ON FILIFORM LIE ALGEBRAS. GEOMETRIC AND ALGEBRAIC STUDIES

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A finite dimensional filiform $\mathbb{K}$-Lie algebra is a nilpotent Lie algebra $\mathfrak{g}$ whose nil index is maximal, that is equal to $\dim \mathfrak{g} - 1$. We describe necessary and sufficient conditions for a filiform algebra over an algebraically closed field of characteristic 0 to admit a contact linear form (in odd dimension) or a symplectic structure (in even dimension). If we fix a Vergne's basis, the set of filiform n-dimensional Lie algebras is a closed Zariski subset of an affine space generated by the structure constants associated with this fixed basis. Then this subset is an algebraic variety and we describe in small dimensions the algebraic components.

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Conventions: All Lie algebras considered in this paper will be defined over an algebraically closed fixed field $\mathbb{K}$ of characteristic 0.

INTRODUCTION

The problem of classification up to isomorphism is a substantial problem in the study of finite dimensional Lie algebras, even over an algebraically closed field $\mathbb{K}$ of characteristic 0. This problem has a solution if we consider simple or semisimple algebras, that is, non abelian with no non-trivial ideals or with no non-zero abelian ideals. Indeed, in 1884, Elie Cartan gave the classification of the complex and real simple finite dimensional Lie algebras. His work is based on the works of Killing. He shows that this classification is reduced to 4 classes and 5 exceptional Lie algebras. The Levi decomposition, which was a conjecture of Killing and Cartan and was proved by Eugenio Elia Levi (1905), states that any finite-dimensional real or complex Lie algebra is the semidirect product of a solvable ideal and a semisimple subalgebra. From this result, the problem of classification is reduced to the classification of solvable Lie algebras and to the problem of representation of semisimple Lie algebras. Thus we are led to classify the solvable Lie algebras. But there are only few results on this topic. We know this classification up to the dimensions 5 or 6.

Without effective solutions to these problems, since the structure of a solvable Lie algebra is determined by its nilradical, that is the maximal nilpotent ideal, we are interested by the classification of nilpotent Lie algebras. This is one of the major aims of the present paper. Nowadays, only the classifications of complex or real nilpotent Lie algebras of dimension lower or equal to 7 was established. Even this partial result was presented after numerous attempts by various authors and in often different approaches. It is surely unrealistic to hope for classifications in higher dimension. Indeed, in dimension 7, we find non isomorphic families of one parameter of nilpotent Lie algebras but also a very large number (more than 100) of non parametrized and non isomorphic Lie algebras. In higher dimension, this number of Lie algebras will become very big and will become very difficult to verify (it is enough to see the different attempts of classifications in dimension 7 and the story of the dimension 7 to realize it). Moreover, when we have a given Lie algebra, it is often difficult to find this algebra in the official list of the classification because there are not the same invariants which are used, and finding the change of basis is a little bit tedious. Thus to pursue the work of classification up to isomorphism in higher dimension seems utopian. Some works showed that particular families of nilpotent Lie algebras were parametrized by tensor spaces. This means that it is equivalent to classify Lie algebras and arbitrary bilinear maps. Therefore, it does not seem wise to study particular properties of Lie algebras by starting on existing classifications. For example, the properties that we consider in this paper are the existence of contact forms or of symplectic forms, and also topological and algebraic properties based on the deformation theory and on the rigidity property. Thus the approach is more original. It was introduced in a previous paper concerning the study of k-step nilpotent Lie algebras. We globally study some reduced families which are invariant by isomorphism and which are closed, that is defined by a finite polynomial system. For these families, we can define an adapted cohomology and then introduce a notion of local rigidity, that is, we consider only deformations of elements of a given family which stay in this family. In [13], we have for the first time studied some geometrical properties of k-step nilpotent Lie algebras by considering families adapted to the characteristic sequence of a nilpotent Lie algebra (see *e.g.* [19] for a presentation of this invariant).

We consider in this work filiform Lie algebras, that is nilpotent Lie algebras whose nilindex is maximal, that is $n-1$ if n is the dimension of the Lie algebra under consideration. Of course, for this family we know the classification up to the dimension 7. In [10], classifications are also given for the dimensions 8 and 9 for this particular family of nilpotent Lie algebras. But there was no general consensus on these classifications. Thus we begin our work with these dimensions. We are interested in topological properties, as the rigidity, which we study by developing this notion of restricted cohomology. In particular we come back on a result of [3] concerning the local rigidity. We describe also in dimension 8 the family (which is neither open nor closed) of symplectic Lie algebras, that is Lie algebras provided with a bilinear symplectic form. In dimension 9, we show that there are no rigid filiform Lie algebras and we determine all the contact Lie algebras. We show also that any symplectic 8-filiform Lie algebra is obtained as a quotient of a contact filiform Lie algebra by its center and we find again the first given description of symplectic Lie algebra. We do the same thing for the dimensions 10 and 11, giving the description of the contact 11-dimensional filiform Lie algebras and consequently the description of the symplectic 10-dimensional filiform Lie algebras. We remark also, that in these dimensions, none of the Lie algebras are rigid. For a general dimension, we determine the family of contact $(2p+1)$-dimensional filiform Lie algebras. We expose also a model for this geometrical property, that is a family of contact Lie algebras such that any filiform contact Lie algebra is a deformation of an algebra of this model. Let us recall also that the reduction of the polynomial Jacobi system, that is the system of polynomial equations given by the Jacobi conditions, is the fundamental problem. We don't have many tools to find the generators of the ideal generated by these equations in the ring $\mathbb{K}[C_{i,j}^k, 1 \leq i < j \leq n, 1 \leq k \leq n]$. In general this ideal I is not equal to $\sqrt{I}$ and the associated affine scheme is not reduced. We described for the family model (parametrized by $(p-1)$ parameters) a process of reduction and we give the associated reduced system. To end this study, we determine from the family of $(2p+1)$-dimensional contact filiform Lie algebras the family of symplectic $(2p)$-dimensional filiform Lie algebras and we propose a notion of deformation of symplectic Lie algebra based on deformations of the contact Lie algebra which is a one-dimensional central extension. Let us note also, that in [5], we have studied filiform Lie algebras admitting a G-grading, where G is an abelian group.

1. GENERALITIES ON FILIFORM LIE ALGEBRAS

1.1. A Vergne's basis

Let $\mathfrak{g}$ be a n-dimensional Lie algebra over the field $\mathbb{K}$. The ascending central series $\{\mathcal{C}_i\mathfrak{g}\}$ of $\mathfrak{g}$ is defined by

$$\mathcal{C}_0\mathfrak{g} = \{0\}, \quad \mathcal{C}_i\mathfrak{g} = \{X \in \mathfrak{g} \,/\, [X, \mathfrak{g}] \subset \mathcal{C}_{i-1}\mathfrak{g}\},\ i > 0,$$

and the descending central series $\{\mathcal{C}^i\mathfrak{g}\}$ of $\mathfrak{g}$ is defined by

$$\mathcal{C}^0\mathfrak{g} = \mathfrak{g}, \quad \mathcal{C}^i\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{i-1}\mathfrak{g}], \quad i > 0.$$

Definition 1. The n-dimensional Lie algebra $\mathfrak{g}$ is called filiform if we have $\dim \mathcal{C}_i\mathfrak{g} = i$ for $0 \leq i \leq n-2$.

A filiform Lie algebra is nilpotent and we have

$$\mathcal{C}_i\mathfrak{g} = \mathcal{C}^{n-i-1}\mathfrak{g}, \quad 0 \leq i \leq n-1.$$

PROPOSITION 2. *Let $\mathfrak{g}$ be a $(n+1)$-dimensional filiform Lie algebra. There exists a basis $\{X_0, X_1, \cdots, X_n\}$ called a Vergne basis of $\mathfrak{g}$ such that*

$$\begin{cases} [X_0, X_i] = X_{i+1}, \ 1 \leq i \leq n-1, \ [X_1, X_{n-1}] = 0, \\ [X_i, X_j] = \displaystyle\sum_{k \geq i+j} C_{i,j}^k X_k. \end{cases}$$

Another characterization of a filiform Lie algebra is given by its characteristic sequence. In fact if X is a vector of the nilpotent Lie algebra $\mathfrak{g}$, the characteristic sequence $c(X)$ of the adjoint operator adX is the decreasing sequence of the dimensions of the Jordan blocks of the nilpotent operator adX. The characteristic sequence $c(\mathfrak{g})$ of $\mathfrak{g}$ is the following sequence $max\{c(X), X \in \mathfrak{g} - \mathcal{C}^1(\mathfrak{g})\}$, the maximum corresponding to the lexicographic order. Any vector X whose characteristic sequence $c(X)$ of adX is equal to $c(\mathfrak{g})$ is called characteristic vector of the nilpotent Lie algebra $\mathfrak{g}$ (so according to the lexicographic ordering, we have $c(Y) \leq c(X)$ for any $Y \in \mathfrak{g}$ if X is a characteristic vector). The n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is filiform if and only if $c(\mathfrak{g}) = n-1$. If $\{X_0, X_1, \cdots, X_n\}$ is a Vergne basis of a filiform Lie algebra $\mathfrak{g}$, the characteristic sequence $c(X_0)$ of the adjoint operator adX_0 is equal to $(n, 1)$. An interesting example of $(n+1)$-dimensional filiform Lie algebra is the Lie algebra L_{n+1} often called the model filiform Lie algebra ([11]) whose Lie bracket μ_0 is

$$\begin{cases} \mu_0(X_0, X_i) = X_{i+1}, \ 1 \leq i \leq n-1, \\ \mu_0(X_i, X_j) = 0, \ 1 \leq i < j \leq n. \end{cases} \tag{1}$$

1.2. **Geometric structure on filiform Lie algebras**

1.2.1. CONTACT AND SYMPLECTIC STRUCTURES

Let $\mathfrak{g}$ be a $(2p)$-dimensional $\mathbb{K}$-Lie algebra. A symplectic form on $\mathfrak{g}$ is a closed 2-form θ, that is satisfying

$$\theta([X, Y], Z) + \theta([Y, Z], X) + \theta([Z, X], Y) = 0$$

for any $X, Y, Z \in \mathfrak{g}$ and which is also nondegenerate that is

$$\theta^p = \theta \wedge \cdots \wedge \theta \neq 0.$$

A Lie algebra provided with a symplectic form θ is called a symplectic Lie algebra and denoted by the pair $(\mathfrak{g}, \theta)$. There exist filiform Lie algebras without symplectic structures. For example, in [6], one finds the list of all filiform Lie algebras up to dimension 6. When the symplectic form is exact, that is if there exists ω in $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ such that $\theta = d\omega$ where $d\omega$ is the bilinear form defined by $d\omega(X, Y) = -\omega([X, Y])$ for any $X, Y \in \mathfrak{g}$, the symplectic Lie algebra is called frobeniusian. But, from [14], there are no frobeniusian nilpotent Lie algebras since one can easily check that the center of any frobeniusian Lie algebra is trivial.

Let $\mathfrak{g}$ be a $n = 2p+1$-dimensional $\mathbb{K}$-Lie algebra. A contact form on $\mathfrak{g}$ is a non zero linear form ω on $\mathfrak{g}$ satisfying

$$\omega \wedge (d\omega)^p \neq 0$$

where $d\omega$ is the bilinear form defined by $d\omega(X, Y) = -\omega([X, Y])$ for any $X, Y \in \mathfrak{g}$ and $(d\omega)^p = d\omega \wedge \cdots \wedge d\omega$ p-times. In case of nilpotent Lie algebras, there is an obstruction to the existence of contact form ([14]), the center of $\mathfrak{g}$ can be of dimension 1. But the center of any filiform Lie algebra is always of dimension 1, then this necessary condition is always satisfied. Let us note that this does not imply that any odd-dimensional filiform Lie algebras admit a contact form.

PROPOSITION 3. *Let $(\mathfrak{g}, \theta)$ be a $2p$-dimensional filiform symplectic Lie algebra. Then the one dimensional central extension $\mathfrak{g}_\theta = \mathfrak{g} \oplus \mathbb{K}Z$ whose bracket is given by*

$$\begin{cases} [X, Y]_{\mathfrak{g}_\theta} = [X, Y] + \theta(X, Y)Z, \ \forall X, Y \in \mathfrak{g}\,, \\ [X, Z]_{\mathfrak{g}_\theta} = 0, \ \forall X \in \mathfrak{g}\,, \end{cases}$$

is a $(2p+1)$-dimensional filiform contact Lie algebra.

