)], \tag{83}$$

The two remaining equations are obtained in a similar manner, resulting in the following:

$$S^{1'}_2 = -U^{1'}_2\, [\det(U)], \qquad S^{2'}_2 = U^{2'}_2\, [\det(U)]. \tag{84}$$

The determinant of the 2×2 complex matrix $U^{A'}_B$ appears everywhere on the right-hand side. Taking the determinant of the matrix $\Lambda^{\alpha'}_\beta$ one gets immediately

$$\det(S) = [\det(U)]^3. \tag{85}$$

However, the U-matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations and they can take on three different values: 1, j or j^2, *i.e.* the matrices $j\, U^{A'}_B$ or $j^2\, U^{A'}_B$ satisfy the same relations as the matrices $U^{A'}_B$ defined above.

The determinant of U can take on the values 1, j or j^2 if $\det(\Lambda) = 1$.

But for the time being, we have no reason yet to impose the unitarity condition. It can be derived from the conditions imposed on the invariance and duality.

In the Hilbert space of spinors the $SL(2, \mathbf{C})$ action conserved naturally two anti-symmetric tensors,

$$\varepsilon_{\alpha\beta} \quad \text{and} \quad \varepsilon_{\dot{\alpha}\dot{\beta}} \text{ and their duals } \varepsilon^{\alpha\beta} \quad \text{and} \quad \varepsilon^{\dot{\alpha}\dot{\beta}}.$$

Spinorial indices thus can be raised or lowered using these fundamental $SL(2, \mathbf{C})$ tensors:

$$\psi_\beta = \epsilon_{\alpha\beta}\, \psi^\alpha, \quad \psi^{\dot{\delta}} = \varepsilon^{\dot{\delta}\dot{\beta}}\, \psi_{\dot{\beta}}.$$

In the space of quark states similar invariant form can be introduced, too. There is only one alternative: either the Kronecker delta, or the anti-symmetric 2-form ε.

Supposing that our cubic combinations of quark states behave like fermions, there is no choice left: if we want to define the duals of cubic forms ρ^α_{ABC} displaying the same symmetry properties, we must impose the covariance principle as follows:

$$\epsilon_{\alpha\beta}\, \rho^\alpha_{ABC} = \varepsilon_{AD}\varepsilon_{BE}\varepsilon_{CG}\, \rho^{DEG}_\beta.$$

The requirement of the invariance of tensor ε_{AB}, $A, B = 1, 2$ with respect to the change of basis of quark states leads to the condition $\det U = 1$, *i.e.* again to the $SL(2, \mathbf{C})$ group.

9. A Z_3 COLOR DYNAMICS

According to present knowledge, the ultimate undivisible and undestructible constituents of matter, called *atoms* by ancient Greeks, are in fact the *quarks*, carrying fractional electric charges and baryonic numbers, two features that appear to be undestructible and conserved under any circumstances.

Taking into account that quarks evolve inside nucleons as almost point-like objects, one may wonder how the notions of space and time still apply in these conditions? Perhaps in this case, too, the Lorentz invariance can be derived from some more fundamental *discrete* symmetries underlying the interactions between quarks? If this is the case, then the symmetry Z_3 must play a fundamental role.

In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$. Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin. Besides, they must belong to different *colors*, also a three-valued set. There are two quarks in the first generation, u and d ("up" and "down"), which may be considered as two states of a more general object, just like proton and neutron in $SU(2)$ symmetry are two isospin components of a nucleon doublet.

This suggests that a convenient generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can be present in a nucleon. The cubic commutation relations realizing a representation of the Z_3 cyclic group introduced above provide such statistics, and exclude the states with four or more quarks at once.

Our aim is to find a generalization of Dirac's equation which would ensure not only the symmetry between particles and anti-particles, realized via introduction of negative mass term, but would describe adequately also the mixing of three different "colors". The overall symmetry of such generalized system should contain two Z_2 groups and one Z_3 group. The two Z_2 groups correspond to two fundamental symmetries: the spin $\frac{1}{2}$ representation, with Pauli's Z_2 exclusion principle, another Z_2 representing the symmetry between particles and anti-particles (here quarks and anti-quarks), and Z_3 representing the symmetry between the colors.

