SOME CONSIDERATIONS ABOUT COMMUTATIVE MULTIPLICATIVE UNITARIES

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Let $\mathcal{H}$ be a Hilbert space. An unitary operator W in $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ is said to be multiplicative if it satisfies the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$, moreover, it is said to be commutative if it satisfies the equation $W_{13}W_{23} = W_{23}W_{13}$. A well-known result of Baaj and Skandalis asserts that every commutative multiplicative unitary on a separable Hilbert space is equivalent, up to tensoring with a n-dimensional Hilbert space, to the commutative multiplicative unitary V_G induced from a locally compact group G constructed in a suitable manner from W. In this paper, we prove that it is possible to remove the condition on the separability of the Hilbert space provided that the commutative multiplicative unitary has a regularity condition. We also study the case where the von Neumann algebras $\mathscr{L}(W)$ and $\widehat{\mathscr{L}}(W)$ generated from W are standard.

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1. INTRODUCTION

One of the purposes of this paper is to examine how the regularity condition for the commutative multiplicative unitaries allows the construction of a locally compact group even in the absence of the separability of underlying Hilbert space.

We recall the following definitions [1, 8]:

• A **multiplicative unitary** on the Hilbert space $\mathcal{H}$ is a unitary operator W belonging to $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ and verifying the *pentagonal rule*: $W_{12}W_{13}W_{23} = W_{23}W_{12}$ [where, if Σ is the unitary "flip" defined by $\Sigma(\xi\otimes\eta) = \eta\otimes\xi$, $W_{12} = W\otimes I$, $W_{13} = \Sigma_{12}W_{23}\Sigma_{12}$ $(\Sigma_{12} = \Sigma\otimes I)$, $W_{23} = I\otimes W$].

• A **commutative multiplicative unitary** is a multiplicative unitary W verifying the condition: $W_{13}W_{23} = W_{23}W_{13}$.

• A multiplicative unitary is said to be **regular** when the norm closure of the operator algebra $\{(id\otimes\omega)(\Sigma W) \,/\, \omega\in\mathcal{B}(\mathcal{H})_*\}$ coincides with the ideal $\mathcal{K}(\mathcal{H})$ of the compact operators on $\mathcal{H}$.

We prove that, if the commutative multiplicative unitary W on a Hilbert space $\mathcal{H}$ is regular, then, even if $\mathcal{H}$ is not separable, the Gelfand spectrum G of the commutative C*-algebra $\mathcal{C}^*(W)$ is a locally compact group.

Subsequently, we prove that, for a given commutative multiplicative unitary W, if the abelian von Neumann algebra $\mathscr{L}(W)$ bicommutant of $\mathcal{C}^*(W)$ and the von Neumann algebra $\widehat{\mathscr{L}}(W)$ dual of $\mathscr{L}(W)$ (generally non commutative) are standard and if their modular conjugations satisfy certain properties (certainly satisfied for the multiplicative unitary induced by a locally compact group), then the unitary W is regular and the Gelfand spectrum of the commutative C*-algebra $\mathcal{C}^*(W)$ is a locally compact group.

2. DEFINITIONS, RECALLS AND NOTATIONS

Let's set H a Hilbert space and W a unitary operator on $\mathcal{H}\otimes\mathcal{H}$.

We let $L_\omega = (\omega\otimes id)(W)$ for all $\omega\in\mathcal{B}(\mathcal{H})_*$ and $\mathcal{A}_0(W) = \{L_\omega \,/\, \omega \in \mathcal{B}(\mathcal{H})_*\}$.

$\mathcal{A}_0(W)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ and it is also an algebra with:

$$L_{\omega_1}\cdot L_{\omega_2} = L_{\omega_1 * \omega_2} \text{ and } \omega_1 * \omega_2(X) = (\omega_1\otimes\omega_2)(W^*(I\otimes X)W) \quad \text{for all } X\in\mathcal{B}(\mathcal{H}).$$

We denote with $\mathcal{C}^*(W)$ the C*-algebra generated by $\mathcal{A}_0(W)$ in $\mathcal{B}(\mathcal{H})$. This algebra is non degenerate in $\mathcal{B}(\mathcal{H})$.

