Dedicated to Professor Philippe G. Ciarlet on the occasion of his 80th birthday

TWO-LEVEL METHODS FOR VARIATIONAL INEQUALITIES OF THE SECOND KIND AND QUASI-VARIATIONAL INEQUALITIES

LORI BADEA

Communicated by Vasile Brînzănescu

In this paper we propose and analyze new two-level methods, one of multiplicative type and another one of additive type, for variational inequalities of the second kind and for quasi-variational inequalities. The iterations of these methods have an optimal computing complexity. Besides that, the convergence conditions of the methods which are introduced for the quasi-variational inequalities are similar with the existence and uniqueness of the solution of the inequality and therefore, do not depend on the number of subdomains.

AMS 2010 Subject Classification: 65N55, 65K15, 65J15.

Key words: domain decomposition methods, multilevel methods, subspace correction methods, variational inequalities of the second kind, quasi-variational inequalities.

1. INTRODUCTION

Literature on the Schwarz methods is very large, and it is motivated by their capability in providing robust and efficient algorithms for large scale problems. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting in 1987 with [14] or those cited in the books [10, 18, 21, 22] and [23]. Naturally, most of the papers dealing with these methods are dedicated to the linear problems. However, their generalization to non-linear problems is not straightforward, in particular for variational inequalities of the second kind or for quasi-variational inequalities. The convergence of the projected Gauss–Seidel relaxation (or successive coordinate minimization) for variational inequalities of the second kind in $\mathbf{R}^d$ has been proved in [13]. There, the non-differentiable term has been decomposed as a sum of terms, each of them depending only on one vector component. The projected Gauss-Seidel method is a particular case of a Schwarz method in which the domain is decomposed into the interior of the supports of the nodal basis functions. Consequently, the above representation of the non-differentiable term can be viewed as a decomposition in concordance with the domain decomposition. A generalization of the convergence proof in [13] to more general decompositions can be obtained using this idea, but it fails if, in order to get a faster convergence, a two-level or multilevel method is considered. This is due to the fact that the nonlinearities are not decoupled on the coarser levels.

In [5], one- and two-level multiplicative Schwarz methods have been proposed for variational and quasi-variational inequalities of the second kind, and they have been applied to frictional contact problems. It is proved there that the convergence rates of the two-level methods are almost independent of the mesh and overlapping parameters. However, the original convex set, which is defined on the fine grid, is used to find the corrections on the coarse grid, too, and this leads to a suboptimal computing complexity. A remedy can be found in adapting minimization techniques for the construction and analysis of multigrid and multilevel methods. To avoid visiting the fine grid, some level convex set for the corrections on the coarse levels have been constructed in [12, 16–20] for complementarity problems and in [6] and [7] for two two-obstacle problems. In this paper, we have adopted the construction of the level convex sets introduced in [6] and [7] for the constrained minimization of the differentiable functionals. By introduction of the level convex sets, the additional interpolations to check the fine-grid constraints are avoided and consequently, the iterations of the proposed algorithms have an optimal computing complexity. We shall see that this construction holds in the case of the two-level methods applied to the variational inequalities of the second kind and quasi-variational inequalities, but the generalization to the multilevel methods for these inequalities, to our knowledge, is an open problem so far because of the difficulties introduced by the domain decompositions on the coarse levels.

In [3], some additive and multiplicative algorithms for inequalities containing a contraction operator have been introduced. These algorithms are globally convergent if a convergence condition, which depends on the number of subdomains in the domain decomposition, is satisfied. Also, similar convergence conditions are needed for the algorithms in [5]. In [8], a multigrid algorithm has been introduced to solve the same type of inequalities as in [3]. This multigrid algorithm has an optimal computing complexity of iterations and moreover, unlike the methods in [3] or [5], the convergence condition is independent of the number of subdomains or the number of levels. The algorithms and the convergence results for the quasi-variational inequalities introduced in this section can be considered to be similar with those in [8], but the quasi-variational inequalities are more complicated than those containing a contraction operator. Moreover, unlike [8], in this paper, we consider algorithms of additive type, too.

To conclude, in this paper, we introduce and analyze two-level methods, one of multiplicative type and another one of additive type, for variational inequalities of the second kind and quasi-variational inequalities whose convex set is of two-obstacle type. Suitable constraints for the corrections computed on the coarse mesh are provided in order to ensure an optimal computing complexity of the iterations. Moreover, in comparison with the two-level multiplicative methods introduced in [5] or [3], the convergence condition of the new algorithms for quasi-variational inequalities is less restrictive and does not depend anymore on the number of the subdomains in the decomposition of the domain, *i.e.* the algorithms converge for any number of subdomains.

The paper is organized as follows. Section 2 is devoted to a general framework in a reflexive Banach space. We introduce here an assumption on the construction of the level convex sets where we look for the corrections. Another two hypotheses, which will be necessary in the convergence proofs, will be introduced, one for the multiplicative algorithm and the other for the additive algorithm. Mainly, these hypotheses refer to the decomposition of the elements in the convex set, and introduce a constant C_0 which will play an important role in the writing of the convergence rate. In Section 3, we introduce subspace correction algorithms for variational inequalities of the second kind, and prove that, under the above assumptions, they are globally convergent. We also estimate their convergence rates. In Section 4, we introduce subspace correction algorithms for the quasi-variational inequalities. As in the previous section, we prove their convergence and estimate the convergence rate, using the assumptions introduced in Section 2. Section 5 is devoted to the two-level methods. If we associate finite element subspaces to the domain decomposition and to the coarse grid, the abstract algorithms introduced in Sections 3 and 4 become two-level Schwarz methods. We show that the assumptions introduced in the previous sections hold for two-obstacle convex sets and we explicitly write the constant C_0 depending on the mesh and domain decomposition parameters. In this way, we get that convergence rates of the two-level methods we have introduced are almost independent of these parameters.

2. GENERAL FRAMEWORK

Let V be a reflexive Banach space and $V_0, V_{11}, \cdots, V_{1m}$ be some closed subspaces of V. Subspace V_0 will correspond to the coarse discretization, and $V_{11}, \cdots, V_{1m}$ corresponds to the decomposition of the domain. Also, let $K \subset V$

be a non empty closed convex set of V. To introduce the algorithms, we make an assumption on choice of the convex sets where we look for the level corrections. These level convex sets depend on the current approximation in the algorithms.

Assumption 2.1. *We assume that for a given $w \in K$, we can recursively introduce the convex sets K_1 and K_0 as:*
$0 \in K_1,\ K_1 \subset \{v_1 \in V : w + v_1 \in K\}$ *and, for a* $w_1 \in K_1$,
$0 \in K_0,\ K_0 \subset \{v_0 \in V_0 : w + w_1 + v_0 \in K\}$.

As we already said, we shall analyze both types of algorithms, multiplicative and additive. In the case of the multiplicative algorithms we make the following

Assumption 2.2. *There exists a constant $C_0 > 0$ such that for any $u, w \in K$, any $w_{1i} \in V_{1i}$, $w_{11} + \ldots + w_{1i} \in K_1$, $i = 1, \ldots, m$, and any $w_0 \in K_0$, there exist $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$, which satisfy*

$$u_{11} \in K_1 \text{ and } w_{11} + \ldots + w_{1i-1} + u_{1i} \in K_1,\ i = 2, \ldots, m,\ u_0 \in K_0$$
$$u - w = \textstyle\sum_{i=1}^m u_{1i} + u_0 \text{ and}$$
$$\textstyle\sum_{i=1}^m ||u_{1i}|| \le C_0(||u - w|| + \sum_{i=1}^m ||w_{1i}|| + ||w_0||).$$

The convex sets K_1 and K_0 are constructed as in Assumption 2.1 using w and $w_1 = w_{11} + \ldots + w_{1m}$.