Let us first underline the Z_2 symmetry of Maxwell and Dirac equations, which implies their hyperbolic character, which makes the propagation possible. Maxwell's equations *in vacuo* can be written as follows:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \wedge \mathbf{B}, \quad -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \mathbf{E}. \tag{86}$$

These equations can be decoupled by applying the time derivation twice, which in vacuum, where $\mathrm{div}\mathbf{E} = 0$ and $\mathrm{div}\mathbf{B} = 0$ leads to the d'Alembert equation for both components separately:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0.$$

Nevertheless, neither of the components of the Maxwell tensor, be it $\mathbf{E}$ or $\mathbf{B}$, can propagate separately alone. It is also remarkable that although each of the fields $\mathbf{E}$ and $\mathbf{B}$ satisfies a second-order propagation equation, due to the coupled system (86) there exists a quadratic combination satisfying the first-order equation, the Poynting four-vector:

$$P^\mu = \left[P^0, \mathbf{P}\right], \quad P^0 = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \quad \mathbf{P} = \mathbf{E} \wedge \mathbf{B},$$

$$\partial_\mu P^\mu = 0. \tag{87}$$

The Dirac equation for the electron displays a similar Z_2 symmetry, with two coupled equations which can be put in the following form:

$$i\hbar\frac{\partial}{\partial t}\psi_+ - mc^2\psi_+ = i\hbar\sigma \cdot \nabla\psi_-,$$

$$-i\hbar\frac{\partial}{\partial t}\psi_- - mc^2\psi_- = -i\hbar\sigma\cdot\nabla\psi_+, \tag{88}$$

where ψ_+ and ψ_- are the positive and negative energy components of the Dirac equation; this is visible even better in the momentum representation:

$$\left[E - mc^2\right]\psi_+ = c\sigma\cdot\mathbf{p}\psi_-,$$

$$\left[-E - mc^2\right]\psi_- = -c\sigma\cdot\mathbf{p}\psi_+. \tag{89}$$

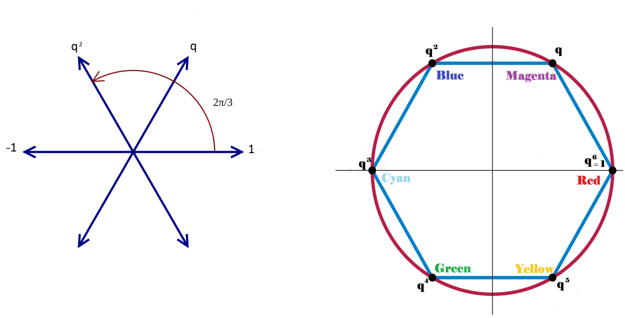
The same effect (negative energy states) can be obtained by changing the direction of time, and putting the minus sign in front of the time derivative, as suggested by Feynman [26].

Each of the components satisfies the Klein-Gordon equation, obtained by successive application of the two operators and diagonalization:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - m^2\right]\psi_\pm = 0$$

As in the electromagnetic case, neither of the components of this complex entity can propagate by itself; only all the components as a whole can.

Apparently, the two types of quarks, u and d, cannot propagate freely, but can form a freely propagating particle perceived as a fermion, only under an extra condition: they must belong to three *different* species called *colors*; short of this they will not form a propagating entity. Also the quark-antiquark combinations could become colorless and propagate freely, too. These simple algebraical rules can be illustrated by the following scheme using the product group $Z_2 \times Z_3$, which is nothing else but the cyclic group Z_6, realized by powers of -1 and j [12], or else, generated by single primitive sixth root of unity, $q = e^{\frac{2\pi i}{6}}$ in the complex plane:



The six complex numbers q^k can be put into correrspondence with three colors and three anti-colors.

The powers of complex generator of the Z_6 cyclic group can be put into correspondence with three colors and three anti-colors, as shown in the figure:

$q^6 = 1 \to$ Red , $\quad q^2 = j \to$ Blue, $\quad q^4 = j^2 \to$ Green;

$q = -j^2 \to$ Magenta, $\quad q^3 = -1 \to$ Cyan, $\quad q^5 = -j \to$ Yellow;

"White" color corresponds to vanishing linear combinations

$q^6 + q^4 + q^2 = 0$, (Red + Blue + Green);

$q^5 + q^3 + q = 0$, (Yellow + Cyan + Magenta);

$q + q^4 = -j^2 + j^2 = 0$, (Magenta + Green),

$q^2 + q^5 = j + (-j) = 0$, (Blue + Yellow),

$q^3 + q^6 = -1 + 1 = 0$, (Cyan + Red),

In fact, the totally "colorless" quadratic combinations (gluons) do not interact strongly with quarks. This means that the combination

$$R\bar{R} + B\bar{B} + G\bar{G} = 0.$$

does vanish, so that only *eight* linear combinations out of nine are independent.

These are:

$$\begin{array}{cc}
\frac{1}{\sqrt{2}}\,(R\bar{B} + B\bar{R}), & \frac{1}{\mathrm{i}\sqrt{2}}(R\bar{B} - B\bar{R}), \\
\frac{1}{\sqrt{2}}\,(R\bar{G} + G\bar{R}), & \frac{1}{\mathrm{i}\sqrt{2}}(R\bar{G} - G\bar{R}), \\
\frac{1}{\sqrt{2}}\,(B\bar{G} + G\bar{B}), & \frac{1}{\mathrm{i}\sqrt{2}}(B\bar{G} - G\bar{B}), \\
\frac{1}{\sqrt{2}}\,(R\bar{R} - B\bar{B}), & \frac{1}{\sqrt{6}}(R\bar{R} + B\bar{B} - 2G\bar{G}),
\end{array}$$

These combinations form the basis of eight traceless Gell-Mann 3×3 matrices, forming the $SU(3)$ Lie algebra. This set of fields form the basis of *quantum chromodynamics*, known under the abridged name as "QCD". Three quarks forming a nucleon are treated as Dirac spinors, whose wave functions account for $3 \times 4 = 12$ components.

