REMARKS OF EXPONENTIALLY HARMONIC MAPS

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Communicated by Marian Aprodu

Let (M, g_0) be a complete Riemannian manifold with a pole x_0 and (N, h) be a Riemannian manifold. We show that if $f : (M, \eta^2 g_0) \to (N, h)$ is an exponentially harmonic map such that η (a smooth function on M) satisfies some condition $(\star)$, then certain monotonicity formula is derived. We study the monotonicity of exponentially harmonic maps under a few different circumstances and discuss their vanishing.

AMS 2010 Subject Classification: 58E20, 58G11, 35J99.

Key words: exponential energy, exponentially harmonic map, monotonicity.

1. INTRODUCTION

In 1990, exponentially harmonic maps were first explored by Eells and Lemaire [11]. Afterwards, Duc and Eells [10], M. Hong [15], J. Hong and Yang [16], Chiang, Pan, Wolak and Yang [2–7], Cheung and Leung [8], Liu [14], Zhang, Wang and Liu [20], and others also investigated exponentially harmonic maps. In 2002, Kanfon, Füzfa and Lambert [17] discovered the applications of exponentially harmonic maps on Friedmann-Lemaître universe, and constructed some new models of exponentially harmonic maps which were coupled with gravity based on a generalization of Lagrangian for bosonic strings coupled with diatonic field. In 2011–2012, Omori [18, 19] obtained Eells-Sampson's existence theorem and Sacks-Uhlenbeck's existence theorem for harmonic maps via exponentially harmonic map. Exponentially harmonic maps and harmonic maps are different. There are exponentially harmonic maps which are not harmonic maps, and there are harmonic maps which are not exponentially harmonic maps either (cf. [16]).

Let (M, g_0) be a complete m-dimensional Riemannian manifold with a pole x_0, (N, h) be an n-dimensional Riemannian manifold, and $f : (M, \eta^2 g_0) \to (N, h)$ be an exponentially harmonic map (where η is a smooth function on M). Denote by $r(x) = \mathrm{dist}_{g_0}(x, x_0)$ the g_0-distance function relative to the pole x_0. Set $B(r) = \{x \in M : r(x) \le r\}$. It is known that $\frac{\partial}{\partial r}$ is an eigenvector of

$Hess_{g_0}(r^2)$ associated with the eigenvalue 2. Denote by μ_{max} (resp. μ_{min}) the maximum (resp. minimal) eigenvalues of $Hess_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$. We obtain a theorem as follows: Suppose that η satisfies the condition $(\star)$ $\frac{\partial \log \eta}{\partial r} \geq 0$ and there is a positive constant C such that

$$(m-2)r\frac{\partial \log \eta}{\partial r} + \frac{m-1}{2}\mu_{min} + 1 - max\{2, \mu_{max}\} \geq C.$$

Then

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v}{\sigma_1^C} \leq \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v}{\sigma_2^C}$$

for any $0 < \sigma_1 \leq \sigma_2$. In particular, if $\int_{B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v = o(R^C)$, then f is constant. As a corollary, if $f : (M, \eta^2 g_0) \to (N, h)$ is a non-constant exponentially harmonic map and η satisfies the condition $(\star)$, then

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \mathrm{d}v}{\sigma_1^C} \leq \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \mathrm{d}v}{\sigma_2^C}$$

for any $0 < \sigma_1 \leq \sigma_2$. We also investigate the monotonicity of an exponentially harmonic map with the radial curvature K_r of M. We finally study the monotonicity of exponentially harmonic maps on a star-like domain.

2. EXPONENTIALLY HARMONIC MAPS

An exponentially harmonic map $f : (M, g) \to (N, h)$ from an m-dimensional Riemannian manifold into an n-dimensional Riemannian manifold is a critical point of the exponential energy functional

$$E_e(f) = \int_M \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \mathrm{d}v_g,$$

where $\mathrm{d}v_g$ is the volume form of M determined by the metric g. More precisely, a C^2 map $f : M \to N$ is *exponentially harmonic* if it satisfies the Euler-Lagrange equation of the exponential energy

$$\frac{\mathrm{d}}{\mathrm{d}t}E_e(f_t)\big|_{t=0} = 0,$$

for any compactly supported variations $f_t : M \to N$ with $f_0 = f$.

