To the memory of Majdi Ben Halima

ON THE CONTINUITY OF THE LIPSMAN MAPPING OF SEMIDIRECT PRODUCTS

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We consider the semidirect product $G = K \ltimes V$ where K is a connected compact Lie group acting by automorphisms on a finite dimensional vector space V equipped with an inner product $\langle , \rangle$. We denote by $\widehat{G}$ the unitary dual of G and by $\mathfrak{g}^\ddagger/G$ the space of admissible coadjoint orbits, where $\mathfrak{g}$ is the Lie algebra of G. It was pointed out by Lipsman that the correspondence between $\widehat{G}$ and $\mathfrak{g}^\ddagger/G$ is bijective. In this paper, we explicitly determine the topology of the spaces $\widehat{G}$. Also we prove that the Lipsman mapping $\Theta : \mathfrak{g}^\ddagger/G \longrightarrow \widehat{G}$ is continuous.

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1. INTRODUCTION

Let G be a second countable locally compact group and $\widehat{G}$ the unitary dual of G, *i.e.*, the set of all equivalence classes of irreducible unitary representations of G. It is well-known that $\widehat{G}$ is equipped with the Fell topology [8]. The description of the dual topology is a good candidate for some aspects of harmonic analysis on G (for example, see [4, 7]). For a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = exp(\mathfrak{g})$, its dual space $\widehat{G}$ is homeomorphic to the space of coadjoint orbits $\mathfrak{g}^*/G$ through the Kirillov mapping (see [16]). In the context of semidirect products $G = K \ltimes N$ of compact connected Lie group K acting on simply connected nilpotent Lie group N, then it was pointed out by Lipsman in [17], that we have again an orbit picture of the dual space of G. The unitary dual space of Euclidean motion groups is homeomorphic to the admissible coadjoint orbits [7]. This result was generalized in [4], for a class of Cartan motion groups.

In this paper, we consider the semidirect product $G = K \ltimes V$ where K is a connected compact Lie group acting by automorphisms on a finite dimensional vector space V equipped with an inner product $\langle , \rangle$. In the spirit of the orbit method due to Kirillov, R. Lipsman established a bijection between a class of coadjoint orbits of G and the unitary dual $\widehat{G}$. For every admissible linear form ψ of the Lie algebra $\mathfrak{g}$ of G, we can construct an irreducible unitary representation π_ψ by holomorphic induction and according to Lipsman (see [17]), every irreducible representation of G arises in this manner. Then we get a map from the set $\mathfrak{g}^\ddagger$ of the admissible linear forms onto the dual space $\widehat{G}$ of G. Note that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' are on the same G-orbit, finally we obtain a bijection between the space $\mathfrak{g}^\ddagger/G$ of admissible coadjoint orbits and the unitary dual $\widehat{G}$. The preceding discussion motivates our main result:

THEOREM 1.1. *The Lipsman mapping*

$$\Theta : \mathfrak{g}^\ddagger/G \longrightarrow \widehat{G}$$

is continuous.

The present work is organized as follows: Section 2 is devoted to the description of the unitary dual $\widehat{G}$ of G. Section 3 deals with the space of admissible coadjoint orbits $\mathfrak{g}^\ddagger/G$ of G. Theorem 1.1 is proved below in Section 4.

2. DUAL SPACES OF SEMIDIRECT PRODUCT

Throughout this paper, K will denote a connected compact Lie group acting by automorphisms on a finite dimensional vector space $(V, \langle , \rangle)$. We write $k.v$ and $A.v$ (resp. $k.\ell$ and $A.\ell$) for the result of applying elements $k \in K$ and $A \in \mathfrak{k} := Lie(K)$ to $v \in V$ (resp. to $\ell \in V^*$).

Now, one can form the semidirect product $G := K \ltimes V$ which is the so-called generalized motion group. As a set $G = K \times V$ and the multiplication in this group is given by

$$(k, v)(h, u) = (kh, v + k.u), \forall (k, v), (h, u) \in G.$$

The Lie algebra of G is $\mathfrak{g} = \mathfrak{k} \oplus V$ (as a vector space) and the Lie algebra structure is given by the bracket

$$[(A, a), (B, b)] = ([A, B], A.b - B.a), \forall (A, a), (B, b) \in \mathfrak{g}.$$