It is known [1] that $\mathcal{C}^*(W)$ is commutative if and only if W is commutative. In this case, G denotes the Gelfand spectrum of $\mathcal{C}^*(W)$.

Henceforth, we denote with W a commutative multiplicative unitary on the Hilbert space $\mathcal{H}$ (not necessarily separable). In this context, we denote with $\mathscr{L}(W)$ the von Neumann algebra bicommutant of $\mathcal{C}^*(W)$ (which is commutative).

Moreover, we let $R_\omega = (id\otimes\omega)(W)$ for all $\omega\in\mathcal{B}(\mathcal{H})_*$ and we denote with $\widehat{\mathcal{C}}^*(W)$ the C^*-algebra (generally non commutative) generated by the operators R_ω. This algebra is non degenerate and $\widehat{\mathscr{L}}(W)$ denotes the von Neumann algebra bicommutant of $\widehat{\mathcal{C}}^*(W)$.

3. REGULAR COMMUTATIVE MULTIPLICATIVE UNITARIES

Theorem 1. *If the commutative multiplicative unitary W on the Hilbert space $\mathcal{H}$ is regular, then:*

(1) *The Gelfand spectrum G of $\mathcal{C}^*(W)$ is a locally compact group.*
(2) *The von Neumann algebra $\mathscr{L}(W)'$ is of type I_n where n is a cardinal number.*

(3) *In particular,* W *is equivalent to the multiplicative unitary tensor product* $V_G \boxtimes I_{\mathcal{K}}$ *with* $\mathcal{K}$ *Hilbert space of dimension* n.

Proof. As $\mathcal{B}(\mathcal{H})^*$ can be identified with $(\mathcal{B}(\mathcal{H})_*)^{**}$ we have that the unit ball $(\mathcal{B}(\mathcal{H})_*)_1$ is dense for the weak*-topology in the unit ball $(\mathcal{B}(\mathcal{H})^*)_1$ in virtue of the Goldstine theorem [3].

From [1], if $g \in G$ and if $\widehat{X}$ denotes the Gelfand transform of X in $\mathcal{C}^*(W)$, then:

$$\text{there exists unique } T_g \in \mathcal{B}(\mathcal{H}) \text{ such that } \omega(T_g) = g(L_\omega) = \widehat{L_\omega}(g) \quad \text{for all } \omega \in \mathcal{B}(\mathcal{H})_*.$$

Moreover, it is known (always from [1]) that: $T_g L_\omega = L_{T_g\omega} T_g$.

We prove that T_g is a unitary operator. Indeed T_g has norm 1. Since g is a state on $\mathcal{C}^*(W)$, then there exists γ state on $\mathcal{B}(\mathcal{H})$ which extends g.

Hence, it is possible to construct a net $\{\omega_\iota\}_{\iota\in I}$ in the unit ball of $(\mathcal{B}(\mathcal{H})_*)_+$ weak*-convergent to γ. But W is regular, therefore $W \in M(\widehat{\mathcal{C}^*}(W)\otimes\mathcal{C}^*(W))$ (as $\mathcal{C}^*(W)$ is commutative, the C^*-tensor product is unique).

Thus, it is possible to apply $\mathrm{id}\otimes\gamma$ to W.

On the other hand, $\mathrm{id}\otimes g$ is a *-homomorphism between $\widehat{\mathcal{C}^*}(W)\otimes\mathcal{C}^*(W)$ and $\mathcal{B}(\mathcal{H})$. Therefore $\mathrm{id}\otimes g$ extends to a unique *-homomorphism $\widetilde{\mathrm{id}\otimes g}$ between $M(\widehat{\mathcal{C}^*}(W)\otimes\mathcal{C}^*(W))$ and $\mathcal{B}(\mathcal{H})$. From such a uniqueness and as $\mathrm{id}\otimes\gamma$ extends $\mathrm{id}\otimes g$, it follows that $(\mathrm{id}\otimes\gamma)(W) = (\widetilde{\mathrm{id}\otimes g})(W)$.