Proof. Let $\{X_0, X_1, \cdots, X_{2p-1}\}$ be a Vergne basis of $\mathfrak{g}$ and let $\{\omega_0, \cdots, \omega_{2p-1}\}$ be its dual basis. If θ is a symplectic form on $\mathfrak{g}$ then

$$d\theta(X_0, X_i, X_{2p-1}) = 0 = \theta(X_{i+1}, X_{2p-1}), \quad i = 1, \cdots, 2p-1.$$

This implies that $\theta = \omega_{2p-1} \wedge (a_0\omega_0 + a_1\omega_1) + \theta_1$. □

Conversely, if $\mathfrak{g}$ is a $(2p+1)$-dimensional contact nilpotent Lie algebra, then its center $Z(\mathfrak{g})$ is one-dimensional [14] and the factor algebra $\mathfrak{g}/Z(\mathfrak{g})$ is a symplectic $2p$-dimensional nilpotent Lie algebra. If $\mathfrak{g}$ is filiform, then $\mathfrak{g}/Z(\mathfrak{g})$ is also filiform. In [16], one proves that $\mathfrak{g}$ admits a contact form if and only if the linear form ω_{2p} is a contact form where $\{\omega_0, \cdots, \omega_{2p}\}$ is the dual basis of the Vergne basis of $\mathfrak{g}$. We deduce

Proposition 4. *A $(2p)$-dimensional Lie algebra $(\mathfrak{g}, \theta)$ is symplectic if and only if in the central extension $\mathfrak{g}_\theta$, the linear form ω_{2p} is a contact form, where $\{\omega_0, \cdots, \omega_{2p}\}$ is the dual basis of the Vergne basis of $\mathfrak{g}_\theta$.*

1.2.2. COMPLEX STRUCTURES

Definition 5. A complex structure on a $(2p)$-dimensional $\mathbb{R}$-Lie algebra $\mathfrak{g}$ is a linear endomorphism J of $\mathfrak{g}$ such that:

(1) $J^2 = -Id$;

(2) $[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0, \qquad \forall X, Y \in \mathfrak{g}.$

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ denote the complexification of $\mathfrak{g}$, and

$$\sigma : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$$

the corresponding conjugation. The above condition (2) is equivalent to the splitting

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$$

where $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are complex Lie subalgebras of $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}^{0,1} = \sigma(\mathfrak{g}^{1,0})$.

Proposition 6 ([15]). *There are no filiform Lie algebra admitting a complex structure.*

1.2.3. AFFINE STRUCTURES

Definition 7. An affine structure on a Lie algebra $\mathfrak{g}$ is a $\mathbb{K}$-bilinear multiplication, denoted $X \cdot Y$ which is left-symmetric, that is

$$X \cdot (Y \cdot Z) - (X \cdot Y) \cdot Z = Y \cdot (X \cdot Z) - (Y \cdot X) \cdot Z$$

for all $X, Y, Z \in \mathfrak{g}$ and satisfies

$$[X, Y] = X \cdot Y - Y \cdot X$$

where $[X, Y]$ denotes the Lie bracket of $\mathfrak{g}$.

The problem, which concerns also the linear representations of Lie algebras [7], is not completely solved even for filiform Lie algebras. We know that, as soon as the dimension is greater or equal to 10, there exist filiform Lie algebras without affine structure. However, let us recall this classical result:

Proposition 8. *Any symplectic Lie algebra is affine.*

Proof. Let $(\mathfrak{g}, \theta)$ be a symplectic Lie algebra. We consider the product XY given by $XY = f(X)Y$ where $f : \mathfrak{g} \to End(\mathfrak{g})$ is defined implicitly by $\theta(f(X)(Y), Z) = -\theta(Y, [X, Z])$ for any $X, Y, Z \in \mathfrak{g}$. Since θ is symplectic, this product XY is well defined and provides $\mathfrak{g}$ with an affine structure. □

In the case of contact Lie algebras, see for example [18].

2. FILIFORM LIE ALGEBRAS OF DIMENSION 8

2.1. Topological description

The classification of filiform Lie algebras over $\mathbb{K}$ of dimension less than or equal to 7 is well known [11]. The aim of this section is to come back to the classification proposed in [3] and to correct some inaccuracies of this last paper.

Let $\mathfrak{g}$ be a 8-dimensional filiform Lie algebra. If we denote by μ its Lie bracket and $\{X_0, \cdots, X_7\}$ a Vergne basis, then the Jacobi identity implies

$$\mu(X_0, \mu(X_i, X_j)) = \mu(X_i, X_{j+1}) + \mu(X_{i+1}, X_j).$$

These identities imply that the structure constants $C^k_{i,j}$ for $k < 7$ are linear combinations of C^7_{ij}. We deduce

PROPOSITION 9 ([3]). *Any 8-dimensional filiform Lie algebra over $\mathbb{K}$ is given in a Vergne basis by*

$$(2) \quad \begin{cases} \mu(X_0, X_i) = X_{i+1}, \ 1 \leq i \leq 6, \\ \mu(X_2, X_5) = a_1 X_7, \qquad \mu(X_1, X_5) = a_1 X_6 + a_2 X_7, \\ \mu(X_3, X_4) = -a_1 X_7, \quad \mu(X_2, X_4) = a_4 X_7, \\ \mu(X_1, X_4) = a_1 X_5 + (a_2 + a_4) X_6 + a_5 X_7, \\ \mu(X_2, X_3) = a_4 X_6 + a_6 X_7, \\ \mu(X_1, X_3) = a_1 X_4 + (a_2 + 2a_4) X_5 + (a_5 + a_6) X_6 + a_7 X_7, \\ \mu(X_1, X_2) = a_1 X_3 + (a_2 + 2a_4) X_4 + (a_5 + a_6) X_5 + a_7 X_6 + a_8 X_7, \end{cases}$$

with $a_1(5a_4 + 2a_2) = 0$.

Consequence. Let $\mathcal{L}ie_8$ be the algebraic variety over $\mathbb{K}$ of 8^3-uple $(C^k_{i,j})$ with $0 \leq i, j \leq 7$ and $0 \leq k \leq 7$ satisfying

$$\begin{cases} C^k_{i,j} = -C^k_{j,i}, \\ \sum_{l=0}^{7} C^l_{i,j} C^s_{l,k} + C^l_{j,k} C^s_{l,i} + C^l_{k,i} C^s_{l,j} = 0, \ \forall s = 0, \cdots, 7. \end{cases}$$

A 8-dimensional Lie algebra with Lie bracket μ is identified with a point of $\mathcal{L}ie_8$ considering the structural constants of μ in a given basis. An action of the algebraic group $GL(8, \mathbb{K})$ on $\mathcal{L}ie_8$ corresponds to the changes of basis and the orbit of a Lie algebra is its isomorphism class. Let $\mathcal{N}il_8$ be the algebraic subvariety of $\mathcal{L}ie_8$ whose elements correpond to the 8-dimensional nilpotent Lie algebra and $\mathcal{F}il_8$ be the set of 8-dimensional filiform Lie algebras. This is a Zariski open subset of $\mathcal{N}il_8$ and from Proposition 9 it is the orbit of the subvariety Fil_8 of $\mathcal{N}il_8$ whose elements are the Lie algebras defined in (2). We deduce that the study of the open set $\mathcal{F}il_8$ can be deduced directly from the study of Fil_8.

The set Fil_8 is an algebraic variety embedded in $\mathbb{K}^8$ and parametrized by the structural constants $a_1, a_2, a_4, a_5, a_6, a_7, a_8$. It is the union of two irreducible connected algebraic components

(1) $Fil_8(1)$ defined by $a_1 = 0$ which is a 6-dimensional plane,

(2) $Fil_8(2)$ defined by $5a_4 + 2a_2 = 0$ which is also a 6-dimensional plane.

We deduce that $\mathcal{F}il_8$ is the union of two irreducible algebraic components, $\mathcal{F}il_8(1)$ and $\mathcal{F}il_8(2)$ which are respectively the orbits of $Fil_8(1)$ and $Fil_8(2)$.

2.2. Deformations

In the following, we identify the Lie bracket of a 8-dimensional nilpotent (resp. filiform) Lie algebra with the point (C_{ij}^k) of $\mathcal{N}il_8$ (resp. $\mathcal{F}il_8$) where the $C_{i,j}^k$ are its structural constants related to the given Vergne basis $\{X_0, \cdots, X_7\}$.

Definition 10. Let μ_0 be a Lie bracket belonging to $\mathcal{F}il_8$. A deformation μ in $\mathcal{F}il_8$ of μ_0 is a formal deformation in the Gerstenhaber sense such that $\mu \in \mathcal{F}il_8 \otimes \mathbb{K}[[t]]$.

From [9], any deformation in $\mathcal{F}il_8$ is isomorphic to a linear deformation $\mu_0 + t\psi$ where ψ is a bilinear skew-symmetric application which is a 2-cocycle for the Chevalley-Eilenberg cohomology of μ_0 and satisfying also the Jacobi identity. Moreover since $\mu_0 + t\psi$ is filiform, ψ is a nilpotent Lie bracket.

The description of deformations in $\mathcal{F}il_8$ of $\mu_0 \in \mathcal{F}il_8$ reduces to the study of bilinear skew-symmetric applications ψ such that $\mu_0 + t\psi$ is in $\mathcal{F}il_8$.

LEMMA 11. *Let μ_0 be in Fil_8. Then $\mu_0 + t\psi$ is a linear deformation in Fil_8 of μ_0 if and only if ψ is given by*

$$(3)\quad \begin{cases} \psi(X_2, X_5) = u_1 X_7, \qquad \psi(X_1, X_5) = u_1 X_6 + u_2 X_7, \\ \psi(X_3, X_4) = -u_1 X_7, \quad \psi(X_2, X_4) = u_4 X_7, \\ \psi(X_1, X_4) = u_1 X_5 + (u_2 + u_4) X_6 + u_5 X_7, \\ \psi(X_2, X_3) = u_4 X_6 + u_6 X_7, \\ \psi(X_1, X_3) = u_1 X_4 + (u_2 + 2u_4) X_5 + (u_5 + u_6) X_6 + u_7 X_7, \\ \psi(X_1, X_2) = u_1 X_3 + (u_2 + 2u_4) X_4 + (u_5 + u_6) X_5 + u_7 X_6 + u_8 X_7, \end{cases}$$

with $u_1(5a_4+2a_2)+a_1(5u_4+2u_2)+tu_1(5u_4+2u_2) = 0$ where $(a_1, a_2, a_4, a_5, a_6, a_7, a_8)$ are the parameters of μ_0.

2.3. Study of the component $\mathcal{F}il_8(1)$

Any Lie algebra in $\mathcal{F}il_8(1)$ is isomorphic to a Lie algebra of $Fil_8(1)$ whose Lie bracket is defined by

$$(4)\quad \begin{cases} \mu_0(X_0, X_i) = X_{i+1}, \ 1 \leq i \leq 6, \\ \mu_0(X_1, X_5) = a_2 X_7, \quad \mu_0(X_2, X_4) = a_4 X_7, \\ \mu_0(X_1, X_4) = (a_2 + a_4) X_6 + a_5 X_7, \\ \mu_0(X_2, X_3) = a_4 X_6 + a_6 X_7, \\ \mu_0(X_1, X_3) = (a_2 + 2a_4) X_5 + (a_5 + a_6) X_6 + a_7 X_7, \\ \mu_0(X_1, X_2) = (a_2 + 2a_4) X_4 + (a_5 + a_6) X_5 + a_7 X_6 + a_8 X_7. \end{cases}$$

We have seen that $Fil_8(1)$ is an algebraic subvariety of $\mathcal{N}il_8$. In [13], we defined for each set k-$\mathcal{N}il_n$ of k-step nilpotent n-dimensional Lie algebras a cochain complex whose associated cohomology parametrizes the "internal" deformations, that is deformations of k-step nilpotent Lie algebras which are also k-step nilpotent. When k is maximal, that is for the filiform case, this cohomology is the Vergne cohomology because any nilpotent deformation of a filiform Lie algebra is always filiform. We consider here the same approach, considering the cohomology adapted to the internal deformations in $Fil_8(1)$ that is deformations of elements of $Fil_8(1)$ which remain in this variety. Since we are only concerned by the second space of this cohomology, we shall describe it. Let μ be in $Fil_8(1)$. We denote by $Z^2_{CR}(\mu,\mu)$ the space of 2-cochains, that is bilinear skew-symmetric maps ψ on $\mathbb{K}^8$ with values on $\mathbb{K}^8$ which are defined by

$$(5)\qquad \begin{cases} \psi(X_1,X_5)=u_2X_7, \quad \psi(X_2,X_4)=u_4X_7,\\ \psi(X_1,X_4)=(u_2+u_4)X_6+u_5X_7,\\ \psi(X_2,X_3)=u_4X_6+u_6X_7,\\ \psi(X_1,X_3)=(u_2+2u_4)X_5+(u_5+u_6)X_6+u_7X_7,\\ \psi(X_1,X_2)=(u_2+2u_4)X_4+(u_5+u_6)X_5+u_7X_6+u_8X_7. \end{cases}$$