Under the identification of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{k}^* \oplus V^*$, we can express the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ as $F(A, a) = f(A) + \ell(a)$, for all $F = (f, \ell) \in \mathfrak{g}^*$ and $(A, a) \in \mathfrak{g}$. The adjoint representation Ad_G and coadjoint representation $Ad_G^*$of G are given respectively, by the following relations

$$Ad_G(k, v)(A, a) = (Ad_K(k)A, k.a - Ad_K(k)A.v), \forall (k, v) \in G, (A, a) \in \mathfrak{g},$$

$$Ad^*_G(k,v)(f,\ell) \;=\; (Ad^*_K(k)f + k.\ell \odot v, k.\ell), \forall (k,v) \in G, (f,\ell) \in \mathfrak{g}^*,$$

where $\ell \odot v$ is the element of $\mathfrak{k}^*$ defined by

$$\ell \odot v(A) = \ell(A.v) = -(A.\ell)(v), \forall A \in \mathfrak{k}, \ell \in V^*, v \in V.$$

Note that the map $\odot : V^* \times V \longrightarrow \mathfrak{k}^*$ defined by $(\ell \odot v)(A) = \ell(A.v)$, $v \in V$, $A \in \mathfrak{k}$ satisfies a fundamental equivariance property:

$$Ad^*_K(k)(\ell \odot v) \;=\; (k.\ell) \odot (k.v),\; k \in K.$$

Therefore, the coadjoint orbit of G passing through $(f,\ell) \in \mathfrak{g}^*$ is given by

$$\mathcal{O}^G_{(f,\ell)} \;=\; \Big\{ \Big(Ad^*_K(k)f + k.\ell \odot v, k.\ell \Big), k \in K, v \in V \Big\}. \tag{2.1}$$

For $\ell \in V^*$, we define $K_\ell := \{k \in K;\; k.\ell = \ell\}$ the isotropy subgroup of ℓ in K and the Lie algebra of K_ℓ is given by the vector space $\mathfrak{k}_\ell = \{A \in \mathfrak{k};\; A.\ell = 0\}$. Let $\imath_\ell : \mathfrak{k}_\ell \hookrightarrow \mathfrak{k}$ be the injection map, then $\imath^*_\ell : \mathfrak{k}^* \longrightarrow \mathfrak{k}^*_\ell$ is the projection map and we have

$$\mathfrak{k}^\circ_\ell \;=\; \mathrm{Ker}(\imath^*_\ell) \tag{2.2}$$

where $\mathfrak{k}^\circ_\ell$ is the annihilator of $\mathfrak{k}_\ell$. If we define the linear map $h_\ell : \mathfrak{k} \longrightarrow V^*$ by

$$h_\ell(A) := -A.\ell,\; \forall A \in \mathfrak{k},$$

then we have $\mathfrak{k}_\ell = \mathrm{Ker}(h_\ell)$. The dual $h^*_\ell : V \longrightarrow \mathfrak{k}^*$ of h_ℓ is given by the relation $h^*_\ell(v)(A) = h_\ell(A)(v) = -(A.\ell)(v)$, and so $h^*_\ell(v) = \ell \odot v$, $\forall \ell \in V^*$, $\forall v \in V$ (for more details see [3]).

The following is a useful lemma from [3], giving a characterization of the annihilator $\mathfrak{k}^\circ_\ell$ in terms of the linear map h_ℓ.

LEMMA 2.1. *Using the previous notations, then we have the equality*

$$\mathfrak{k}^\circ_\ell = \mathrm{Im}(h^*_\ell).$$

Here we recall briefly the description of the unitary dual of G via Mackey's little group theory (see [18]). For every non-zero linear form ℓ on V, we denote by χ_ℓ the unitary character of the vector Lie group V given by $\chi_\ell = \mathrm{e}^{i\ell}$. Let ρ be an irreducible unitary representation of K_ℓ on some Hilbert space $\mathcal{H}_\rho$. The map

$$\rho \otimes \chi_\ell : (k,v) \longmapsto \mathrm{e}^{i\ell(v)}\rho(k)$$

is a representation of the Lie group $K_\ell \ltimes V$ such that one induces, in order to get a unitary representation of G. We denote by $\mathcal{H}_{(\rho,\ell)} := L^2(K,\mathcal{H}_\rho)^\rho$ the subspace of $L^2(K,\mathcal{H}_\rho)$ consisting of all the maps ξ which satisfy the covariance condition