What we propose here is to see the same set of wave functions as a collection of three Pauli spinors corresponding to three colors (six degrees of freedom), and three Pauli spinors corresponding to three anti-colors (also six degrees of freedom), satisfying the system of twelve linear equations of first order, intertwining not only particles with anti-particles, but also the three colors and three anti-colors, thus displaying the full symmetry group $Z_2 \times Z_2 \times Z_3$, because we want to maintain both Z_2 symmetries present in Dirac's equations in order to take into account the half-integer spin of quarks as well as to incorporate the quark-anti-quark symmetry.

We shall follow the logical scheme that leads from Pauli equation to the Dirac equation by introducing a negative mass term and doubling the number of Pauli spinors.

The inclusion of spin variable, subjected to Pauli's exclusion principle, into a Schroedinger-like equation can be done by replacing the usual complex wave function by a column vector containing two complex components. The energy, momentum and mass operators should be represented by 2×2 matrices. The simplest linear equation considered by Pauli at first had the following form:

$$E \mathbb{1}_2 \, \psi = mc^2 \mathbb{1}_2 \, \psi + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi, \tag{90}$$

where according to the correspondence principle, E stays for the operator $-i\hbar\partial_t$, $\mathbf{p}$ stays for the operator-valued vector $-i\hbar\boldsymbol{\nabla}$, and where ψ stays now for the two-component *Pauli spinor* $\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$, the 3-dimensional momentum vector $\mathbf{p}$ is scalarly multiplied by $\boldsymbol{\sigma}$ representing the three hermitian traceless Pauli's matrices $\boldsymbol{\sigma} = [\sigma_x, \ \sigma_y, \ \sigma_z]$, and$\mathbb{1}_2$ stays for the 2×2 unit matrix. But this equation fails to satisfy the Lorentz invariance criterion: it suffices to take the square of the energy operator to discover that (90) leads to the following quadratic relation

$$E^2 = m^2c^4 + 2mc^3 \boldsymbol{\sigma} \cdot \mathbf{p} + c^2 \, \mathbf{p}^2 \tag{91}$$

instead of the desired Lorentz-invariant relation $E^2 = m^2c^4 + c^2\mathbf{p}^2$. At this stage the Lorentz invariance could be recovered by introducing another Pauli spinor entangled with the first one via equations similar with (90), but with a negative mass term for the second Pauli spinor:

$$E \mathbb{1}_2 \, \psi_+ = mc^2 \mathbb{1}_2 \, \psi_+ + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi_-,$$

$$E \mathbb{1}_2 \, \psi_- = -mc^2 \mathbb{1}_2 \, \psi_- + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi_+, \tag{92}$$

where $\psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}$. It is easy to see now that by simple iteration we get the right relation satisfied simultaneously by both components:

$$E^2 \psi_+ - c^2 \mathbf{p}^2 \, \psi_+ = m^2c^4 \psi_+, \quad E^2 \psi_- - c^2 \mathbf{p}^2 \, \psi_- = m^2c^4 \psi_-.$$

The four equations (92) are just one of the representations of the equation of the electron discovered shortly after by Dirac, but in a totally different manner, derived as a "square root" of the Klein-Gordon equation; but at the moment the idea of introducing a negative mass seemed physically unacceptable.

The two equations (92) can be re-written using a matrix notation:

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} + \begin{pmatrix} 0 & c\,\boldsymbol{\sigma}\mathbf{p} \\ \boldsymbol{\sigma}\mathbf{p} & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{93}$$

where the entries in the energy operator and the mass matrix are in fact 2×2 identity matrices, as well as the σ-matrices appearing in the last matrix, so that in reality the above equation represents the 4×4 Dirac equation, only in a different basis [24].

The system of linear equations (93) displays two important discrete Z_2 symmetries: the space reflection consisting in simultaneous change of the direction of spin and momentum, $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}, \mathbf{p} \to -\mathbf{p}$, and the particle-antiparticle symmetry realized by the transfromation $m \to -m$, $\psi_+ \to \psi_-$, $\psi_- \to \psi_+$. Our next aim is to extend the $Z_2 \times Z_2$ symmetry by including the Z_3 group which will mix not only the two spin states and particles with anti-particles, but also the three colors.