This assumption is simpler in the case of the additive algorithms

Assumption 2.3. *There exist a constant $C_0 > 0$ such that for any $u, w \in K$, there exist $u_{1i} \in V_{1i} \cap K_1$, $i = 1, \ldots, m$, and $u_0 \in K_0$, which satisfy*

$$u - w = \textstyle\sum_{i=1}^m u_{1i} + u_0 \text{ and } \sum_{i=1}^m ||u_{1i}|| + ||u_0|| \le C_0||u - w||.$$

The convex sets K_1 and K_0 are constructed as in Assumption 2.1 with the above w and $w_1 = 0$.

Usually, the above conditions containing the constant C_0 are named stability conditions of the decomposition. Now, we consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$, and, like in [5], we assume that there exist two real numbers $p,\ q > 1$ such that for any real number $M > 0$ there exist two constants $\alpha_M,\ \beta_M > 0$ for which

$$\alpha_M\|v - u\|^p \le \langle F'(v) - F'(u), v - u\rangle, \text{ and} \tag{2.1}$$
$$\|F'(v) - F'(u)\|_{V'} \le \beta_M\|v - u\|^{q-1}, \tag{2.2}$$

for any $u, v \in V$ with $\|u\|, \|v\| \le M$. Above, we have denoted by F' the Gâteaux derivative of F, and we have marked that the constants α_M and β_M may depend on M. It is evident that if (2.1) and (2.2) hold, then for any

$u, v \in V$, $\|u\|, \|v\| \le M$, we have

$$\alpha_M \|v-u\|^p \le \langle F'(v) - F'(u), v-u \rangle \le \beta_M \|v-u\|^q. \tag{2.3}$$

Following the way in [15], we can prove that for any $u, v \in V$, $\|u\|, \|v\| \le M$, we have

$$\begin{aligned} &\langle F'(u), v-u \rangle + \tfrac{\alpha_M}{p} \|v-u\|^p \le F(v) - F(u) \\ &\le \langle F'(u), v-u \rangle + \tfrac{\beta_M}{q} \|v-u\|^q. \end{aligned} \tag{2.4}$$

We point out that since F is Gâteaux differentiable and satisfies (2.4), F is a strictly convex functional (see Proposition 5.4 in [11], pag. 24). Also, we can prove that $q \le 2 \le p$.

3. SUBSPACE CORRECTION ALGORITHM FOR VARIATIONAL INEQUALITIES OF THE SECOND KIND

Let $\varphi : V \to \mathbf{R}$ be a convex lower semicontinuous functional and we assume that $F + \varphi$ is coercive in the sense that

$$F(v) + \varphi(v) \to \infty, \text{ as } \|v\| \to \infty, \ v \in K, \tag{3.1}$$

if K is not bounded. In the multiplicative case, in addition to the hypotheses of Assumption 2.2, we suppose that

$$\begin{aligned} &\textstyle\sum_{i=1}^m [\varphi(w + \sum_{j=1}^{i-1} w_{1j} + u_{1i}) - \varphi(w + \sum_{j=1}^{i-1} w_{1j} + w_{1i})] \\ &+\varphi(w + w_1 + u_0) - \varphi(w + w_1 + w_0) \\ &\le \varphi(u) - \varphi(w + \textstyle\sum_{i=1}^m w_{1i} + w_0) \end{aligned} \tag{3.2}$$

for $u, w \in K$, $u_{1i}, w_{1i} \in V_{1i}$ and $u_0, w_0 \in V_0$ as in Assumption 2.2, and $w_1 = \sum_{j=1}^m w_{1j}$. Also, in addition to Assumption 2.3, for the additive case, we suppose that

$$\sum_{i=1}^m \varphi(w + u_{1i}) + \varphi(w + u_0) \le m\varphi(w) + \varphi(u) \tag{3.3}$$

for any $u, w \in K$, $u_{1i} \in V_{1i}$, $i = 1, \ldots, m$, and $u_0 \in V_0$ which satisfy Assumption 2.3.

Now, we consider the problem

$$u \in K : \langle F'(u), v-u \rangle + \varphi(v) - \varphi(u) \ge 0, \text{ for any } v \in K, \tag{3.4}$$

which is equivalent with the minimization problem

$$u \in K : F(u) + \varphi(u) \le F(v) + \varphi(v), \text{ for any } v \in K. \tag{3.5}$$

These problems have a unique solution (see [11], Proposition 1.2, pag. 34). From (2.4) we see that, for a given $M > 0$ such that the solution u of (3.4) satisfies $\|u\| \leq M$, we have

$$
\begin{aligned}
(3.6)\qquad & \frac{\alpha_M}{p}\|v-u\|^p \leq F(v)-F(u)+\varphi(v)-\varphi(u), \\
& \text{for any } v \in K,\ \|v\| \leq M.
\end{aligned}
$$

Now, we introduce the algorithm which is of the multiplicative type

Algorithm 3.1. We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we successively perform the following steps:

– at the level 1, we construct the convex set K_1 as in Assumption 2.1 with $w = u^n$. Then, we first write $w_1^n = 0$, and, for $i = 1, \dots, m$, we successively calculate $w_{1i}^{n+1} \in V_{1i}$, $w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1} \in K_1$, the solution of the inequalities

$$
\begin{aligned}
(3.7)\qquad & \langle F'(u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1}\rangle \\
& +\varphi(u^n + w_1^{n+\frac{i-1}{m}} + v_{1i}) - \varphi(u^n + w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}) \geq 0,
\end{aligned}
$$

for any $v_{1i} \in V_{1i}$, $w_1^{n+\frac{i-1}{m}} + v_{1i} \in K_1$, and write $w_1^{n+\frac{i}{m}} = w_1^{n+\frac{i-1}{m}} + w_{1i}^{n+1}$,

– at the level 0, we construct the convex set K_0 as in Assumption 2.1 with $w = u^n$ and $w_1 = w_1^{n+1}$. Then, we calculate $w_0^{n+1} \in K_0$, the solution of the inequality

$$
\begin{aligned}
(3.8)\qquad & \langle F'(u^n + w_1^{n+1} + w_0^{n+1}), v_0 - w_0^{n+1}\rangle \\
& +\varphi(u^n + w_1^{n+1} + v_0) - \varphi(u^n + w_1^{n+1} + w_0^{n+1}) \geq 0,
\end{aligned}
$$

for any $v_0 \in K_0$,

– we write $u^{n+1} = u^n + w_1^{n+1} + w_0^{n+1}$.

The proposed additive algorithm is written as follows

Algorithm 3.2. We start the algorithm with an $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we simultaneously perform, the following steps:

– we construct the convex sets K_1 and K_0 as in Assumption 2.1 with $w = u^n$ and $w_1 = 0$,

– we simultaneously calculate:

(*a*) $w_{1i}^{n+1} \in V_{1i} \cap K_1$, the solutions of the inequalities

$$
(3.9)\qquad \langle F'(u^n + w_{1i}^{n+1}), v_{1i} - w_{1i}^{n+1}\rangle + \varphi(u^n + v_{1i}) - \varphi(u^n + w_{1i}^{n+1}) \geq 0,
$$

for any $v_{1i} \in V_{1i} \cap K_1$, write $w_1^{n+1} = \sum_{i=1}^m w_{1i}^{n+1}$,

(*b*) $w_0^{n+1} K_0$, the solution of the inequality

$$
(3.10)\qquad \langle F'(u^n + w_0^{n+1}), v_0 - w_0^{n+1}\rangle + \varphi(u^n + v_0) - \varphi(u^n + w_0^{n+1}) \geq 0,
$$

for any $v_0 \in K_0$,

Then, we write $u^{n+1} = u^n + \frac{r}{m+1}(w_1^{n+1} + w_0^{n+1})$, with a fixed $0 < r \leq 1$.

These algorithms do not suppose a decomposition of the convex set K depending on the subspaces of V. Moreover, by introduction of the level convex sets, the additional interpolations to check the fine-grid constraints are avoided and consequently, the iterations have an optimal computing complexity. Like problem (3.4), problems (3.7)–(3.10) have unique solutions, and they are equivalent with minimization problems. We have the following general convergence result.