$$\xi(kh) = \rho(h^{-1})\xi(k), \forall k \in K, h \in K_\ell.$$

The induced representation

$$\pi_{(\rho,\ell)} := \mathrm{Ind}_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell)$$

is defined on $\mathcal{H}_{(\rho,\ell)}$ by

$$\pi_{(\rho,\ell)}(k,v)\xi(h) = \mathrm{e}^{i\ell(h^{-1}.v)}\xi(k^{-1}h)$$

where $(k,v) \in G, h \in K$ and $\xi \in \mathcal{H}_{(\rho,\ell)}$. By Mackey's theory we can say that the induced representation $\pi_{(\rho,\ell)}$ is irreducible and every infinite dimensional irreducible unitary representation of G is equivalent to one of $\pi_{(\rho,\ell)}$. Moreover, tow representations $\pi_{(\rho,\ell)}$ and $\pi_{(\rho',\ell')}$ are equivalent if and only if ℓ and ℓ' are contained in the same K-orbit and the representation ρ and ρ' are equivalent under the identification of the conjugate subgroups K_ℓ and $K_{\ell'}$. All irreducible representations of G which are not trivial on the normal subgroup V, are obtained by this manner. On the other hand, we denote also by τ the extension of every unitary irreducible representation τ of K on G, which are simply defined by $\tau(k,v) := \tau(k)$ for $k \in K$ and $v \in V$. Let Ω be a K-orbit in V^*. We fix $\ell \in \Omega$ and we define the subset $\widehat{G}(\Omega)$ of $\widehat{G}$ by

$$\widehat{G}(\Omega) = \left\{\mathrm{Ind}_{K_\ell \ltimes V}^{K \ltimes V}(\rho \otimes \chi_\ell); \rho \in \widehat{K_\ell}\right\}.$$

Then we conclude that

$$\widehat{G} = \widehat{K} \bigcup \Big(\bigcup_{\Omega \in \Lambda} \widehat{G}(\Omega) \Big)$$

where Λ is the set of the non-trivial orbits in V^*/K.

In the remainder of this paper, we shall assume that G is exponential, *i.e.*, K_ℓ is connected for all $\ell \in V^*$. Let ρ_μ be an irreducible representation of K_ℓ with highest weight μ. For simplicity, we shall write $\pi_{(\mu,\ell)}$ instead of $\pi_{(\rho_\mu,\ell)}$ and $\mathcal{H}_{(\mu,\ell)}$ instead of $\mathcal{H}_{(\rho_\mu,\ell)}$.

We close this section by presenting two results which are being used in the description of the dual topology of G. These are required for our proof of Theorem 1.1.

Let N be an abelian group, and assume that the compact Lie group K acts on the left on N by automorphisms. As sets, the semidirect product $K \ltimes N$ is the Cartesian product $K \times N$ and the group multiplication is given by

$$(k_1, x_1) \cdot (k_2, x_2) = (k_1 k_2, x_1 + k_1 x_2).$$

Let χ be a unitary character of N, and let K_χ be the stabilizer of χ under the action of K on $\widehat{N}$ defined by

$$(k \cdot \chi)(x) = \chi(k^{-1}x).$$

If ρ is an element of $\widehat{K_\chi}$, then the triple $(\chi,(K_\chi,\rho))$ is called a cataloguing triple. From the notations of [2], we denote by $\pi(\chi,K_\chi,\rho)$ the induced representation $\mathrm{Ind}_{K_\chi\ltimes N}^{K\ltimes N}(\rho\otimes\chi)$. Referring to [2, p. 187], we have

Proposition 1. *The mapping* $(\chi,(K_\chi,\rho))\longrightarrow\pi(\chi,K_\chi,\rho)$ *is onto* $\widehat{K\ltimes N}$.

We denote by $\mathcal{A}(K)$ the set of all pairs (K',ρ'), where K' is a closed subgroup of K and ρ' is an irreducible representation of K'. We equip $\mathcal{A}(K)$ with the Fell topology (see [8]). Therefore, every element in $\widehat{K\ltimes N}$ can be catalogued by elements in the topological space $\widehat{N}\times\mathcal{A}(K)$. Larry Baggett has given an abstract description of the topology of the dual space of a semidirect product of a compact group with an abelian group in terms of the Mackey parameters of the dual space (see [2, Theorem 6.2-A]). The following result provides a precise and neat description of the topology of $\widehat{K\ltimes N}$.

