ON THE CONVERGENCE OF SEQUENCES OF PROBABILITY MEASURES

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Our goal in this note is to obtain an extension of a result known as the Portmanteau theorem (see, for instance, Theorem 2.1 of Chapter 1, pp. 16–17 of P. Billingsley, *Convergence of Probability Measures,* Second Edition, Wiley, New York and Toronto, 1999, or Theorem 3.1 of Chapter 3, pp. 108–110, of S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence,* Wiley, Hoboken, New Jersey, 2005).

As in the Portmanteau theorem, our result consists of several equivalent assertions.

The theorem that we obtain is stated using the vector space of all continuous bounded functions with bounded supports and the vector space of all uniformly continuous bounded functions with bounded supports.

Like in the Portmanteau theorem, we deal with sequences of probability measures. However, when dealing with our more general type of convergence, the limit of the sequence is not necessarily a probability measure.

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1. INTRODUCTION

As usual, by a *Polish space* we mean a metric space whose topology defined by the metric is separable and complete.

Let (X, d) be a Polish space.

Let $\mathcal{B}(X)$ be the Borel σ-algebra defined by the metric topology on X, and let $\mathcal{M}(X)$ be the real Banach space of all real-valued signed measures on $(X, \mathcal{B}(X))$, where $\mathcal{M}(X)$ is endowed with the total variation norm.

Like in many of our earlier works, we let $B_b(X)$ and $C_b(X)$ be the real Banach spaces of all real-valued bounded Borel measurable functions on X and of all continuous bounded real-valued functions on X, respectively, the norms on both $B_b(X)$ and $C_b(X)$ being the uniform (sup) norm.

As in Szarek and Zaharopol [7], we will use the notations

$$\langle f, \mu \rangle = \int_X f(x)\, \mathrm{d}\mu(x)$$

for $f \in B_b(X)$ and $\mu \in \mathcal{M}(X)$;

$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\} = \text{the open ball with center } x_0 \text{ and radius } r,$$

$$\text{for } x_0 \in X \text{ and } r \in \mathbb{R},\ r > 0;$$

$\bar{A}$ and $\overset{\circ}{A}=$ the closure and the interior of a subset A of X, respectively.

As usual, we say that a subset A of X is bounded if A can be included in an open ball (of finite radius); that is, if there exist $x_0 \in X$ and $r \in \mathbb{R}$, $r > 0$, such that $A \subseteq B(x_0, r)$.

Given $f : X \to \mathbb{R}$, we will use the notation $\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}} =$ the support of f.

We will denote by $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$ the vector space of all bounded continuous functions $f : X \to \mathbb{R}$ with bounded support.

Also, let $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$ be the vector space of all real-valued uniformly continuous bounded functions on X that have bounded supports.

Now, let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}(X)$, and let $\mu \in \mathcal{M}(X)$.

We say that $(\mu_n)_{n\in\mathbb{N}}$ *converges to* μ *along* $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$ if, for every $f \in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, the sequence $(\langle f, \mu_n \rangle)_{n\in\mathbb{N}}$ converges to $\langle f, \mu \rangle$.

Similarly, we say that $(\mu_n)_{n\in\mathbb{N}}$ *converges to* μ *along* $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$ if, for every $f \in C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$, the sequence $(\langle f, \mu_n \rangle)_{n\in\mathbb{N}}$ converges to $\langle f, \mu \rangle$.

As usual, given a subset A of X, we use the notation $\partial A = \bar{A} \setminus \overset{\circ}{A}=$ the boundary of A.

Following Ethier and Kurtz's monograph [5] (see p. 108), or Gugushvili [6], given a Borel subset A of X, and $\nu \in \mathcal{M}(X)$, $\nu \geq 0$, we say that A *is a* ν*-continuity subset of* X if $\nu(\partial A) = 0$.

Our goal in this paper is to prove the following theorem:

THEOREM 1.1. *Let* $\mu_n)_{n\in\mathbb{N}}$ *be a sequence of probability measures,* $\mu_n \in \mathcal{M}(X)$ *for every* $n \in \mathbb{N}$. *Also, let* $\mu \in \mathcal{M}(X)$, $\mu \geq 0$.

The following assertions are equivalent:

(a) *The sequence* $(\mu_n)_{n\in\mathbb{N}}$ *converges to* μ *along* $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$.

(b) *The sequence* $(\mu_n)_{n\in\mathbb{N}}$ *converges to* μ *along* $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$.

(c) *The following two statements hold true:*

(c1) *For every closed bounded subset* F *of* X, *we have*

$$\limsup_{n\to\infty} \mu_n(F) \leq \mu(F).$$

(c2) *For every open bounded subset G of X, we have*

$$\liminf_{n\to\infty} \mu_n(G) \geq \mu(G).$$

(d) *For every bounded Borel μ-continuity subset A of X, we have that the sequence $(\mu_n(A))_{n\in\mathbb{N}}$ converges to $\mu(A)$.*

Note that Theorem 1.1 is obviously true if X is a finite set. Thus, we will assume from now on that X is infinite; if D is a countable dense subset of X, then D is an infinite set, as well (such a countable dense subset of X exists because X is separable).

Theorem 1.1 is an extension of a well-known result referred to as the Portmanteau theorem (see, for example, Theorem 2.1, pp. 16–17, in Chapter 1 of Billingsley's monograph [1], or Theorem 2.1, pp. 3–4, of Billingsley [2], or Theorem 3.1, pp. 108–110, in Chapter 3 of Ethier and Kurtz's monograph [5], or Theorem 6 of Gugushvili [6]).

The convergence of sequences of probability measures that appears at (a) and at (b) of Theorem 1.1 in this paper is significantly more general than the convergence in the $C_b(X)$-weak topology of $\mathcal{M}(X)$ that appears in the Portmanteau theorem (for details on the $C_b(X)$-weak topology of $\mathcal{M}(X)$, see Section 1 of Szarek and Zaharopol [7], see also Section 1.1 of [8]; for various kinds of topologies on spaces of measures, see Section 4.6, Section 4.7, and Chapter 8 of the impressive monograph of Bogachev [3]).

Indeed, it is well-known and easy to prove that if a sequence $(\nu_n)_{n\in\mathbb{N}}$ of Borel probability measures converges in the $C_b(X)$-weak topology of $\mathcal{M}(X)$ to an element ν of $\mathcal{M}(X)$, $\nu \geq 0$, then, necessarily, ν is a probability measure; by contrast, if a sequence $(\nu_n)_{n\in\mathbb{N}}$ converges to an element ν of $\mathcal{M}(X)$, $\nu \geq 0$, in the sense described at (a) or (b) of Theorem 1.1 of this paper, then ν may fail to be a probability measure.

The paper is organized as follows: in the next section (Section 2), we discuss several families of continuous functions that are used in the work; in Section 3, we present several properties of continuous functions; in the last section (Section 4), we use all the results discussed earlier in order to prove Theorem 1.1.

2. USEFUL FAMILIES OF CONTINUOUS FUNCTIONS

We will employ the notations introduced so far. Thus, we assume given a Polish space (X, d).

As usual, if A is a nonempty subset of X and $x \in X$, we use the notation $d(x, A) = \inf_{y\in A} d(x, y)$.

Let $A \in X$, $A \neq \emptyset$. We will denote by g_A the function $g_A : X \to \mathbb{R}$ defined by $g_A(x) = d(x, A)$ for every $x \in X$.

LEMMA 2.1. *Given a nonempty subset A of X, the function g_A is uniformly continuous.*

Proof. Let A be a nonempty subset of X.

Using Theorem 5.1.6, pp. 50–51 of Dixmier [4], we obtain that

$$|d(x, A) - d(y, A)| \leq d(x, y),$$

or

$$|g_A(x) - g_A(y)| \leq d(x, y) \tag{2.1}$$

for every $x \in X$ and $y \in X$.

Clearly, in view of the inequalities (2.1), we obtain that g_A is uniformly continuous. $\square$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(z) = \begin{cases} 1 & \text{if} \quad z \leq 0 \\ 1 - z & \text{if} \quad 0 < z < 1 \\ 0 & \text{if} \quad z \geq 1 \end{cases} \tag{2.2}$$

LEMMA 2.2. *The function ϕ is uniformly continuous.*

Proof. We have to prove that for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $\delta \in \mathbb{R}$, $\delta > 0$, such that $|\phi(x) - \phi(y)| < \varepsilon$ whenever $|x - y| < \delta$.

To this end, let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and set $\delta = \dfrac{\varepsilon}{2}$.

We will assume that $\varepsilon < 1$ (obviously, we can make this assumption).

Now let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be such that $|x - y| < \delta$.