In Proposition 2.1, we derive the first variation of the exponential energy of f in a different way but equivalent to [2–4]. Let ∇ and ∇^N be the Levi-Civita connections of M and N, respectively. Denote $\tilde{\nabla}$ the induced connection on $f^{-1}TN$ given by $\tilde{\nabla}_X W = \nabla^N_{\mathrm{d}f(X)} W$, where X is a tangent vector field of M and W is a section of $f^{-1}TN$.

PROPOSITION 2.1. *If $f:(M,g)\to(N,h)$ is a C^2 map, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_e(f_t)|_{t=0}=-\int_M(\tau_e(f),\,V)\,\mathrm{d}v_g,$$

where $V=\frac{\mathrm{d}}{\mathrm{d}t}f_t|_{t=0}$.

Proof. Let $\phi:(-\epsilon,\epsilon)\times M\to N$ be defined by $\phi(t,x)=f_t(x)$, where $(-\epsilon,\epsilon)\times M$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$ on $(-\epsilon,\epsilon)$, and X on M to vector fields on $(-\epsilon,\epsilon)\times M$, and still denote those by $\frac{\partial}{\partial t}$, X. Then $V=\mathrm{d}\phi(\frac{\partial}{\partial t})|_{t=0}$. We use the same notations ∇ and $\tilde\nabla$ for the Levi-Civita connection on $(-\epsilon,\epsilon)\times M$ and the induced connection $\phi^{-1}TN$. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M. We calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_e(f_t)|_{t=0} &= \int_M\Big[\mathrm{e}^{\frac{|\mathrm{d}f_t|^2}{2}}\frac{\partial}{\partial t}\frac{|\mathrm{d}f_t|^2}{2}\Big]|_{t=0}\,\mathrm{d}v\\ &= \int_M\Big[\mathrm{e}^{\frac{|\mathrm{d}f_t|^2}{2}}\frac{1}{2}\frac{\partial}{\partial t}\Big(\sum_{i=1}^m(\mathrm{d}f_t(e_i),\,\mathrm{d}f_t(e_i)\Big)\Big]|_{t=0}\,\mathrm{d}v\\ &= \int_M\Big[\mathrm{e}^{\frac{|\mathrm{d}f_t|^2}{2}}\sum_i\Big(\tilde\nabla_{\frac{\partial}{\partial t}}\mathrm{d}\phi(e_i),\,d\phi(e_i)\Big)\Big]|_{t=0}\mathrm{d}v\\ &= \int_M\Big[\mathrm{e}^{\frac{|\mathrm{d}f_t|^2}{2}}\sum_i(\tilde\nabla_{e_i}\mathrm{d}\phi(\frac{\partial}{\partial t}),\,d\phi(e_i))\Big]|_{t=0}\mathrm{d}v\\ &= \int_M\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\sum_i\Big[e_i\Big(\mathrm{d}\phi(\frac{\partial}{\partial t}),\mathrm{d}f_t(e_i)\Big)-\Big(\mathrm{d}\phi(\frac{\partial}{\partial t}),\tilde\nabla_{e_i}\mathrm{d}f_t(e_i)\Big)\Big]|_{t=0}\mathrm{d}v\end{aligned}$$

where we used

$$\tilde\nabla_{\frac{\partial}{\partial t}}\mathrm{d}\phi(e_i)-\tilde\nabla_{e_i}\mathrm{d}\phi(\frac{\partial}{\partial t})=\phi([\frac{\partial}{\partial t},\,e_i])=0,$$

in the fourth equality.

Let X_t be a compactly supported vector field on M such that $(X_t,Y)_g=(\mathrm{d}\phi(\frac{\partial}{\partial t}),\mathrm{d}f_t(Y))_h$ for any vector filed Y on M. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_e(f_t)|_{t=0} = &\int_M\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\sum_i\Big[e_i(X_t,e_i)_g-\Big(\mathrm{d}\phi(\frac{\partial}{\partial t}),\tilde\nabla_{\frac{\partial}{\partial t}}\mathrm{d}f_t(e_i)\Big)_h\Big]|_{t=0}\,\mathrm{d}v\\ = &\int_M\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\Big[\mathrm{div}(X_t)-\sum_i\Big(\mathrm{d}\phi(\frac{\partial}{\partial t}),\tilde\nabla_{\frac{\partial}{\partial t}}\mathrm{d}f_t(e_i)-\mathrm{d}f_t(\nabla_{e_i}e_i)\Big)\Big]|_{t=0}\mathrm{d}v\\ = &\int_M\Big[\mathrm{div}(\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}X_t)-\Big(\mathrm{d}\phi(\frac{\partial}{\partial t}),\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}(\tau(f_t)+\mathrm{d}f_t\cdot\mathrm{grad}e(f)\Big)\Big]_{t=0}\mathrm{d}v\\ = &-\int_M(\tau_e(f),V)\,\mathrm{d}v,\end{aligned}$$