The subscript 'CR' comes from the "restricted Chevalley complex", whose cohomology spaces will also be used later. If ∂_μ is the coboundary operator of the Chevalley-Eilenberg complex of μ, then $\partial_\mu(\psi)=0$ and any cochain is closed. Let $B^2_{CR}(\mu,\mu)$ the space of $\partial_\mu(f)$ for $f\in End(\mathbb{K}^8)$ such that $f(X_0)=\sum_{i=0}^7\alpha_iX_i$ and $f(X_1)=\sum_{i=1}^7\beta_iX_i$. We have

$$(6)\qquad \begin{cases} \delta f(X_1,X_5)=v_2X_7, \quad \delta f(X_2,X_4)=v_4X_7,\\ \delta f(X_1,X_4)=(v_2+v_4)X_6+v_5X_7,\\ \delta f(X_2,X_3)=v_4X_6+v_6X_7,\\ \delta f(X_1,X_3)=(v_2+2v_4)X_5+(v_5+v_6)X_6+v_7X_7,\\ \delta f(X_1,X_2)=(v_2+2v_4)X_4+(v_5+v_6)X_5+v_7X_6+v_8X_7, \end{cases}$$

with

$$(7)\qquad \begin{cases} v_2=a_2(\beta_1-2\alpha_0), \quad v_4=a_4(\beta_1-2\alpha_0),\\ v_5=a_5(\beta_1-3\alpha_0)+\alpha_1(-2a_2^2-5a_2a_4-5a_4^2),\\ v_6=a_6(\beta_1-3\alpha_0)+\alpha_1(-3a_2a_4-3a_4^2),\\ v_7=a_7(\beta_1-4\alpha_0)-2a_4\beta_3-\alpha_1(a_5+a_6)(5a_2+9a_4),\\ v_8=a_8(\beta_1-5\alpha_0)-3a_7\alpha_1(2a_2+3a_4)-2a_6\beta_3-3a_4\beta_4\\ \qquad +3\alpha_3a_4(a_2+2a_4)-\alpha_1(a_5+a_6)(3a_5+2a_6). \end{cases}$$

Then $B^2_{CR}(\mu,\mu)$ is a linear subspace of $Z^2_{CR}(\mu,\mu)$ and the quotient space

$$H^2_{CR}(\mu,\mu)=Z^2_{CR}(\mu,\mu)/B^2_{CR}(\mu,\mu)$$

parametrizes the deformations in $Fil_8(1)$. We deduce, using the classical theory of Nijenhuis-Richardson, that a Lie algebra $\mathfrak{g}$ with bracket μ such that

$H^2_{CR}(\mu,\mu) = 0$ is rigid in $Fil_8(1)$, that is, its orbit in Fil_8 is open. But, since $Fil_8(1)$ is isomorphic to a linear 6-dimensional space, its associated affine scheme is naturally reduced and the converse is also true: if $\mathfrak{g}$ is rigid, then $\dim H^2_{CR}(\mu,\mu) = 0$. But any cocycle ψ with $u_2u_4 \neq 0$ cannot be cohomologous with a cocycle where $u_2 = u_4 = 0$. We deduce that $\dim H^2_{CR}(\mu,\mu) \geq 1$. We deduce

PROPOSITION 12. *No Lie algebra belonging to $Fil_8(1)$ is rigid in $\mathcal{F}il_8$ and also in $\mathcal{N}il_8$ and in $\mathcal{L}ie_8$ where $\mathcal{N}il_8$ (respectively $\mathcal{L}ie_8$) is the algebraic variety of 8-dimensional nilpotent Lie algebras (respectively the algebraic variety of 8-dimensional Lie algebras).*

Let us determine the Lie algebras of this component which satisfy $\dim H^2_{CR}(\mu,\mu) = 1$. Let $\mathfrak{g}$ be such a Lie algebra. From the previous remark, any cocycle $\psi \in Z^2_{CR}(\mu,\mu)$ must be cohomologous to a cocycle with $u_5 = u_6 = u_7 = u_8 = 0$ and a_2 or a_4 non zero. Suppose $a_4 \neq 0$. The coefficients β_3 and β_4 can always be chosen such that $u_7 - v_7 = 0$ and $u_8 - v_8 = 0$.

(1) If $a_2 + a_4 \neq 0$ and $a_5 \neq 0$ we can choose $\alpha_1, \beta_1 - 3\alpha_0$ and $\beta_1 - 2\alpha_0$ such that $u_6 - v_6 = u_5 - v_5 = u_2 - v_2 = 0$. The corresponding Lie algebra satisfies $\dim H^2_{CR}(\mu,\mu) = 1$.

(2) If $a_2 + a_4 \neq 0$, $a_5 = 0, a_6 \neq 0$ and $2a_2^2 + 5a_2a_4 + 5a_4^2 \neq 0$ that is $\dfrac{a_2}{a_4} \neq \dfrac{-5 \pm i\sqrt{15}}{4}$ (here $\mathbb{K} = \mathbb{C}$) we can choose $\alpha_1, \beta_1 - 3\alpha_0$ and $\beta_1 - 2\alpha_0$ such that $u_6 - v_6 = u_5 - v_5 = u_2 - v_2 = 0$. The corresponding Lie algebra satisfies $\dim H^2_{CR}(\mu,\mu) = 1$.

(3) $a_2 + a_4 = 0$ and $a_6 \neq 0$ we can choose $\alpha_1, \beta_1 - 3\alpha_0$ and $\beta_1 - 2\alpha_0$ such that $u_6 - v_6 = u_5 - v_5 = u_2 - v_2 = 0$. The corresponding Lie algebra satisfies $\dim H^2_{CR}(\mu,\mu) = 1$.

In all other cases $\dim H^2_{CR}(\mu,\mu) > 1$.

To simplify denote by $\mu(a_2, a_4, a_5, a_6, a_7, a_8)$ a Lie bracket of a Lie algebra belonging to $Fil_8(1)$. The previous computations shows that the Lie algebras $\mu(\alpha, 1, 0, 1, 0, 0)$ with α such that $2\alpha^2 + 5\alpha + 5 \neq 0$ and $\mu(\alpha, 1, 1, 0, 0, 0)$ with $\alpha \neq -1$ satisfy $\dim H^2_{CR}(\mu,\mu) = 1$. From [3], these two Lie algebras are isomorphic. So consider the one-parameter family $\mathcal{T}^1_\alpha$ constituted of $\mu(\alpha, 1, 1, 0, 0, 0)$ with $\alpha \neq -1$. Any deformation in $Fil_8(1)$ of an algebra of this family belongs to this family. Since $Fil_8(1)$ is a 6-dimensional plane, a reduced algebraic variety, we deduce that the closure of $\mathcal{T}^1_\alpha$ is $Fil_8(1)$.

PROPOSITION 13. *The family $\mathcal{T}^1_\alpha$*

$$(8)\quad \begin{cases} \mu(X_0, X_i) = X_{i+1},\ 1 \leq i \leq 6, & \mu(X_1, X_5) = \alpha X_7, \quad \mu(X_2, X_4) = X_7, \\ \mu(X_1, X_4) = (\alpha+1)X_6 + X_7, & \mu(X_2, X_3) = X_6, \\ \mu(X_1, X_3) = (\alpha+2)X_5 + X_6, & \mu(X_1, X_2) = (\alpha+2)X_4 + X_5. \end{cases}$$

is rigid in $Fil_8(1)$ and $\mathcal{F}il_8(1) = \overline{\mathcal{O}(\mathcal{T}^1_\alpha)}$.

Remarks.

(1) This generalized notion of rigidity which concerns a one-parameter family of Lie algebras has already been defined in [9].

(2) To compare these results with [3], we give the complex classification, up to isomorphism, of elements of Fil_8. Recall that such results would be utopic to establish for greater dimensions.

PROPOSITION 14. *Let us write* $(a_2, a_4, a_5, a_6, a_7, a_8)$ *a Lie algebra of* $Fil_8(1)$. *Then any Lie algebra of* $Fil_8(1)$ *is isomorphic to one of the following:*

$$\begin{array}{llll} (\lambda,1,-1,1,0,0) & (\lambda,1,0,0,0,0) & (-2,1,1,0,0,0) & (1,0,-1,1,\lambda,0) \\ (0,0,\lambda,1,1,0) & (0,0,\lambda,1,0,0) & (\lambda,0,0,0,1,1) & (1,0,0,0,1,0) \\ (1,0,0,0,0,1) & (1,0,0,0,0,0) & (0,0,1,0,1,0) & (0,0,1,0,0,0) \\ (0,0,0,0,1,0) & (0,0,0,0,0,1) & (0,0,0,0,0,0). & \end{array}$$

2.4. Study of the component $\mathcal{F}il_8(2)$

We consider the Lie algebras of (2) with $a_4 = -\frac{2}{5}a_2$. Any Lie algebra of $\mathcal{F}il_8(2)$ is isomorphic to a Lie algebra of $Fil_8(2)$ with Lie bracket defined by:

$$(9)\quad \begin{cases} \mu(X_0, X_i) = X_{i+1},\ 1 \leq i \leq 6, \\ \mu(X_2, X_5) = a_1X_7, \qquad \mu(X_1, X_5) = a_1X_6 + a_2X_7, \\ \mu(X_3, X_4) = -a_1X_7, \quad \mu(X_2, X_4) = -\dfrac{2}{5}a_2X_7, \\ \mu(X_1, X_4) = a_1X_5 + \dfrac{3}{5}a_2X_6 + a_5X_7, \quad \mu(X_2, X_3) = -\dfrac{2}{5}a_2X_6 + a_6X_7, \\ \mu(X_1, X_3) = a_1X_4 + \dfrac{1}{5}a_2X_5 + (a_5 + a_6)X_6 + a_7X_7, \\ \mu(X_1, X_2) = a_1X_3 + \dfrac{1}{5}a_2X_4 + (a_5 + a_6)X_5 + a_7X_6 + a_8X_7, \end{cases}$$

Then $Fil_8(2)$ is a 6-dimensional plane parametrized by $(a_1,a_2,a_5,a_6,a_7,a_8)$. We consider similarly to the previous section the linear deformations in $Fil_8(2)$ of the Lie brackets belonging to $Fil_8(2)$. We denote always by $H^2_{CR}(\mu,\mu)$ the space which parametrizes these deformations. The space of 2-cocycles $Z^2_{CR}(\mu,\mu)$ is constituted of the skew-symmetric bilinear applications ψ given by

$$(10)\quad \begin{cases} \psi(X_2, X_5) = u_1X_7, \qquad \psi(X_1, X_5) = u_1X_6 + u_2X_7, \\ \psi(X_3, X_4) = -u_1X_7, \quad \psi(X_2, X_4) = -\dfrac{2}{5}u_2X_7, \\ \psi(X_1, X_4) = u_1X_5 + \dfrac{3}{5}u_2X_6 + u_5X_7,\ \psi(X_2, X_3) = -\dfrac{2}{5}u_2X_6 + u_6X_7, \\ \psi(X_1, X_3) = u_1X_4 + \dfrac{1}{5}u_2X_5 + (u_5 + u_6)X_6 + u_7X_7, \\ \psi(X_1, X_2) = u_1X_3 + \dfrac{1}{5}u_2X_4 + (u_5 + u_6)X_5 + u_7X_6 + u_8X_7. \end{cases}$$