We may and do assume that $x \leq y$ (because if $x > y$, we just switch the roles of x and y).

Clearly, it is enough to prove that $|\phi(x) - \phi(y)| < \varepsilon$ in the following five cases:

Case I: $x \leq y \leq 0$;

Case II: $x \leq 0$ and $y \in (0, 1)$;

Case III: $x \in (0, 1)$ and $y \in (0, 1)$;

Case IV: $x \in (0, 1)$ and $y \geq 1$;

Case V: $1 \leq x \leq y$.

Case I and Case V: If $x \leq y \leq 0$, or if $1 \leq x \leq y$, then using the definition of ϕ (see the equality (2.2)), we obtain that $|\phi(x) - \phi(y)| = 0 < \varepsilon$.

Case II: Using the definition of ϕ, we obtain that

$$|\phi(x) - \phi(y)| = |1 - (1 - y)| = y. \tag{2.3}$$

Since $x \leq 0$ and $|x - y| = y - x < \frac{\varepsilon}{2}$, it follows that

$$y < x + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} < \varepsilon. \tag{2.4}$$

In view of (2.3) and (2.4), we further obtain that $|\phi(x) - \phi(y)| < \varepsilon$.

Case III: Since $x \in (0,1)$ and $y \in (0,1)$, using (2.2), we obtain that

$$|\phi(x) - \phi(y)| = |(1 - x) - (1 - y)| = |y - x| < \frac{\varepsilon}{2} < \varepsilon.$$

Case IV: Since $x \in (0,1)$ and $y \geq 1$, using (2.2), we obtain that

$$|\phi(x) - \phi(y)| = |1 - x - 0| = 1 - x \leq y - x < \frac{\varepsilon}{2} < \varepsilon. \quad \square$$

LEMMA 2.3. *Let F be a closed nonempty subset of X and let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.*

(a) *The function $g_F^{(\varepsilon)} : X \to \mathbb{R}$ defined by $g_F^{(\varepsilon)}(x) = \left(1 - \frac{d(x,F)}{\varepsilon}\right) \vee 0$ for every $x \in X$, is bounded and uniformly continuous.*

(b) *If, in addition, F is also a bounded subset of X, then $g_F^{(\varepsilon)}$ is a bounded uniformly continuous function and has a bounded support.*

Proof. (a) Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and let F be a closed nonempty subset of X. Set $F_\varepsilon = \bigcup_{y \in F} B(y, \varepsilon)$.

It is easy to see that

$$g_F^{(\varepsilon)}(x) = \begin{cases} 1 - \dfrac{d(x,F)}{\varepsilon} & \text{if} \quad x \in F_\varepsilon \setminus F \\ 1 & \text{if} \quad x \in F \\ 0 & \text{if} \quad x \in X \setminus F_\varepsilon \end{cases} \tag{2.5}$$

for every $x \in X$.

Using (2.5), we obtain that $0 \leq g_F^{(\varepsilon)}(x) \leq 1$ for every $x \in X$. Therefore, $g_F^{(\varepsilon)}$ is a bounded function.

We will now prove that $g_F^{(\varepsilon)}$ is uniformly continuous.

To this end, we will show that

$$g_F^{(\varepsilon)}(x) = \phi\left(\frac{d(x,F)}{\varepsilon}\right) \tag{2.6}$$

for every $x \in X$.

Note that if we prove that (2.6) holds true for every $x \in X$, then we obtain that $g_F^{(\varepsilon)}$ is uniformly continuous. Indeed, using Lemma 2.1, we obtain that the function $x \mapsto \dfrac{d(x,F)}{\varepsilon}$, $x \in X$, is uniformly continuous; since, by Lemma 2.2, the function ϕ is uniformly continuous, we obtain the uniform continuity of $g_F^{(\varepsilon)}$, as well.

In order to prove that (2.6) holds true for every $x \in X$, it is enough to prove that (2.6) is true in the following three situations:

(*i*) $x \in X \setminus F_\varepsilon$;

(*ii*) $x \in F_\varepsilon \setminus F$;

(*iii*) $x \in F$.

(*i*) Let $x \in X \setminus F_\varepsilon$. Using the equality (2.5), we obtain that $g_F^{(\varepsilon)}(x) = 0$.

On the other hand, $d(x,F) \geq \varepsilon$ because $x \in X \setminus F_\varepsilon$, so $\dfrac{d(x,F)}{\varepsilon} \geq 1$; therefore, using the definition of ϕ (the equality (2.2)), we obtain that $\phi\left(\dfrac{d(x,F)}{\varepsilon}\right) = 0$.

(*ii*) We now assume that $x \in F_\varepsilon \setminus F$.

In view of (2.5), we obtain that $g_F^{(\varepsilon)}(x) = 1 - \dfrac{d(x,F)}{\varepsilon}$.

Since $x \in F_\varepsilon \setminus F$, it follows that $0 < d(x,F) < \varepsilon$, so $\dfrac{d(x,F)}{\varepsilon} \in (0,1)$; therefore, using the definition of ϕ (see (2.2)), we obtain that

$$\phi\left(\frac{d(x,F)}{\varepsilon}\right) = 1 - \frac{d(x,F)}{\varepsilon}.$$

Thus, (2.6) holds true for $x \in F_\varepsilon \setminus F$.

(*iii*) Let $x \in F$.

Using the equality (2.5), we obtain that $g_F^{(\varepsilon)}(x) = 1$.

Since $x \in F$, it follows that $d(x,F) = 0$, so $\phi\left(\dfrac{d(x,F)}{\varepsilon}\right) = \phi(0) = 1$ (see the equality (2.2)).

(*b*) Assume that (in addition to being a closed nonempty subset of X) F is also bounded.

We have to prove that $g_F^{(\varepsilon)}$ has bounded support.

To this end, we will prove that:

(1) $\operatorname{supp}\left(g_F^{(\varepsilon)}\right) = \bar{F}_\varepsilon$

and

(2) $\bar{F}_\varepsilon$ is a bounded subset of X.

(1) We have to prove that

$$\text{supp}\ \left(g_F^{(\varepsilon)}\right) \subseteq \bar{F}_\varepsilon \tag{2.7}$$

and

$$\bar{F}_\varepsilon \subseteq \text{supp}\ \left(g_F^{(\varepsilon)}\right). \tag{2.8}$$

Proof of the Inclusion (2.7). Let $g \in \text{supp}\left(g_F^{(\varepsilon)}\right)$. Then, there exists a convergent sequence $(z_n)_{n\in\mathbb{N}}$ of elements of X such that $g_F^{(\varepsilon)}(z_n) \neq 0$ for every $n \in \mathbb{N}$ and such that $(z_n)_{n\in\mathbb{N}}$ converges to z.

Using the equality (2.5), we obtain that $z_n \notin X \setminus F_\varepsilon$, so $z_n \in F_\varepsilon$ for every $n \in \mathbb{N}$. Since $(z_n)_{n\in\mathbb{N}}$ converges to z, it follows that $z \in \bar{F}_\varepsilon$.

We have therefore proved that if $z \in \text{supp}\left(g_F^{(\varepsilon)}\right)$, then $z \in \bar{F}_\varepsilon$. Thus, (2.7) holds true.

Proof of the Inclusion (2.8). Let $z \in \bar{F}_\varepsilon$. Thus, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n \in F_\varepsilon$ for every $n \in \mathbb{N}$, and such that $(z_n)_{n\in\mathbb{N}}$ converges to z.

Since $z_n \in F_\varepsilon$, it follows that $d(z_n, F) < \varepsilon$, so $\dfrac{d(z_n, F)}{\varepsilon} < 1$ for every $n \in \mathbb{N}$.

Thus, using the equality (2.5), we obtain that $g_F^{(\varepsilon)}(z_n) \neq 0$ for every $n \in \mathbb{N}$. Since $(z_n)_{n\in\mathbb{N}}$ converges to z, it follows that $z \in \text{supp}\left(g_F^{(\varepsilon)}\right)$. Thus, $z \in \text{supp}\left(g_F^{(\varepsilon)}\right)$ whenever $z \in \bar{F}_\varepsilon$. Accordingly, (2.8) holds true.

(2) Since F is bounded, it follows that there exists $x_0 \in X$ and $\eta \in \mathbb{R}$, $\eta > 0$, such that $F \subseteq B(x_0, \eta)$. We now note that

$$F_\varepsilon \subseteq B(x_0, \eta + \varepsilon). \tag{2.9}$$

Indeed, if $z \in F_\varepsilon$, then there exists $x \in F$ such that $d(z, x) < \varepsilon$; therefore, $d(z, x_0) \leq d(z, x) + d(x, x_0) < \varepsilon + \eta$.