THEOREM 3.1. *Let V be a reflexive Banach, $V_0, V_{11}, \cdots, V_{1m}$ some closed subspaces of V, and K a non empty closed convex subset of V which satisfies Assumption 2.1, Assumption 2.2 when we apply Algorithm 3.1, and Assumption 2.3 in the case of Algorithm 3.2. Also, we assume that F is Gâteaux differentiable and satisfies (2.1) and (2.2), the functional φ is convex and lower semicontinuous, satisfies (3.2) for Algorithm 3.1, (3.3) for Algorithm 3.2, and $F + \varphi$ is coercive if K is not bounded. Let*

$$M = \sup\{||v|| \,:\, F(v) + \varphi(v) \leq F(u^0) + \varphi(u^0)\} \tag{3.11}$$

where u^0 is the starting point in Algorithms 3.1 or 3.2. Then, the norms of the approximations of the solution u of problem (3.4) obtained from these algorithms are bounded by M and we have the following error estimations:

(i) if $p = q = 2$ we have

$$\begin{aligned} &F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \\ &\leq (\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)], \end{aligned} \tag{3.12}$$

$$\|u^n - u\|^2 \leq \frac{2}{\alpha_M}(\tfrac{C_1}{C_1+1})^n [F(u^0) + \varphi(u^0) - F(u) - \varphi(u)]. \tag{3.13}$$

(ii) if $p > q$ we have

$$\begin{aligned} &F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \\ &\leq \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}, \end{aligned} \tag{3.14}$$

$$\|u - u^n\|^p \leq \frac{p}{\alpha_M} \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}]^{\frac{q-1}{p-q}}}. \tag{3.15}$$

The constants $C_1 > 0$ and $C_2 > 0$ depend on the functional F, the solution u, the initial approximation u^0, m, and the constant C_0.

Remark 3.1. For Algorithm 3.1, constants C_1 and C_2 can be written as,

$$\begin{aligned} C_1 = \beta_M(1 + 2C_0)(m+1)^{2-\frac{q}{p}}(\tfrac{p}{\alpha_M})^{\frac{q}{p}}(F(u^0) - F(u) \\ +\varphi(u^0) - \varphi(u))^{\frac{p-q}{p(p-1)}} + \beta_M C_0 (m+1)^{\frac{p-q+1}{p}} \frac{1}{\varepsilon^{\frac{1}{p-1}}}(\tfrac{p}{\alpha_M})^{\frac{q-1}{p-1}} \end{aligned} \tag{3.16}$$

$$C_2 = \frac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}} + (q-1)C_1^{\frac{p-1}{q-1}}} \tag{3.17}$$

where

$$\varepsilon = \alpha_M/(p\beta_M C_0 (m+1)^{\frac{p-q+1}{p}}). \tag{3.18}$$

Also, in the case of Algorithm 3.2, these constants can be written as,

$$\begin{aligned} C_1 = {} & \tfrac{m+1}{r}[1 - \tfrac{r}{m+1} + (1+C_0)(m+1)\tfrac{\beta_M}{\frac{\alpha_M}{2}} \\ & + C_0^2(m+1)(\tfrac{\beta_M}{\frac{\alpha_M}{2}})^2] \end{aligned} \tag{3.19}$$

$$\begin{aligned} C_2 = {} & \\ & \tfrac{p-q}{(p-1)(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}+(q-1)C_3^{\frac{p-1}{q-1}}}. \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} C_3 = {} & \tfrac{m+1-r}{r}[F(u^0)-F(u)+\varphi(u^0)-\varphi(u)]^{\frac{p-q}{p-1}} \\ & +(\tfrac{m+1}{r})^{\frac{q}{p}} \tfrac{\beta_M(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \\ & \cdot(F(u^0)-F(u)+\varphi(u^0)-\varphi(u))^{\frac{p-q}{p(p-1)}} \\ & +(\tfrac{m+1}{r})^{\frac{q-1}{p-1}} \tfrac{\beta_M^{\frac{p}{p-1}} C_0^{\frac{p}{p-1}} (m+1)^{q-1}}{(\frac{\alpha_M}{p})^{\frac{q}{p-1}}} \end{aligned} \tag{3.21}$$

Proof of Theorem 3.1. Except the changes of notation due to the introduction of the convex sets K_1 and K_0, the proof in the case of the multiplicative Algorithm 3.1 is identical with that of Theorem 1 in [5] and will be omitted. The proof for the additive Algorithm 3.2 especially uses the convexity of the functionals F and φ. The proof is divided into several steps.

Step 1. The existence of M defined in (3.11) follows from the coercivity of $F+\varphi$. In view of the convexity of F, we get

$$\begin{aligned} & F(u^{n+1}) = F(u^n + \tfrac{r}{m+1}(\textstyle\sum_{i=1}^m w_{1i}^{n+1} + w_0^{n+1})) \\ & \le (1-r)F(u^n) + \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m F(u^n + w_{1i}^{n+1}) + F(u^n + w_0^{n+1})] \end{aligned} \tag{3.22}$$

A similar result can be obtained for φ, *i.e.*, we have

$$\begin{aligned} & \varphi(u^{n+1}) \le (1-r)\varphi(u^n) \\ & + \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m \varphi(u^n + w_{1i}^{n+1}) + \varphi(u^n + w_0^{n+1})] \end{aligned} \tag{3.23}$$

From (3.9), (3.10) and the above two inequalities, we get

$$F(u^{n+1}) + \varphi(u^{n+1}) \le F(u^n) + \varphi(u^n)$$

Therefore, for any $n \geq 0$ and $i = 1, \cdots, m$, we get

$$\begin{aligned} &\max\{F(u^n + w_{1i}^{n+1}) + \varphi(u^n + w_{1i}^{n+1}), \\ &F(u^n + w_0^{n+1}) + \varphi(u^n + w_0^{n+1})\} \\ &\leq F(u^n) + \varphi(u^n) \leq F(u^0) + \varphi(u^0). \end{aligned} \tag{3.24}$$

Step 2. Now, from (3.9), (3.10) and (2.4), for any $n \geq 0$ and $i = 1, \cdots, m$, we have

$$\begin{aligned} &F(u^n) - F(u^n + w_{1i}^{n+1}) + \varphi(u^n) - \varphi(u^n + w_{1i}^{n+1}) \\ &\geq \tfrac{\alpha_M}{p} \|w_{1i}^{n+1}\|^p \text{ and} \\ &F(u^n) - F(u^n + w_0^{n+1}) + \varphi(u^n) - \varphi(u^n + w_0^{n+1}) \\ &\geq \tfrac{\alpha_M}{p} \|w_0^{n+1}\|^p \end{aligned} \tag{3.25}$$

In view of (3.22) and (3.25), we get

$$F(u^{n+1}) \leq (1-r)F(u^n) + \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m F(u^n + w_{1i}^{n+1}) + F(u^n + w_0^{n+1})]$$

$$\leq F(u^n) - \tfrac{r}{m+1}\tfrac{\alpha_M}{p}[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p]$$

$$+\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u^n) - \varphi(u^n + w_{1i}^{n+1})) + \varphi(u^n) - \varphi(u^n + w_0^{n+1})]$$

Consequently, we have

$$\begin{aligned} &\tfrac{r}{m+1}\tfrac{\alpha_M}{p}[\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p] \leq F(u^n) - F(u^{n+1}) + \\ &\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u^n) - \varphi(u^n + w_{1i}^{n+1})) + \varphi(u^n) - \varphi(u^n + w_0^{n+1})] \end{aligned} \tag{3.26}$$

But, in view of (3.23), we have

$$\tfrac{r}{m+1}[\textstyle\sum_{i=1}^m (\varphi(u^n) - \varphi(u^n + w_{1i}^{n+1})) + \varphi(u^n) - \varphi(u^n + w_0^{n+1})]$$

$$\leq \varphi(u^n) - \varphi(u^{n+1}),$$

and consequently,

$$\begin{aligned} &\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p \\ &\leq \tfrac{m+1}{r}\tfrac{p}{\alpha_M}[F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})] \end{aligned} \tag{3.27}$$