For all $\omega \in \mathcal{B}(\mathcal{H})_*$ we have:

$$\omega((\mathrm{id}\otimes\gamma)(W)) = \omega(\lim_\iota(\mathrm{id}\otimes\omega_\iota)(W)) = \lim_\iota \omega_\iota(L_\omega) = \gamma(L_\omega) = g(L_\omega) = \omega(T_g).$$

Whence, we have $(\mathrm{id}\otimes\gamma)(W) = T_g$. Since $\mathrm{id}\otimes\gamma$ is a *-homomorphism and W is a unitary operator then $(\mathrm{id}\otimes\gamma)(W)$ is unitary and therefore T_g is unitary.

If $g \in G$, we define $\alpha_g : \mathcal{C}^*(W) \to \mathcal{C}^*(W)$ such that $\alpha_g(X) = T_g X T_g^*$. Since $T_g\mathcal{C}^*(W)T_g^* = \mathcal{C}^*(W)$ we have that α_g is an *-automorphism. (Indeed, $T_g L_\omega T_g^* = L_{T_g\omega}$ and $L_\omega = L_{T_g T_g^*\omega} = T_g L_{T_g^*\omega} T_g^*$.)

Taking advantage of α_g is an *-automorphism and using an argument similar to that used in [2], we can conclude that G is a locally compact group.

We set $\pi : \mathscr{C}_0(G) \to \mathscr{L}(W)$ the inverse of the Gelfand transformation of $\mathcal{C}^*(W)$.

As $\alpha_g(f) = f(\cdot g)$, then $\pi(\alpha_g(f)) = T_g\pi(f)T_g^*$ for all $g \in G$, $f \in \mathscr{C}_0(G)$. [For the sake of simplicity we denote with α_g also the *-automorphism of $\mathscr{C}_0(G)$ such that $\alpha_g(f) = f(\cdot g)$].

If $P(\Delta)$ with Δ Borel set of G is the spectral measure induced from π, then the following identity holds:

$$T_g P(\Delta) T_g^* = P(\Delta g^{-1}) \quad \text{for all } g \in G,\ \Delta \text{ Borel set of } G.$$

Indeed, if $\mu_{\xi,\eta}(\Delta) = \langle P(\Delta)\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$, then we have:

$$\int_G f(t) \mathrm{d}\mu_{T_g^*\xi, T_g^*\eta}(t) = \langle \pi(f) T_g^*\xi, T_g^*\eta\rangle$$

$$= \langle \pi(\alpha_g(f))\xi, \eta\rangle = \int_G f(tg) \mathrm{d}\mu_{\xi,\eta}(t), \quad f \in \mathscr{C}_0(G).$$

That implies:

$$\int_G \chi_\Delta(t) \mathrm{d}\mu_{T_g^*\xi, T_g^*\eta}(t) = \int_G \chi_\Delta(tg) \mathrm{d}\mu_{\xi,\eta}(t) = \int_G \chi_{\Delta g^{-1}}(t) \mathrm{d}\mu_{\xi,\eta}(t)$$

that is:

$$\mu_{T_g^*\xi, T_g^*\eta}(\Delta) = \mu_{\xi,\eta}(\Delta g^{-1}) \quad \text{for all } g \in G,\ \Delta \text{ Borel set of } G.$$

From that we have: $T_g P(\Delta) T_g^* = P(\Delta g^{-1})$. In this case, it is possible to apply a result of L.H. Loomis [4] that extends the Stone-von Neumann-Mackey theorem to the non-separable case [5], and therefore conclude completely the proof. □

In the previous theorem, we assumed that the commutative multiplicative unitary W is regular. But, for our purposes, it is sufficient to assume that the W is semi-regular, namely $W \in M(\widehat{\mathcal{C}}^*(W) \otimes \mathcal{C}^*(W))$.

4. CASE STUDY: $\mathscr{L}(W)$ AND $\widehat{\mathscr{L}}(W)$ ARE STANDARD

If G is a locally compact group, $\mathrm{d}t$ is the right invariant Haar measure on G, then we denote with $\mathcal{R}(G)$ the von Neumann algebra generated by the right translations $\{\rho_s\}_{s\in G}$ $[\rho_s(t) = st]$. If δ is the modular function of G and if J_G is the conjugation on $L^2(G, \mathrm{d}t)$ defined by $J_G\xi(t) := \delta(t)^{\frac{1}{2}}\overline{\xi(t^{-1})}$ for all $\xi \in L^2(G, \mathrm{d}t)$, then $\mathcal{R}(G)$ is standard with modular conjugation J_G [6].