Using the inclusion (2.9), we obtain that $\bar{F}_\varepsilon \subseteq \overline{B(x_0, \eta + \varepsilon)}$, so $\bar{F}_\varepsilon$ is a bounded set.

Since (1) and (2) hold true, we obtain that $g_F^{(\varepsilon)}$ has bounded support. $\square$

Let D be a countable dense subset of X. As pointed out in Introduction, we may and do assume that D is infinite.

Since D is an infinite countable subset of X, there exists a sequence $(x_i)_{n\in\mathbb{N}}$ of elements of D such that:

(i) $x_i \neq x_j$ for every $i \in \mathbb{N}$ and $j \in \mathbb{N}$, $i \neq j$, and

(ii) $D = \{x_i \,|\, i \in \mathbb{N}\}$; that is, the range of the sequence $(x_i)_{i\in\mathbb{N}}$ is the entire set D.

For every $i \in \mathbb{N}$ and $j \in \mathbb{N}$, let $f_{ij} : X \to \mathbb{R}$ be defined by $f_{ij}(x) = 2(1 - jd(x, x_i)) \vee 0$ for every $x \in X$.

Since $f_{ij}(x) = 2 \times \left(\left(1 - \frac{d(x, x_i)}{\frac{1}{j}}\right) \vee 0\right)$ for every $i \in \mathbb{N}$, $j \in \mathbb{N}$, and $x \in X$, and since the singletons $\{x_i\}$, $i \in \mathbb{N}$, are closed bounded subsets of X, it follows that $f_{ij} = g^{\left(\frac{1}{j}\right)}_{\{x_i\}}$ where $g^{\left(\frac{1}{j}\right)}_{\{x_i\}}$ is the function that appears in Lemma 2.3 for $F = \{x_i\}$ and $\varepsilon = \frac{1}{j}$ for every $i \in \mathbb{N}$ and $j \in \mathbb{N}$.

Using Lemma 2.3, we obtain that f_{ij}, $i \in \mathbb{N}$, $j \in \mathbb{N}$, are uniformly continuous functions with bounded supports (actually, from the proof of (b) of Lemma 2.3, we obtain that $\operatorname{supp} f_{ij} = \overline{B\left(x_i, \frac{1}{j}\right)}$ for every $i \in \mathbb{N}$ and $j \in \mathbb{N}$).

Next, for every open bounded subset G of X, and every $m \in \mathbb{N}$, we define $L^{(G)}_m = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} \,\middle|\, i \le m, j \le m, B\left(x_i, \frac{1}{j}\right) \subseteq G\right\}$.

Now, for every $m \in \mathbb{N}$ and every open bounded subset G of X, let $g^{(G)}_m : X \to \mathbb{R}$ be defined by $g^{(G)}_m(x) = 0$ for every $x \in X$ if $L^{(G)}_m = \emptyset$, and $g^{(G)}_m(x) = \left(\sum_{(i,j)\in L^{(G)}_m} f_{ij}(x)\right) \wedge 1$ if $L^{(G)}_m$ is nonempty and $x \in X$.

LEMMA 2.4. *Let G be an open bounded subset of X, and let $m \in \mathbb{N}$. Then the function $g^{(G)}_m$ is bounded, has bounded support, and is uniformly continuous.*

Proof. It is easy to see that

$$0 \le g^{(G)}_m(x) \le \mathbf{1}_G(x) \tag{2.10}$$

for every $x \in X$, so $g^{(G)}_m$ is obviously a bounded function.

Taking into consideration that G is a bounded subset of X, we obtain that $\bar{G}$ is bounded as well. Thus, using again the inequalities (2.10), we obtain that the function $g^{(G)}_m$ has bounded support.

We now prove that $g^{(G)}_m$ is uniformly continuous. We first note that if $L^{(G)}_m = \emptyset$, then $g^{(G)}_m(x) = 0$ for every $x \in X$, so, obviously, $g^{(G)}_m$ is uniformly continuous in this case. Thus, we may and do assume that $L^{(G)}_m$ is nonempty.

Set $h^{(G)}_m = \sum_{(i,j)\in L^{(G)}_m} f_{ij}$.

Since $L^{(G)}_m$ is a finite set, and since f_{ij} is uniformly continuous for every i and j, it follows that $h^{(G)}_m$ is uniformly continuous, as well.

Clearly,

$$g_m^{(G)} = h_m^{(G)} \wedge 1. \tag{2.11}$$

Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Since $h_m^{(G)}$ is uniformly continuous, it follows that there exists $\delta \in \mathbb{R}$, $\delta > 0$, such that $\left|h_m^{(G)}(x) - h_m^{(G)}(y)\right| < \varepsilon$ whenever $x \in X$ and $y \in X$ satisfy the inequality $d(x, y) < \delta$.

Our goal now is to prove that, for every $x \in X$ and $y \in X$ such that $d(x, y) < \delta$, we have that

$$\left|g_m^{(G)}(x) - g_m^{(G)}(y)\right| < \varepsilon. \tag{2.12}$$

To this end, let $x \in X$ and $y \in X$ be such that $d(x, y) < \delta$. We have to study the following five cases:

(a) $h_m^{(G)}(x) = 0$ and $h_m^{(G)}(y) = 0$.
(b) $h_m^{(G)}(x) \cdot h_m^{(G)}(y) = 0$ and $\left(h_m^{(G)}(x)\right)^2 + \left(h_m^{(G)}(y)\right)^2 > 0$.
(c) $0 < h_m^{(G)}(x) < 1$ and $0 < h_m^{(G)}(y) < 1$.
(d) $0 < h_m^{(G)}(x) < 1$ and $h_m^{(G)}(y) \geq 1$, or $h_m^{(G)}(x) \geq 1$ and $0 < h_m^{(G)}(y) < 1$.
(e) $h_m^{(G)}(x) \geq 1$ and $h_m^{(G)}(y) \geq 1$.

We now discuss the five cases.

(a) and (e). If x and y are in case (a) or in case (e), using the equality (2.11), we obtain that $\left|g_m^{(G)}(x) - g_m^{(G)}(y)\right| = 0$, so, obviously, the inequality (2.12) holds true.

(b) Since we assume that $h_m^{(G)}(x) \cdot h_m^{(G)}(y) = 0$ and

$$\left(h_m^{(G)}(x)\right)^2 + \left(h_m^{(G)}(y)\right)^2 > 0,$$

we distinguish two situations:

((b)-1) $h_m^{(G)}(x) = 0$ and $h_m^{(G)}(y) > 0$,

and

((b)-2) $h_m^{(G)}(x) > 0$ and $h_m^{(G)}(y) = 0$.

Note that it is enough to prove that (2.12) holds true only for x and y in the situation ((b)-1) because if x and y are in the situation ((b)-2), then, by switching the roles of x and y, we obtain that x and y are in the situation ((b)-1).

Proof of (2.12) for x and y in the situation ((b)-1). Since we assume that $h_m^{(G)}(x) = 0$ and $h_m^{(G)}(y) > 0$, we obtain that $g_m^{(G)}(x) = 0$ and

$$g_m^{(G)}(y) = \left(h_m^{(G)} \wedge 1\right)(y) \leq h_m^{(G)}(y).$$

Since we also assume that $d(x,y) < \delta$, we further obtain that

$$\left|h_m^{(G)}(x) - h_m^{(G)}(y)\right| < \varepsilon.$$

Since $\left|h_m^{(G)}(x) - h_m^{(G)}(y)\right| = h_m^{(G)}(y)$, it follows that $h_m^{(G)}(y) < \varepsilon$.

Accordingly, $\left|g_m^{(G)}(x) - g_m^{(G)}(y)\right| = g_m^{(G)}(y) \le h_m^{(G)}(y) < \varepsilon$.

Thus, the inequality (2.12) holds true whenever

$$h_m^{(G)}(x) = 0 \quad \text{and} \quad h_m^{(G)}(y) > 0.$$

(c) Since we now assume that $0 < h_m^{(G)}(x) < 1$ and $0 < h_m^{(G)}(y) < 1$, we obtain that $g_m^{(G)}(x) = h_m^{(G)}(x)$ and $g_m^{(G)}(y) = h_m^{(G)}(y)$.

Since we also assume that $d(x,y) < \delta$ (so, $\left|h_m^{(G)}(x) - h_m^{(G)}(y)\right| < \varepsilon$), we obtain that (2.12) holds true.

(d) We now assume that

($d1$) $0 < h_m^{(G)}(x) < 1$ and $h_m^{(G)}(y) \ge 1$,

or that

($d2$) $h_m^{(G)}(x) \ge 1$ and $0 < h_m^{(G)}(y) < 1$.