Step 3. Writing

$$\tilde{u}^{n+1} = u^n + \sum_{i=1}^m w_{1i}^{n+1} + w_0^{n+1}, \tag{3.28}$$

from the convexity of F, we get

$$F(u^{n+1}) \leq (1 - \frac{r}{m+1})F(u^n) + \frac{r}{m+1}F(\tilde{u}^{n+1}) \tag{3.29}$$

Applying Assumption 2.3 for $w = u^n$ and $v = u$, we get a decomposition

$u_{11}^n, \cdots, u_{1m}^n$, u_0^n, of $u - u^n$, and we can replace v_{1i} and v_0 by u_{1i}^n and u_0^n in (3.9) and (3.10), respectively. From (3.29), (2.4), (3.9) and (3.10), we obtain

$$\begin{aligned}
& F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \tfrac{r}{m+1}\tfrac{\alpha_M}{p}||u - \tilde{u}^{n+1}||^p \\
& \le (1 - \tfrac{r}{m+1})[F(u^n) - F(u)] \\
& + \tfrac{r}{m+1}[F(\tilde{u}^{n+1}) - F(u) + \tfrac{\alpha_M}{p}||u - \tilde{u}^{n+1}||^p] + \varphi(u^{n+1}) - \varphi(u) \\
& \le (1 - \tfrac{r}{m+1})[F(u^n) - F(u)] \\
& + \tfrac{r}{m+1}\langle F'(\tilde{u}^{n+1}), \tilde{u}^{n+1} - u\rangle + \varphi(u^{n+1}) - \varphi(u) \\
& \le (1 - \tfrac{r}{m+1})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] \\
& + \tfrac{r}{m+1}\textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle \\
& + \tfrac{r}{m+1}\langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle \\
& + \tfrac{r}{m+1}\textstyle\sum_{i=1}^m [\varphi(u^n + u_{1i}^n) - \varphi(u^n + w_{1i}^{n+1})] \\
& + \tfrac{r}{m+1}[\varphi(u^n + u_0^n) - \varphi(u^n + w_0^{n+1})] \\
& + \tfrac{r}{m+1}[\varphi(u^n) - \varphi(u)] + \varphi(u^{n+1}) - \varphi(u^n)
\end{aligned} \tag{3.30}$$

Now, using (2.2) and Assumption 2.3, we get, as in [4], for instance,

$$\begin{aligned}
& \textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle \\
& + \langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle \\
& \le \beta_M(\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}|| + ||w_0^{n+1}||)^{q-1}[\textstyle\sum_{i=1}^m ||u_{1i}^n - w_{1i}^{n+1}|| + ||u_0^n - w_0^{n+1}||] \\
& \le \beta_M C_0 (m+1)^{\frac{(p-1)(q-1)}{p}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q-1}{p}} ||u - \bar{u}^{n+1}|| \\
& + \beta_M (1 + C_0)(m+1)^{\frac{(p-1)q}{p}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q}{p}}
\end{aligned}$$

But, for any $\varepsilon > 0$, $r > 1$ and $x, y \ge 0$, we have $x^{\frac{1}{r}} y \le \varepsilon x + \frac{1}{\varepsilon^{\frac{1}{r-1}}} y^{\frac{r}{r-1}}$. Therefore, we get

$$\begin{aligned}
& \textstyle\sum_{i=1}^m \langle F'(u^n + w_{1i}^{n+1}) - F'(\tilde{u}^{n+1}), u_{1i}^n - w_{1i}^{n+1}\rangle \\
& + \langle F'(u^n + w_0^{n+1}) - F'(\tilde{u}^{n+1}), u_0^n - w_0^{n+1}\rangle \\
& \le \beta_M (1 + C_0)(m+1)^{\frac{(p-1)q}{p}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q}{p}} \\
& + \beta_M C_0 \frac{(m+1)^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q-1}{p-1}} \\
& + \beta_M C_0 \varepsilon (m+1)^{\frac{(p-1)(q-1)}{p}} ||u - \bar{u}^{n+1}||^p
\end{aligned} \tag{3.31}$$

for any $\varepsilon > 0$. Also, using (3.23) and (3.3), we get

$$\begin{aligned}
& \tfrac{r}{m+1}\textstyle\sum_{i=1}^m [\varphi(u^n + u_{1i}^n) - \varphi(u^n + w_{1i}^{n+1})] \\
& + \tfrac{r}{m+1}[\varphi(u^n + u_0^n) - \varphi(u^n + w_0^{n+1})] \\
& + \tfrac{r}{m+1}[\varphi(u^n) - \varphi(u)] + \varphi(u^{n+1}) - \varphi(u^n) \\
& \le \tfrac{r}{m+1}[\textstyle\sum_{i=1}^m \varphi(u^n + u_{1i}^n) + \varphi(u^n + u_0^n) - m\varphi(u^n) - \varphi(u)] \le 0
\end{aligned}$$

Consequently, from (3.30) and (3.31), we have

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) \\
&+\tfrac{r}{m+1}[\tfrac{\alpha_M}{p} - \beta_M C_0 \varepsilon (m+1)^{\frac{(p-1)(q-1)}{p}}]||u - \tilde{u}^{n+1}||^p \\
&\le (1 - \tfrac{r}{m+1})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] \\
&+\tfrac{r}{m+1}\beta_M[(1 + C_0)(m+1)^{\frac{(p-1)q}{p}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q}{p}} \\
&+C_0 \frac{(m+1)^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}} (\textstyle\sum_{i=1}^m ||w_{1i}^{n+1}||^p + ||w_0^{n+1}||^p)^{\frac{q-1}{p-1}}]
\end{aligned}
\tag{3.32}
$$

for any $\varepsilon > 0$.

Step 4. From (3.32) and (3.27), we get

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) \\
&+\tfrac{r}{m+1}[\tfrac{\alpha_M}{p} - \beta_M C_0 \varepsilon (m+1)^{\frac{(p-1)(q-1)}{p}}]||u - \tilde{u}^{n+1}||^p \\
&\le (1 - \tfrac{r}{m+1})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] \\
&+\tfrac{r}{m+1}\beta_M[(\tfrac{m+1}{r})^{\frac{q}{p}} \frac{(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \\
&\cdot(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q}{p}} \\
&+(\tfrac{m+1}{r})^{\frac{q-1}{p-1}} \frac{C_0(m+1)^{\frac{(p-1)(q-1)}{p}}}{(\frac{\alpha_M}{p})^{\frac{q-1}{p-1}} \varepsilon^{\frac{1}{p-1}}} \\
&\cdot(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q-1}{p-1}}]
\end{aligned}
$$

With

$$
\varepsilon = \frac{\alpha_M}{p} \frac{1}{\beta_M C_0 (m+1)^{\frac{(p-1)(q-1)}{p}}},
$$

the above equation becomes,

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) \\
&\le \tfrac{m+1-r}{r}[F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})] \\
&+\beta_M[(\tfrac{m+1}{r})^{\frac{q}{p}} \frac{(1+C_0)(m+1)^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \\
&\cdot(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q}{p}} \\
&+(\tfrac{m+1}{r})^{\frac{q-1}{p-1}} \frac{\beta_M^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}} (m+1)^{q-1}}{(\frac{\alpha_M}{p})^{\frac{q}{p-1}}} \\
&\cdot(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1}))^{\frac{q-1}{p-1}}]
\end{aligned}
\tag{3.33}
$$

Using (3.6), we see that error estimations in (3.13) and (3.15) can be obtained from (3.12) and (3.14), respectively.

Now, if $p = q = 2$, from the above equation, we easily get equation (3.12), where C_1 is given in (3.19).