Now we want to describe three different two-component fields (which can be incidentally given the names of three colors, the "red" one φ_+, the "blue" one χ_+, and the "green" one ψ_+); more explicitly,

$$\varphi_+ = \begin{pmatrix} \varphi_+^1 \\ \varphi_+^2 \end{pmatrix}, \ \chi_+ = \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix}, \ \psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix}, \tag{94}$$

We follow the minimal scheme taking into account the existence of spin by using only Pauli spinors on which the 3-momentum operator acts through the scalar product $\boldsymbol{\sigma} \cdot \mathbf{p}$. In order to satisfy the required existence of anti-particles, we should also introduce three "anti-colors", denoted by a "minus" underscript, corresponding to the opposite colors: "cyan" for φ_-, "yellow" for χ_- and "magenta" for ψ_-; here, too, we have to do with two-component columns:

$$\varphi_- = \begin{pmatrix} \varphi_-^1 \\ \varphi_-^2 \end{pmatrix}, \ \chi_- = \begin{pmatrix} \chi_-^1 \\ \chi_-^2 \end{pmatrix}, \ \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}, \tag{95}$$

all in all *twelve* components. A somewhat similar construction, but with three Dirac spinors, can be found in [28].

This leaves little space for the choice of the system of intertwined equations; here is the ternary generalization of Dirac's equation, intertwining not only particles with antiparticles, but also the three "colors", in such a way that the entire system becomes invariant under the action of the $Z_2 \times Z_2 \times Z_3$ group.

The set of linear equations for three Pauli spinors endowed with colors, and another three Pauli spinors corresponding to their anti-particles endowed with "anti-colors" involves altogether twelve complex functions. The twelve components could describe three independent Dirac particles, but here they will be intertwined in a particular manner, mixing together not only spin states and particle-antiparticle states, but also the three colors.

We shall follow the logic that led from Pauli's to Dirac's equation extending it to the colors acted upon by the Z_3-group. In the expression for the energy operator (*i.e.* the Hamiltonian), mass terms is positive when acting on particles, and acquires negative sign acting on anti-particles, *i.e.* it changes sign while intertwining particle-antiparticle components. We shall also assume that the mass term acquires the factor j when we switch from the red component φ to the blue component ξ, and j^2 for the green component ψ. The momentum operator will be non-diagonal, as in the Dirac equation, systematically intertwining not only particles with antiparticles, but also colors with anti-colors (see *e.g.* [34, 35]).

The system that satisfies all these assumptions is as follows:

$$E\,\varphi_+ = mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-$$

$$\begin{gathered} E\,\chi_- = -j\,mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+ \\ E\,\psi_+ = j^2\,mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_- \\ E\,\varphi_- = -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+ \\ E\,\chi_+ = j\,mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_- \end{gathered}$$

$$E\,\psi_- = -j^2\,mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_+ \tag{96}$$

where the Pauli spinors $\varphi_\pm$, $\chi_\pm$, $\psi_\pm$ are as in (94) and (95), on which Pauli sigma-matrices act in a natural way.

On the right-hand side, the mass terms form a diagonal matrix whose entries follow an ordered row of powers of the sixth root of unity $q = e^{\frac{2\pi i}{6}}$. Indeed, we have

$$m = q^6 m, \;\; -jm = q^5 m, \;\; j^2 m = q^4 m,$$

$$-m = q^3 m, \;\; jm = q^2 m, \;\; -j^2 m = qm.$$

The diagonalisation of our system requires the sixth-order iteration, in contrast with the Dirac equation, which needs only the second-order iteration: the square of the Dirac operator results in the Klein-Gordon equation satisfied simultaneously by all components.

In the case of our ternary generalization, the final result is extremely simple: all the components satisfy the same sixth-order equation,

$$\begin{gathered} E^6\,\varphi_+ = m^6 c^{12}\,\varphi_+ + c^6 \mid \mathbf{p} \mid^6\;\varphi_+, \\ E^6\,\varphi_- = m^6 c^{12}\,\varphi_- + c^6 \mid \mathbf{p} \mid^6\;\varphi_-. \end{gathered} \tag{97}$$

and similarly for all other components.

The energy operator is obviously diagonal, and its action on the spinor-valued column-vector can be represented as a 6×6 operator valued unit matrix. The mass operator is diagonal, too, but its elements represent all powers of the sixth root of unity q, which are $q = -j^2$, $q^2 = j$, $q^3 = -1$, $q^2 = j^2$, $q^5 = -j$ and $q^6 = 1$.

Finally, the momentum operator is proportional to a *circulant matrix* which mixes up all the components of the column vector. We shall choose the basis in which the twelve components form the column in which the six Pauli spinors (three colors and three anti-colors) are organized in the following order (from top to bottom): $\varphi_+, \chi_+, \psi_+, \varphi_-, \chi_-, \psi_-$. With respect to this basis our matrix operators acquire the following form:

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & jm & 0 & 0 & 0 & 0 \\ 0 & 0 & j^2 m & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 0 & 0 \\ 0 & 0 & 0 & 0 & -jm & 0 \\ 0 & 0 & 0 & 0 & 0 & -j^2 m \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In fact, the dimension of the two matrices M and P displayed above is 12×12: all the entries in the first one are proportional to the 2×2 identity matrix, so that in the definition one should read $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ instead of m, $\begin{pmatrix} jm & 0 \\ 0 & jm \end{pmatrix}$ instead of $j\, m$, etc.