where we applied the divergence theorem in the last equality, $\tau(f) = \sum_{i=1}^{m}\{\tilde{\nabla}_{e_i} \mathrm{d}f(e_i) - \mathrm{d}f(\nabla_{e_i}e_i)\}$ is the tension field of f. $\square$

Definition 2.2. A map $f : (M,\, g_{ij}) \to (N,\, h_{\alpha\beta})$ is exponentially harmonic if the exponential tension field of f

$$\tau_e(f) = \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\Big(\tau(f) + \mathrm{d}f \cdot \mathrm{grad}\, e(f)\Big) = 0,$$

where $e(f) = \frac{|\mathrm{d}f|^2}{2}$ is the energy density of f. In terms of local coordinates, f satisfies

$$\begin{aligned} & g^{ij}\Big(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f^\alpha}{\partial x^k} + \Gamma'^{\alpha}_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\Big) + g^{il}g^{jm}h_{\beta\gamma}\frac{\partial f^\alpha}{\partial x^l}\frac{\partial f^\gamma}{\partial x^m}\frac{\partial^2 f^\beta}{\partial x^i \partial x^j} \\ - \quad & g^{il}g^{jm}h_{\beta\gamma}\Gamma_{ij}^k\frac{\partial f^\alpha}{\partial x^l}\frac{\partial f^\beta}{\partial x^m}\frac{\partial f^\gamma}{\partial x^k} + g^{ij}g^{lm}h_{\beta\gamma}\Gamma'^{\beta}_{\mu\nu}\frac{\partial f^\mu}{\partial x^i}\frac{\partial f^\nu}{\partial x^l}\frac{\partial f^\gamma}{\partial x^m}\frac{\partial f^\alpha}{\partial x^j} = 0, \end{aligned}$$

where $\tau^\alpha(f) = g^{ij} f^\alpha_{i|j} = g^{ij}(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \Gamma'^{\alpha}_{\beta\gamma} f^\beta_i f^\gamma_j)$ is the tension field of f, and Γ^k_{ij} and $\Gamma'^{\alpha}_{\beta\gamma}$ are the Christoffel symbols on M and N.

For a 2-tensor $Q \in \Gamma(T^*M \otimes T^*M)$, its divergence $\mathrm{div}\, Q \in \Gamma(T^*M)$ is given by

$$\mathrm{div}\, Q(X) = \sum_{i=1}^{m} (\nabla_{e_i} Q)(e_i,\, X),$$

where X is any smooth vector field on M. For two 2-tensors $Q_1,\, Q_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product is defined by

$$\langle Q_1,\, Q_2 \rangle = \sum_{i,j=1}^{m} Q_1(e_i,\, e_j) Q_2(e_i,\, e_j),$$

where $\{e_i\}$ is an orthonormal frame on M with respect to g. For a vector field $X \in \Gamma(TM)$, denote by θ_X its dual one form, i.e., $\theta_X(Y) = (X, Y)_g$, where $Y \in \Gamma(TM)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$:

$$\nabla\theta_X(Y,\, Z) = \nabla_Y \theta_X(Z) = (\nabla_Y X,\, Z)_g.$$

If $X = \nabla\eta$ is the gradient field of a C^2 function η on M, then $\theta_X = d\eta$ and $\nabla\theta_X = Hess\,\eta$.

LEMMA 2.3. *Let Q be a symmetric (0,2)-type tensor field and let X be a vector field. Then*

$$\mathrm{div}(i_X Q) = (\mathrm{div} Q)(X) + \langle Q, \nabla\theta_X \rangle = (\mathrm{div} Q)(X) + \frac{1}{2}\langle Q,\, L_X g \rangle, \tag{2.1}$$

where L_X is the Lie derivative of the metric g in the direction of X. Let $\{e_1, \cdots, e_m\}$ be a local orthonormal frame on M. We obtain

$$\begin{aligned}\frac{1}{2}\langle Q,\, L_X g\rangle &= \sum_{i,j=1}^{m} \frac{1}{2}\langle Q(e_i, e_j),\, L_X g(e_i, e_j)\rangle \\ &= \sum_{i,j=1}^{m} Q(e_i,\, e_j)(\nabla e_i X,\, e_j)_g = \langle Q,\, \nabla\theta_X\rangle.\end{aligned}$$

(cf. [1,9]).