Finally, if $q < p$, from (3.5), (3.24) and (3.33), we get

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) \\ &\le C_3[F(u^n) + \varphi(u^n) - F(u^{n+1}) - \varphi(u^{n+1})]^{\frac{q-1}{p-1}}. \end{aligned} \tag{3.34}$$

where C_3 is given in (3.21). Now, from (3.34), we get

$$F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) + \frac{1}{C_3^{\frac{p-1}{q-1}}}[F(u^{n+1}) + \varphi(u^{n+1})$$
$$-F(u) - \varphi(u)]^{\frac{p-1}{q-1}} \le F(u^n) + \varphi(u^n) - F(u) - \varphi(u),$$

and, like in [5], for instance, we have

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) \\ &\le [(n+1)C_2 + (F(u^0) + \varphi(u^0) - F(u) - \varphi(u))^{\frac{q-p}{q-1}}]^{\frac{q-1}{q-p}}, \end{aligned} \tag{3.35}$$

where C_2 is given in (3.20). Equation (3.35) is another form of (3.14). □

4. SUBSPACE CORRECTION ALGORITHMS FOR QUASI-VARIATIONAL INEQUALITIES

Let $\varphi : V \times V \to \mathbf{R}$ be a functional such that, for any $u \in V$, $\varphi(u, \cdot) : V \to \mathbf{R}$ is convex and lower semicontinuous. We assume that $F + \varphi$ is coercive in the sense that

$$F(v) + \varphi(u, v) \to \infty, \text{ as } \|v\| \to \infty, \ v \in K, \text{ for any } u \in K \tag{4.1}$$

if K is not bounded.

In this section we assume that $p = q = 2$ in (2.1) and (2.2). Also, we assume that for any $M > 0$ there exists $c_M > 0$ such that

$$\begin{aligned} &|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \\ &\le c_M\|v_1 - v_2\|\|w_1 - w_2\| \end{aligned} \tag{4.2}$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $\|v_1\|,\ \|v_2\|,\ \|w_1\|\ \|w_2\| \le M$. As in the previous section, we introduce additional conditions concerning φ. In the multiplicative case, we suppose that

$$\begin{aligned} &\textstyle\sum_{i=1}^m[\varphi(u, w + \sum_{j=1}^{i-1} w_{1j} + u_{1i}) - \varphi(u, w + \sum_{j=1}^{i-1} w_{1j} + w_{1i})] \\ &+\varphi(u, w + w_1 + u_0) - \varphi(u, w + w_1 + w_0) \\ &\le \varphi(u, v) - \varphi(u, w + \textstyle\sum_{i=1}^m w_{1i} + w_0) \end{aligned} \tag{4.3}$$

for $u, w \in K$, $u_{1i}, w_{1i} \in V_{1i}$ and $u_0, w_0 \in V_0$ satisfying Assumption 2.2, and $w_1 = \sum_{j=1}^m w_{1j}$. Also, for the additive case, we suppose that

$$\sum_{i=1}^m \varphi(u, w + u_{1i}) + \varphi(u, w + u_0) \le m\varphi(u, w) + \varphi(u, u) \tag{4.4}$$

for any $u, w \in K$, $u_{1i} \in V_{1i}$, $i = 1, \dots, m$, and $u_0 \in V_0$ which satisfy Assumption 2.3. We notice that conditions (4.1), (4.3) and (4.4) imposed on the second argument of $\varphi(u, v)$ are similar with conditions (3.1), (3.2) and (3.3) of $\varphi(v)$ in the case of the variational inequalities of the second kind.

Now, we consider the quasi-variational inequality

$$u \in K : \langle F'(u), v - u\rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \text{ for any } v \in K. \tag{4.5}$$

Since φ is convex in the second variable and F is differentiable and satisfies (2.1), problem (4.5) is equivalent with the minimization problem

$$u \in K : F(u) + \varphi(u, u) \leq F(v) + \varphi(u, v), \text{ for any } v \in K. \tag{4.6}$$

As in [5], we can show that problem (4.5) has a unique solution if there exists a constant $\varkappa < 1$ such that

$$\frac{c_M}{\alpha_M} \leq \varkappa \text{ for any } M > 0. \tag{4.7}$$

In view of (2.4) we see that, for a given $M > 0$ such that the solution u of (4.5) satisfies $\|u\| \leq M$, we have

$$\begin{aligned} &\tfrac{\alpha_M}{2}\|v - u\|^2 \leq F(v) - F(u) + \varphi(u, v) - \varphi(u, u), \\ &\text{for any } v \in K, \ \|v\| \leq M. \end{aligned} \tag{4.8}$$

To solve problem (4.5), we now introduce two algorithms, one of multiplicative type and another one of the additive type. The convergence conditions of these algorithms are independent of the number of subdomains and, as we already said, their iterations have an optimal computing complexity. In the following algorithm, which is of multiplicative type, we keep unchanged the first argument of φ for several iterations.

Algorithm 4.1. We start the algorithm with an arbitrary $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we write $\tilde{u}^n = u^n$ and carry out the following two steps:

1. We perform $\kappa \geq 1$ multiplicative iterations, keeping the first argument of φ equal with u^n. We start with $\tilde{u}^n$ and having $\tilde{u}^{n+k-1}$ at iteration $1 \leq k \leq \kappa$, we successively calculate level corrections and compute $\tilde{u}^{n+k}$:

– at the level 1 we construct the convex set K_1 as in Assumption 2.1 with $w = \tilde{u}^{n+k-1}$. Then, we first write $w_1^k = 0$, and, for $i = 1, \dots, m$, we successively calculate $w_{1i}^{k+1} \in V_{1i}$, $w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1} \in K_1$, the solution of the inequalities

$$\begin{aligned} &\langle F'(\tilde{u}^{n+k-1} + w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1}), v_{1i} - w_{1i}^{k+1}\rangle \\ &+\varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+\frac{i-1}{m}} + v_{1i}) \\ &-\varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1}) \geq 0, \end{aligned} \tag{4.9}$$

for any $v_{1i} \in V_{1i}$, $w_1^{k+\frac{i-1}{m}} + v_{1i} \in K_1$, and write $w_1^{k+\frac{i}{m}} = w_1^{k+\frac{i-1}{m}} + w_{1i}^{k+1}$.

– at the level 0, we construct the convex set K_0 as in Assumption 2.1 with $w = \tilde{u}^{n+k-1}$ and $w_1 = w_1^{k+1}$. Then, we calculate $w_0^{k+1} \in K_0$, the solution of the inequality

$$\begin{aligned} &\langle F'(\tilde{u}^{n+k-1} + w_1^{k+1} + w_0^{k+1}), v_0 - w_0^{k+1} \rangle \\ &+\varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+1} + v_0) \\ &-\varphi(u^n, \tilde{u}^{n+k-1} + w_1^{k+1} + w_0^{k+1}) \geq 0, \end{aligned} \tag{4.10}$$

for any $v_0 \in K_0$.

– we write $\tilde{u}^{n+k} = \tilde{u}^{n+k-1} + w_1^{k+1} + w_0^{k+1}$.

2. We write $u^{n+1} = \tilde{u}^{n+\kappa}$.

In the second algorithm, which is of additive type, we also keep unchanged for several iterations the first argument of φ.

Algorithm 4.2. We start the algorithm with an $u^0 \in K$. Assuming that at iteration $n \geq 0$ we have $u^n \in K$, we write $\tilde{u}^n = u^n$ and carry out the following two steps:

1. We perform $\kappa \geq 1$ additive iterations, keeping the first argument of φ equal with u^n. We start with $\tilde{u}^n$ and having $\tilde{u}^{n+k-1}$ at iteration $1 \leq k \leq \kappa$, we simultaneously calculate level corrections and compute $\tilde{u}^{n+k}$:

– we construct the convex sets K_1 and K_0 as in Assumption 2.1 with $w = \tilde{u}^{n+k-1}$ and $w_1 = 0$,

– we simultaneously calculate:

(*a*) $w_{1i}^{k+1} \in V_{1i} \cap K_1$, the solutions of the inequalities

$$\begin{aligned} &\langle F'(\tilde{u}^{n+k-1} + w_{1i}^{k+1}), v_{1i} - w_{1i}^{k+1} \rangle \\ &+\varphi(u^n, \tilde{u}^{n+k-1} + v_{1i}) - \varphi(u^n, \tilde{u}^{n+k-1} + w_{1i}^{k+1}) \geq 0, \end{aligned} \tag{4.11}$$

for any $v_{1i} \in V_{1i} \cap K_1$, write $w_1^{k+1} = \sum_{i=1}^m w_{1i}^{k+1}$, and

(*b*) $w_0^{k+1} K_0$, the solution of the inequality

$$\begin{aligned} &\langle F'(\tilde{u}^{n+k-1} + w_0^{k+1}), v_0 - w_0^{k+1} \rangle \\ &+\varphi(u^n, \tilde{u}^{n+k-1} + v_0) - \varphi(u^n, \tilde{u}^{n+k-1} + w_0^{k+1}) \geq 0, \end{aligned} \tag{4.12}$$

for any $v_0 \in K_0$.