Theorem 2.2. *Let Y be a subset of $\widehat{K\ltimes N}$ and π an element of $\widehat{K\ltimes N}$. Then π is weakly contained in Y if and only if there exist: a cataloguing triple $(\chi,(K_\chi,\rho))$ for π, an element (K',ρ') of $\mathcal{A}(K)$, and a net $\{(\chi_n,(K_{\chi_n},\rho_n))\}$ of cataloguing triples such that:*

(i) *for each n, the irreducible unitary representation $\pi(\chi_n,K_{\chi_n},\rho_n)$ of $K\ltimes N$ is an element of Y;*

(ii) *the net $\{(\chi_n,(K_{\chi_n},\rho_n))\}$ converges to $(\chi,(K',\rho'))$;*

(iii) *K_χ contains K', and the induced representation $\mathrm{Ind}_{K'}^{K_\chi}(\rho')$ contains ρ.*

3. ADMISSIBLE COADJOINT ORBITS OF SEMIDIRECT PRODUCT

We keep the notations of Section 2. Fix a non-zero linear form $\ell\in V^*$, and we consider an irreducible representation ρ_μ of K_ℓ with highest weight μ. Then the stabilizer G_ψ of $\psi=(\mu,\ell)$ in G is given by

$$\begin{aligned} G_\psi &= \Big\{(k,v)\in G;\ (Ad_K^*(k)\mu+k.\ell\odot v,k.\ell)=(\mu,\ell)\Big\}\\ &= \Big\{(k,v)\in G;\ k\in K_\ell, Ad_K^*(k)\mu+\ell\odot v=\mu\Big\}\\ &= \Big\{(k,v)\in G;\ k\in K_\ell,\ Ad_K^*(k)\mu=\mu\Big\}\end{aligned}$$

since $i_\ell^*(\ell\odot v)=0$ (see Lemma 2.1 and the equality (2.2)). Thus, we have $G_\psi=K_\psi\ltimes V_\psi$, then ψ is aligned (see [17]). A linear form $\psi\in\mathfrak{g}^*$ is called admissible if there exists a unitary character χ of the identity component of G_ψ such that $d\chi=i\psi_{|\mathfrak{g}_\psi}$. According to Lipsman (see [17]), the representation of G obtained by holomorphic induction from (μ, ℓ) is equivalent to the representation $\pi_{(\mu,\ell)}$. Let τ_λ be an irreducible representation of K with highest weight λ, then the representation of G obtained by holomorphic induction from $(\lambda, 0)$ is equivalent to τ_λ. The coadjoint orbit of G through $(\lambda, 0) \in \mathfrak{g}^*$ is denoted by $\mathcal{O}_\lambda^G$. It is clear that $\mathcal{O}_\lambda^G$ is an admissible coadjoint orbit of G. We denote by $\mathfrak{g}^\ddagger \subset \mathfrak{g}^*$ the set of all admissible linear forms on $\mathfrak{g}$. The quotient space $\mathfrak{g}^\ddagger/G$ is called the space of admissible coadjoint orbits of G. Moreover, one can check that $\mathfrak{g}^\ddagger/G$ is the union of the set of all orbits $\mathcal{O}_{(\mu,\ell)}^G$ and the set of all orbits $\mathcal{O}_\lambda^G$.

We conclude this section by recalling needed results. Let L be a closed subgroup of K. Let T_K and T_L be maximal tori respectively in K and L such that $T_L \subset T_K$. Their corresponding Lie algebras are denoted by $\mathfrak{t}_\mathfrak{k}$ and $\mathfrak{t}_\mathfrak{l}$. We denote by W_K and W_L the Weyl groups of K and L associated respectively to the tori T_K and T_L. Notice that every element $\lambda \in P_K$ takes pure imaginary values on $\mathfrak{t}_\mathfrak{k}$, where P_K is the integral weight lattice of T_K. Hence such an element $\lambda \in P_K$ can be considered as an element of $(i\mathfrak{t}_\mathfrak{k})^*$. Let C_K^+ be a positive Weyl chamber in $(i\mathfrak{t}_\mathfrak{k})^*$, and we define the set P_K^+ of dominant integral weights of T_K by $P_K^+ := P_K \cap C_K^+$. For $\lambda \in P_K^+$, denote by $\mathcal{O}_\lambda^K$ the K-coadjoint orbit passing through the vector $-i\lambda$. It was proved by Kostant in [15], that the projection of $\mathcal{O}_\lambda^K$ on $\mathfrak{t}_\mathfrak{k}^*$ is a convex polytope with vertices $-i(w.\lambda)$ for $w \in W_K$, and that is the convex hull of $-i(W_K.\lambda)$. For the same manner, we fix a positive Weyl chamber C_L^+ in $\mathfrak{t}_\mathfrak{l}^*$ and we define the set P_L^+ of dominant integral weights of T_L.