The entries in the second matrix P contain 2×2 Pauli's sigma-matrices, so that P is also a 12×12 matrix. The energy operator E is proportional to the 12×12 identity matrix. It is not difficult to recognize tensor products of certain Pauli's matrices with certain 3×3 matrices B and Q introduced in one of the previous sections dealing with ternary generalization of Clifford algebras; using these matrices we can write down our 12×12 matrix operator (96) as follows:

$$E \mathbb{1}_{12}\, \Psi = mc^2\, \sigma_3 \otimes B \otimes \mathbb{1}_2\, \Psi + c\, \sigma_1 \otimes Q_3 \otimes \boldsymbol{\sigma}\mathbf{p}\, \Psi. \tag{98}$$

By multiplying on the left by the matrix

$$\sigma_3 \otimes B^\dagger \otimes \mathbf{1}_2$$

we arrive at the following form of ternary generalization of Dirac's equation:

(99) $$\Big[E\,\sigma_3 \otimes B^\dagger \otimes \mathbf{1}_2 - i\,c\sigma_2 \otimes j^2\,Q_2 \otimes \boldsymbol{\sigma}\cdot\mathbf{p}\Big]\;\Psi = mc^2\;\mathbf{1}_2 \otimes \mathbf{1_3} \otimes \mathbf{1}_2\;\Psi$$

where we used the fact that under matrix multiplication, $\sigma_3\sigma^3 = \mathbb{1}_2$, $B^\dagger B = \mathbb{1}_3$ and $B^\dagger Q_3 = j^2\,Q_2$.

One can check by direct computation that the sixth power of this operator gives the same result as before,

(100)
$$\Big[E\,\sigma_3 \otimes B^\dagger \otimes \mathbb{1}_2 - i\sigma_2 \otimes j^2\,Q_2 \otimes c\,\boldsymbol{\sigma}\cdot\mathbf{p}\Big]^6 = \left[E^6 - c^6\mathbf{p}^6\right]\mathbb{1}_{12} = m^6c^{12}\mathbb{1}_{12}$$

The ternary Dirac equation can be written in a concise manner using the Minkowskian indices and the usual pseudo-scalar product of two four-vectors as follows:

(101) $$\Gamma^\mu p_\mu = mc^2\mathbb{1}_{12}$$

with 12×12 matrices $\Gamma^\mu, \mu = 0,1,2,3$ defined as follows:

$$\Gamma^0 = \sigma_3 \otimes B^\dagger \otimes \mathbb{1}_2, \qquad\qquad \Gamma^k = -i\sigma_2 \otimes j^2\,Q_2 \otimes \sigma^k.$$

It is also worthwhile to note that not only taking the sixth power of our operator yields the simple algebraic relation (100), but the similar relation exists between the determinants:

(102) $$\det\Big(E\sigma_3 \otimes B^\dagger \otimes \mathbb{1}_2 - i\sigma_2 \otimes j^2 Q_2 \otimes c\boldsymbol{\sigma}\cdot\mathbf{p}\Big) = \left(E^6 - c^6 \mid \mathbf{p}\mid^6\right)^2$$
$$= \det\left(mc^2\mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_2\right) = m^{12}c^{24}.$$

The eigenvalues of the generalized Dirac operator have all the same absolute value equal to $R = \mid (E^6 - c^6 \mid \mathbf{p}\mid^6)^{\frac{1}{6}} \mid$, and are given by:

(103) $$R,\; -R,\; jR,\; -jR,\; j^2R,\; -j^2R.$$

They are double degenerate, *i.e.* although the characteristic equation is of twelfth order, it has only six distinct eigenvalues. This result will be important for the subsequent discussion of the generalized Lorentz invariance.

Although the four 12×12 matrices do not satisfy usual anti-commutation relations similar to those of the 4×4 Dirac matrices γ^μ, *i.e.* $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}\;\mathbf{1}_4$. Nevertheless, the system of equations satisfied by the 12-dimensional wave function Ψ,

(104) $$-i\hbar\,\Gamma^\mu\,\partial_\mu\,\Psi = mc\Psi$$

is a hyperbolic one, and has the same light cone as the Klein-Gordon equation. To corroborate this statement, let us first consider the massless case,