– we write $\tilde{u}^{n+k} = \tilde{u}^{n+k-1} + \frac{r}{m+1}(w_1^{k+1} + w_0^{k+1})$, with a fixed $0 < r \leq 1$.

2. We write $u^{n+1} = \tilde{u}^{n+\kappa}$

Now, we can prove

THEOREM 4.1. *Let V be a reflexive Banach, $V_0, V_{11}, \cdots, V_{1m}$ some closed subspaces of V, and K a non empty closed convex subset of V which satisfies Assumption 2.1, Assumption 2.2 when we apply Algorithms 4.1, and Assumption 2.3 in the case of Algorithms 4.2. Also, we assume that F is Gâteaux* *differentiable and satisfies (2.1) and (2.2) with* $p = q = 2$, *the functional* φ *is convex and lower semicontinuous in the second variable, satisfies (4.2), (4.3) for Algorithm 4.1, (4.4) for Algorithm 4.2, and* $F + \varphi$ *satisfies the coercivity condition (4.1) if* K *is not bounded. Let*

$$M = \sup\{||v|| \,:\, F(v) + \varphi(u, v) \le F(u^0) + \varphi(u, u^0)\} \tag{4.13}$$

where u *is the solution of problem (4.5) and* u^0 *is its initial approximation in Algorithms 4.1 or 4.2. On these conditions, if*

$$\frac{c_M}{\alpha_M} < \frac{1}{2} \text{ for any } M > 0 \tag{4.14}$$

and κ *satisfies*

$$\left(\frac{C_1}{C_1 + 1}\right)^{\kappa} < \frac{1 - 2\frac{c_M}{\alpha_M}}{1 + 3\frac{c_M}{\alpha_M} + 4\frac{c_M^2}{\alpha_M^2} + \frac{c_M^3}{\alpha_M^3}} \tag{4.15}$$

then, the norms of the approximations of the solution u *of problem (4.5) obtained from these algorithms are bounded by* M, *the two algorithms are convergent and we have the following error estimations:*

$$\begin{aligned} &F(u^{n+1}) + \varphi(u, u^{n+1}) - F(u) - \varphi(u, u) \\ &\le [2\tfrac{c_M}{\alpha_M} + (\tfrac{C_1}{C_1+1})^{\kappa}(1 + 3\tfrac{c_M}{\alpha_M} + 4\tfrac{c_M^2}{\alpha_M^2} + \tfrac{c_M^3}{\alpha_M^3})]^n \\ &\cdot[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)] \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\|u^n - u\|^2 \le \tfrac{2}{\alpha_M}[2\tfrac{c_M}{\alpha_M} + (\tfrac{C_1}{C_1+1})^{\kappa}(1 + 3\tfrac{c_M}{\alpha_M} + 4\tfrac{c_M^2}{\alpha_M^2} + \tfrac{c_M^3}{\alpha_M^3})]^n \\ &\cdot[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)]. \end{aligned} \tag{4.17}$$

The constant $C_1 > 0$ *depends on the functionals* F *and* φ, *the solution* u, *the initial approximation* u^0, m, *and the constant* C_0.

Remark 4.1. Constant C_1 can be written as

$$C_1 = (m + 1)[(1 + 2C_0)\frac{\beta_M}{\frac{\alpha_M}{2}} + C_0^2(\frac{\beta_M}{\frac{\alpha_M}{2}})^2] \tag{4.18}$$

for Algorithm 4.1, and

$$\begin{aligned} C_1 = &\tfrac{m+1}{r}[1 - \tfrac{r}{m+1} + (1 + C_0)(m + 1)\frac{\beta_M}{\frac{\alpha_M}{2}} \\ &+ C_0^2(m + 1)(\frac{\beta_M}{\frac{\alpha_M}{2}})^2] \end{aligned} \tag{4.19}$$

for Algorithm 4.2.

Proof of Theorem 4.1. First, we see that in view of (4.8), (4.17) can be obtained from (4.16).

As in Theorem 3.1, the existence of $M > 0$ defined in (4.13) follows from the coercivity condition (4.1). Now, we show that this M has the properties in the statement of the theorem. In this proof, equations (2.1), (2.2) and (4.2) will be used with u, v, v_1, v_2, w_1 and w_2 replaced only with the solution u of problem (4.5) or its approximations obtained from Algorithms 4.1 or 4.2. Let us assume that M_n is the maximum of the norms of these approximations obtained after n iterations. With this M_n, we shall get that error estimation (4.16) holds until the iteration n. Even if C_1 depends on M_n, this error estimation implies $F(u^n)+\varphi(u,u^n) \le F(u^0)+\varphi(u,u^0)$. Moreover, using the minimization problems equivalent with the inequalities in the algorithms we get that the other approximations of u satisfy a similar equation, *i.e.* $M_n \le M$.

For a fixed $n \ge 0$, let us consider the problem

$$\tilde{u} \in K : \langle F'(\tilde{u}), v-\tilde{u}\rangle + \varphi(\tilde{u}^n, v) - \varphi(\tilde{u}^n, \tilde{u}) \ge 0, \text{ for any } v \in K, \tag{4.20}$$

where $\tilde{u}^n = u^n \in K$ is the approximation obtained from one of Algorithms 4.1 or 4.2 after n iterations. Variational inequality of the second kind (4.20) is similar with (3.4), and, in view of (4.1), (4.3) and (4.4), $\varphi(\tilde{u}^n, v)$ satisfies (3.1)–(3.3). Therefore, in view of Theorem 3.1, we have

$$\begin{aligned}&F(\tilde{u}^{n+\kappa}) + \varphi(\tilde{u}^n, \tilde{u}^{n+\kappa}) - F(\tilde{u}) - \varphi(\tilde{u}^n, \tilde{u})\\ &\le (\tfrac{C_1}{C_1+1})^\kappa[F(\tilde{u}^n) + \varphi(\tilde{u}^n, \tilde{u}^n) - F(\tilde{u}) - \varphi(\tilde{u}^n, \tilde{u})]\end{aligned}$$

or

$$\begin{aligned}&F(u^{n+1}) + \varphi(u^n, u^{n+1}) - F(\tilde{u}) - \varphi(u^n, \tilde{u})\\ &\le (\tfrac{C_1}{C_1+1})^\kappa[F(u^n) + \varphi(u^n, u^n) - F(\tilde{u}) - \varphi(u^n, \tilde{u})]\end{aligned} \tag{4.21}$$

Using (3.16) and (3.19), constant C_1 can be written as in (4.18) for Algorithm 4.1 and as in (4.19) in the case of Algorithm 4.2. From (2.4), (4.20) and (4.2), we have

$$\begin{aligned}&F(\tilde{u}) + \varphi(u, \tilde{u}) - F(u) - \varphi(u,u) + \tfrac{\alpha_M}{2}||\tilde{u}-u||^2\\ &\le \langle F'(\tilde{u}), \tilde{u}-u\rangle + \varphi(u^n, \tilde{u}) - \varphi(u^n, u)\\ &+\varphi(u,\tilde{u}) - \varphi(u,u) - \varphi(u^n,\tilde{u}) + \varphi(u^n, u)\\ &\le \varphi(u,\tilde{u}) - \varphi(u,u) - \varphi(u^n,\tilde{u}) + \varphi(u^n, u)\\ &\le c_M||u-u^n||||u-\tilde{u}|| \le \tfrac{c_M}{2}||u-u^n||^2 + \tfrac{c_M}{2}||u-\tilde{u}||^2\end{aligned}$$