Also we denote by $i_\mathfrak{l}^*$ the $\mathbb{C}$-linear extension of both the natural projection of $\mathfrak{k}^*$ onto $\mathfrak{l}^*$ and the natural projection of $\mathfrak{t}_\mathfrak{k}^*$ onto $\mathfrak{t}_\mathfrak{l}^*$. Consider tow irreducible representations $\tau_\lambda \in \widehat{K}$ and $\rho_\mu \in \widehat{L}$ with respective highest weights $\lambda \in P_K^+$ and $\mu \in P_L^+$. We have the following result.

Lemma 3.1. *If $\mu = i_\mathfrak{l}^*(s.\lambda)$ with $s \in W_K$, then τ_λ occurs in the induced representation* $\mathrm{Ind}_L^K(\rho_\mu)$.

We refer to [1], for the proof of this Lemma.

4. MAIN RESULTS

Let us now return to the context and notations of the previous Sections. To an irreducible representation ρ_μ of K_ℓ with highest weight μ and a non-zero linear form ℓ on V, we associate the representation $\pi_{(\mu,\ell)}$ of G and its corresponding cataloguing triple $(\ell, (K_\ell, \rho_\mu))$. Also for an irreducible representation τ_λ of K with highest weight λ, we denote by $(0, (K, \tau_\lambda))$ the cataloguing of the trivial extension of τ_λ to G. By $\mathcal{C}(K)$ we mean the space of all closed subgroups of K equipped with the compact-open topology [8]. Its well-known that $\mathcal{C}(K)$ is a compact space and an important fact worth mentioning here is that:

Remark 4.1. If we have the following convergence

$$\ell_m \longrightarrow \ell \tag{4.1}$$

$$K_{\ell_m} \longrightarrow L \tag{4.2}$$

where L is a subgroup of K, then K_ℓ contains L.

Thanks to Baggett's theorem (Theorem 2.2), we have the following tow first Propositions.

PROPOSITION 4.2. *Let* $(\pi_{(\mu^n,\ell_n)})_n$ *be a sequence of elements in* $\widehat{G}$. *Then we have:* $(\pi_{(\mu^n,\ell_n)})_n$ *converges to* $\pi_{(\mu,\ell)}$ *in* $\widehat{G}$, *if and only if for every subsequence* $(K_{\ell_{nm}})_m$, *for which the sequence of subgroup-representation pairs* $\big((K_{\ell_{nm}}, \rho_{\mu^{nm}})\big)_m$ *converges to* (L,ρ) *in* $\mathcal{A}(K)$, *we have that* $(\ell_{nm})_m$ *converges to* ℓ *and* $\rho_\mu \in \mathrm{Ind}_L^{K_\ell}(\rho)$.

PROPOSITION 4.3. *Let* $(\pi_{(\mu^n,\ell_n)})_n$ *be a sequence of elements in* $\widehat{G}$. *Then* $(\pi_{(\mu^n,\ell_n)})_n$ *converges to* τ_λ *in* $\widehat{G}$, *if and only if for every subsequence* $(K_{\ell_{nm}})_m$, *for which the sequence of subgroup-representation* $\big((K_{\ell_{nm}}, \rho_{\mu^{nm}})\big)_m$ *converges to* (L,ρ) *in* $\mathcal{A}(K)$, *we have that* $(\ell_{nm})_m$ *converges to* 0 *and* $\tau_\lambda \in \mathrm{Ind}_L^{K}(\rho)$.

To study the convergence in the quotient space $\mathfrak{g}^\ddagger/G$, we need the following result (see [16. p. 135] for the proof).

LEMMA 4.4. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. We denote $\mathfrak{g}^*/G$ the space of coadjoint orbits and by $p_G : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e., a subset V in $\mathfrak{g}^*/G$ is open if and only if $p_G^{-1}(V)$ is open in $\mathfrak{g}^*$. Therefore, a sequence $(\mathcal{O}_n^G)_n$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}^G$ in $\mathfrak{g}^*/G$ if and only if for any $l \in \mathcal{O}^G$, there exist $l_n \in \mathcal{O}_n^G$, $n \in \mathbb{N}$, such that $l = \lim\limits_{n\longrightarrow+\infty} l_n$.*

Now, we can prove the following propositions.