Let f be a linear endomorphism of $\mathfrak{g}$ such that $\delta f \in Z^2_{CR}(\mu,\mu)$. If $f(X_0) = \sum_{i=0}^{7} \alpha_i X_i$ and $f(X_1) = \sum_{i=1}^{7} \beta_i X_i$ then

$$(11)\qquad \left\{\begin{array}{ll} \delta f(X_2,X_5) = v_1 X_7, & \delta f(X_1,X_5) = v_1 X_6 + v_2 X_7, \\ \delta f(X_3,X_4) = -v_1 X_7, & \delta f(X_2,X_4) = -\dfrac{2}{5} v_2 X_7, \\ \multicolumn{2}{l}{\delta f(X_1,X_4) = v_1 X_5 + \dfrac{3}{5} v_2 X_6 + v_5 X_7,} \\ \multicolumn{2}{l}{\delta f(X_2,X_3) = -\dfrac{2}{5} v_2 X_6 + v_6 X_7,} \\ \multicolumn{2}{l}{\delta f(X_1,X_3) = v_1 X_4 + \dfrac{1}{5} v_2 X_5 + (v_5+v_6) X_6 + v_7 X_7,} \\ \multicolumn{2}{l}{\delta f(X_1,X_2) = v_1 X_3 + \dfrac{1}{5} v_2 X_4 + (v_5+v_6) X_5 + v_7 X_6 + v_8 X_7,} \end{array}\right.$$

with
(12)

$$\left\{\begin{array}{l} v_1 = a_1(\beta_1 - \alpha_0 - \alpha_1 a_1), \\ v_2 = a_2(\beta_1 - 2\alpha_0 - 3\alpha_1 a_1), \\ v_5 = a_5(\beta_1 - 3\alpha_0 - 5a_1\alpha_1) - \alpha_1(2a_1a_6 + \frac{4}{5}a_2^2) + 2\alpha_3 a_1^2 - 2\beta_3 a_1, \\ v_6 = a_6(\beta_1 - 3\alpha_0 - 2a_1\alpha_1) + \alpha_1(a_1a_5 + \frac{18}{25}a_2^2) - 2\alpha_3 a_1^2 + 2\beta_3 a_1, \\ v_7 = a_7(\beta_1 - 4\alpha_0 - 5a_1\alpha_1) - \frac{7}{5}\alpha_1 a_2(a_5+a_6) - \frac{4}{5}\alpha_3 a_1 a_2 + \frac{4}{5}\beta_3 a_2, \\ v_8 = a_8(\beta_1 - 5\alpha_0 - 5a_1\alpha_1) - \alpha_1(\frac{12}{5}a_2a_7 + (a_5+a_6)(3a_5+2a_6)) + \\ \qquad \alpha_3(2a_1(a_5+2a_6) - \frac{6}{25}a_2^2) - \frac{4}{5}\alpha_4 a_1 a_2 + 2\alpha_5 a_1^2 - 2\beta_3 a_6 + \frac{6}{5}\beta_4 a_2 - 2\beta_5 a_1, \end{array}\right.$$

(1) If $a_1 = a_2 = 0$ then $\dim H^2_{CR}(\mu,\mu) \geq 3$.
(2) If $a_1 = 0$ and $a_2 \neq 0$, then $\dim H^2_{CR}(\mu,\mu) \geq 2$.
(3) If $a_2 = 0, a_1 \neq 0$ then $\dim H^2_{CR}(\mu,\mu) \geq 1$.
(4) Assume now $a_2 \neq 0, a_1 \neq 0$, We will develop this case because it is the part of [3] which presents an inaccuaracy. Let us compute the kernel of the linear system $\{v_i = 0\}$. Since $a_1a_2 \neq 0$ then $v_1 = v_2 = 0$ is equivalent to

$$\beta_1 = -\alpha_1 a_1,\ \alpha_0 = -2\alpha_1 a_1.$$

We can also choose β_5 to obtain $v_8 = 0$. Then the system is reduced to

$$\left\{\begin{array}{l} v_5 = -\alpha_1(2a_6a_1 + \frac{4}{5}a_2^2) + 2\alpha_3 a_1^2 - 2\beta_3 a_1, \\ v_6 = (3a_6a_1 + a_5a_1 + \frac{18}{25}a_2^2)\alpha_1 - 2\alpha_3 a_1^2 + 2\beta_3 a_1, \\ v_7 = (2a_7a_1 - \frac{7}{5}a_2(a_5+a_6))\alpha_1 - \frac{4}{5}\alpha_3 a_1 a_2 + \frac{4}{5}\beta_3 a_2, \end{array}\right.$$

The matrix of this system is

$$M = \begin{pmatrix} -2a_6a_1 - \frac{4}{5}a_2^2 & 2a_1^2 & -2a_1 \\ 3a_6a_1 + a_5a_1 + \frac{18}{25}a_2^2 & -2a_1^2 & 2a_1 \\ 2a_7a_1 - \frac{7}{5}a_2(a_5+a_6) & -\frac{4}{5}a_1a_2 & \frac{4}{5}a_2 \end{pmatrix}$$

Since this matrix is singular, then $\dim \mathrm{Ker} M \geq 1$ and $\dim H^2_{CR}(\mu,\mu) \geq 1$.

Since the affine scheme associated with this component is reduced, the Lie algebras with $a_1a_2 \neq 0$ are not rigid. We can now study the conditions to have $\dim H^2_{CR}(\mu,\mu) = 1$, this is equivalent to $\mathrm{rank}(M) = 2$ that is

$$a_1(a_5 + a_6) - \frac{2}{25}a_2^2 \neq 0$$

or

$$a_1a_2\left(\frac{18}{5}a_5 + \frac{26}{5}a_6\right) + \frac{72}{125}a_2^3 - 4a_1^2a_7 \neq 0.$$

In particular, we can take $a_1 = a_2 = 1, a_5 = -a_6 = t$. For each value of t, the dimension of $H_{2,r}(\mathfrak{g},\mathfrak{g})$ of the corresponding Lie algebra is equal to 1. We deduce

Proposition 15. *None of the Lie algebras of $Fil_8(2)$ are rigid in $\mathcal{F}il_8$. This component is the closure of the one-dimensional rigid family $\mathcal{T}_t^2(8)$ of Lie algebras isomorphic to*

$$(13)\qquad \begin{cases} \mu(X_0,X_i) = X_{i+1},\ 1 \leq i \leq 6, & \\ \mu(X_2,X_5) = X_7, & \mu(X_1,X_5) = X_6 + X_7, \\ \mu(X_3,X_4) = -X_7, & \mu(X_2,X_4) = -\frac{2}{5}X_7, \\ \mu(X_1,X_4) = X_5 + \frac{3}{5}X_6 + tX_7, & \mu(X_2,X_3) = -\frac{2}{5}X_6 - tX_7, \\ \mu(X_1,X_3) = X_4 + \frac{1}{5}X_5, & \mu(X_1,X_2) = X_3 + \frac{1}{5}X_4. \end{cases}$$

Remark. The problem of classification of 8-dimensional filiform Lie algebras has been already solved. We can find this classification in [9]. A lot of the results previously obtained of course are direct consequences of this classification. But to obtain a general result of the classification problem is certainly utopian. This implies to develop another way. We began a new approach in [13] by considering subfamilies of k-step nilpotent Lie algebras and defining an adapted cohomology of deformations. The previous calculus is performed in this way. Nevertheless, since this classification is known and since we have given this classification for the algebras of the first component, it would be surprising not to give it for the second component.

Proposition 16. *Let $\mathfrak{g}$ be a filiform Lie algebra belonging to $\mathcal{F}il_8(2)$. Then any Lie algebra of $Fil_8(2)$ is isomorphic to one of the following corresponding to*

$$(a_1,a_2,a_5,a_6,a_7,a_8) \in \{(1,0,0,0,0,0),(1,0,0,0,1,0),(1,0,1,0,\lambda,0),(1,1,\lambda,-2,0,0)\}.$$

2.5. Symplectic structures

We determine all filiform 8-dimensional symplectic Lie algebras. A direct approach consists to write the conditions related to the existence of a symplectic form for Lie algebras belonging to each component. Let $\mathfrak{g} \in Fil_8$ and θ be a closed 2-form on $\mathfrak{g}$. Let $\{X_0, \cdots, X_7\}$ be a Vergne basis (2). Computing $d\theta(X_0, X_i, X_7) = 0$, we obtain that $\theta(X_i, X_7) = 0$ for $i = 2, \cdots, 6$. As a consequence, $\theta(X_i, X_j) = 0$ as soon as $i + j \geq 9$ and $d\theta(X_i, X_j, X_k) = 0$ is always satisfied for $i+j+k \geq 9$. We call the weight of the equation $d\theta(X_i, X_j, X_k) = 0$ the integer $p = i + j + k$ and we solve this equation for $p = 8, 7, \cdots, 3$.

(1) $p = 8$.

$$\begin{cases} 0 = & (3a_1 + 2a_3)\theta(X_3, X_5) - a_1\theta(X_1, X_7) + a_1\theta(X_2, X_6), \\ 0 = & (2a_1 + a_3)\theta(X_3, X_5) - a_3\theta(X_1, X_7). \end{cases}$$

(2) $p = 7$.

$$\begin{cases} 0 = & \theta(X_2, X_6) + \theta(X_1, X_7), \\ 0 = & \theta(X_3, X_5) - a_1\theta(X_0, X_7) + \theta(X_2, X_6), \\ 0 = & a_3\theta(X_0, X_7) - \theta(X_3, X_5), \\ 0 = & (3a_1 + 2a_3)\theta(X_3, X_4) - (a_1 + a_3)\theta(X_1, X_6) - a_4\theta(X_1, X_7) \\ & +(2a_1 + a_3)\theta(X_2, X_5) + (a_2 + a_4)\theta(X_2, X_6). \end{cases}$$

Then, we have to consider the matrix

$$M = \begin{pmatrix} 0 & -a_1 & a_1 & 3a_1 + 2a_3 \\ 0 & -a_3 & 0 & 2a_1 + a_3 \\ 0 & 1 & 1 & 0 \\ -a_1 & 0 & 1 & 1 \\ a_3 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -a_1 & a_1 & a_1 \\ 0 & a_1 & 0 & a_1 \\ 0 & 1 & 1 & 0 \\ -a_1 & 0 & 1 & 1 \\ -a_1 & 0 & 0 & -1 \end{pmatrix}$$

If θ is symplectic, one of the scalar $\theta(X_0, X_7)$ or $\theta(X_1, X_7)$ is non zero which is equivalent to say that the rank of M is less than 4. But $rank M = 4$ if and only if $a_1 \neq 0$ so any 8-dimensional filiform symplectic Lie algebra $\mathfrak{g}$ belongs to $Fil_8(1)$. Moreover, the symplectic form satisfies $\theta(X_i, X_j) = 0$ for $i + j \geq 8$. If we compute the relations of weight 6 we obtain

$$\begin{cases} 0 = & \theta(X_2, X_5) + \theta(X_1, X_6) - a_2\theta(X_0, X_7), \\ 0 = & \theta(X_3, X_4) + \theta(X_2, X_5) - a_4\theta(X_0, X_7), \\ 0 = & (a_2 + 2a_4)\theta(X_3, X_4) - (a_2 + 2a_4)\theta(X_2, X_5) + a_4\theta(X_1, X_6). \end{cases}$$

But θ is non degenerate if and only if $\theta(0, 7)\theta(1, 6)\theta(2, 5)\theta(3, 4) \neq 0$. This implies

(1) $2a_2 + 5a_4 \neq 0$, that is $\mathfrak{g} \in Fil_8(1)$ and $\mathfrak{g} \notin Fil_8(2)$ and $a_4(a_2 + 2a_4)(2a_2 - a_4) \neq 0$

(2) or $a_2 = a_4 = 0$.

PROPOSITION 17. *An 8-dimensional filiform Lie algebra is symplectic if and only if it is isomorphic to a Lie algebra* $\mathfrak{g} \in Fil_8(1) - Fil_8(2)$ *and* $a_4(a_2 + a_4)(2a_2 - a_4)(a_2 + 2a_4) \neq 0$ *or* $\mathfrak{g} \in Fil_8(1) \cap Fil_8(2)$ *and* $a_2 = a_4 = 0$.

Remark that the symplectic Lie algebras in $Fil_8(1) - Fil_8(2)$ form a Zariski dense subset of $Fil_8(1) - Fil_8(2)$. We can also note that any Lie algebra of the rigid family $\mathcal{T}^1_\alpha$ is symplectic, except for three values of α which are $-2, \frac{1}{2}, -\frac{5}{2}$.

2.6. **Determination of symplectic 8-dimensional filiform Lie algebras using contact 9-dimensional filiform Lie algebras.**

If $(\mathfrak{g}, \theta)$ is a $2p$-dimensional symplectic Lie algebra, then the Lie algebra $\mathfrak{g}_\theta$ the one-dimensional central extension

$$\mathfrak{g}_\theta = \mathfrak{g} \oplus_\theta \mathbb{K}Z.$$

Recall that the Lie bracket $\mu_1$of $\mathfrak{g}_\theta$ is given by

$$\begin{cases} \mu_1(X,Y) = \theta(X,Y)Z + \mu(X,Y), \\ \mu_1(X,Z) = 0 \end{cases}$$

for any $X, Y \in \mathfrak{g}$, $\mathfrak{g}_\theta$ is a contact $(2p+1)$-dimensional Lie algebra and Z generates the center. From Proposition 4, $\mathfrak{g}$ is a factor algebra $\mathfrak{g}_1/Z(\mathfrak{g}_1)$ of a filiform contact algebra $\mathfrak{g}_1$ and the linear form ω_{2p} is a contact form. Then we can determine all the symplectic filiform algebras in dimension 8 starting from the contact filiform 9-dimensional Lie algebras. This study is the aim of the last section, but we can already use these results, all the proofs are given in the following section.