Using again (2.4) and (4.5), we get

$$\begin{aligned}&\tfrac{\alpha_M}{2}||u-u^n||^2 \le \langle F'(u), u-u^n\rangle + \varphi(u,u) - \varphi(u,u^n)\\ &+F(u^n) - F(u) - \varphi(u,u) + \varphi(u,u^n)\\ &\le F(u^n) + \varphi(u,u^n) - F(u) - \varphi(u,u)\end{aligned} \tag{4.22}$$

From the last two equations, in view of (4.14), we get

$$\begin{aligned}&F(\tilde{u}) + \varphi(u,\tilde{u}) - F(u) - \varphi(u,u)\\ &\le \tfrac{c_M}{\alpha_M}[F(u^n) + \varphi(u,u^n) - F(u) - \varphi(u,u)]\end{aligned} \tag{4.23}$$

Now, we have

$$
\begin{aligned}
&F(u^{n+1})+\varphi(u,u^{n+1})-F(u)-\varphi(u,u)\\
&=F(u^{n+1})+\varphi(u^n,u^{n+1})-F(\tilde u)-\varphi(u^n,\tilde u)\\
&+F(\tilde u)+\varphi(u,\tilde u)-F(u)-\varphi(u,u)\\
&+\varphi(u,u^{n+1})-\varphi(u^n,u^{n+1})+\varphi(u^n,\tilde u)-\varphi(u,\tilde u)
\end{aligned}
\tag{4.24}
$$

But, in view of (4.21), we get

$$
\begin{aligned}
&F(u^{n+1})+\varphi(u^n,u^{n+1})-F(\tilde u)-\varphi(u^n,\tilde u)\\
&\le (\tfrac{C_1}{C_1+1})^\kappa[F(u^n)+\varphi(u^n,u^n)-F(\tilde u)-\varphi(u^n,\tilde u)]\\
&=(\tfrac{C_1}{C_1+1})^\kappa[F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)\\
&+F(u)+\varphi(u,u)-F(\tilde u)-\varphi(u,\tilde u)]\\
&+(\tfrac{C_1}{C_1+1})^\kappa[\varphi(u^n,u^n)-\varphi(u^n,\tilde u)-\varphi(u,u^n)+\varphi(u,\tilde u)]
\end{aligned}
\tag{4.25}
$$

It follows from (4.24), (4.25), (4.23) and (4.2) that

$$
\begin{aligned}
&F(u^{n+1})+\varphi(u,u^{n+1})-F(u)-\varphi(u,u)\\
&\le (\tfrac{C_1}{C_1+1})^\kappa[F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]\\
&+[1-(\tfrac{C_1}{C_1+1})^\kappa][F(\tilde u)+\varphi(u,\tilde u)-F(u)-\varphi(u,u)]\\
&+(\tfrac{C_1}{C_1+1})^\kappa[\varphi(u^n,u^n)-\varphi(u^n,\tilde u)-\varphi(u,u^n)+\varphi(u,\tilde u)]\\
&+\varphi(u,u^{n+1})-\varphi(u^n,u^{n+1})+\varphi(u^n,\tilde u)-\varphi(u,\tilde u)\\
&\le [(\tfrac{C_1}{C_1+1})^\kappa-\tfrac{c_M}{\alpha_M}(\tfrac{C_1}{C_1+1})^\kappa+\tfrac{c_M}{\alpha_M}][F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]\\
&+c_M(\tfrac{C_1}{C_1+1})^\kappa||u^n-u||||u^n-\tilde u||+c_M||u^n-u||||u^{n+1}-\tilde u||
\end{aligned}
$$

and we have

$$
\begin{aligned}
&(\tfrac{C_1}{C_1+1})^\kappa||u^n-u||||u^n-\tilde u||+||u^n-u||||u^{n+1}-\tilde u||\\
&\le (\tfrac{C_1}{C_1+1})^\kappa(||u^n-u||^2+||u^n-u||||u-\tilde u||)+||u^n-u||||u^{n+1}-\tilde u||\\
&\le \tfrac12[3(\tfrac{C_1}{C_1+1})^\kappa+1]||u^n-u||^2+\tfrac12(\tfrac{C_1}{C_1+1})^\kappa||u-\tilde u||^2+\tfrac12||u^{n+1}-\tilde u||^2
\end{aligned}
$$

Therefore, from the last two equation, we have

$$
\begin{aligned}
&F(u^{n+1})+\varphi(u,u^{n+1})-F(u)-\varphi(u,u)\\
&\le [(\tfrac{C_1}{C_1+1})^\kappa-\tfrac{c_M}{\alpha_M}(\tfrac{C_1}{C_1+1})^\kappa+\tfrac{c_M}{\alpha_M}]\\
&\cdot[F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]\\
&+\tfrac{c_M}{2}[3(\tfrac{C_1}{C_1+1})^\kappa+1]||u^n-u||^2\\
&+\tfrac{c_M}{2}(\tfrac{C_1}{C_1+1})^\kappa||u-\tilde u||^2+\tfrac{c_M}{2}||u^{n+1}-\tilde u||^2
\end{aligned}
\tag{4.26}
$$

From (2.4), (4.5) and (4.23) we have

$$
\begin{aligned}
&\tfrac{\alpha_M}{2}||\tilde u-u||^2\le\langle F'(u),u-\tilde u\rangle+\varphi(u,u)-\varphi(u,\tilde u)\\
&+F(\tilde u)+\varphi(u,\tilde u)-F(u)-\varphi(u,u)\\
&\le F(\tilde u)+\varphi(u,\tilde u)-F(u)-\varphi(u,u)\\
&\le \tfrac{c_M}{\alpha_M}[F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]
\end{aligned}
\tag{4.27}
$$

In view of (2.4), (4.20), (4.25) and (4.2), we get

$$\begin{aligned}
&\frac{\alpha_M}{2}||u^{n+1}-\tilde{u}||^2 \\
&\le \langle F'(\tilde{u}), \tilde{u}-u^{n+1}\rangle + \varphi(u^n,\tilde{u}) - \varphi(u^n,u^{n+1}) \\
&+F(u^{n+1}) + \varphi(u^n,u^{n+1}) - F(\tilde{u}) - \varphi(u^n,\tilde{u}) \\
&\le (\tfrac{C_1}{C_1+1})^\kappa [F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u) \\
&+F(u)+\varphi(u,u)-F(\tilde{u})-\varphi(u,\tilde{u})] \\
&+c_M(\tfrac{C_1}{C_1+1})^\kappa ||u^n-u|| ||u^n-\tilde{u}||
\end{aligned}$$

As previously, using (4.22) and (4.27), we get

$$\begin{aligned}
&||u^n-u|| ||u^n-\tilde{u}|| \le \tfrac{3}{2}||u^n-u||^2 + \tfrac{1}{2}||u-\tilde{u}||^2 \\
&\le [\tfrac{3}{\alpha_M}+\tfrac{c_M}{\alpha_M^2}][F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]
\end{aligned}$$

From the last two equations, since $F(u)+\varphi(u,u)-F(\tilde{u})-\varphi(u,\tilde{u}) \le 0$, we have

$$\begin{aligned}
&\frac{\alpha_M}{2}||u^{n+1}-\tilde{u}||^2 \le (\tfrac{C_1}{C_1+1})^\kappa [1+3\tfrac{c_M}{\alpha_M}+\tfrac{c_M^2}{\alpha_M^2}] \\
&\cdot[F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]
\end{aligned} \tag{4.28}$$

Finally, from (4.26), (4.22), (4.27) and (4.28), we get

$$\begin{aligned}
&F(u^{n+1})+\varphi(u,u^{n+1})-F(u)-\varphi(u,u) \\
&\le [2\tfrac{c_M}{\alpha_M}+(\tfrac{C_1}{C_1+1})^\kappa(1+3\tfrac{c_M}{\alpha_M}+4\tfrac{c_M^2}{\alpha_M^2}+\tfrac{c_M^3}{\alpha_M^3})] \\
&\cdot[F(u^n)+\varphi(u,u^n)-F(u)-\varphi(u,u)]
\end{aligned}$$

i.e., (4.16) holds. □

Remark 4.2. 1. Theorem 4.1 shows that if the convergence condition (4.14) is satisfied and the number κ of the intermediate iterations is sufficiently large then Algorithms 4.1 and 4.2 converge and we have error estimation (4.17).