As in case (b), it is enough to prove that (2.12) holds true for x and y in the situation ($d1$), because if x and y are in the situation ($d2$), we can prove that (2.12) is true by switching the roles of x and y.

Proof of (2.12) in the situation (d1). In this situation, we obtain that $g_m^{(G)}(x) = h_m^{(G)}(x)$ and $g_m^{(G)}(y) = 1$.

Since we also assume that $d(x,y) < \delta$, we obtain that

$$\left|h_m^{(G)}(x) - h_m^{(G)}(y)\right| < \varepsilon.$$

It follows that (2.12) holds true in this case because

$$\left|g_m^{(G)}(x) - g_m^{(G)}(y)\right| = g_m^{(G)}(y) - g_m^{(G)}(x) = 1 - h_m^{(G)}(x) \le h_m^{(G)}(y) - h_m^{(G)}(x) < \varepsilon.$$

Since the inequality (2.12) holds true in all cases (a)-(e), it follows that (2.12) holds true for all $x \in X$ and $y \in X$ such that $d(x,y) < \delta$; therefore, $g_m^{(G)}$ is uniformly continuous. □

PROPOSITION 2.5. *For every open bounded subset G of X, and for every $m \in \mathbb{N}$, we have $g_m^{(G)} \in C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$.*

Proof. Use Lemma 2.4. □

3. **PROPERTIES OF CONTINUOUS FUNCTIONS**

Our goal in this section is to discuss several properties of certain constructions, constructions which involve continuous functions.

As usual in this work, we assume given a Polish space (X, d).

We will need the following (known) lemma:

LEMMA 3.1. *Let F be a nonempty closed subset of X. Then:*

(a) *For every sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of real numbers that is decreasing (i.e., $\varepsilon_k \geq \varepsilon_{k+1}$ for every $k \in \mathbb{N}$) and that converges to zero, we have that $\bigcap\limits_{k=1}^{\infty} F_{\varepsilon_k} = F$, where F_{ε_k}, $k \in \mathbb{N}$ are the subsets of X defined in the proof of (a) of Lemma 2.3.*

(b) *For every sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of strictly positive real numbers that converges to zero, we have that $\bigcap\limits_{k=1}^{\infty} F_{\varepsilon_k} = F$.*

(c) $\bigcap\limits_{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}} F_{\varepsilon} = F$.

The proof of the lemma appears implicitly in the proof of the implication $(a) \Longrightarrow (b)$ of Theorem 2.1 of Billingsley [2], or in the proof of the implication $(i) \Longrightarrow (ii)$ of Theorem 6 of Gugushvili [6].

LEMMA 3.2. *Let F be a closed nonempty subset of X, and let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $(\varepsilon_k)_{k\in\mathbb{N}}$ converges to zero. Then the sequence $\left(g_F^{(\varepsilon_k)}(x)\right)_{k\in\mathbb{N}}$ converges pointwise to $\mathbf{1}_F$ on X (here, $g_F^{(\varepsilon_k)}$, $k \in \mathbb{N}$, are the functions defined at (a) of Lemma 2.3; that is, $g_F^{(\varepsilon_k)}(x) = \left(1 - \dfrac{d(x,F)}{\varepsilon_k}\right) \vee 0$ for every $x \in X$ and $k \in \mathbb{N}$).*

Proof. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers that converges to zero. We have to prove that the sequence $\left(g_F^{(\varepsilon_k)}\right)_{k\in\mathbb{N}}$ satisfies the following condition: for every $x \in X$, the sequence $\left(g_F^{(\varepsilon_k)}\right)_{k\in\mathbb{N}}$ converges and $\lim\limits_{k\to+\infty} g_F^{(\varepsilon_k)}(x) = \begin{cases} 1 & \text{if } \ x \in F \\ 0 & \text{if } \ x \in X \setminus F \end{cases}$.

Thus, it is enough to prove that the following two assertions are true:

(A) If $x \in F$, then the sequence (of real numbers) $\left(g_F^{(\varepsilon_k)}(x)\right)_{k\in\mathbb{N}}$ converges to 1.

(B) If $x \in X \setminus F$, then the sequence $\left(g_F^{(\varepsilon_k)}(x)\right)_{k\in\mathbb{N}}$ converges to zero.

Proof of (A). Let $x \in F$. Using the equality (2.5), we obtain that $g_F^{(\varepsilon_k)}(x) = 1$ for every $k \in \mathbb{N}$. Thus, obviously, $\left(g_F^{(\varepsilon_k)}(x)\right)_{k\in\mathbb{N}}$ converges to 1.

Proof of (B). Let $x \in X \setminus F$. Set $\alpha_x = d(x, F)$. Since F is a closed subset of X, it follows that $\alpha_x > 0$.

Taking into consideration that $(\varepsilon_k)_{k\in\mathbb{N}}$ converges to zero, we obtain that there exists $k_x \in \mathbb{N}$ such that $\varepsilon_k < \alpha_x$ for every $k \in \mathbb{N}$, $k \geq k_x$.

We obtain that $d(x, F) > \varepsilon_k$, so $x \notin F_{\varepsilon_k}$; using the equality (2.5), we further obtain that $g_F^{(\varepsilon_k)}(x) = 0$ for every $k \in \mathbb{N}$, $k \geq k_x$.

We have therefore proved that all the terms of the sequence $\left(g_F^{(\varepsilon_k)}(x)\right)_{k\in\mathbb{N}}$ are equal to zero except possibly a finite number of them. Hence, $\left(g_F^{(\varepsilon_k)}(x)\right)_{k\in\mathbb{N}}$ converges to zero. □

If F is a closed nonempty subset of X, if $\mu \in \mathcal{M}(X)$ is a positive element (a finite measure), and if $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, then the function $g_F^{(\varepsilon)}$ is μ-integrable (because, by (a) of Lemma 2.3, $g_F^{(\varepsilon)}$ is uniformly continuous and bounded, so $g_F^{(\varepsilon)}$ is a bounded Borel measurable function; since μ is a finite measure, it follows that $g_F^{(\varepsilon)}$ is μ-integrable). Accordingly, the assertion of the following lemma makes sense:

Lemma 3.3. *As before, let F be a closed nonempty subset of X. Also, let $\mu \in \mathcal{M}(X)$, $\mu \geq 0$. Then $\lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \left\langle g_F^{(\varepsilon)}, \mu \right\rangle$ does exist and is equal to $\mu(F)$.*

Proof. In order to prove the lemma, it is enough to prove that for every sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of real numbers such that $\varepsilon_k > 0$ for every $k \in \mathbb{N}$, and such that $(\varepsilon_k)_{k\in\mathbb{N}}$ converges to zero, we have that the sequence $\left(\left\langle g_F^{(\varepsilon_k)}, \mu \right\rangle\right)_{k\in\mathbb{N}}$ converges to $\mu(F)$.

To this end, let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_k > 0$ for every $k \in \mathbb{N}$, and such that $(\varepsilon_k)_{k\in\mathbb{N}}$ converges to zero.

We now note that:

(i) As explained before the lemma, the functions $g_F^{(\varepsilon_k)}$, $k \in \mathbb{N}$, are μ-integrable.

(ii) Using the equality (2.5), we obtain that $0 \leq g_F^{(\varepsilon_k)}(x) \leq \mathbf{1}_X(x)$ for every $x \in X$ and $k \in \mathbb{N}$.

(iii) The function $\mathbf{1}_X$ is μ-integrable because μ is a finite measure.

(iv) The sequence $\left(g_F^{(\varepsilon_k)}\right)_{k\in\mathbb{N}}$ converges pointwise to $\mathbf{1}_F$ on X (by Lemma 3.2).

In view of the fact that the above assertions (i), (ii), (iii), and (iv) hold true, we obtain that we can apply the Lebesgue dominated convergence theorem to the sequence $\left(g_F^{(\varepsilon_k)}\right)_{k\in\mathbb{N}}$ with respect to the measure μ.

Using the Lebesgue dominated convergence theorem, we obtain that $\lim\limits_{k\to+\infty}\int g_F^{(\varepsilon_k)}(x)\,\mathrm{d}\mu(x)$ does exist and is equal to $\int\limits_X \mathbf{1}_F\,\mathrm{d}\mu=\mu(F)$.

Thus, the sequence $\left(\left\langle g_F^{(\varepsilon_k)},\mu\right\rangle\right)_{k\in\mathbb{N}}$ converges to $\mu(F)$. $\square$

We will now discuss several facts that will be used in the proof of the implication $(d)\Longrightarrow(a)$ of our main result in this paper (Theorem 1.1).