PROPOSITION 18. *Any* 9*-dimensional filiform Lie algebra provided with a contact form is isomorphic to a Lie algebra of the following family:*

$$\begin{cases} \mu(X_0, X_i) = X_{i+1}, \ 1 \leq i \leq 7, \\ \mu(X_1, X_6) = a_2 X_8, & \mu(X_2, X_5) = a_4 X_8, \\ \mu(X_1, X_5) = (a_2 + a_4) X_7 + a_5 X_8, & \mu(X_3, X_4) = a_6 X_8 \\ \mu(X_2, X_4) = (a_4 + a_6) X_7 + a_7 X_8, \\ \mu(X_1, X_4) = (a_2 + 2a_4 + a_6) X_6 + (a_5 + a_7) X_7 + a_8 X_8, \\ \mu(X_2, X_3) = (a_4 + a_6) X_6 + a_7 X_7 + a_9 X_8, \\ \mu(X_1, X_3) = (a_2 + 3a_4 + 2a_6) X_5 + (a_5 + 2a_7) X_6 + (a_8 + a_9) X_7 + a_{10} X_8, \\ \mu(X_1, X_2) = (a_2 + 3a_4 + 2a_6) X_4 + (a_5 + 2a_7) X_5 + (a_8 + a_9) X_6 \\ \qquad + a_{10} X_7 + a_{11} X_8, \end{cases}$$

with $-3a_4^2 + 2a_6^2 + 2a_2a_6 + a_4a_6 = 0$ *and* $a_2a_4a_6 \neq 0$.

Since the center is generated by X_8, we deduce

Proposition 19. *Any symplectic 8-dimensional filiform Lie algebra is isomorphic to a Lie algebra of the following family*

$$\left\{\begin{array}{l}\mu(X_0,X_i)=X_{i+1},\ 1\leq i\leq 6,\\ \mu(X_1,X_5)=(a_2+a_4)X_7,\\ \mu(X_2,X_4)=(a_4+a_6)X_7,\\ \mu(X_1,X_4)=(a_2+2a_4+a_6)X_6+(a_5+a_7)X_7,\\ \mu(X_2,X_3)=(a_4+a_6)X_6+a_7X_7,\\ \mu(X_1,X_3)=(a_2+3a_4+2a_6)X_5+(a_5+2a_7)X_6+(a_8+a_9)X_7,\\ \mu(X_1,X_2)=(a_2+3a_4+2a_6)X_4+(a_5+2a_7)X_5+(a_8+a_9)X_6+a_{10}X_7\end{array}\right.$$

with $-3a_4^2+2a_6^2+2a_2a_6+a_4a_6=0$ *and* $a_2a_4a_6\neq 0$. *This is equivalent to say*

We come back to the notations (2) and we put

$$b_2=a_2+a_4,\ b_4=a_4+a_6, b_5=a_5+a_7,\ b_6=a_7, b_7=a_8+a_9, b_8=a_{10}.$$

Then the conditions $-3a_4^2+2a_6^2+2a_2a_6+a_4a_6=0$ and $a_2a_4a_6\neq 0$ imply that

$$b_2=b_4=0,\ \text{or}\ b_4(b_2+b_4)(2b_2-b_4)(b_2+2b_4)\neq 0.$$

Any symplectic 8-dimensional filiform Lie algebra is isomorphic to a Lie algebra of the following family

$$\left\{\begin{array}{l}\mu(X_0,X_i)=X_{i+1},\ 1\leq i\leq 6,\\ \mu(X_1,X_5)=b_2X_7,\\ \mu(X_2,X_4)=b_4X_7,\\ \mu(X_1,X_4)=(b_2+b_4)X_6+b_5X_7,\\ \mu(X_2,X_3)=b_4X_6+b_6X_7,\\ \mu(X_1,X_3)=(b_2+2b_4)X_5+(b_5+b_6)X_6+b_7X_7,\\ \mu(X_1,X_2)=(b_2+2b_4)X_4+(b_5+b_6)X_5+b_7X_6+b_8X_7,\end{array}\right.$$

with $b_2=b_4=0$ or $b_4(b_2+b_4)(b_2-b_4)(b_2+2b_4)\neq 0$. We find again all the conditions of Proposition 17.

This last way to determine the symplectic structures permits also the introduce of a notion of symplectic deformation. Recall that a deformation of a symplectic Lie algebra can be non symplectic. The simplest example is given by the even dimensional abelian Lie algebra. This algebra is symplectic and any Lie algebra is isomorphic to a deformation of this abelian algebra and it is clear that non symplectic Lie algebras exist as soon as the dimension is strictly greater than 2. Likewise if a symplectic Lie algebra $\mathfrak{g}_1$ is a deformation of a Lie algebra $\mathfrak{g}_0$, this last is not necessarily symplectic. Then the classical notion of deformation is not well adapted to the notion of symplectic structures. But it is not the case for the contact structures. Any deformation of a contact Lie algebra is still a contact Lie algebra (see also [14]). This remark leads to introduce a restricted notion of deformation that we shall call **symplectic deformation**:

Definition 20. Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be two symplectic 8-dimensional filiform Lie algebras and let $\mathfrak{g}_0^9$ and $\mathfrak{g}_1^9$ be 9-dimensional contact filiform Lie algebras such that $\mathfrak{g}_i = \pi(\mathfrak{g}_i^9)$, $i = 1, 2$ where π is the canonical projection $\pi : \mathfrak{g} \to \mathfrak{g}/Z(\mathfrak{g})$. We say that $\mathfrak{g}_1$ is a symplectic deformation of $\mathfrak{g}_0$ if $\mathfrak{g}_1^9$ is a (classical) deformation of $\mathfrak{g}_0^9$.

2.7. **Affine structures**

From [7], we know that any 8-filiform Lie algebra admits an affine structure. To prove this, we construct affine structures of adjoint type, that is, if L_i denote the linear map $L_i(X) = X_i X$ then $L_0 = adX_0$. Since L_i for $i \geq 2$ is given by $L_i = [L_0, L_{i-1}]$, such affine structure is completely determinate by L_1. For exemple if we consider the rigid family $\mathcal{T}_t^2(8)$ in $Fil_8(2)$, we consider for L_1 the linear map whose matrix in a Vergne's basis is

$$\begin{pmatrix}
0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2(70\alpha_6-25t-42)}{375} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\alpha_3 & \frac{70\alpha_6-25t-42}{375} & -\frac{2}{25}(5\alpha_6-3) & \frac{1}{5} & 0 & 0 & 0 \\
0 & \alpha_4 & \alpha_3 & \alpha_5 & \frac{t}{2}-\frac{3}{25}(5\alpha_6-3) & \alpha_6 & -\frac{1}{2} & 0
\end{pmatrix}$$

or

$$\begin{pmatrix}
0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2(-210\alpha_6+125t-42)}{375} & 0 & -\frac{2}{5} & 0 & 0 & 0 & 0 \\
0 & \frac{2(-210\alpha_6+125t-42)}{375} & \frac{-210\alpha_6+125t-42}{375} & -\frac{14}{25}(5\alpha_6+1) & -\frac{3}{5} & 0 & 0 & 0 \\
0 & \alpha_4 & \alpha_3 & \alpha_5 & \frac{t}{2}-\frac{21}{25}(5\alpha_6+1) & \alpha_6 & -\frac{1}{2} & 0
\end{pmatrix}$$

Note that all these affine structures are complete, that is the linear map $R_Y : X \to X \cdot Y$ is nilpotent for any Y, or equivalently the trace of R_Y is zero.

Note also that the linear representation of $\mathfrak{g}$:

$$\rho : \mathfrak{g} \to End(\mathfrak{g})$$

given by $\rho(X) = L_X$ is not faithful because in all the previous cases $L_{X_7} = 0$.

Remark. Let us consider the polarization of the product $\nabla(X,Y) = X \cdot Y$. We put

$$\mu(X,Y) = \nabla(X,Y) - \nabla(Y,X), \quad s(X,Y) = \nabla(X,Y) + \nabla(Y,X).$$

Since $\mathbb{K}$ is of characteristic not 2, then

$$\nabla(X,Y) = \frac{s(X,Y) + \mu(X,Y)}{2}$$

and μ is a Lie bracket. The applications μ and s are related by the affine condition

$$\begin{aligned} A(X,Y,Z) = \mu(\mu(X,Y),Z) + \mu(s(Y,Z),X) - \mu(s(Z,X),Y) + 2s(\mu(X,Y),Z) \\ - s(\mu(Y,Z),X) + s(\mu(X,Z),Y) - s(s(Y,Z),X) + s(s(X,Z),Y) = 0. \end{aligned}$$

We have also

$$\begin{aligned} &A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y) \\ &\qquad = 2(\mu(s(X,Y),Z) + \mu(s(Y,Z),X) + \mu(s(Z,X),Y)). \end{aligned}$$

3. FILIFORM LIE ALGEBRAS OF DIMENSION 9

3.1. The variety $\mathcal{F}il_9$

Using a similar approach as in dimension 8, we obtain

Proposition 21. *Any 9-dimensional filiform Lie algebra over $\mathbb{K}$ is given in a Vergne basis by*

(14)

$$\begin{cases} \mu(X_0,X_i) = X_{i+1}, \ 1 \leq i \leq 7, \\ \mu(X_1,X_6) = a_2X_8, \qquad \mu(X_2,X_5) = a_4X_8, \\ \mu(X_1,X_5) = (a_2+a_4)X_7 + a_5X_8, \quad \mu(X_3,X_4) = a_6X_8, \\ \mu(X_2,X_4) = (a_4+a_6)X_7 + a_7X_8, \\ \mu(X_1,X_4) = (a_2+2a_4+a_6)X_6 + (a_5+a_7)X_7 + a_8X_8, \\ \mu(X_2,X_3) = (a_4+a_6)X_6 + a_7X_7 + a_9X_8, \\ \mu(X_1,X_3) = (a_2+3a_4+2a_6)X_5 + (a_5+2a_7)X_6 + (a_8+a_9)X_7 + a_{10}X_8, \\ \mu(X_1,X_2) = (a_2+3a_4+2a_6)X_4 + (a_5+2a_7)X_5 \\ \qquad +(a_8+a_9)X_6 + a_{10}X_7 + a_{11}X_8, \end{cases}$$

with $-3a_4^2 + 2a_6^2 + 2a_2a_6 + a_4a_6 = 0.$

We denote by Fil_9 the set of Lie algebras described above. It is clear that $\mathcal{F}il_9$, the open set of 9-dimensional filiform Lie algebras, is the orbit of Fil_9 in $\mathcal{N}ilp_9$ associated with the action of the linear group $GL(9,\mathbb{K})$. This reduces the study of $\mathcal{F}il_9$ to Fil_9. Let V^9 be the 9-dimensional $\mathbb{K}$-vector space characterized by the structure constants $\{a_i\}_{2\leq i\leq 11}, i\neq 3$.