Let D be a bounded domain of M with C^1 boundary. Applying the Stokes' theorem, we have

$$\int_{\partial D} Q(X,n)\mathrm{d}s_g = \int_D \Big(\mathrm{div}(Q)(X) + \langle Q,\, \frac{1}{2}L_X g\rangle\Big)\mathrm{d}v_g, \tag{2.2}$$

where n is the unit outward normal vector field along ∂D.

The exponential stress-energy tensor of a C^2 map $f : M \to N$ between Riemannian manifolds is defined by

$$S_e(f) = \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\big(\frac{|\mathrm{d}f|^2}{2}g - f^*h\big).$$

The exponential stress-energy tensor of f is conserved if $\mathrm{div}\, S_e(f) = 0$. The following proposition was mentioned by Eells and Lemaire [11], and Chiang provided a different proof in [2].

Proposition 2.4. *If $f : (M, g) \to (N, h)$ an exponentially harmonic map, then*

$$\mathrm{div}S_e(f) = -\Big(\tau_e(f),\, \mathrm{d}f(X)\Big) = 0,$$

where X is a vector field on M. Hence, the associated exponential stress-energy tensor of f is conserved.

If f is an exponentially harmonic map, then we obtain

$$\int_{\partial D} S_e(f)(X,n)\mathrm{d}s_g = \int_D \langle S_e(f),\, \frac{1}{2}L_X g\rangle \mathrm{d}v_g, \tag{2.3}$$

by applying (2.2) to $Q = S_e(f)$ and Proposition 2.4.

3. MONOTONICITY

Let $(M,\, g_0)$ be a complete m-dimensional Riemannian manifold with a pole x_0 and $(N,\, h)$ be an n-dimensional Riemannian manifold. Denote by $r(x) = \mathrm{dist}_{g_0}(x,\, x_0)$ the g_0-distance function relative to the pole x_0. Put $B(r) =$

$\{x \in M : r(x) \leq r\}$. It is known that $\frac{\partial}{\partial r}$ is an eigenvector of $Hess_{g_0}(r^2)$ associated with the eigenvalue 2. Denote by μ_{max} (resp. μ_{min}) the maximum (resp. minimal) eigenvalues of $Hess_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$. Suppose that $f : (M, g) \to (N, h)$ is a stationary map (via exponential energy) with $g = \eta^2 g_0$, $0 < \eta \in C^\infty(M)$. It is obvious that the vector field $n = \eta^{-1}\frac{\partial}{\partial r}$ is an outer normal vector field along $\partial B(r) \subset (M, g)$.

THEOREM 3.1. *Let $f : (M, \eta^2 g_0) \to (N, h)$ be an exponentially harmonic map. Suppose that η satisfies the condition ($\star$): $\frac{\partial \log \eta}{\partial r} \geq 0$ and there is a constant $C > 0$ such that*

$$(m-2)r\frac{\partial \log \eta}{\partial r} + \frac{m-1}{2}\mu_{min} + 1 - max\{2, \mu_{max}\} \geq C.$$

Then

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v}{\sigma_1^C} \leq \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v}{\sigma_2^C} \tag{3.1}$$

for any $0 < \sigma_1 \leq \sigma_2$. In particular, if $\int_{B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v = o(R^C)$, then f is constant.

Proof. In (2.3), take $D = B(r)$ and $X = r\frac{\partial}{\partial r} = \frac{1}{2}\nabla^0 r^2$ (the covariant derivative ∇^0 determined by g_0), we have

$$\int_{B(r)} \langle S_e(f), \frac{1}{2}L_X g\rangle \mathrm{d}v_g = \int_{\partial B(r)} S_e(f)\langle X, n\rangle \mathrm{d}s_g. \tag{3.2}$$