In our approach, we will use an idea of the proof of $(iv)\Longrightarrow(i)$ of Theorem 6 (Portmanteau Lemma) of Gugushvili [6] (see also the proof of $(d)\Longrightarrow$ (a) of Theorem 2.1, pp. 3–4 of Billingsley [2]).

Let $\mu\in\mathcal{M}(X)$ be a probability measure. Also, let $g\in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, $g\geq 0$, $g\neq 0$. Since g is a bounded function, and since $g\geq 0$, there exists $b_g\in\mathbb{R}$, $b_g>0$, such that $0\leq g(x)\leq b_g$ for every $x\in X$.

Set $D_\mu^{(g)}=\{x\in(0,b_g]\,|\,\mu(\{z\in X\,|\,g(z)=x\})>0\}$. Note that the set $D_\mu^{(g)}$ is at most countable (empty, finite, or countable) because μ is a finite measure.

Now, let $\varepsilon\in\mathbb{R}$, $\varepsilon>0$. Clearly, there exists a partition $P_{b_g,\mu}^{(\varepsilon)}:0=x_{0,b_g}^{(\mu,g)}<x_{1,b_g}^{(\mu,g)}<\cdots<x_{k,b_g}^{(\mu,g)}=b_g$ of the interval $[0,b_g]$ such that $x_{i,b_g}^{\mu,g}$, $i=1,2,\ldots,k$, do not belong to $D_\mu^{(g)}$, and such that $\sigma\left(P_{b_g,\mu}^{(\varepsilon)}\right)<\varepsilon$ (where $\sigma\left(P_{b_g,\mu}^{(\varepsilon)}\right)$ is the size (or the norm) of $P_{b_g,\mu}^{(\varepsilon)}$; for details on the size of a partition, see Section 4 of Szarek and Zaharopol [7]).

Thus, let $P_{b_g,\mu}^{(\varepsilon)}:0=x_{0,b_g}^{(\mu,g)}<x_{1,b_g}^{(\mu,g)}<\cdots<x_{k,b_g}^{(\mu,g)}=b_g$ be such a partition of $[0,b_g]$ (that is, $P_{b_g,\mu}^{(\varepsilon)}$ has the following properties: $\sigma\left(P_{b_g,\mu}^{(\varepsilon)}\right)<\varepsilon$, and $x_{i,b_g}^{(\mu,g)}\notin D_\mu^{(g)}$ for every $i=1,2,\ldots,k$).

Set $A_{i,P_{b_g,\mu}^{(\varepsilon)}}=\left\{z\in X\,\middle|\,x_{i-1,b_g}^{(\mu,g)}\leq g(z)<x_{i,b_g}^{(\mu,g)}\right\}=g^{-1}\left(\left[x_{i-1,b_g}^{(\mu,g)},x_{i,b_g}^{(\mu,g)}\right)\right)$ for every $i=1,2,\ldots,k$.

Next, let $v_{P_{b_g,\mu}^{(\varepsilon)}}:X\to\mathbb{R}$ be defined by $v_{P_{b_g,\mu}^{(\varepsilon)}}(z)=\sum\limits_{i=1}^{k}x_{i-1,b_g}^{(\mu,g)}\mathbf{1}_{A_i,P_{b_g,\mu}^{(\varepsilon)}}(z)$ for every $z\in X$.

Also, let $w_{P_{b_g,\mu}^{(\varepsilon)}}:X\to\mathbb{R}$ be defined by $w_{P_{b_g,\mu}^{(\varepsilon)}}(z)=\sum\limits_{i=1}^{k}x_{i,b_g}^{(\mu,g)}\mathbf{1}_{A_i,P_{b_g,\mu}^{(\varepsilon)}}(z)$ for every $z\in X$.

LEMMA 3.4. *The functions* $v_{P^{(\varepsilon)}_{b_g,\mu}}$ *and* $w_{P^{(\varepsilon)}_{b_g,\mu}}$ *are Borel measurable, and* $v_{P^{(\varepsilon)}_{b_g,\mu}} \leq g \leq w_{P^{(\varepsilon)}_{b_g,\mu}}$ *(in the sense that*

$$v_{P^{(\varepsilon)}_{b_g,\mu}}(z) \leq g(z) \leq w_{P^{(\varepsilon)}_{b_g,\mu}}(z) \tag{3.1}$$

for every $z \in X$*).*

Proof. Since g is a continuous function, and since $\left[x^{(\mu,g)}_{i-1,b_g}, x^{(\mu,g)}_{i,b_g}\right)$ is a Borel subset of real numbers, it follows that $A_{i,P^{(\varepsilon)}_{b_g,\mu}} \in \mathcal{B}(X)$ for every $i = 1, 2, \ldots, k$, so both $v_{P^{(\varepsilon)}_{b_g,\mu}}$ and $w_{P^{(\varepsilon)}_{b_g,\mu}}$ are Borel measurable functions.

We now prove that the inequalities (3.1) hold true for every $z \in X$. To this end, let $z \in X$.

Taking into consideration the manner in which the partition $P^{(\varepsilon)}_{b_g,\mu}$ is defined, we obtain that there exists a unique $i \in \{1, 2, \ldots, k\}$ such that $x^{(\mu,g)}_{i-1,b_g} \leq g(z) < x^{(\mu,g)}_{i,b_g}$.

Thus, $v_{P^{(\varepsilon)}_{b_g,\mu}}(z) = x^{(\mu,g)}_{i-1,b_g}$ and $w_{P^{(\varepsilon)}_{b_g,\mu}}(z) = x^{(\mu,g)}_{i,b_g}$, so the inequalities (3.1) hold true for z. □

We will use the notation $g^{-1}(x) = \{z \in X \mid g(z) = x\}$, $x \in \mathbb{R}$.

LEMMA 3.5. (a)

$$\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}} \subseteq g^{-1}\left(x^{(\mu,g)}_{i-1,b_g}\right) \cup g^{-1}\left(x^{(\mu,g)}_{i,b_g}\right) \tag{3.2}$$

for every $i = 2, 3, \ldots, k$.

(b) $\mu\left(\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right) = 0$ *for every* $i = 2, 3, \ldots, k$.

Proof. (a) Let $i \in \{2, 3, \ldots, k\}$. Clearly, the inclusion (3.2) holds true if $\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}}$ is empty. Thus, we may and do assume that $\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}} \neq \emptyset$.

Let $z \in \partial A_{i,P^{(\varepsilon)}_{b_g,\mu}}$. Since $\partial\left(A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right) = \overline{\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}}} \setminus \overbrace{\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}}}^{\circ}$, and since X is a metric space, we note that there exist:

– a sequence $(\rho_n)_{n\in\mathbb{N}}$ of elements of $A_{i,P^{(\varepsilon)}_{b_g,\mu}}$, such that $(\rho_n)_{n\in\mathbb{N}}$ converges to z,

and

– a sequence $(\eta_n)_{n\in\mathbb{N}}$ of elements of $X \setminus A_{i,P^{(\varepsilon)}_{b_g,\mu}}$, such that $(\eta_n)_{n\in\mathbb{N}}$ converges to z, as well.

Since g is continuous, and since the sequences $(\rho_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ converge to z, it follows that $(g(\rho_n))_{n\in\mathbb{N}}$ and $(g(\eta_n))_{n\in\mathbb{N}}$ converge to $g(z)$.

Since $\rho_n \in A_{i,P^{(\varepsilon)}_{b_g,\mu}}$, it follows that $g(\rho_n) \in \left[x^{(\mu,g)}_{i-1,b_g}, x^{(\mu,g)}_{i,b_g}\right)$ for every $n \in \mathbb{N}$, so $g(z) \in \left[x^{(\mu,g)}_{i-1,b_g}, x^{(\mu,g)}_{i,b_g}\right]$.

Since $\eta_n \in X \setminus A_{i,P^{(\varepsilon)}_{b_g,\mu}}$, it follows that $g(\eta_n) \notin \left[x^{(\mu,g)}_{i-1,b_g}, x^{(\mu,g)}_{i,b_g}\right)$ for every $n \in \mathbb{N}$, so, since $(g(\eta_n))_{n\in\mathbb{N}}$ converges to $g(z)$, and since $g(z)$ belongs to $\left[x^{(\mu,g)}_{i-1,b_g}, x^{(\mu,g)}_{i,b_g}\right]$, we obtain that $g(z) = x^{(\mu,g)}_{i-1,b_g}$, or else $g(z) = x^{(\mu,g)}_{i,b_g}$; that is, $g(z) \in \left\{x^{(\mu,g)}_{i-1,b_g}, x^{(\mu,g)}_{i,b_g}\right\}$.

Accordingly, $z \in g^{-1}\left(x^{(\mu,g)}_{i-1,b_g}\right) \cup g^{-1}\left(x^{(\mu,g)}_{i,b_g}\right)$. Thus, the inclusion (3.2) holds true.