(105) $$-i\hbar\,\Gamma^\mu\,\partial_\mu\,\Psi = 0.$$

Assuming the general solution of the form $e^{k_\mu x^\mu}$, we can replace the derivations by the components of the wave 4-vector k^μ, and take the sixth power of the matrix $\Gamma^\mu k_\mu$. The resulting dispersion relation was shown to be

$$k_0^6 - \mid \mathbf{k} \mid^6 = \left(k_0^2 - \mid \mathbf{k} \mid^2\right)\left(k_0^2 - j \ \mid \mathbf{k} \mid^2\right)\left(k_0^2 - j^2 \ \mid \mathbf{k} \mid^2\right) =$$
$$= \left(k_0^2 - \mid \mathbf{k} \mid^2\right) \ \left(k_0^4 + k_0^2 \mid \mathbf{k} \mid^2 + \mid \mathbf{k} \mid^4\right) = 0.$$

The first factor defines the usual light cone, while the factor of degree four is strictly positive (besides the origin 0). The system has only one characteristic surface which is the same for all massless fields. Each of the three factors remains invariant under a different representation of the $SL(2, \mathbf{C})$ group.

Let us introduce the following three matrices representing the same four-vector k^μ:

$$K_3 = \begin{pmatrix} k_0 & k_x \\ k_x & k_0 \end{pmatrix}, \ K_1 = \begin{pmatrix} k_0 & jk_x \\ jk_x & k_0 \end{pmatrix}, \ K_2 = \begin{pmatrix} k_0 & j^2 k_x \\ j^2 k_x & k_0 \end{pmatrix}, \tag{106}$$

whose determinants are, respectively,

$$\det K_1 = k_0^2 - j^2 k_x^2, \quad \det K_2 = k_0^2 - j k_x^2, \ \det K_3 = k_0^2 - k_x^2. \tag{107}$$

Note that only the third matrix K_3 is hermitian, and corresponds to a *real* space-time vector k^μ, while neither of the remaining two matrices K_1 and K_2 is hermitian; however, one is the hermitian conjugate of another.

In what follows, we shall replace the absolute value of the wave vector $\mid \mathbf{k} \mid$ by a single spatial component, say k_x, because for any given 4-vector $k^\mu = [k_0, \mathbf{k}]$ we can choose the coordinate system in such a way that its x-axis should be aligned along the vector $\mathbf{k}$. Then it is easy to check that one has:

$$\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} k_0 \\ k_x \end{pmatrix} = \begin{pmatrix} k'_0 \\ k'_x \end{pmatrix}$$
$$\begin{pmatrix} \cosh u & j^2 \sinh u \\ j \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} k_0 \\ j \ k_x \end{pmatrix} = \begin{pmatrix} k'_0 \\ j \ k'_x \end{pmatrix}$$
$$\begin{pmatrix} \cosh u & j \sinh u \\ j^2 \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} k_0 \\ j^2 k_x \end{pmatrix} = \begin{pmatrix} k'_0 \\ j^2 k'_x \end{pmatrix} \tag{108}$$

The transformed vectors are given by the following expressions:

$$\text{i)} \ k_0' = k_0 \cosh u + k_x \ \sinh u, \quad k_x' = k_0 \sinh u + k_x \ \cosh u$$
$$\text{ii)} \ k_0' = k_0 \cosh u + j^2 \ k_x \ \sinh u, \quad k_x' = j \ k_0 \sinh u + k_x \ \cosh u$$
$$\text{iii)} \ k_0' = k_0 \cosh u + j \ k_x \ \sinh u, \quad k_x' = j^2 \ k_0 \sinh u + k_x \ \cosh u.$$

Let us now introduce a 6×6 matrix composed out of the above three 2×2

matrices:

$$\begin{pmatrix} 0 & k_0 \mathbb{1}_2 + \mathbf{k}\cdot\boldsymbol{\sigma} & 0 \\ 0 & 0 & k_0 \mathbb{1}_2 + j\,\mathbf{k}\cdot\boldsymbol{\sigma} \\ k_0 \mathbb{1}_2 + j^2\,\mathbf{k}\cdot\boldsymbol{\sigma} & 0 & 0 \end{pmatrix} \tag{109}$$

or, more explicitly,

$$K = \begin{pmatrix} 0 & 0 & k_0 & k_x & 0 & 0 \\ 0 & 0 & k_x & k_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_0 & jk_x \\ 0 & 0 & 0 & 0 & jk_x & k_0 \\ k_0 & j^2k_x & 0 & 0 & 0 & 0 \\ j^2k_x & k_0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{110}$$

It is easy to check that

$$\mathrm{detK} = (\mathrm{det}K_1)\cdot(\mathrm{det}K_2)\cdot(\mathrm{det}K_3) \tag{111}$$

$$= (k_0^2 - k_x^2)(k_0^2 - j^2k_x^2)(k_0^2 - jk_x^2) = k_0^6 - k_x^6. \tag{112}$$