Firstly, we calculate the left-hand side of the above equation and obtain

$$\begin{aligned} \langle S_e(f), \frac{1}{2}L_X g\rangle &= \langle S_e(f), r\frac{\partial \log \eta}{\partial r} g\rangle + \langle S_e(f), \frac{1}{2}\eta^2 L_X g_0\rangle \\ &= r\frac{\partial \log \eta}{\partial r}\langle S_e(f), g\rangle + \frac{1}{2}\eta^2 \langle S_e(f), Hess_{g_0}(r^2)\rangle, \end{aligned} \tag{3.3}$$

by a straightforward computation. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g_0 and $e_m = \frac{\partial}{\partial r}$. We may assume that $Hess_{g_0}(r^2)$ is a diagonal matrix with respect to $\{e_i\}$. Note that $\{\hat{e}_i = \eta^{-1}e_i\}$ is an orthonormal frame with respect to g. Therefore,

$$\begin{aligned} \frac{1}{2}\eta^2\langle S_e(f), Hess_{g_0}(r^2)\rangle &= \frac{1}{2}\eta^2 \sum_{i,j=1}^m S_e(f)(\hat{e}_i, \hat{e}_j) Hess_{g_0}(r^2)(\hat{e}_i, \hat{e}_j) \\ &= \frac{1}{2}\eta^2 \Big[\sum_{i=1}^m \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} Hess_{g_0}(r^2)(\hat{e}_i, \hat{e}_j) \\ &\quad - \sum_{i,j=1}^m \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} (\mathrm{d}f(\hat{e}_i), \mathrm{d}f(\hat{e}_j)) Hess_{g_0}(r^2)(\hat{e}_i, \hat{e}_j) \Big] \end{aligned}$$

$$= \frac{1}{2}\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}\sum_{i=1}^{m} Hess_{g_0}(r^2)(e_i,\, e_i)$$

$$-\frac{1}{2}\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\sum_{i=1}^{m}(\mathrm{d}f(\hat{e}_i),\, \mathrm{d}f(\hat{e}_i))Hess_{g_0}(r^2)(e_i,\, e_i)$$

$$\geq \frac{1}{2}\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}[(m-1)\mu_{min}+2] - \frac{1}{2}max\{2,\, \mu_{max}\}\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\sum_{i=1}^{m}(\mathrm{d}f(\hat{e}_i),\, \hat{e}_i))$$

$$= \frac{1}{2}\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}[(m-1)\mu_{min}+2] - \frac{1}{2}max\{2,\, \mu_{max}\}\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}|\mathrm{d}f|^2$$

$$\geq \frac{1}{2}\Big[(m-1)\mu_{min}+2-2max\{2, \mu_{max}\}\Big]\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}, \tag{3.4}$$

and

$$\langle S_e(f),\, g\rangle = m\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2} - \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}(\mathrm{d}f(\hat{e}_i),\, \mathrm{d}f(\hat{e}_j))_h(\hat{e}_i, \hat{e}_j)_g$$

$$= m\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2} - \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}|\mathrm{d}f|^2 \geq (m-2)\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}. \tag{3.5}$$

It follows from (3.3), (3.4), (3.5) and the condition ($\star$) that

$$\langle S_e(f),\, \frac{1}{2}L_X g\rangle \geq \Big[r\frac{\partial \log\eta}{\partial r}(m-2) + \frac{m-1}{2}\mu_{min} + 1 - max\{2, \mu_{max}\}\Big]$$

$$\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2} \geq C\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}. \tag{3.6}$$

Secondly, applying the co-area and following fact:

$$|\nabla r|_g^2 = \sum_{i=1}^{m}(\hat{e}_i r)^2 = \sum_{i=1}^{m-1}\eta^{-2}(e_i r)^2 + \eta^{-2}$$

$$= \sum_{i=1}^{m-1}\eta^{-2}\Big[(e_i, \frac{\partial}{\partial r})_{g_0}\Big]^2 + \eta^{-2} = \eta^{-2}\ (\text{i.e., } |\nabla r|_g = \eta^{-1}),$$

we have

$$\int_{\partial B(r)} S_e(f)(X,\, n)\mathrm{d}s_g = \int_{\partial B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\Big[\frac{|\mathrm{d}f|^2}{2}(X,\, n) - (\mathrm{d}f(X),\, \mathrm{d}f(n))_h\Big]\mathrm{d}s_g$$

$$= r\int_{\partial B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}\eta\mathrm{d}s_g - \int_{\partial B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} r\eta^{-1}(\mathrm{d}f(\frac{\partial}{\partial r}),\, \mathrm{d}f(\frac{\partial}{\partial r}))_h\mathrm{d}s_g$$

$$\leq r\int_{\partial B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}\eta\mathrm{d}s_g = r\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r\int_{\partial B(t)}\Big[\frac{\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}}{|\nabla r|}\mathrm{d}s_g\Big]\mathrm{d}t$$

$$= r\frac{\mathrm{d}}{\mathrm{d}r}\int_{B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}\mathrm{d}v. \tag{3.7}$$