(b) Using (a) and the fact that $x^{(\mu,g)}_{i,b_g} \notin D^{(g)}_{\mu}$ for every $i = 1, 2, \ldots, k$, we obtain that (b) is true. □

LEMMA 3.6. *Let $\mu \in \mathcal{M}(X)$, $g \in C^{(\mathrm{b})}_{\mathrm{bs}}(X)$, $g \geq 0$, $P^{(\varepsilon)}_{b_g,\mu}$, and $A_{i,P^{(\varepsilon)}_{b_g,\mu}}$, $i = 2, 3, \ldots, k$, be as before. Assume also that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{M}(X)$ such that $\mu_n \geq 0$ and $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$, and such that the following condition is satisfied:*

(C) The sequence $(\mu_n(A))_{n\in\mathbb{N}}$ converges to $\mu(A)$ whenever A is a bounded Borel measurable subset of X such that $\mu(\partial A) = 0$.

Then, for every $i = 2, 3, \ldots, k$, the sequence $\left(\mu_n\left(A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right)\right)_{n\in\mathbb{N}}$ converges to $\mu\left(A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right)$.

Proof. Let $i \in \{2, 3, \ldots, k\}$. Using (b) of Lemma 3.5, we obtain that $\mu\left(\partial A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right) = 0$. Thus, taking into consideration that $A_{i,P^{(\varepsilon)}_{b_g,\mu}}$ is a Borel bounded set, we obtain that we can use condition (C), and we conclude that $\left(\mu_n\left(A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right)\right)_{n\in\mathbb{N}}$ converges to $\mu\left(A_{i,P^{(\varepsilon)}_{b_g,\mu}}\right)$. □

4. WEAK TYPE CONVERGENCE OF SEQUENCES OF MEASURES

In this section, we prove the main result of the paper (Theorem 1.1) and we state an open problem which was our motivation for obtaining the result.

Proof of Theorem 1.1. $(a) \Longrightarrow (b)$. Since $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X) \subseteq C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, it follows that, if $(\mu_n)_{n\in\mathbb{N}}$ converges to μ along $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, then, obviously, the sequence $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$ whenever $g \in C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$.

Accordingly, $(\mu_n)_{n\in\mathbb{N}}$ converges to μ along $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$.

$(b) \Longrightarrow (c)$. Assume that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ along $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$.

We have to prove that $(b) \Longrightarrow (c\text{-}1)$ and that $(b) \Longrightarrow (c\text{-}2)$.

Proof of $(b) \Longrightarrow (c\text{-}1)$. Assume that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ along $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$, and let F be a closed bounded subset of X. We have to prove that the inequality (1.1) holds true.

Clearly, (1.1) is true whenever F is the empty set. Thus, we may and do assume that $F \neq \emptyset$.

For every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, let $g_F^{(\varepsilon)} : X \to \mathbb{R}$ be defined by

$$g_F^{(\varepsilon)}(x) = \left(1 - \frac{d(x, F)}{\varepsilon}\right) \wedge 0$$

for every $x \in X$ (note that the functions $g_F^{(\varepsilon)}$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, appear also at (a) of Lemma 2.3).

Using Lemma 2.3, we obtain that $g_F^{(\varepsilon)} \in C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$ for every $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

Since we assume that (b) holds true, we obtain that, for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, we have that the sequence $\left(\left\langle g_F^{(\varepsilon)}, \mu_n\right\rangle\right)_{n\in\mathbb{N}}$ converges to $\left\langle g_F^{(\varepsilon)}, \mu\right\rangle$.

We now define a function $\eta : (0, +\infty) \to \mathbb{R}$ as follows: $\eta(\varepsilon) = \left\langle g_F^{(\varepsilon)}, \mu\right\rangle$ for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Next, we note that η is increasing. Indeed, let $\varepsilon_1 \in \mathbb{R}$ and $\varepsilon_2 \in \mathbb{R}$ be such that $0 < \varepsilon_1 < \varepsilon_2$. Then, $\dfrac{d(x, F)}{\varepsilon_1} \geq \dfrac{d(x, F)}{\varepsilon_2}$, so $-\dfrac{d(x, F)}{\varepsilon_1} \leq -\dfrac{d(x, F)}{\varepsilon_2}$ for every $x \in X$.

Using the equality (2.5), we obtain that $0 \leq g_F^{(\varepsilon_1)}(x) \leq g_F^{(\varepsilon_2)}(x)$ for every $x \in X$.

Thus, η is increasing because we have $\eta(\varepsilon_1) \leq \eta(\varepsilon_2)$ for every $\varepsilon_1 \in \mathbb{R}$ and $\varepsilon_2 \in \mathbb{R}$ such that $0 < \varepsilon_1 < \varepsilon_2$.

By Lemma 3.3, $\lim\limits_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \eta(\varepsilon)$ does exist and

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \eta(\varepsilon) = \mu(F). \tag{4.1}$$

Using the equality (2.5), we obtain that

$$\mathbf{1}_F \leq g_F^{(\varepsilon)} \tag{4.2}$$

for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Since μ_n, $n \in \mathbb{N}$, are probability measures, using (4.2), we obtain that

$$\mu_n(F) \leq \left\langle g_F^{(\varepsilon)}, \mu_n \right\rangle \tag{4.3}$$

for every $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Using (4.3), we obtain that

$$\limsup_{n\to+\infty} \mu_n(F) \leq \limsup_{n\to+\infty} \left\langle g_F^{(\varepsilon)}, \mu_n \right\rangle = \lim_{n\to+\infty} \left\langle g_F^{(\varepsilon)}, \mu_n \right\rangle = \left\langle g_F^{(\varepsilon)}, \mu \right\rangle = \eta(\varepsilon) \tag{4.4}$$

for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Finally, using (4.4) and (4.1), we obtain that

$$\limsup_{n\to+\infty} \mu_n(F) \leq \lim_{\substack{\varepsilon\to+\infty \\ \varepsilon>0}} \eta(\varepsilon) = \mu(F).$$

Thus, the inequality (1.1) holds true.

Proof of $(b) \implies (c\text{-}2)$. Let G be an open bounded subset of X.

We have to prove that the inequality (1.2) holds true.

To this end, let $g_m^{(G)}$, $m \in \mathbb{N}$, be the functions defined before Lemma 2.4.

Since, as pointed out in the proof of Lemma 2.4, $0 \leq g_m^{(G)} \leq \mathbf{1}_G$, it follows that

$$\int_X \mathbf{1}_G \,\mathrm{d}\mu_n \geq \int_X g_m^{(G)} \,\mathrm{d}\mu_n \tag{4.5}$$

for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

The inequalities (4.5) can be restated as follows:

$$\mu_n(G) \geq \int_X g_m^{(G)} \,\mathrm{d}\mu_n \tag{4.6}$$

for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

Using (4.6), we obtain that for every $k \in \mathbb{N}$, we have

$$\mu_k(G) \geq \inf_{n\geq k} \int_X g_m^{(G)} \,\mathrm{d}\mu_n. \tag{4.7}$$

Accordingly,

$$\mu_l(G) \geq \inf_{n\geq k} \int_X g_m^{(G)} \,\mathrm{d}\mu_n \tag{4.8}$$

for every $k \in \mathbb{N}$ and $l \in \mathbb{N}$ such that $k \leq l$.

Using (4.8), we obtain that

$$\inf_{l\geq k}\mu_l(G)\geq\inf_{n\geq k}\int_X g_m^{(G)}\,\mathrm{d}\mu_n \tag{4.9}$$

for every $k\in\mathbb{N}$ and $m\in\mathbb{N}$.

Since $\inf\limits_{l\geq k}\mu_l(G)\leq\sup\limits_{j\in\mathbb{N}}\inf\limits_{l\geq j}\mu_l(G)$ for every $k\in\mathbb{N}$, using (4.9), we obtain that $\sup\limits_{j\in\mathbb{N}}\inf\limits_{l\geq j}\mu_l(G)\geq\inf\limits_{n\geq k}\int\limits_X g_m^{(G)}\,\mathrm{d}\mu_n$ for every $k\in\mathbb{N}$, so, for every $m\in\mathbb{N}$, we have

$$\sup_{j\in\mathbb{N}}\inf_{l\geq j}\mu_l(G)\geq\sup_{k\in\mathbb{N}}\inf_{n\geq k}\int_X g_m^{(G)}\,\mathrm{d}\mu_n.$$

We can restate the above inequality as follows:

$$\liminf_{n\to+\infty}\mu_n(G)\geq\liminf_{n\to+\infty}\int_X g_m^{(G)}\,\mathrm{d}\mu_n \tag{4.10}$$

for every $m\in\mathbb{N}$.