PROPOSITION 4.5. *Let* $\big(\mathcal{O}^G_{(\mu^n,\ell_n)}\big)_n$ *be a sequence in* $\mathfrak{g}^\ddagger/G$. *If* $\big(\mathcal{O}^G_{(\mu^n,\ell_n)}\big)_n$ *converges to* $\mathcal{O}^G_{(\mu,\ell)}$ *in* $\mathfrak{g}^\ddagger/G$, *then for every subsequence* $(K_{\ell_{nm}})_m$, *for which the sequence of subgroup-representation pairs* $\big((K_{\ell_{nm}}, \rho_{\mu^{nm}})\big)_m$ *converges to* (L,ρ) *in* $\mathcal{A}(K)$, *we have that* $(\ell_{nm})_m$ *converges to* ℓ *and* $\rho_\mu \in \mathrm{Ind}_L^{K_\ell}(\rho)$.

Proof. We assume that the sequence of admissible coadjoint orbits $\big(\mathcal{O}^G_{(\mu^n,\ell_n)}\big)_n$ converges to $\mathcal{O}^G_{(\mu,\ell)}$ in $\mathfrak{g}^\ddagger/G$. By referring to [3], we show that the coadjoint orbit $\mathcal{O}^G_{(\mu,\ell)}$ is always obtained by symplectic induction from the coadjoint orbit $M = \mathcal{O}^H_{(\mu,\ell)}$ of $H := K_\ell \ltimes V$ passing through $(\mu,\ell) \in \mathfrak{k}^*_\ell \oplus V^*$ ($\mathfrak{k}_\ell \ltimes V := Lie(H)$), *i.e.*,

$$\mathcal{O}^G_{(\mu,\ell)} = M_{ind} := J^{-1}_{\widetilde{M}}(0)/H, \tag{4.3}$$

where $J_{\widetilde{M}} : \widetilde{M} = M \times T^*G \longrightarrow \mathfrak{k}^*_\ell \ltimes V^*$ is the momentum map of $\widetilde{M}$ and the zero level set $J^{-1}_{\widetilde{M}}(0)$ is given by

$$J^{-1}_{\widetilde{M}}(0) = \Big\{\Big((Ad^*_K(k)\mu,\ell), g, (Ad^*_K(k)\mu + \ell \odot v, \ell)\Big),\ k \in K_\ell, g \in G, v \in V\Big\}.$$

Let φ_M be the action of H on M, hence H acts on $\widetilde{M} = M \times T^*G$ by $\varphi_{\widetilde{M}}$ as follows

$$\varphi_{\widetilde{M}}(h)(m,g,f) = \Big(\varphi_M(h)(m), gh^{-1}, Ad^*_H(h)f\Big), \tag{4.4}$$

for all $h \in H, (m,g,f) \in M \times T^*G$. By identifying $\mathfrak{g}^*$ with the left-invariant 1-form on G. Then we can write $T^*G \cong G \times \mathfrak{g}^*$.

Using Lemma 4.4 and by combining (4.3) with (4.4), then we deduce that for every subsequence $(K_{\ell_{n_m}})_m$, for which the sequence of subgroup-representation pairs $(K_{\ell_{n_m}}, \rho_{\mu^{n_m}})_m$ converges to (L,ρ) in $\mathcal{A}(K)$, there exist sequences $k_m, h_m \in K_{\ell_{n_m}}$, $v_m, w_m \in V$, and $g_m \in G$ such that the sequence $(\phi_m)_m$ defined by

$$\begin{aligned}
\phi_m &= \varphi_{\widetilde{M}}(k_m,v_m)\big((Ad^*_K(h_m)\mu^{n_m}, \ell_{n_m}), g_m, (Ad^*_K(h_m)\mu^{n_m} \\
&+ \ell_{n_m} \odot w_m, \ell_{n_m})\big) \\
&= \Big(Ad^*_K(k_m h_m)\mu^{n_m} + i^*_{\ell_{n_m}}(\ell_{n_m} \odot v_m), \ell_{n_m}\Big), g_m(k_m,v_m)^{-1}, \\
&\quad (Ad^*_K(k_m h_m)\mu^{n_m} + Ad^*_K(k_m)(\ell_{n_m} \odot w_m) + \ell_{n_m} \odot v_m, \ell_{n_m})\Big)
\end{aligned}$$