PROPOSITION 22. *Fil_9 is a 8-dimensional irreducible algebraic subvariety of V^9.*

This algebraic variety Fil_9 has only one singular point corresponding to $a_i=0$ for all i. In all the other points, the tangent space $T_\mu(Fil_9)$ to Fil_9 is identified to the vector space of 2-cocycles:
(15)

$$\begin{cases}
\varphi(X_0,X_i)=0,\ 1\leq i\leq 7,\\
\varphi(X_1,X_6)=u_2X_8, \qquad \varphi(X_2,X_5)=u_4X_8,\\
\varphi(X_1,X_5)=(u_2+u_4)X_7+u_5X_8, \quad \varphi(X_3,X_4)=u_6X_8,\\
\varphi(X_2,X_4)=(u_4+u_6)X_7+u_7X_8,\\
\varphi(X_1,X_4)=(u_2+2u_4+u_6)X_6+(u_5+u_7)X_7+u_8X_8,\\
\varphi(X_2,X_3)=(u_4+u_6)X_6+u_7X_7+u_9X_8,\\
\varphi(X_1,X_3)=(u_2+3u_4+2u_6)X_5+(u_5+2u_7)X_6+(u_8+u_9)X_7+u_{10}X_8,\\
\varphi(X_1,X_2)=(u_2+3u_4+2u_6)X_4+(u_5+2u_7)X_5+(u_8+u_9)X_6\\
\qquad +u_{10}X_7+u_{11}X_8,
\end{cases}$$

with $u_2a_6+u_4(\frac{1}{2}a_6-3a_4)+u_6(2a_6+a_2+\frac{1}{2}a_4)=0$. Let $f\in gl(9,\mathbb{K})$ be an endomorphism. We put $f(X_0)=\sum_{i=0}^8\alpha_iX_i$ and $f(X_1)=\sum_{i=1}^8\beta_iX_i$. Assume that $\delta f(X_0,X_i)=0$. Then $f(X_{i+1})=\mu(f(X_0),X_i)+\mu(X_0,f(X_i))$ for $i=1,\cdots,7$. The other components of δf are

$$\begin{cases}
v_2=a_2(\beta_1-2\alpha_0), \quad v_4=a_4(\beta_1-2\alpha_0), \quad v_6=a_6(\beta_1-2\alpha_0),\\
v_5=a_5(\beta_1-3\alpha_0)-\alpha_1(2a_2^2+9a_4^2+6a_2a_4+5a_4a_6),\\
v_7=a_7(\beta_1-3\alpha_0)-\alpha_1(7a_6^2+3a_2a_4+7a_2a_6+11a_4a_6),\\
v_8=a_8(\beta_1-4\alpha_0)-\alpha_1((5a_2+11a_4+5a_6)a_5+(6a_2+19a_4+10a_6)a_7)-2\beta_3a_4,\\
v_9=a_9(\beta_1-4\alpha_0)-\alpha_1((3a_4+4a_6)a_5+(4a_2+9a_4+8a_6)a_7)-2\beta_3a_6,\\
v_{10}=a_{10}(\beta_1-5\alpha_0)-\alpha_1(P_1^1(a_8,a_9)P_2^1(a_2,a_4,a_6)+P_3^2(a_5,a_7))\\
\qquad +\alpha_3P_4^2(a_2,a_4,a_6)-3\beta_4(a_4+a_6)-2\beta_3a_7,\\
v_{11}=a_{11}(\beta_1-6\alpha_0)-\alpha_1P_5^2(a_2,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})\\
\qquad +\alpha_3P_6^2(a_2,a_4,a_5,a_6,a_7)+\alpha_4P_7^2(a_2,a_4,a_6)-2\beta_3a_9-3\beta_4a_7-2\beta_5(2a_4+a_6),
\end{cases}$$

where P_i^k is an homogeneous polynomial of degree k. If we denote by H^*_{CR} the restricted Chevalley cohomology of Lie algebras belonging to Fil_9, we have $\dim H^2_{CR}(\mu,\mu)\geq 1$ for any $\mu\in Fil_9$. We deduce

PROPOSITION 23. *None of the 9-dimensional filiform $\mathbb{K}$-Lie algebras is rigid in $\mathcal{F}il_9$ and also in $\mathcal{N}ilp_9$, and also in $\mathcal{L}ie_9$.*

Proof. In fact for any $\mu \in \mathcal{F}il_9$, $\dim H^2_{CR}(\mu,\mu) \neq 0$. Since the affine scheme associated with Fil_9 is reduced, any rigid Lie algebra in this variety has a trivial cohomology. Then no $\mu \in Fil_9$ is rigid. □

Now we determine the Lie algebras μ such that $\dim H^2_{CR}(\mu,\mu)$ is the smallest one. Assume that $a_6(a_4+a_6)(2a_4+a_6) \neq 0$. Then we can find δf such that

$$u_2+v_2 = u_5+v_5 = u_9+v_9 = u_{10}+v_{10} = u_{11}+v_{11} = 0.$$

In this case, the parameters $\alpha_0, \alpha_1, \beta_3, \beta_4, \beta_5$ are fixed and the other relations $u_i + v_i$ cannot be reduced to 0. Then there exists a representative φ in the cohomological class with $u_2 = u_5 = u_9 = u_{10} = u_{11} = 0$. Since $u_2a_6 + u_4(\frac{1}{2}a_6 - 3a_4) + u_6(2a_6 + a_2 + \frac{1}{2}a_4) = 0$, for such a Lie algebra we have $\dim H^2_{CR}(\mu,\mu) = 2$ and it is the lower bound. We deduce

THEOREM 24. *The variety $\mathcal{F}il_9$ is the closure of the orbit of a rigid 2-parameters family.*

Let us consider the family $\mathcal{T}^9_t$ of Lie algebras μ defined by ($a_2 = \frac{3t^2-t-2}{2}$, $a_4 = t, a_5 = 1, a_6 = 1, a_8 = u, a_9 = 0, a_{10} = 0, a_{11} = 0$) with $t \neq 0, -1, -\frac{1}{2}$. This family answers to this theorem.

3.2. Contact 9-dimensional filiform Lie algebras

Let $\mathfrak{g}$ be a 9-dimensional filiform Lie algebra. Let $\{\omega_0, \cdots, \omega_8\}$ be the dual basis of $\{X_0, \cdots, X_8\}$. From Proposition 4, $\mathfrak{g}$ is a contact Lie algebra if and only if ω_8 is a contact form. This is equivalent to

$$a_2a_4a_6 \neq 0$$

where a_i are the constant structures of $\mathfrak{g}$ described in (14). We deduce:

PROPOSITION 25. *A 9-dimensional filiform Lie algebra admits a contact form if and only it is isomorphic to a Lie algebra of Fil_9 whose structure constants given in (14) satisfy $a_2a_4a_6 \neq 0$.*

3.3. Come back on 8-dimensional symplectic filiform Lie algebras

The previous theorem is the result announced in Proposition 18. As we have said, the determination of contact 9-dimensional filiform Lie algebras permits a quick determination of the class of symplectic filiform 8-dimensional Lie algebras.

From Proposition 25, we can highlight a model of 9-dimensional filiform contact Lie algebra, that is a Lie algebra such as any 9-dimensional filiform contact Lie algebra is isomorphic to a deformation of this model. We consider the Lie algebras $\mathfrak{g}_{a_2,a_4,a_6}$ given by

$$a_5 = a_7 = a_8 = a_9 = a_{10} = a_{11} = 0$$

and

$$-3a_4^2 + 2a_6^2 + 2a_2a_6 + a_4a_6 = 0, \quad a_2a_4a_6 \neq 0.$$

Since $a_6 \neq 0$, $a_2 = \dfrac{(a_4 - a_6)(3a_4 + 2a_6)}{2a_6}$ and the condition $a_2a_4a_6 \neq 0$ is equivalent to $a_4a_6(a_4 - a_6)(3a_4 + 2a_6) \neq 0$.

PROPOSITION 26. *Any 9-dimensional filiform contact Lie algebra is isomorphic to a linear deformation of a Lie algebra* $\mathfrak{g}_{a_2,a_4,a_6}$.

COROLLARY 27. *Any 8-dimensional symplectic filiform Lie algebra is isomorphic to a linear symplectic deformation of a Lie algebra of the following family*

$$\begin{cases} \mu(X_0, X_i) = X_{i+1}, \ 1 \leq i \leq 6, \\ \mu(X_1, X_5) = b_2X_7, \quad \mu(X_2, X_4) = b_4X_7, \\ \mu(X_1, X_4) = (b_2 + b_4)X_6, \quad \mu(X_2, X_3) = b_4X_6, \\ \mu(X_1, X_3) = (b_2 + 2b_4)X_5, \quad \mu(X_1, X_2) = (b_2 + 2b_4)X_4, \end{cases}$$

with $b_2 = b_4 = 0$ *or* $b_4(b_2 + b_4)(2b_2 - b_4)(b_2 + 2b_4) \neq 0$.

3.4. The varieties $\mathcal{F}il_{10}$ and $\mathcal{F}il_{11}$

PROPOSITION 28. *Any 10-dimensional filiform Lie algebra over* $\mathbb{K}$ *is given in a Vergne basis by*

(16)

$$\begin{cases} \mu(X_0, X_i) = X_{i+1}, \ 1 \leq i \leq 8, \\ \mu(X_2, X_7) = a_1X_9, \quad \mu(X_1, X_7) = a_1X_8 + a_2X_9, \quad \mu(X_3, X_6) = -a_1X_9, \\ \mu(X_2, X_6) = a_4X_9, \quad \mu(X_1, X_6) = a_1X_7 + (a_2 + a_4)X_8 + a_5X_9, \\ \mu(X_4, X_5) = a_1X_9, \quad \mu(X_3, X_5) = a_7X_9, \quad \mu(X_2, X_5) = (a_4 + a_7)X_8 + a_8X_9, \\ \mu(X_1, X_5) = a_1X_6 + (a_2 + 2a_4 + a_7)X_7 + (a_5 + a_8)X_8 + a_9X_9, \\ \mu(X_3, X_4) = a_7X_8 + a_{10}X_9 \\ \mu(X_2, X_4) = (a_4 + 2a_7)X_7 + (a_8 + a_{10})X_8 + a_{11}X_9, \\ \mu(X_1, X_4) = a_1X_5 + (a_2 + 3a_4 + 3a_7)X_6 + (a_5 + 2a_8 + a_{10})X_7 \\ \qquad +(a_9 + a_{11})X_8 + a_{12}X_9, \\ \mu(X_2, X_3) = (a_4 + 2a_7)X_6 + (a_8 + a_{10})X_7 + a_{11}X_8 + a_{13}X_9, \\ \mu(X_1, X_3) = a_1X_4 + (a_2 + 4a_4 + 5a_7)X_5 + (a_5 + 3a_8 + 2a_{10})X_6 \\ \qquad +(a_9 + 2a_{11})X_7 + (a_{12} + a_{13})X_8 + a_{14}X_9, \\ \mu(X_1, X_2) = a_1X_3 + (a_2 + 4a_4 + 5a_7)X_4 + (a_5 + 3a_8 + 2a_{10})X_5 \\ \qquad +(a_9 + 2a_{11})X_6 + (a_{12} + a_{13})X_7 + a_{14}X_8 + a_{15}X_9, \end{cases}$$

with the conditions

$$\begin{cases} a_1(2a_2+7a_4+7a_7)=0, \\ 3a_4^2+3a_4a_7-2a_2a_7=0, \\ a_1(2a_9+5a_{11})-2a_2a_{10}+a_4(7a_8-2a_{10})+a_7(-3a_5+2a_8-7a_{10})=0. \end{cases}$$

We denote the set of this multiplications by Fil_{10}. If $2a_2+7a_4+7a_7=0$, then

$$3a_4^2+3a_4a_7-2a_2a_7=3a_4^2+10a_4a_7+7a_7^2=(a_4+a_7)(3a_4+7a_7).$$

We deduce

PROPOSITION 29. *The set $\mathcal{F}il_{10}$ of 10-dimensional filiform Lie algebras is the union of the algebraic components*

(1) $\mathcal{F}il_{10}(1)=\mathcal{O}(Fil_{10}(1))$ *where $Fil_{10}(1)$ is the set of multiplication $\mu \in Fil_{10}$ satisfying*

$$a_1=0,\ 3a_4^2+3a_4a_7-2a_2a_7=0,\ -2a_2a_{10}+a_4(7a_8-2a_{10}) \\ +a_7(-3a_5+2a_8-7a_{10})=0.$$

(2) $\mathcal{F}il_{10}(2)=\mathcal{O}(Fil_{10}(2))$ *where $Fil_{10}(2)$ is the set of multiplication $\mu \in Fil_{10}$ satisfying*

$$a_1,a_2=0,\ a_4=-a_7,\ a_1(2a_9+5a_{11})+a_4(3a_5+5a_8+5a_{10})=0,$$

(3) $\mathcal{F}il_{10}(3)=\mathcal{O}(Fil_{10}(3))$ *where $Fil_{10}(3)$ is the set of multiplication $\mu \in Fil_{10}$ satisfying*

$$a_2=-2a_4,\ 3a_4=-7a_7,\ a_1(2a_9+5a_{11})+a_4\left(\frac{9}{7}a_5+\frac{43}{7}a_8+5a_{10}\right)=0.$$

PROPOSITION 30. *Any 11-dimensional filiform Lie algebra over $\mathbb{K}$ is given in a Vergne basis by*