We obtain from (3.3), (3.6) and (3.7) that

$$0 \le C \int_{B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v \le r \frac{\mathrm{d}}{\mathrm{d}r} \int_{B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v,$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{\int_{B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v}{r^C} \ge 0.$$

Consequently,

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v}{\sigma_1^C} \le \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v}{\sigma_2^C} \tag{3.8}$$

for any $0 < \sigma_1 \le \sigma_2$. □

COROLLARY 3.2. *Let $f : (M, \eta^2 g_0) \to (N, h)$ be a non-constant exponentially harmonic map. Suppose that η satisfies the condition ($\star$). Then*

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \mathrm{d}v}{\sigma_1^C} \le \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \mathrm{d}v}{\sigma_2^C} \tag{3.9}$$

for any $0 < \sigma_1 \le \sigma_2$.

LEMMA 3.3. *Let (M, g) be a complete Riemannian manifold with a pole x_0 and K_r the radial curvature of M.*

(i) *If $-a^2 \le K_r \le -b^2$ with $a \ge b > 0$ and $(m-1)b - 2a \ge 0$, then*

$$\Big[(m-1)\mu_{min} + 2 - 2max\{2,\, \mu_{max}\}\Big] \ge 2(m - \frac{2a}{b}).$$

(ii) *If $-\frac{E}{(1+r^2)^{1+\epsilon}} \le K_r \le \frac{F}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $E \ge 0$ and $0 \le F < 2\epsilon$, then*

$$\Big[(m-1)\mu_{min} + 2 - 2\,max\{2,\, \mu_{max}\}\Big] \ge 2\Big[1 + (m-1)(1 - \frac{F}{2\epsilon}) - 2\mathrm{e}^{\frac{E}{2\epsilon}}\Big].$$

(iii) *If $-\frac{\alpha^2}{\gamma^2+r^2} \le K_r \le \frac{\beta^2}{\gamma^2+r^2}$ with $\alpha \ge 0$, $\beta^2 \in [0,\, 1/4]$ and $\gamma \ge 0$, then*

$$\Big[(m-1)\mu_{min} + 2 - 2max\{2,\, \mu_{max}\}\Big] \ge 2\Big[1 + (m-1)\frac{1+\sqrt{1-4\beta^2}}{2} - 2\frac{1+\sqrt{1+4\alpha^2}}{2}\Big].$$

(cf. [9,12,13]).

THEOREM 3.4. *Let (M, g) be a complete Riemannian manifold with a pole x_0. Suppose that the radial curvature K_r of M satisfies one of three conditions* *(i), (ii) or (iii) in* Lemma 3.3. *If $f:(M,g)\to(N,h)$ is an exponentially harmonic map with $A>0$, then*

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v}{\sigma_1^A} \le \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v}{\sigma_2^A} \tag{3.10}$$

for any $0<\sigma_1\le\sigma_2$, where

$$A=\begin{cases} m-\frac{2a}{b} & \text{if } K_r \text{ satisfies (i)},\\ 1+(m-1)(1-\frac{F}{2\epsilon})-2\mathrm{e}^{\frac{E}{2\epsilon}} & \text{if } K_r \text{ satisfies (ii)}\\ [1+(m-1)\frac{1+\sqrt{1-4\beta^2}}{2}-2\frac{1+\sqrt{1+4\alpha^2}}{2}] & \text{if } K_r \text{ satisfies (iii)}. \end{cases} \tag{3.11}$$

In particular, if $\int_{B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v = o(R^A)$, then f is constant.