It is also remarkable that the determinant remains the same in the basis in which the ternary Dirac operator was proposed, namely when we consider the matrix

$$K = \begin{pmatrix} k_0 & 0 & 0 & k_x & 0 & 0 \\ 0 & k_0 & k_x & 0 & 0 & 0 \\ 0 & 0 & k_0 & 0 & 0 & jk_x \\ 0 & 0 & 0 & k_0 & jk_x & 0 \\ 0 & j^2k_x & 0 & 0 & k_0 & 0 \\ j^2k_x & 0 & 0 & 0 & 0 & k_0 \end{pmatrix} \tag{113}$$

Let us show now that the spinorial representation of Lorentz boosts can be applied to each of the three matrices K_1, K_2 and K_3 separately, keeping their determinants unchanged. As a matter of fact, besides the well-known formula:

$$\begin{pmatrix} \cosh\frac{u}{2} & \sinh\frac{u}{2} \\ \sinh\frac{u}{2} & \cosh\frac{u}{2} \end{pmatrix} \begin{pmatrix} k_0 & k_x \\ k_x & k_0 \end{pmatrix} \begin{pmatrix} \cosh\frac{u}{2} & \sinh\frac{u}{2} \\ \sinh\frac{u}{2} & \cosh\frac{u}{2} \end{pmatrix} = \begin{pmatrix} k'_0 & k'_x \\ k'_x & k'_0 \end{pmatrix}, \tag{114}$$

with

$$k'_0 = k_0\ \cosh u + k_x\ \sinh u, \qquad k'_x = k_0\ \sinh u + k_x\ \cosh u. \tag{115}$$

which becomes apparent when we remind that

$$\cosh^2\frac{u}{2} + \sinh^2\frac{u}{2} = \cosh u \ \text{ and }\ 2\sinh\frac{u}{2}\ \cosh\frac{u}{2} = \sinh u,$$

keeping unchanged the Minkowskian scalar product: $k'^2_0 - k'^2_x = k_0^2 - k_x^2$, we have also two transformations of the same kind which keep invariant the "complexified" Minkowskian squares appearing as factors in the sixth-order expression

$k_0^6 - k_x^6$, namely

$$k_0^2 - j\, k_x^2 \quad \text{and} \quad k_0^2 - j^2\, k_x^2.$$

The above expressions can be identified as the determinants of the following 2×2 matrices:

$$(116) \qquad k_0^2 - j\, k_x^2 = \det \begin{pmatrix} k_0 & j^2 k_x^2 \\ j^2\, k_x & k_0 \end{pmatrix}, \quad k_0^2 - j^2\, k_x^2 = \det \begin{pmatrix} k_0 & j k_x^2 \\ j\, k_x & k_0 \end{pmatrix}.$$

It is easy to check that we have:

$$(117) \qquad \begin{pmatrix} \cosh \frac{u}{2} & \sinh \frac{u}{2} \\ \sinh \frac{u}{2} & \cosh \frac{u}{2} \end{pmatrix} \begin{pmatrix} k_0 & j k_x \\ j k_x & k_0 \end{pmatrix} \begin{pmatrix} \cosh \frac{u}{2} & \sinh \frac{u}{2} \\ \sinh \frac{u}{2} & \cosh \frac{u}{2} \end{pmatrix} = \begin{pmatrix} k'_0 & j k'_x \\ j k'_x & k'_0 \end{pmatrix},$$

with $k'_0 = k_0\, \cosh u + j\, k'_x\, \sinh u$, so that $k'^2_0 - j k'^2_x = k_0^2 - j k_x^2$, as well as
(118)

$$\begin{pmatrix} \cosh \frac{u}{2} & \sinh \frac{u}{2} \\ \sinh \frac{u}{2} & \cosh \frac{u}{2} \end{pmatrix} \begin{pmatrix} k_0 & j^2\, k_x \\ j^2\, k_x & k_0 \end{pmatrix} \begin{pmatrix} \cosh \frac{u}{2} & \sinh \frac{u}{2} \\ \sinh \frac{u}{2} & \cosh \frac{u}{2} \end{pmatrix} = \begin{pmatrix} k'_0 & j^2 k'_x \\ j^2 k'_x & k'_0 \end{pmatrix},$$

Therefore, we can draw the following conclusion: the 12×12 matrix formed by the tensor product of σ_3 with the 6×6 matrix K defined above, has the same determinant and the same eigenvalues (102, 103) as the generalized Dirac operator 99, if we replace k_0 by E and $\mathbf{k}$ by $c\mathbf{p}$. We have shown that the determinant of the matrix $\sigma_3 \otimes K$ (equal to $(k_0^6 - k_x^6)^2$) remains invariant under the generalized Lorentz transformation composed of three representations, the usual unitary one and two complex ones. Therefore there exists a similarity between the two matrices, which preserves the invariance under the generalized Lorentz group intertwined with Z_3. This does not contradict the no-go theorems by O'Raifeartaigh [36] and Coleman and Mandula [37].