(17)

$$\begin{cases} \mu(X_0,X_i)=X_{i+1},\ 1\leq i\leq 9, \\ \mu(X_1,X_8)=a_2X_{10}, \qquad \mu(X_2,X_7)=a_4X_{10}, \\ \mu(X_1,X_7)=(a_2+a_4)X_9+a_5X_{10}, \\ \mu(X_3,X_6)=a_7X_{10}, \\ \mu(X_2,X_6)=(a_4+a_7)X_9+a_8X_{10}, \\ \mu(X_1,X_6)=(a_2+2a_4+a_7)X_8+(a_5+a_8)X_9+a_9X_{10}, \\ \mu(X_4,X_5)=a_{10}X_{10}, \\ \mu(X_3,X_5)=(a_7+a_{10})X_9+a_{11}X_{10}, \\ \mu(X_2,X_5)=(a_4+2a_7+a_{10})X_8+(a_8+a_{11})X_9+a_{12}X_{10}, \\ \mu(X_1,X_5)=(a_2+3a_4+3a_7+a_{10})X_7+(a_5+2a_8+a_{11})X_8+(a_9+a_{12})X_9 \\ \qquad +a_{13}X_{10}, \end{cases}$$

$$\left\{\begin{array}{l}
\mu(X_3,X_4)=(a_7+a_{10})X_8+a_{11}X_9+a_{14}X_{10},\\
\mu(X_2,X_4)=(a_4+3a_7+2a_{10})X_7+(a_8+2a_{11})X_8+(a_{12}+a_{14})X_9+a_{15}X_{10},\\
\mu(X_1,X_4)=(a_2+4a_4+6a_7+3a_{10})X_6+(a_5+3a_8+3a_{11})X_7+(a_9+2a_{12}\\
\qquad\qquad +a_{14})X_8+(a_{13}+a_{15})X_9+a_{16}X_{10},\\
\mu(X_2,X_3)=(a_4+3a_7+2a_{10})X_6+(a_8+2a_{11})X_7+(a_{12}+a_{14})X_8+a_{15}X_9\\
\qquad\qquad +a_{17}X_{10},\\
\mu(X_1,X_3)=(a_2+5a_4+9a_7+5a_{10})X_5+(a_5+4a_8+5a_{11})X_6+(a_9+3a_{12}\\
\qquad\qquad +2a_{14})X_7+(a_{13}+2a_{15})X_8+(a_{16}+a_{17})X_9+a_{18}X_{10},\\
\mu(X_1,X_2)=(a_2+5a_4+9a_7+5a_{10})X_4+(a_5+4a_8+5a_{11})X_5+(a_9+3a_{12}\\
\qquad\qquad +2a_{14})X_6+(a_{13}+2a_{15})X_7+(a_{16}+a_{17})X_8+a_{18}X_9+a_{19}X_{10},
\end{array}\right.$$

with the conditions

$$\left\{\begin{array}{l}
(J_1):3z_2^2+3z_2z_3-2z_1z_3=0,\\
(J_2):z_7(2z_1+2z_2+z_3)+z_3(3z_5+z_6)-7z_2z_6=0,\\
(J_3):z_4(2z_1+7(z_2+z_3))-2z_3(2z_2+z_3)=0,\\
(J_4):z_4(2z_8+5z_9)-z_{10}(2z_1+9z_2+12z_3)-z_7(3z_5+7z_6-z_7)+4z_6^2\\
\qquad\qquad -2z_3(2z_8+7z_9)+8z_9(z_2+2z_3)=0.
\end{array}\right.$$

where $z_1=a_2+a_4, z_2=a_4+a_7, z_3=a_7+a_{10}, z_4=a_{10}, z_5=a_5+a_{11}, z_6=a_8+a_{11}, z_7=a_{11}, z_8=a_9+a_{12}, z_9=a_{12}+a_{14}, z_{10}=a_{14}$.

We shall determine the open set of filiform contact Lie algebras.

3.5. **Contact and symplectic structures**

Let $\mathfrak{g}$ be a 11-dimensional filiform Lie algebra belonging to Fil_{11}. Let $\{\omega_1,\cdots,\omega_{10}\}$ be the dual basis of a Vergne basis of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is a contact Lie algebra. Then, from [16], the form ω_{10} is also a contact form. Then $\mathfrak{g}$ is a contact algebra if and only if ω_{10} is a contact form in $\mathfrak{g}_1$. We deduce

PROPOSITION 31. *An 11-dimensional filiform Lie algebra is a contact Lie algebra if and only if it is isomorphic to a Lie algebra of Fil_{11} with $a_2a_4a_7a_{10}\neq 0$.*

We deduce that a model of contact 11-dimensional filiform Lie algebra is given by the family $\mathfrak{g}_{a_2,a_4,a_7,a_{10}}$ of Lie algebras of Fil_{11} corresponding to

$$a_5=a_8=a_9=a_{11}=a_{12}=a_{13}=a_{14}=a_{15}=a_{16}=a_{17}=a_{18}=a_{19}=0$$

with the conditions,

$$\left\{\begin{array}{l}
3z_2^2+3z_2z_3-2z_1z_3=0,\\
z_4(2z_1+7z_2+7z_3)-2z_3(2z_2+z_3)=0,\\
a_2a_4a_7a_{10}=z_4(z_3-z_4)(z_2-z_3+z_4)(z_1-z_2+z_3-z_4)\neq 0
\end{array}\right.$$

and $z_1 = a_2 + a_4, z_2 = a_4 + a_7, z_3 = a_4 + a_{10}, z_4 = a_{10}$. Let us note that the algebraic variety determined by the two first equations is not reduced to 0, for example the point $(a_2, a_4, a_7, a_{10}) = (1, -1, 1, -1)$ belongs to this algebraic set. Let us note also that the linear form ω_{10} where $\{\omega_i\}$ is the dual basis of $\{X_i\}$ satisfies

$$d\omega_{10} = -\omega_0 \wedge \omega_9 - a_2\omega_1 \wedge \omega_8 - a_4\omega_2 \wedge \omega_7 - a_7\omega_3 \wedge \omega_6 - a_{10}\omega_4 \wedge \omega_5$$

and is a contact form.

Let $\mathcal{A}$ be the open set of 11-dimensional filiform contact Lie algebras. Then this open set is the orbit of the family of Lie algebras $\mathfrak{g}_{a_2,a_4,a_7,a_{10}}$ and satisfying $a_2a_4a_7a_{10} \neq 0$. Moreover, $\mathcal{A}$ is not connected, is the union of two algebraic irreducible sets and

$$\mathcal{F}il_{11} = \overline{\mathcal{A}}.$$

so $\mathcal{F}il_{11}$ is the union of two algebraic irreducible components.

Proposition 32. *None of the* 11*-dimensional filiform Lie algebras is rigid.*

Proof. The proof is similar to the 9-dimensional case. Since the affine scheme is reduced, it is sufficient to prove that the dimension of the second space of cohomology $H^2_{CR}(\mu, \mu)$ of any Lie algebra of (17) is not zero. It is clear that, if μ is a Lie algebra of (17), the dimension of the 2-cocycles ψ such that $\mu + t\psi$ belongs to (17) is of dimension 16 parametrized by the a_i. If f is an endomorphism of $\mathbb{K}^{11}$, then putting $f(e_0) = \sum_0^{10} \alpha_i X_i$ and $f(e_1) = \sum_1^{10} \beta_i X_i$, $\delta f(X_1, X_8) = v_2 X_{10}$, $\delta f(X_2, X_7) = v_4 X_{10}$, $\delta f(X_3, X_6) = v_7 X_{10}$, $\delta f(X_4, X_5) = v_{10} X_{10}$, we have

$$\begin{cases} v_2 = a_2(\beta_1 - 2\alpha_0), \\ v_4 = a_4(\beta_1 - 2\alpha_0), \\ v_7 = a_7(\beta_1 - 2\alpha_0), \\ v_{10} = a_{10}(\beta_1 - 2\alpha_0). \end{cases}$$

We deduce that the dimension of the space of deformations is greater or equal to 3. □

Consequence. Determination of the symplectic 10-dimensional filiform Lie algebras. From the Proposition 4 we deduce:

Proposition 33. *Any* 10*-dimensional symplectic filiform Lie algebra is isomorphic to*

$$\begin{cases} \mu(X_0, X_i) = X_{i+1},\ 1 \leq i \leq 8, \\ \mu(X_1, X_7) = (a_2 + a_4)X_9, \quad \mu(X_2, X_6) = (a_4 + a_7)X_9, \\ \mu(X_1, X_6) = (a_2 + 2a_4 + a_7)X_8 + (a_5 + a_8)X_9, \\ \mu(X_3, X_5) = (a_7 + a_{10})X_9, \quad \mu(X_2, X_5) = (a_4 + 2a_7 + a_{10})X_8 + (a_8 + a_{11})X_9, \\ \mu(X_1, X_5) = (a_2 + 3a_4 + 3a_7 + a_{10})X_7 + (a_5 + 2a_8 + a_{11})X_8 + (a_9 + a_{12})X_9, \\ \mu(X_3, X_4) = (a_7 + a_{10})X_8 + a_{11}X_9, \\ \mu(X_2, X_4) = (a_4 + 3a_7 + 2a_{10})X_7 + (a_8 + a_{11})X_8 + (a_{12} + a_{14})X_9, \\ \mu(X_1, X_4) = (a_2 + 4a_4 + 6a_7 + 3a_{10})X_6 + (a_5 + 3a_8 + a_{11})X_7 \\ \qquad\qquad + (a_9 + a_{12} + a_{14})X_8 + (a_{13} + a_{15})X_9, \\ \mu(X_2, X_3) = (a_4 + 3a_7 + 2a_{10})X_6 + (a_8 + 2a_{11})X_7 + (a_{12} + a_{14})X_8 + a_{15}X_9, \\ \mu(X_1, X_3) = (a_2 + 5a_4 + 9a_7 + 5a_{10})X_5 + (a_5 + 4a_8 + 5a_{11})X_6 \\ \qquad\qquad + (a_9 + 3a_{12} + 2a_{14})X_7 + (a_{13} + 2a_{15})X_8 + (a_{16} + a_{17})X_9, \\ \mu(X_1, X_2) = (a_2 + 5a_4 + 9a_7 + 5a_{10})X_4 + (a_5 + 4a_8 + 5a_{11})X_5 + (a_9 + 3a_{12} \\ \qquad\qquad + 2a_{14})X_6 + (a_{13} + 2a_{15})X_7 + (a_{16} + a_{17})X_8 + a_{18}X_9 \end{cases}$$

with $a_2a_4a_7a_{10} \neq 0$ *and if* $z_1 = a_2 + a_4, z_2 = a_4 + a_7, z_3 = a_7 + a_{10}, z_4 = a_{10}, z_5 = a_5 + a_{11}, z_6 = a_8 + a_{11}, z_7 = a_{11}$,

$$\begin{cases} 3z_2^2 + 3z_2z_3 - 2z_1z_3 = 0, \\ z_7(2z_1 + 2z_2 + z_3) + z_3(3z_5 + z_6) - 7z_2z_6 = 0. \end{cases}$$

We deduce, from the definition of a symplectic model:

COROLLARY 34. *The symplectic models of* 10*-dimensional filiform symplectic Lie algebras are the Lie algebras corresponding to*

$$a_5 = a_8 = a_{11} = a_9 = a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17} = a_{18} = 0.$$

4. CONTACT AND SYMPLECTIC FILIFORM LIE ALGEBRAS

4.1. $(2p+1)$-dimensional contact filiform Lie algebras

Let $\{X_0, \cdots, X_{2n}\}$ be a Vergne basis of a $(2p+1)$-dimensional filiform Lie algebra $\mathfrak{g}$ and let us denote by $C_{i,j}^k$ the structure constants related to this basis. We have seen in dimension 11 or smaller, but this remains trivially true for greater dimension, that the structure constants of $\mathfrak{g}$ related to this basis are linear combinations of the $(p-1)^2$ structure constants $a_{i,j} = C_{i,j}^{2p}$ for $1 \leq i < j \leq 2p-1-i$. Using the same notations as in the previous section, we deduce that Fil_{2p+1} is an algebraic variety embedded in $\mathbb{K}^{(p-1)^2}$. In fact, all the other structure constants are defined by the linear equation

$$[X_0, [X_i, X_j]] = [X_{i+1}, X_j] + [X_i, X_{j+1}]$$

as soon as $j+1 \leq 2p$. More precisely, we have

$$\begin{cases} C^{2p-k}_{2p-1-j-k,j} = \sum_j \alpha_j a_{2p-1-j,j}, \quad 2j > 2p-1-k, \quad k = 1, \cdots, 2p-4 \\ C^{2p-k}_{2p-2-j-k,j} = \sum_j \beta_j a_{2p-2-j,j}, \quad 2j > 2p-2-k, \quad k = 1, \cdots, 2p-5 \\ \cdots \\ C^{2p-k}_{2p-(2p-3)-j-k,j} = C^{2p-k}_{3-j-k,j} = a_{1,2}, \end{cases}$$

The coefficients α_j, β_j are described in [9]. Let $\{\omega_0, \omega_1, \cdots, \omega_{2p}\}$ be the dual basis. All the Jacobi conditions are given by $d(d\omega_{2p}) = 0$. This gives

$$(p-3)^2 + (p-4)(p-5) + (p-6)^2 + \cdots + \varepsilon \tag{18}$$

equations where $\varepsilon = 2$ if $p \equiv 0 \pmod 3$, $\varepsilon = 1$ if $p \equiv 1 \pmod 3$ and $\varepsilon = 2^2$ if $p \equiv 2 \pmod 3$. This shows that, as soon as the dimension exceeds 19 the number of polynomial equations is greater than the number of parameters $a_{i,j}$.