Also $L^\infty(G, \mathrm{d}t)$ identified as maximal abelian von Neumann algebra on the Hilbert space $L^2(G, \mathrm{d}t)$ is standard and its modular conjugation is given by $C_G\xi(s) = \overline{\xi(s)}$ for all $\xi \in L^2(G, \mathrm{d}t)$.

The following equation:

$$(V_G\xi)(s,t) = \xi(st, t) \qquad \text{for all } \xi \in L^2(G\times G, \mathrm{d}t\times \mathrm{d}t) \quad \text{and} \quad \text{for all } s, t \in G$$

defines a commutative and regular multiplicative unitary

$$V_G : L^2(G, \mathrm{d}t) \otimes Ł^2(G, \mathrm{d}t) \to L^2(G, \mathrm{d}t) \otimes Ł^2(G, \mathrm{d}t).$$

The corresponding operators L_ω and R_ω are given by [8]:

$$L_{\omega_{\xi,\eta}}\zeta(s) = \langle\rho(s)\xi,\eta\rangle_2\ \zeta(s) R_{\omega_{\xi,\eta}}\zeta(s) = \int_G \xi(t)\overline{\eta(t)}\zeta(st)\mathrm{d}(t)$$
$$\text{for all } \xi,\eta,\zeta\in L^2(G,\mathrm{d}t).$$

The following identities express a link between the multiplicative unitary, the right translations and the conjugations:

$$C_G\rho(s)C_G = \rho(s) \quad \text{for all } s\in G$$
$$(C_G\otimes J_G)V_G(C_G\otimes J_G) = V_G^* \quad [7].$$

In according to the previous considerations, starting from a commutative multiplicative unitary whose related objects satisfy similar identities, we want to obtain the regularity of the unitary. Therefore, we prove the following:

Theorem 2. *Let's W a commutative multiplicative unitary on the Hilbert space $\mathcal{H}$. Suppose that the von Neumann algebras $\mathscr{L}(W)$ and $\widehat{\mathscr{L}}(W)$ are standard with modular conjugations respectively J and $\widehat{J}$.*

If G is the Gelfand spectrum of $\mathcal{C}^*(W)$ and if the following relations are satisfied:

1. $JT_gJ = T_g$ *for all* $g\in G$
2. $(J\otimes\widehat{J})W(J\otimes\widehat{J}) = W^*$

then W is regular and G is a locally compact group.

In order to prove the theorem, we need the following results:

Lemma 3. *The following statements are equivalent:*

a) $(J\otimes\widehat{J})W(J\otimes\widehat{J}) = W^*$

b) $\widehat{J}L_{\omega_{\xi,\eta}}{}^*\widehat{J} = L_{\omega_{J\eta,J\xi}}$ *for all* $\xi,\eta\in\mathcal{H}$

c) $JR_{\omega_{\xi,\eta}}{}^*J = R_{\omega_{\widehat{J}\eta,\widehat{J}\xi}}$ *for all* $\xi,\eta\in\mathcal{H}$.

Proof. a) $\iff$ b) For all $\xi,\eta,\zeta,\theta\in\mathcal{H}$ we have:

$$\begin{aligned}\langle\widehat{J}L_{\omega_{\xi,\eta}}{}^*\widehat{J}\zeta,\theta\rangle &= \langle\widehat{J}\theta, L_{\omega_{\xi,\eta}}{}^*\widehat{J}\zeta\rangle = \langle L_{\omega_{\xi,\eta}}\widehat{J}\theta,\widehat{J}\zeta\rangle = \omega_{\widehat{J}\theta,\widehat{J}\zeta}(L_{\omega_{\xi,\eta}})\\ &= \omega_{\xi,\eta}\otimes\omega_{\widehat{J}\theta,\widehat{J}\zeta}(W) = \omega_{\xi\otimes\widehat{J}\theta,\eta\otimes\widehat{J}\zeta}(W) = \langle W(\xi\otimes\widehat{J}\theta),\eta\otimes\widehat{J}\zeta\rangle\\ &= \langle W(J\otimes\widehat{J})(J\xi\otimes\theta),(J\otimes\widehat{J})(J\eta\otimes\zeta)\rangle =\\ &= \langle J\eta\otimes\zeta),(J\otimes\widehat{J})W(J\otimes\widehat{J})(J\xi\otimes\theta))\rangle.\end{aligned}$$