10. INTERACTION WITH GAUGE FIELDS

The matrix representation of the system (99) is by no means unique. In the form which most closely resembles the classical Dirac equation, we chose the following representation for our ternary Dirac operator (designed be $\mathcal{D}$ for convenience):

$$(119) \qquad \mathcal{D} = E\, \sigma_3 \otimes B^\dagger \otimes \mathbb{1}_2 - i\sigma_2 \otimes j Q_2 \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p}\,.$$

Obviously, the essential sixth order diagonalized system resulting from the sixth iteration of this operator, as well as its characteristic equation and eigenvalues remain unchanged under an arbitrary similarity transformation, $\mathcal{D} \to P^{-1} \mathcal{D} P$. Taking into account the particular tensorial structure of ternary Dirac operator, the matrices P should display similar structure in order to keep the three factors separated. This reduces the allowed similarity matrices to the following family:

$$P = R \otimes S \otimes U,$$

with R being a 2×2 matrix, S denoting a 3×3 matrix, and U proportional to the 2×2 unit matrix in order not to change the scalar product $\boldsymbol{\sigma} \cdot \mathbf{p}$ in the last tensorial factor in $\mathcal{D}$.

The minimal coupling between the Dirac particles (electrons and positrons) with the electromagnetic field is obtained by inserting the four-potential A_μ into the Dirac equation:

$$\gamma^\mu (p_\mu - e\, A_\mu)\, \psi = m\, \psi. \tag{120}$$

Ternary generalization of Dirac's equation, when expressed with explicit Minkowskian indices, offers a similar possibility of introducing gauge fields. The particular structure of 12×12 matrices Γ_μ makes possible the accomodation of three types of gauge fields, corresponding to three factors from which the tensor product results.

The overall gauge field can be decomposed into a sum of three contributions: the $SU(3)$ gauge field $\lambda_a B^a_\mu$, with $\lambda_a, \quad a = 1, 2, ...8$ denoting the eight 3×3 traceless Gell-Mann matrices, the $SU(2)$ gauge field $\sigma_k\, A^k_\mu, \quad k = 1, 2, 3$ and the electric field potential A_μ. We propose to insert each of these gauge potentials into a common 12×12 matrix as follows: The strong interaction gauge potential is aligned on the $SU(3)$ matrix basis:

$$B_\mu = \mathbb{1}_2 \otimes \lambda_a B^a_\mu \otimes \mathbb{1}_2, \quad , a, b = 1, 2, ...8\,,$$

appearing as the second factor in the tensor product;

The $SU(2)$ weak interaction potential A^i_μ aligned along the three σ-matrices of the first tensorial factor

$$\sigma_k\, A^k_\mu \otimes \mathbb{1}_3 \otimes \mathbb{1}_2, \quad i, k, .. = 1, 2, 3\,,$$

and the electromagnetic potential A^{em}_μ aligned along the unit 2×2 matrix appearing as the third factor in the tensor product

$$\mathbb{1}_2 \otimes \mathbb{1}_3 \otimes A_\mu \mathbb{1}_2\,,$$

so that the overall expression for the gauge potential becomes:

$$\mathcal{A}_\mu = \mathbb{1}_2 \otimes \lambda_a B^a_\mu \otimes \mathbb{1}_2 + \sigma_k\, A^k_\mu \otimes \mathbb{1}_3 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes A^{em}_\mu \mathbb{1}_2\,. \tag{121}$$

The proposed ternary generalization of Dirac's equation including color degrees of freedom contains naturally not only the $SU(3)$-invariant strong interactions, but leads automatically to another type of gauge fields to which quarks are also sensitive: these are the gauge fields generated by the $SU(2)$ and $U(1)$ symmetries incorporated in the system.

There is an extra bonus here: namely, one can look at the same system (99) in the limit when the color interaction is switched off. This amounts to replacing the 3×3 matrices B and Q_3 by unit matrices$\mathbb{1}_3$. The resulting system is equivalent with a cartesian product of three identical Dirac equations:

$$E\mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_2 - \sigma_1 \otimes \mathbb{1}_3 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2\, \sigma_3 \otimes \mathbb{1}_3 \otimes \mathbb{1}_2\,. \tag{122}$$

Without any symmetry breaking, this set of equations describes three identical fermions sensitive exclusively to the $SU(2)\times U(1)$ gauge fields, *i.e.* the electroweak interaction, like the elementary particles known as *leptons* – in this setting they appear as natural *colorless* companions of quarks. This sheds new light on the fact that their number is equal, and even if other families of quarks had to be introduced (which we did not consider here), described by a similar ternary Dirac system, they would also give rise to another set of three leptons. And this is what the experimental data confirmed since the discovery of the families with other "flavors". The gauge fields are obviously common to all families.

In principle, we should have started with zero masses for all particles, quarks and leptons alike, and let the Higgs-Kibble mechanism generate non-zero masses. The Higgs field necessary for this to happen can be introduced like in the model of matrix algebras in the context of *non-commutative geometry*, (see [31–33]; see also [2]).

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