Since $g_m^{(G)}\in C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$ for every $m\in\mathbb{N}$ (see Proposition 2.5), using the assumption (b), we obtain that the sequence $\left(\int\limits_X g_m^{(G)}\,\mathrm{d}\mu_n\right)_{n\in\mathbb{N}}$ converges to $\int\limits_X g_m^{(G)}\,\mathrm{d}\mu$, so (4.10) becomes

$$\liminf_{n\to+\infty}\mu_n(G)\geq\int_X g_m^{(G)}\,\mathrm{d}\mu \tag{4.11}$$

for every $m\in\mathbb{N}$.

We now note that the sequence $\left(g_m^{(G)}\right)_{m\in\mathbb{N}}$ converges pointwise to $\mathbf{1}_G$ (in the sense, of course, that $\left(g_m^{(G)}(x)\right)_{m\in\mathbb{N}}$ converges to $\mathbf{1}_G(x)$ for every $x\in X$).

Since $0\leq g_m^{(G)}\leq\mathbf{1}_G$ for every $m\in\mathbb{N}$, and since μ is a (positive) finite measure (so, the function $\mathbf{1}_G$ is μ-integrable), it follows that we can apply the Lebesgue dominated convergence theorem to the sequence of functions $\left(g_m^{(G)}\right)_{m\in\mathbb{N}}$ with respect to the measure μ.

We obtain that the sequence $\left(g_m^{(G)}\right)_{m\in\mathbb{N}}$ converges, and

$$\lim_{m\to+\infty}\int_X g_m^{(G)}\,\mathrm{d}\mu=\int_X\mathbf{1}_G\,\mathrm{d}\mu=\mu(G). \tag{4.12}$$

In view of (4.12), using (4.11), we obtain that the inequality (1.2) is valid.

$(c) \Longrightarrow (d)$. Let A be a bounded Borel subset of X such that $\mu(\partial A) = 0$.

Since A is bounded, it follows that both $\bar{A}$ and $\overset{\circ}{A}$ are bounded, as well. Therefore, we can use statement (c-1) for $\bar{A}$ and statement (c-2) for $\overset{\circ}{A}$.

We obtain that

$$\limsup_{n\to+\infty}(\mu_n(\bar{A})) \leq \mu(\bar{A}) = \mu(\overset{\circ}{A}) \leq \liminf_{n\to+\infty}(\mu_n(\overset{\circ}{A})). \tag{4.13}$$

Since $\mu_n(\overset{\circ}{A}) \leq \mu_n(\bar{A})$ for every $n \in \mathbb{N}$, it follows that

$$\liminf_{n\to+\infty}\mu_n(\overset{\circ}{A}) \leq \liminf_{n\to+\infty}\mu_n(\bar{A}) \leq \limsup_{n\to+\infty}\mu_n(\bar{A}). \tag{4.14}$$

Combining the inequalities (4.13) and (4.14), we obtain that

$$\liminf_{n\to+\infty}\mu_n(\overset{\circ}{A}) = \liminf_{n\to+\infty}\mu_n(\bar{A}) = \mu(\bar{A}) = \mu(\overset{\circ}{A}) = \limsup_{n\to+\infty}\mu_n(\overset{\circ}{A}) = \limsup_{n\to+\infty}\mu_n(\bar{A}). \tag{4.15}$$

Using the equalities (4.15), we further obtain that

$$\liminf_{n\to+\infty}\mu_n(A) = \mu(A) = \limsup_{n\to+\infty}\mu_n(A).$$

Thus, the sequence $(\mu_n(A))_{n\in\mathbb{N}}$ converges to $\mu(A)$. We have therefore proved that, for every bounded μ-continuity subset A of X, the sequence $(\mu_n(A))_{n\in\mathbb{N}}$ converges to $\mu(A)$.

$(d) \Longrightarrow (a)$. Under the assumption that (d) holds true, we have to prove that, for every $g \in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, the sequence $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$. Thus, assume that (d) holds true.

We will prove the convergence of the sequence $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ in two steps:

Step 1. At this step we will prove that $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$ for every $g \in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, $g \geq 0$.

Step 2. Here we will prove that $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$ for every $g \in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$.

Step 1 (*Full Details*). As pointed out after Lemma 3.3, in our discussion we will use the approach of the proof of $(iv) \Longrightarrow (i)$ of Theorem 6 (Portmanteau Lemma) of Gugushvili [6] (see also the proof of $(d) \Longrightarrow (a)$ of Theorem 2.1, pp. 3-4 of Billingsley [2]).

Let $g \in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, $g \geq 0$. Clearly, $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$ if $g = 0$. Thus, we may and do assume that $g \neq 0$.

Since g is a bounded function and $g \geq 0$, there exists $b \in \mathbb{R}$, $b > 0$, such that $0 \leq g(z) < b$ for every $z \in X$.

Also, let $D_\mu^{(g)}$ be the set defined before Lemma 3.4.

We will show that $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$, by proving that for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|\langle g, \mu_n\rangle - \langle g, \mu\rangle| < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

To this end, let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Next, let $P : 0 = x_0 < x_1 < \cdots < x_k = b$ be a partition of the interval $[0, b]$ such that $\sigma(P) < \dfrac{\varepsilon}{b + 1 + \mu(X)}$ (where $\sigma(P)$ is the norm (or mesh) of P) and such that x_i, $i = 1, 2, \ldots, k$, do not belong to $D_\mu^{(g)}$. (Such a partition does exist because (as pointed out when we defined $D_\mu^{(g)}$ (before Lemma 3.4)), the set $D_\mu^{(g)}$ is at most countable.)

Set $A_i = \{z \in X \,|\, x_{i-1} \leq g(z) < x_i\} = g^{-1}([x_{i-1}, x_i))$ for every $i = 1, 2, \ldots, k$.

Next, let $v : X \to \mathbb{R}$ be defined by $v(z) = \sum_{i=1}^{k} x_{i-1}\mathbf{1}_{A_i}(z)$ for every $z \in X$.

Also, let $w : X \to \mathbb{R}$ be defined by $w(z) = \sum_{i=1}^{k} x_i\mathbf{1}_{A_i}(z)$ for every $z \in X$.

Using Lemma 3.4, we obtain that $v(z) \leq g(z) \leq w(z)$ for every $z \in X$.

Taking into consideration that μ_n, $n \in \mathbb{N}$, and μ are finite (positive) measures, by integrating each side of the above inequalities with respect to μ_n, $n \in \mathbb{N}$, and with respect to μ, and using the definitions of the functions v and w, we obtain that

$$\sum_{i=1}^{k} x_{i-1}\mu_n(A_i) \leq \int_X g(z)\,\mathrm{d}\mu_n(z) \leq \sum_{i=1}^{k} x_i\mu_n(A_i) \tag{4.16}$$

for every $n \in \mathbb{N}$, and

$$\sum_{i=1}^{k} x_{i-1}\mu(A_i) \leq \int_X g(z)\,\mathrm{d}\mu(z) \leq \sum_{i=1}^{k} x_i\mu(A_i). \tag{4.17}$$

Using Lemma 3.6, we obtain that the sequence $(\mu_n(A_i))_{n\in\mathbb{N}}$ converges to $\mu(A_i)$ for every $i = 2, 3, \ldots, k$.

Thus, there exists $n_\varepsilon \in \mathbb{N}$ large enough such that

$$|\mu_n(A_i) - \mu(A_i)| \leq \frac{\varepsilon}{k(b + 1 + \mu(X))}$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $i = 2, 3, \ldots, k$.