converges to $\big((\mu,\ell), e_G, (\mu,\ell)\big)$. It follows that

$$\ell_{n_m} \longrightarrow \ell \tag{4.5}$$

and

$$Ad^*_K(k_m h_m)\mu^{n_m} + i^*_{\ell_{n_m}}(\ell_{n_m} \odot v_m) \longrightarrow \mu \tag{4.6}$$

as $n \longrightarrow +\infty$. By compactness of K we may assume that $(k_m h_m)_m$ converges to $p \in L \subset K_\ell$. Using the fact that $i^*_{\ell_{n_m}}(\ell_{n_m} \odot v_m) = 0$, we obtain from (4.6) that

$$\mu^{n_m} = Ad^*(p^{-1})\mu \tag{4.7}$$

for m large enough. Furthermore, we known that there exists an element $s \in W_{K_\ell}$ such that $Ad^*(p^{-1})\mu = s.\mu$. Hence $\mu^{n_m} = s.\mu$ for m large enough and we conclude by Lemma 3.1 that $\rho_\mu \in \mathrm{Ind}_L^{K_\ell}(\rho)$. This completes the proof of the Proposition. $\square$

PROPOSITION 4.6. *If the sequence* $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ *converges to* $\mathcal{O}^G_\lambda$ *in* $\mathfrak{g}^\ddagger/G$, *then for every subsequence* $(K_{\ell_{n_m}})_m$, *for which the sequence of subgroup-representation pairs* $\left((K_{\ell_{n_m}}, \rho_{\mu^{n_m}})\right)_m$ *converges to* (L,ρ) *in* $\mathcal{A}(K)$, *we have that* $(\ell_{n_m})_m$ *converges to* 0 *and* $\tau_\lambda \in \mathrm{Ind}_L^K(\rho_\mu)$.

Proof. We use the notations and proceedings of the proof of the last proposition. Let us assume that the sequence $\left(\mathcal{O}^G_{(\mu^n,\ell_n)}\right)_n$ converges to $\mathcal{O}^G_\lambda$. Also for every subsequence $(K_{\ell_{n_m}})_m$, for which the sequence of subgroup-representation pairs $\left((K_{\ell_{n_m}}, \rho_{\mu^{n_m}})\right)_m$ converges to (L,ρ) in $\mathcal{A}(K)$, there exist sequences $k_m, h_m \in K_{\ell_{n_m}}$, $v_m, w_m \in V$, and $g_m \in G$ such that the sequence $(\Psi_m)_m$ defined by

$$\begin{aligned}
\Psi_m &= \varphi_{\widetilde{M}}(k_m,v_m)\big((Ad^*_K(h_m)\mu^{n_m},\ell_{n_m}), g_m, (Ad^*_K(h_m)\mu^{n_m} \\
&+ \ell_{n_m}\odot w_m, \ell_{n_m})\big) \\
&= \Big(Ad^*_K(k_m h_m)\mu^{n_m} + i^*_{\ell_{n_m}}(\ell_{n_m}\odot v_m), \ell_{n_m}\big), g_m(k_m,v_m)^{-1}, \\
&\quad (Ad^*_K(k_m h_m)\mu^{n_m} + Ad^*_K(k_m)(\ell_{n_m}\odot w_m) + \ell_{n_m}\odot v_m, \ell_{n_m})\Big)
\end{aligned}$$

converges to $\big((\lambda,0), e_G, (\lambda,0)\big)$. From the above facts, we conclude the following convergence

$$\ell_{n_m} \longrightarrow 0 \tag{4.8}$$

$$Ad^*(k_m h_m)\mu^{n_m} \longrightarrow \lambda. \tag{4.9}$$

By assumption that the sequence $(k_m h_m)_m$ converges to $p \in L$, we obtain from (4.9), that $\mu^{n_m} = Ad^*(p^{-1})\lambda$ for m large enough. Hence there exists $w \in W_K$, such that $\mu^{n_m} = w.\lambda$ for m large enough. Lemma 3.1 allows us to derive that $\tau_\lambda \in \mathrm{Ind}_L^K(\rho_\mu)$. $\square$

We have finished the proof of Theorem 1.1.

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