On the other hand:

$$\begin{aligned}\langle L_{\omega_{J\eta,J\xi}}\zeta,\theta\rangle &= \langle L_{\omega_{J\eta,J\xi}}\zeta,\theta\rangle = \omega_{\zeta,\theta}(L_{\omega_{J\eta,J\xi}}) = \omega_{J\eta,J\xi}\otimes\omega_{\zeta,\theta}(W)\\ &= \omega_{J\eta\otimes\zeta,J\xi\otimes\theta}(W) = \langle W(J\eta\otimes\zeta),J\xi\otimes\theta\rangle\end{aligned}$$

$$= \langle J\eta \otimes \zeta), W^*(J\xi \otimes \theta)\rangle = \langle J\eta \otimes \zeta), W^*(J\xi \otimes \theta)\rangle.$$

Since the vectors ξ, η, ζ, θ are totally arbitrary we obtain the equivalence between a) and b).

a) $\iff$ c) For all $\xi, \eta, \zeta, \theta \in \mathcal{H}$ we have:

$$\begin{aligned}&\langle \eta \otimes \widehat{J}\zeta, (J \otimes \widehat{J})W(J \otimes \widehat{J})(\xi \otimes \widehat{J}\theta))\rangle \\ &\quad = \langle W(J \otimes \widehat{J})(\xi \otimes \widehat{J}\theta), (J \otimes \widehat{J})(\eta \otimes \widehat{J}\zeta)\rangle = \langle W(J\xi \otimes \theta), (J\eta \otimes \zeta)\rangle \\ &\qquad = \omega_{J\xi\otimes\theta, J\eta\otimes\zeta}(W) = \omega_{J\xi, J\eta}(R_{\omega_{\theta,\zeta}}) = \langle R_{\omega_{\theta,\zeta}} J\xi, J\eta\rangle \\ &\qquad\qquad = \langle \eta, JR_{\omega_{\theta,\zeta}} J\xi\rangle = \langle JR_{\omega_{\theta,\zeta}}{}^* J\xi, \eta\rangle.\end{aligned}$$

On the other hand:

$$\begin{aligned}&\langle \eta \otimes \widehat{J}\zeta, W^*(J\xi \otimes \widehat{J}\theta)\rangle = \langle W(\eta \otimes \widehat{J}\zeta)), \xi \otimes \widehat{J}\theta\rangle = \omega_{\eta\otimes\widehat{J}\zeta, \xi\otimes\widehat{J}\theta}(W) \\ &\quad = \omega_{\eta,\xi}(R_{\omega_{\widehat{J}\zeta,\widehat{J}\theta}}) = \langle R_{\omega_{\widehat{J}\zeta,\widehat{J}\theta}}\eta, \xi\rangle = \langle R_{\omega_{\widehat{J}\zeta,\widehat{J}\theta}}\eta, \xi\rangle = \langle R_{\omega_{\widehat{J}\zeta,\widehat{J}\theta}}\eta, \xi\rangle.\end{aligned}$$

Even in this case, given the arbitrariness of the vectors ξ, η, ζ, θ, we obtain the equivalence between a) and c). $\square$

LEMMA 4. *The following statements are equivalent:*

a) $JT_gJ = T_g$ *for all* $g \in G$

b) $L_{\omega_{\xi,\eta}}{}^* = L_{\omega_{J\xi,J\eta}}$ *for all* $\xi, \eta \in \mathcal{H}$

c) $R_{\omega_{\xi,\eta}} = JR_{\omega_{\eta,\xi}}J$ *for all* $\xi, \eta \in \mathcal{H}$

Proof.

a) $\iff$ b) The statement $JT_gJ = T_g$ for all $g \in G$ is equivalent to $\omega_{\xi,\eta}(JT_gJ) = \omega_{\xi,\eta}(T_g)$ for all $g \in G$ and $\xi, \eta \in \mathcal{H}$.