If $\mathfrak{g}$ admits a contact form, then from [16] the form ω_{2p} is also a contact form. We deduce

Proposition 35. *A $(2p+1)$-dimensional filiform Lie algebra admits a contact form if and only if the structure constants related to a Vergne basis satisfy:*

$$a_{1,2p-2} \cdot a_{2,2p-3} \cdots a_{i,2p-1-i} \cdots a_{p-1,p} \neq 0.$$

Since any deformation of a contact Lie algebra is also a contact Lie algebra, we deduce that the set of $(2p+1)$-dimensional filiform contact Lie algebra is a Zariski open set in $\mathcal{F}il_{2p+1}$. Let us consider the family $\mathcal{A}$ contained in this open set and corresponding to Lie algebras whose structure constants satisfy $a_{i,j} = 0$ except $a_{1,2p-2}, a_{2,2p-3}, \cdots, a_{i,2p-1-i}, \cdots, a_{p-1,p}$ which are supposed to be different from 0. It is clear that any contact filiform $(2p+1)$-dimensional Lie algebra is a deformation of a Lie algebra of this family. We remark that this family is parametrized by the $(p-1)$ structure constants $a_{1,2p-2}, a_{2,2p-3}, a_{i,2p-1-i}, a_{p-1,p}$ but the system of polynomial equations deduced from the Jacobi conditions, which is a consequence of $d(d\omega_{2p}) = 0$ is composed, when p is greater than 7, of a number of equations greater than $p-1$. This number depend to p $mod(3)$. For example, if $p = 3k+1$, we have $3k$ parameters and $3k^2 - 3k + 1$ polynomial equations and $3k^2 - 3k + 1 > 3k$ as soon as $k \geq 2$.

Let us note also that the set $\mathcal{A}$ is not empty. In fact the Lie algebra corresponding to

$$a_{1,2p-2} = 1, a_{2,2p-3} = -1, \cdots, a_{i,2p-1-i} = (-1)^{i+1}, \cdots, a_{p-1,p} = (-1)^p \tag{19}$$

belongs to this family and more generally, if

$$a_{1,2p-2} = \lambda, a_{2,2p-3} = -\lambda, \cdots, a_{i,2p-1-i} = (-1)^{i+1}\lambda, \cdots, a_{p-1,p} = (-1)^p\lambda$$

then the corresponding Lie algebras are in $\mathcal{A}$. This implies that $\dim \mathcal{A} \geq 1$. Let us denote by $\mathfrak{g}_0$ the Lie algebra of $\mathcal{A}$ corresponding to (19) and let us compute the space of deformations. As we have seen in previous work [13], it is sufficient to compute the cocycles of $\mathfrak{g}_0$ which preserves the Vergne's basis. For example, in case of dimension 9, from (14), the space of such cocycles is of dimension 8. Let us compute the space of coboundaries. It is generated by the δf satisfying $\delta f(X_0, Y) = 0$ and $f \in gl(9, \mathbb{K})$. This last condition implies that f is determined when we know $f(X_0) = \alpha_0 X_0 + \cdots + \alpha_8 X_8$ and $f(X_1) = \beta_1 X_1 + \cdots + \beta_8 X_8$. We obtain

$$\delta f(X_1, X_6) = (-2\alpha_0 + \beta_1)X_8 = -\delta f(X_2, X_5) = \delta f(X_3 X_4)$$

and

$$\delta f(X_1, X_4) = -\delta f(X_2, X_3) = -2\beta_3 X_8,\ \delta f(X_1, X_2) = -2\beta_5 X_8,$$

in all other cases $\delta f(X_i, X_j) = 0$. We deduce that the space of deformations is of dimension 4. In dimension $2p$, the space of cocycles which preserves the Vergne's basis is embedded in a vector space of dimension $(p-1)^2$ parametrized by the structure constants

$$\{a_{1,2p-2}, \cdots, a_{1,2}, a_{2,p-3}, \cdots, a_{2,3}, \cdots, a_{p-2,p}, a_{p-2,p-1}, a_{p-1,p}\}$$

that is $a_{i,j}$ with $1 \leq i < j \leq 2p-2$, $3 \leq i+j \leq 2p-1$ and with $a_{i,j} = C_{i,j}^{2p}$. If f is a linear endomorphism of $gl(2p, \mathbb{K})$, then $\delta f(X_0, X_i) = 0$ implies that $f(X_i)$ is determined for $2 \leq i \leq 2p$ by $f(X_0) = \alpha_0 X_0 + \cdots + \alpha_{2p} X_{2p}$ and $f(X_1) = \beta_1 X_1 + \cdots + \beta_{2p} X_{2p}$. This implies

$$\delta f(X_1, X_{2p-2}) = (-2\alpha_0 + \beta_1)X_{2p} = -\delta f(X_2, X_{2p-3}) = \cdots = (-1)^p \delta f(X_{p-1} X_p),$$

and the other non zero $\delta f(X_i, X_j)$ are

$$\begin{cases} \delta f(X_1, X_{2i}) = 2\beta_{2p-2i-1} X_{2p},\ 1 \leq i \leq p-2, \\ \delta f(X_2, X_{2i+1}) = -2\beta_{2p-2i-3} X_{2p},\ 1 \leq i \leq p-3, \\ \delta f(X_3, X_{2i}) = 2\beta_{2p-2i-5} X_{2p},\ 2 \leq i \leq p-2, \\ \cdots \\ \delta f(X_{p-2}, X_{p-1}) = \beta_3 X_{2p}. \end{cases}$$

We remark also that the parameters $a_{1,2}, a_{1,3}, a_{2,3}, a_{1,4}, a_{2,4}$, $a_{1,5}$ do not appear in the polynomial Jacobi equations because the forms $\omega_i \wedge \omega_j$ which appear in the expression of $d\omega_{2^p}$ are closed for $(i,j) \in \{(1,2), (1,3), (2,3), (1,4), (2,4), (1,5)\}$. From the previous computations of the coboundaries, we have in particular

$$\begin{cases} \delta f(X_1,X_2) = 2\beta_{2p-3}X_{2p}, & \delta f(X_1,X_4) = 2\beta_{2p-5}X_{2p}, \\ \delta f(X_1,X_3) = \delta f(X_1,X_5) = 0, & \\ \delta f(X_2,X_3) = 2\beta_{2p-5}X_{2p}, & \delta f(X_2,X_4) = 0. \end{cases}$$

Then we can consider that the parameters $a_{1,2}, a_{1,4}$ are orbital parameters and $a_{1,3}, a_{2,3}, a_{2,4}$, $a_{1,5}$ are parameters of non trivial deformations. To end this work, we compute the space of deformation of $\mathfrak{g}_0$. It remains to compute the dimension of the space of cocycles. We have seen that it is embedded in a vector space of dimension $(p-1)^2$ and the number of polynomial equations given by the Jacobi relations was (18). But the affine scheme associated to this polynomial equations is not reduced. We can find relations between these equations from the following remark that we illustrate in dimension 11. In this dimension the Jacobi polynomial equation is constituted of 4 equations corresponding to $J(X_i,X_j,X_k)=0$ (J for the Jacobi condition related to triple (X_i,X_j,X_k)). To simplify we denote by (i,j,k) the polynomial $J(X_i,X_j,X_k)$ and let $p=i+j+k$ be its weight. In this case, we have 4 parameters $(a_{1,8}=a_2, a_{2,7}=a_4, a_{3,6}=a_7, a_{4,5}=a_{10})$ and 4 equations corresponding to $(p=6,(i,j,k)=(1,2,3))$, $(p=7,(i,j,k)=(1,2,4))$, $(p=8,(i,j,k)=(1,2,5),(1,3,4))$. But we have

$$\begin{aligned} &[X_0,(1,2,3)] = (1,2,4) \\ &[X_0,(1,2,4)] = (1,3,4)+(1,2,5). \end{aligned}$$

Thus the system of Jacobi equations can be reduced to the system

$$(1,2,3)=0,(1,3,4)=0$$

and the corresponding affine scheme is reduced. Then we have 4 parameters which have to satisfy 2 independent relations. The space of parameters is then of dimension 2. Let us come back to the general model $\mathfrak{g}_0$. The linear space of parameters is of dimension $(p-1)$ and it is generated by the structure constants

$$a_{1,2p-2}, a_{2,2p-3}, \cdots, a_{p-2,p+1}, a_{p-1,p}.$$

The weights take their values in $(6,7,\cdots,2p-2)$ and concern the Jacobi equation:

$$(1,2,3),(1,2,4),(1,2,5),(1,3,4),\cdots,(1,p-2,p-1),(1,p-3,p),\cdots,(1,2,2p-5),$$

$$(2,p-1,p+1),\cdots,(2,3,2p-7),\cdots\}$$

the last term in this ordered sequence depends on ($p \bmod 3$), more precisely, if $2p-2 \equiv 2 \pmod 3$, then the last term is $(k-1,k+1,k+2)$ with $2p-2=3k+2$, if $2p-2 \equiv 1 \pmod 3$, then the last term is $(k-1,k,k+2)$, and if $2p-2 \equiv 0 \pmod 3$, then the last term is $(k-1,k,k+2)$. We have seen (18) that the number of Jacobi polynomial equations is

$$(p-3)^2+(p-4)(p-5)+(p-6)^2+\cdots+\varepsilon$$

where $\varepsilon = 2$ if $p \equiv 0 \pmod 3$, $\varepsilon = 1$ if $p \equiv 1 \pmod 3$ and $\varepsilon = 2^2$ if $p \equiv 2 \pmod 3$. This scheme is not reduced. To reduce it we consider the relations

$$[X_0,(i,j,k)] = (i+1,j,k)+(i,j+1,k)+(i,j,k+1).$$

Putting $2p-2 = 3m+r$ with $0 \leq r \leq 2$, we can write the reduced number N_r of equations:

- If $m = 2h$ and $r = 0$, then $N_r = 3h^2-3h+1$,
- If $m = 2h+1$ and $r = 1$, then $N_r = 3h^2+3h$,
- If $m = 2h$ and $r = 2$, then $N_r = 3h^2-h$.

We can see than, as soon as $n \geq 14$, that is $p = 7, m = 4, r = 2$ then the number of parameters is 6 and $N_r = 10$. In the same way, we can reduce this new polynomial system using the identity

$$[X_1,(i,j,k)] = ((i+2,j,k)C_{1,i}^{i+2}+(i,j+2,k)C_{1,j}^{j+2}+(i,j,k+2)C_{1,k}^{k+2})X_{i+j+k+2}.$$

which is a direct consequence of the natural grading of $\mathfrak{g}_0$. We deduce in particular

$$[X_1,(1,2,3)] = (-(1,3,4)C_{1,2}^4+(1,2,5)C_{1,3}^5)X_{10}.$$

To end this section, we can look at the case $n = 14$, this case corresponding to $N_r > p-1$. We have 6 coefficients and 7 relations after the reduction of the first type. We can choose as generating relations, the relation $(1,2,i)$ for $i = 3,5,6,7,8,9$ and $(3,4,5)$. The relation of second type concerning these equations are, where $1(i,j,k)$ is the coefficient of $[X_1,(i,j,k)]$,

$$\begin{cases} 1(1,2,3) = (1,2,5)C_{1,3}^5, \\ 1(1,2,4) = (1,2,6)C_{1,4}^6, \\ 1(1,2,5) = (1,4,5)C_{1,2}^4+(1,2,7)C_{1,5}^7, \\ 1(1,2,6) = (1,4,6)C_{1,2}^4+(1,2,8)C_{1,6}^8, \\ 1(1,2,7) = (1,4,7)C_{1,2}^4+(1,2,9)C_{1,7}^9, \\ 1(2,3,5) = -(3,4,5)C_{1,2}^4+(2,3,7)C_{1,5}^7, \end{cases}$$

If $C_{1,2}^4C_{1,3}^5C_{1,4}^6C_{1,5}^7C_{1,6}^8C_{1,7}^9 \neq 0$, then the Jacobi polynomial system is reduced only to one equation $(1,2,3)$. In this open set, the space of parameters of deformations of the models is of dimension greater or equal to 5.

4.2. **Filiform symplectic algebras**

From the previous study, we have that the $(2p)$-dimensional symplectic filiform Lie algebras are isomorphic to a quotient of a contact $(2p+1)$-dimensional filiform Lie algebra $\mathfrak{g}_{2p+1}$ by its center $\mathbb{K}\{X_{2p}\}$. Then it can be written with the structure constants of $\mathfrak{g}_{2p+1}$ with the condition $a_{1,2p-2}a_{2,2p-4}\cdots a_{p-1,p} \neq 0$.

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