Proof. Setting $\eta=1$ in Theorem 3.1 and applying Lemma 3.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\int_{B(r)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}}{r^A}\mathrm{d}v \ge 0.$$

Hence, we obtain

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v}{\sigma_1^A} \le \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v}{\sigma_2^A} \tag{3.12}$$

for any $0<\sigma_1\le\sigma_2$. □

Corollary 3.5. *Let $(M,\, g)$ be a complete Riemannian manifold with a pole x_0. Suppose that the radial curvature K_r of M satisfies one of three conditions* (i), (ii) *or* (iii) *in* Lemma 3.3. *If $f:(M,\,g)\to(N,\,h)$ is a non-constant exponentially harmonic map with $A>0$, then*

$$\frac{\int_{B(\sigma_1)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\mathrm{d}v}{\sigma_1^A} \le \frac{\int_{B(\sigma_2)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\mathrm{d}v}{\sigma_2^A}$$

for any $0<\sigma_1\le\sigma_2$.

Definition 3.6. The energy functional $E_1(f)=\int_M \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\mathrm{d}v$ of a map $f:M\to N$ is *slowly divergent* if there exists a positive function $\phi(r)$ with $\int_{R_0}^{\infty}\frac{dr}{r\phi(r)}=+\infty (R_0>0)$ such that

$$\lim_{R\to\infty}\int_{B(R)}\frac{\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}}{\phi(r(x))}\mathrm{d}v<\infty. \tag{3.13}$$

Theorem 3.7. *Let $f : (M, \eta^2 g_0) \to (N, h)$ be an exponentially harmonic map. Suppose that η satisfies the condition $(\star)$ and $E_1(f)$ is slowly divergent, then f is constant.*

Proof. In the proof of Theorem 3.1, we have

$$C\int_{B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v_g \leq R\frac{\mathrm{d}}{\mathrm{d}r}\int_{\partial B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\eta \mathrm{d}s_g. \tag{3.14}$$

Assume that f is a non-constant map. Then there exists $R_0 > 0$ such that for $R \geq R_0$

$$\int_{B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2} \mathrm{d}v \geq C_1, \tag{3.15}$$

where C_1 is a positive constant. It follows from (3.14) and (3.15) that

$$\int_{\partial B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\eta \mathrm{d}s_g \geq \frac{C_1 \cdot C}{R}, \tag{3.16}$$

for $R \geq R_0$. Consequently,

$$\begin{aligned}\lim_{R\to\infty}\int_{B(R)} \frac{\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}}{\phi(r(x))}\mathrm{d}v &= \int_0^\infty \frac{\mathrm{d}R}{\phi(R)}\int_{\partial B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\eta \mathrm{d}s_g \\ &\geq \int_{R_0}^\infty \frac{\mathrm{d}R}{\phi(R)}\int_{\partial B(R)} \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}} \frac{|\mathrm{d}f|^2}{2}\eta \mathrm{d}s_g \\ &\geq C \cdot C_1 \int_{R_0}^\infty \frac{\mathrm{d}R}{R\phi(R)} = \infty,\end{aligned}$$

which contradicts with (3.13). Hence, f must be constant. □

Definition 3.8. A bounded domain $D \subset M$ with C^1 boundary ∂D is *star – like* if there exists an interior point $x_0 \in D$ such that

$$\langle \frac{\partial}{\partial r_{x_0}}, n\rangle \geq 0,$$

where n is the unit outer normal to ∂D, the vector field $\frac{\partial}{\partial r_{x_0}}$ is a unit vector field such that for any $x \in (D - \{x_0\}) \cup \partial D$, $\frac{\partial}{\partial r_{x_0}}$ is also a unit vector field tangent to the unique geodesic joining x_0 and pointing away from x_0 (cf. [9]).

It is clear that any convex domain is star-like.

Theorem 3.9. *Let $f : (M, \eta^2 g_0) \to (N, h)$ be an exponentially harmonic map and $D \subset M$ be a bounded star-like domain with C^1 boundary with the pole $x_0 \in D$. Suppose that η satisfies the condition $(\star)$ on D. If $f|_{\partial D} \equiv constant \equiv P \in N$, then f is constant in D.*

Proof. In the proof of Theorem 3.1, letting $X = r\frac{\partial}{\partial r}$ with $r = r_{x_0}$ we have

$$\langle S_e(f), \frac{1}{2}L_X g\rangle \geq C\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2} \geq 0. \tag{3.17}$$

Since $f|_{\partial D} = P$ and $\mathrm{d}f(\xi) = 0$ for any tangent vector ξ of ∂D, we obtain the following on ∂D

$$\begin{aligned} S_e(f)(X, n) &= rS_e(f)(\frac{\partial}{\partial r}, n) \\ &= \Big[\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}(\frac{\partial}{\partial r}, n)_g - \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}(\mathrm{d}f(\frac{\partial}{\partial r}), \mathrm{d}f(n))_h\Big] \\ &= r(\frac{\partial}{\partial r}, n)_g\Big[\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2} - \mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}|\mathrm{d}f|^2\Big] \\ &= -r(\frac{\partial}{\partial r}, n)_g\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2} \leq 0. \end{aligned} \tag{3.18}$$