For $n \in \mathbb{N}$, $n \geq n_\varepsilon$, it follows that

$$\left|\int\limits_X g(z)\,\mathrm{d}\mu_n(z) - \int\limits_X g(z)\,\mathrm{d}\mu(z)\right|$$

$$\leq \left|\int\limits_X g(z)\,\mathrm{d}\mu_n(z) - \sum_{i=1}^{k} x_i\mu_n(A_i)\right| + \left|\sum_{i=1}^{k} x_i\mu_n(A_i) - \sum_{i=1}^{k} x_i\mu(A_i)\right|$$

$$+ \left|\sum_{i=1}^{k} x_i\mu(A_i) - \int\limits_X g(z)\,\mathrm{d}\mu(z)\right|$$

$$= \left(\sum_{i=1}^{k} x_i\mu_n(A_i) - \int\limits_X g(z)\,\mathrm{d}\mu_n(z)\right) + \left|\sum_{i=1}^{k} x_i\mu_n(A_i) - \sum_{i=1}^{k} x_i\mu(A_i)\right|$$

$$+ \left(\sum_{i=1}^{k} x_i\mu(A_i) - \int\limits_X g(z)\,\mathrm{d}\mu(z)\right)$$

$$\leq \left(\sum_{i=1}^{k} x_i\mu_n(A_i) - \sum_{i=1}^{k} x_{i-1}\mu_n(A_i)\right) + \sum_{i=1}^{k} x_i|\mu_n(A_i) - \mu(A_i)|$$

$$+ \left(\sum_{i=1}^{k} x_i\mu(A_i) - \sum_{i=1}^{k} x_{i-1}\mu(A_i)\right).$$

Taking into consideration that $x_0 = 0$, that

$$x_1 = x_1 - x_0 \leq \sigma(P) < \frac{\varepsilon}{b + 1 + mu(X)},$$

that $|\mu_n(A_i) - \mu(A_i)| < \dfrac{\varepsilon}{b + 1 + mu(X)}$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $i = 2, 3, \ldots, k$, that $x_i - x_{i-1} \leq \sigma(P)$ for every $i = 1, 2, \ldots, k$, and that $A_1, A_2, \ldots, A_k$ are mutually disjoint Borel subsets of X, we obtain that for $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have

$$\left(\sum_{i=1}^{k} x_i\mu_n(A_i) - \sum_{i=1}^{k} x_{i-1}\mu_n(A_i)\right) + \sum_{i=1}^{k} x_i|\mu_n(A_i) - \mu(A_i)|$$

$$+ \left(\sum_{i=1}^{k} x_i\mu(A_i) - \sum_{i=1}^{k} x_{i-1}\mu(A_i)\right)$$

$$= x_1\mu_n(A_1) + \sum_{i=2}^{k}(x_i - x_{i-1})\mu_n(A_i) + \sum_{i=1}^{k} x_i|\mu_n(A_i) - \mu(A_i)|$$

$$+x_1\mu(A_1)+\sum_{i=2}^{k}(x_i-x_{i-1})\mu(A_i)$$

$$\leq \frac{\varepsilon}{b+1+\mu(X)}\mu_n(A_1)+\sum_{i=2}^{k}(x_i-x_{i-1})\mu_n(A_i)+\sum_{i=1}^{k}x_i\frac{\varepsilon}{k(b+1+\mu(X))}$$

$$+\frac{\varepsilon}{b+1+\mu(X)}\mu(A_1)+\sum_{i=2}^{k}(x_i-x_{i-1})\mu(A_i)$$

$$\leq \frac{\varepsilon}{b+1+\mu(X)}\mu_n(A_1)+\sum_{i=2}^{k}\sigma(P)\mu_n(A_i)+\sum_{i=1}^{k}x_i\frac{\varepsilon}{k(b+1+\mu(X))}$$

$$+\frac{\varepsilon}{b+1+\mu(X)}\mu(A_1)+\sum_{i=2}^{k}\sigma(P)\mu(A_i)$$

$$\leq \frac{\varepsilon}{b+1+\mu(X)}\mu_n(A_1)+\sum_{i=2}^{k}\frac{\varepsilon}{b+1+\mu(X)}\mu_n(A_i)$$

$$+\sum_{i=1}^{k}x_i\frac{\varepsilon}{k(b+1+\mu(X))}+\frac{\varepsilon}{b+1+\mu(X)}\mu(A_1)+\sum_{i=2}^{k}\frac{\varepsilon}{b+1+\mu(X)}\mu(A_i)$$

$$=\frac{\varepsilon}{b+1+\mu(X)}\mu_n\left(\bigcup_{i=1}^{k}A_i\right)+\sum_{i=1}^{k}x_i\frac{\varepsilon}{k(b+1+\mu(X))}$$

$$+\frac{\varepsilon}{b+1+\mu(X)}\mu\left(\bigcup_{i=1}^{k}A_i\right)$$

$$=\frac{\varepsilon}{b+1+\mu(X)}+\sum_{i=1}^{K}x_i\frac{\varepsilon}{k(b+1+\mu(X))}+\frac{\varepsilon}{b+1+\mu(X)}\mu(X)$$

$$\leq \frac{\varepsilon}{b+1+\mu(X)}+b\sum_{i=1}^{k}\frac{\varepsilon}{k(b+1+\mu(X))}+\frac{\varepsilon}{b+1+\mu(X)}\mu(X)$$

$$=\frac{\varepsilon}{b+1+\mu(X)}+kb\frac{\varepsilon}{k(b+1+\mu(X))}+\frac{\varepsilon}{b+1+\mu(X)}\mu(X)$$

$$=\frac{\varepsilon}{b+1+\mu(X)}(1+b+\mu(X))=\varepsilon.$$

Thus, we have proved that for every $\varepsilon\in\mathbb{R}$, $\varepsilon>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $|\langle g,\mu_n\rangle-\langle g,\mu\rangle|<\varepsilon$ for every $n\in\mathbb{N}$, $n\geq n_\varepsilon$.

Therefore, the sequence $(\langle g,\mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g,\mu\rangle$ whenever $g\in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$, $g\geq 0$.

Step 2. We now prove that the sequence $(\langle g,\mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g,\mu\rangle$ whenever $g\in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$.

To this end, let $g\in C_{\mathrm{bs}}^{(\mathrm{b})}(X)$.

Set $g^+ = g \vee 0$ and $g^- = (-g) \vee 0$ (here, we may think of g as an element of $B_b(X)$; therefore, in $B_b(X)$, the functions g^+ and g^- are well-defined and belong to $B_b(X)$).

It is easy to see that both g^+ and g^- are continuous bounded functions and have bounded supports.

Thus, $g^+ \in C_{\mathrm{bs}}^{(b)}(X)$ and $g^- \in C_{\mathrm{bs}}^{(b)}(X)$.

Since $g^+ \geq 0$ and $g^- \geq 0$, using Step 1, we obtain that the sequences $(\langle g^+, \mu_n\rangle)_{n\in\mathbb{N}}$ and $(\langle g^-, \mu_n\rangle)_{n\in\mathbb{N}}$ converge to $\langle g^+, \mu\rangle$ and $\langle g^-, \mu\rangle$, respectively.

Since $\langle g, \mu_n\rangle = \langle g^+, \mu_n\rangle - \langle g^-, \mu_n\rangle$ for every $n \in \mathbb{N}$, and since $\langle g, \mu\rangle = \langle g^+, \mu\rangle - \langle g^-, \mu\rangle$, it follows that the sequence $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle g, \mu\rangle$. □

We will conclude the paper with two open problems.

The terminology we will use here is inspired by the terms introduced by Ethier and Kurtz in Section 4 of Chapter 3 of [5].

A subset $\mathcal{A}$ of $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$ is said to be convergence determining for the convergence along $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$ if every sequence $(\mu_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{M}(X)$ such that μ_n is a probability measure for every $n \in \mathbb{N}$, and every $\mu \in \mathcal{M}(X)$, $\mu \geq 0$, have the property that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ along $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$ whenever the following condition is satisfied:

($C_{\mathrm{bs}}^{(\mathrm{b})}(X), \mathcal{A}$) *The sequence* $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ *converges to* $\langle g, \mu\rangle$ *for every* $g \in \mathcal{A}$.

Similarly, given a subset $\mathcal{S}$ of $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$, we say that $\mathcal{S}$ is convergence determining for the convergence along $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$ if any sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures in $\mathcal{M}(X)$ and any $\mu \in \mathcal{M}(X)$, $\mu \geq 0$, have the property that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ along $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$ if the condition ($C_{\mathrm{bs}}^{(\mathrm{ucb})}(X), \mathcal{S}$) below is satisfied:

($C_{\mathrm{bs}}^{(\mathrm{ucb})}(X), \mathcal{S}$) *The sequence* $(\langle g, \mu_n\rangle)_{n\in\mathbb{N}}$ *converges to* $\langle g, \mu\rangle$ *whenever* $g \in \mathcal{S}$.

Question 1. Is there a countable convergence determining set $\mathcal{A}$ for the convergence along $C_{\mathrm{bs}}^{(\mathrm{b})}(X)$?

Question 2. Is there a countable convergence determining set $\mathcal{S}$ for the convergence along $C_{\mathrm{bs}}^{(\mathrm{ucb})}(X)$?

A positive answer to any of the above questions would allow us to complete the ergodic decomposition defined by transition probabilities in Polish spaces to a decomposition quite similar to the decomposition defined by transition probabilities in locally compact separable metric spaces.

Note that even proofs of the existence of such sets, without explicit methods of obtaining the sets, are enough for the completion of the ergodic decomposition. Of course, having also methods for constructing such sets would be very useful for practical and theoretical purposes.

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