But $\omega_{\xi,\eta}(JT_gJ) = \langle JT_gJ\xi, \eta\rangle = \overline{\langle T_gJ\xi, J\eta\rangle} = \overline{g(L_{\omega_{J\xi,J\eta}})} = g(L_{\omega_{J\xi,J\eta}}{}^*)$, whereas $\omega_{\xi,\eta}(T_g) = \langle T_g\xi, \eta\rangle = g(L_{\omega_{\xi,\eta}})$. Therefore, the statement $JT_gJ = T_g$ for all $g \in G$ is equivalent to $L_{\omega_{J\xi,J\eta}}{}^* = L_{\omega_{\xi,\eta}}$ for all $\xi, \eta \in \mathcal{H}$ namely, $L_{\omega_{\xi,\eta}}{}^* = L_{\omega_{J\xi,J\eta}}$ for all $\xi, \eta \in \mathcal{H}$.

b) $\iff$ c) The statement $L_{\omega_{\xi,\eta}}{}^* = L_{\omega_{J\xi,J\eta}}$ for all $\xi, \eta \in \mathcal{H}$ is equivalent to $\omega_{\zeta,\theta}(L_{\omega_{\xi,\eta}}{}^*) = \omega_{\zeta,\theta}(L_{\omega_{J\xi,J\eta}})$ for all $\xi, \eta, \zeta, \theta \in \mathcal{H}$.

But $\omega_{\zeta,\theta}(L_{\omega_{\xi,\eta}}{}^*) = \overline{\langle L_{\omega_{\xi,\eta}}\theta, \zeta\rangle} = \overline{\omega_{\theta,\zeta}(L_{\omega_{\xi,\eta}})} = \overline{\omega_{\xi,\eta}(R_{\omega_{\theta,\zeta}})} = \overline{\langle R_{\omega_{\theta,\zeta}}\xi, \eta\rangle}$, whereas $\omega_{\zeta,\theta}(L_{\omega_{J\xi,J\eta}}) = \langle L_{\omega_{J\xi,J\eta}}\zeta, \theta\rangle = \omega_{\zeta,\theta}(L_{\omega_{J\xi,J\eta}}) = \omega_{J\xi,J\eta}(R_{\omega_{\zeta,\theta}}) = \langle R_{\omega_{\zeta,\theta}}J\xi, J\eta\rangle = \overline{\langle JR_{\omega_{\zeta,\theta}}J\xi, \eta\rangle}$, namely, $R_{\omega_{\theta,\zeta}} = JR_{\omega_{\zeta,\theta}}J$ for all $\zeta, \theta \in \mathcal{H}$. $\square$

Proof of Theorem 2. In order to prove the theorem, it is sufficient to adapt to the current context a result of Baaj-Skandalis [1] that states that a multiplicative unitary V on the Hilbert space $\mathcal{K}$ is regular if there exists a unitary operator $S : \mathcal{K} \to \overline{\mathcal{K}}$ such that $S^*\overline{L_\omega}S = L_{\omega^*}$ for all $\omega \in \mathcal{B}(\mathcal{H})_*$ [$\overline{\mathcal{K}}$ is the conjugate Hilbert space of $\mathcal{K}$].