It follows from (3.2), (3.17) and (3.18) that

$$0 \leq \int_D C\mathrm{e}^{\frac{|\mathrm{d}f|^2}{2}}\frac{|\mathrm{d}f|^2}{2}\mathrm{d}v \leq 0,$$

which implies that f is constant on D. □

REFERENCES

[1] P. Baird, *Stress-energy tensor and the Lichnerowicz Laplacian.* J. Geom. Phys. **58** (2008), 1329–1342.

[2] Y.J. Chiang, *Exponentially harmonic maps, exponential stress energy and stability.* Commun. Contemp. Math. **18** (2016), *6*, 1–14.

[3] Y.J. Chiang, *Exponentially harmonic maps and their properties.* Math. Nachr. **228** (2015), *7–8*, 1970–1980.

[4] Y.J. Chiang and H. Pan, *On exponentially harmonic maps.* Acta Math. Sinica **58** (2015), *1*, 131–140.

[5] Y.J. Chiang, *Developments of harmonic maps, wave maps and Yang-Mills fields into biharmonic maps, biwave maps and bi-Yang-Mills fields.* Front. Math., Springer, Basel, Birkhäuser, 2013.

[6] Y.J. Chiang and R. Wolak, *Transversal wave maps and transversal exponential wave maps.* J. Geom. **104** (2013), *3*, 443–459.

[7] Y.J. Chiang and Y.H. Yang, *Exponential wave maps.* J. Geom. Phys. **57** (2007), *12*, 2521–2532.

[8] L.-F. Cheung and P.-F. Leung, *The second variation formula for exponentially harmonic maps.* Bull. Aust. Math. Soc. **59** (1999), 509–514.

[9] Y.X. Dong and S.S. Wei, *On vanishing theorems for vector bundles of valued p-forms and their applications.* Comm. Math. Phys. **304** (2011), 329–368.

[10] D.M. Duc and J. Eells, *Regularity of exponentially harmonic functions.* Internat. J. Math. **2** (1991), *1*, 395–398.

[11] J. Eells and L. Lemaire, *Some properties of exponentially harmonic maps.* In: B. Bojarski et al. (Eds.), *Partial Differential Equations. Part 1, 2*, 129–136. Banach Center Publ. **27**, Polish Acad. Sci., Warsaw, 1992.

[12] R.E. Green and H. Wu, *Function theory on manifolds which possess pole.* In: Lecture Notes in Math. **699**, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[13] Y.B. Han, Y. Li, Y.B. Ren and S.S. Wei, *New comparison theorems in Riemannian Geometry.* Bull. Inst. Math. Acad. Sin. (N.S.) **9** (2014), *2*, 163–186.

[14] J. Liu, *Nonexistence of stable exponentially harmonic maps from or into compact convex hypersurfaces in* $\mathbb{R}^{m+1}$. Turkish J. Math. **32** (2008), 117–126.

[15] M.C. Hong, *On the conformal equivalence of harmonic maps and exponentially harmonic maps.* Bull. Lond. Math. Soc. **24** (1992), 488–492.

[16] J.Q. Hong and Y. Yang, *Some results on exponentially harmonic maps.* Chinese Ann. Math. Ser. A **14** (1993), *6*, 686–691.

[17] A.D. Kanfon, A. Füzfa and D. Lambert, *Some examples of exponentially harmonic maps.* J. Phys. A, Math. Gen. (2002), *35*, 7629–7639.

[18] T. Omori, *On Eells-Sampson's existence theorem for harmonic maps via exponentially harmonic maps.* Nagoya Math. J. **201** (2011), 133–146.

[19] T. Omori, *On Sacks-Uhlenbeck's existence theorem for harmonic maps via exponentially harmonic maps.* Internat. J. Math. **23** (2012), *10*, 1–6.

[20] Y. Zhang, Y. Wang and J. Liu, *Negative exponentially harmonic maps.* J. Beijing Normal Univ. (Natur. Sci.) **34** (1998), *10*, 324–329.

Received 1 February 2017

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