In our case, we have: for all $\xi,\eta\in\mathcal{H}$

$$JL_{\omega_{\xi,\eta}}J = {L_{\omega_{\xi,\eta}}}^* \quad \text{because } \mathscr{L}(W) \text{ is standard.}$$

From this it follows that:

$$\widehat{J}JL_{\omega_{\xi,\eta}}J\widehat{J} = \widehat{J}{L_{\omega_{\xi,\eta}}}^*\widehat{J} = L_{\omega_{J\eta,J\xi}} \quad \text{[by b) of the Lemma 3]}$$
$$= {L_{\omega_{\eta,\xi}}}^* \quad \text{[by b) of the Lemma 4]}$$
$$= JL_{\omega_{\eta,\xi}}J \quad \text{[once again, because } \mathscr{L}(W) \text{ is standard.]}$$

Therefore, we have: for all $\xi,\eta\in\mathcal{H}$

$$J\widehat{J}[JL_{\omega_{\xi,\eta}}J]\widehat{J}J = L_{\omega_{\eta,\xi}}.$$

Now, for all $\xi,\eta,\zeta,\theta\in\mathcal{H}$, we have:

$$\langle(\mathrm{id}\otimes\omega_{\eta,\xi})(\Sigma W)\zeta,\theta\rangle = \omega_{\zeta,\theta}\otimes\omega_{\eta,\xi}(\Sigma W) = \omega_{\zeta\otimes\eta,\theta\otimes\xi}(\Sigma W)$$
$$= \langle W(\zeta\otimes\eta),\xi\otimes\theta\rangle.$$

On the other hand:

$$\langle(J\otimes J\widehat{J}J)W(J\otimes J\widehat{J}J)(J\xi\otimes\eta),(J\zeta\otimes\theta)\rangle = \langle\zeta\otimes J\widehat{J}J\theta, W(\xi\otimes J\widehat{J}J\eta)\rangle$$
$$= \overline{\langle W(\xi\otimes J\widehat{J}J\eta),\zeta\otimes J\widehat{J}J\theta\rangle} = \overline{\omega_{\xi\otimes\zeta,J\widehat{J}J\eta\otimes J\widehat{J}J\theta}(W)} = \overline{\omega_{J\widehat{J}J\eta,J\widehat{J}J\theta}(L_{\omega_{\xi,\zeta}})}$$
$$= \overline{\langle L_{\omega_{\xi,\zeta}}J\widehat{J}J\eta, J\widehat{J}J\theta\rangle} = \langle J\widehat{J}(JL_{\omega_{\xi,\zeta}}J)\widehat{J}J\eta,\theta\rangle = \langle L_{\omega_{\zeta,\xi}}\eta,\theta\rangle$$
$$= \omega_{\eta,\theta}(L_{\omega_{\zeta,\xi}}) = \omega_{\zeta,\xi}\otimes\omega_{\eta,\theta}(W) = \omega_{\zeta\otimes\eta,\xi\otimes\theta}(W)$$
$$= \langle W(\zeta\otimes\eta),\xi\otimes\theta\rangle.$$

Thus, we have: for all $\xi,\eta,\zeta,\theta\in\mathcal{H}$

$$\langle(\mathrm{id}\otimes\omega_{\eta,\xi})(\Sigma W)\zeta,\theta\rangle = \langle(J\otimes J\widehat{J}J)W(J\otimes J\widehat{J}J)(J\xi\otimes\eta),(J\zeta\otimes\theta)\rangle.$$

Now, set $V=(J\otimes J\widehat{J}J)W(J\otimes J\widehat{J}J)$, V is a unitary operator on $\mathcal{H}\otimes\mathcal{H}$ and therefore, identifying $\mathcal{H}\otimes\mathcal{H}$ with $L^2(J(\mathcal{H}))$, V can be regarded as a unitary operator on the conjugate Hilbert space $L^2(J(\mathcal{H}))$ and thus, V can be regarded as a conjugate unitary operator on the Hilbert space $L^2(\mathcal{H})$.

Then, we have:

$$\langle(\mathrm{id}\otimes\omega_{\eta,\xi})(\Sigma W)\zeta,\eta\rangle = \langle V(J\xi\otimes\eta),(J\zeta\otimes\theta)\rangle = \langle V(T_{J\xi,\eta})J\zeta,\theta\rangle$$
$$\text{for all } \xi,\eta,\zeta,\theta\in\mathcal{H}.$$

Since $V(T_{J\xi,\eta})J$ is a compact operator as product of two conjugate operators one of which compact, $(\mathrm{id}\otimes\omega_{\eta,\xi})(\Sigma W)$ is compact for all $\xi,\eta\in\mathcal{H}$. That implies the regularity of W. $\square$

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