2. Convergence condition (4.14) is a little more restrictive than the existence and uniqueness condition of the solution (4.7) but they are of the same type.

3. Contrarily to the convergence condition of the algorithms in [5], (4.14) does not depend on the number m of subdomains.

5. CONVERGENCE RATE OF THE TWO-LEVEL METHODS

Algorithms in the previous sections can be viewed as two-level Schwarz methods in a subspace correction variant if we use the finite element spaces. The convergence rates given in Theorems 3.1 and 4.1 depend on the functionals F and φ, the number m of the subspaces and the constant C_0 introduced in Assumption 2.2 or 2.3. Since, in the multiplicative methods, the number of subspaces can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other, and we can use multiprocessor machines for the additive methods, we can conclude that our convergence rates essentially depend on the constant C_0.

We prove in this section that Assumptions 2.2 and 2.3 as well as conditions (3.2), (3.3), (4.3) and (4.4) hold for closed convex sets K of two-obstacle type for which we construct level convex sets K_1 and K_2 as in Assumption 2.1. Also, we are able to explicitly write the dependence of C_0 on the domain decomposition and mesh parameters. Therefore, from Theorems 3.1 and 4.1, we can conclude that the two-level methods globally converge for variational inequalities of the second kind and quasi-variational inequalities. Moreover, the introduced methods have an optimal computing complexity per iteration, in view of the dependence of C_0 on the mesh and domain decomposition parameters, the convergence rate is optimal for the variational inequalities of the second kind. This convergence rate depends very weakly on the mesh and domain decomposition parameters, and it is even independent of them for some particular choices.

We consider two simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ of the domain $\Omega \subset \mathbf{R}^d$ of mesh sizes h and H, respectively. The mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, and we assume that both the families, of fine and coarse meshes, are regular (see [9], p. 124, for instance). We assume that the domain Ω is decomposed as

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{5.1}$$

and that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. The overlapping parameter of this decomposition will be denoted by δ. In addition, we suppose that there exists a constant C, independent of both meshes, such that the diameter of the connected components of each Ω_i is less than CH. We point out that the domain Ω may be different from $\Omega_0 = \cup_{\tau \in \mathcal{T}_H} \tau$, but we assume that if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$ then it also lies on $\partial\Omega$, and there exists a constant C, independent of both meshes, such that $\operatorname{dist}(x, \Omega_0) \le CH$ for any node x of $\mathcal{T}_h$.

We consider the piecewise linear finite element space

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_h,\ v = 0 \text{ on } \partial\Omega\}, \tag{5.2}$$

and also, for $i = 1, \ldots, m$, let

$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega \backslash \Omega_i\} \tag{5.3}$$

be the subspaces of V_h corresponding to the domain decomposition $\Omega_1, \ldots, \Omega_m$. We also introduce the continuous, piecewise linear finite element space corresponding to the H-level,

$$V_H^0 = \{v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0\}, \tag{5.4}$$

where the functions v are extended with zero in $\Omega\backslash\Omega_0$. The spaces V_h and V_h^i, $i = 1, \ldots, m$, and V_H^0 are considered as subspaces of $W^{1,s}$, for some fixed $1 < s < \infty$. We denote by $\|\cdot\|_{0,s}$ the norm in L^s, and by $\|\cdot\|_{1,s}$ and $|\cdot|_{1,s}$ the norm and seminorm in $W^{1,s}$, respectively.

We consider problems (3.4) and (4.5) in the space $V = V_h$ with the convex set of the form

$$K = \{v \in V_h : a \leq v \leq b\}, \tag{5.5}$$

where $a,\ b \in V_h,\ a \leq b$. The two-level methods are obtained from the algorithms in the previous sections with $V_0 = V_H^0$, $V_{11} = V_h^1, \ldots, V_{1m} = V_h^m$.

In general, the functionals φ in the original problems do not satisfy the technical conditions (3.2), (3.3) and (4.4), (4.3). For this reason, they have been replaced in [5] by approximations, obtained by numerical quadrature in V_h. In the case of the variational inequalities of the second kind, we assume that the functional φ is of the form

$$\varphi(v) = \sum_{k \in \mathcal{N}_h} s_k(h)\phi(v(x_k)) \tag{5.6}$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is a continuous and convex function, $\mathcal{N}_h$ is the set of nodes of the mesh partition $\mathcal{T}_h$, and $s_k(h) \geq 0$, $k \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. For the quasi-variational inequalities, we assume that the functional φ is of the form

$$\varphi(u, v) = \sum_{k \in \mathcal{N}_h} s_k(h)\phi(u, v(x_k)) \tag{5.7}$$

where $\phi : V_h \times \mathbf{R} \to \mathbf{R}$ is continuous, and, as above, $s_k(h) \geq 0$, $k \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. Also, we assume that $\phi(u, \cdot) : \mathbf{R} \to \mathbf{R}$ is convex for any $u \in V_h$.

To verify that Assumptions 2.1–2.3, conditions (3.2) and (3.3) (for functionals φ of the form (5.6)) and (4.4) and (4.3) (for the functionals φ in (5.7)) hold for the convex set in (5.5), we use the nonlinear interpolation operator $I_H : V_h \to V_H^0$ which has been introduced in [2].

Now, we define the level convex sets K_1 and K_0, satisfying Assumption 2.1. Let K be the convex set defined in (5.5), and $w \in K$. We consider

$$\begin{aligned} &K_1 = [a_1, b_1],\ a_1 = a - w,\ b_1 = b - w, \\ &K_0 = [a_0, b_0],\ a_0 = I_H(a_1 - w_1),\ b_0 = I_H(b_1 - w_1) \end{aligned} \tag{5.8}$$

where w_1 has been chosen in K_1. Similar level convex sets have been constructed in [6] for a multilevel method applied to the constrained minimization of differentiable functionals. The following proposition is the two-level variant of Proposition 3.1 in [6].

PROPOSITION 5.1. *Assumption 2.1 holds for the convex sets K_1 and K_0 defined in (5.8) for any $w \in K$ and $w_1 \in K_1$.*

Now, let us consider $u, w \in K$ and define

$$u_1 = u - w - I_H(u - w - w_1) \text{ and } u_0 = I_H(u - w - w_1). \tag{5.9}$$

where $w_1 \in K_1$. The following result is a particular case of Lemmas 3.2 and 3.3 in [6],

LEMMA 5.1. *If K_1 and K_2 are defined in (5.8), and u_1 and u_2 are defined in (5.9), then*

$$u_1 \in K_1,\ u_0 \in K_0 \textit{ and } u - w = u_1 + u_0 \tag{5.10}$$

and

$$\begin{aligned} &|u_0|_{1,s},\ |u_1|_{1,s} \le CC_{d,s}(H,h)[|w_1|_{1,s} + |u - w|_{1,s}], \\ &||u_1||_{0,s} \le ||w_1||_{0,s} + CHC_{d,s}(H,h)[|w_1|_{1,s} + |u - w|_{1,s}] \\ &||u_0||_{0,s} \le C[||u - w||_{0,s} + ||w_1||_{0,s}]. \end{aligned} \tag{5.11}$$

where

$$C_{d,s}(H,h) = \begin{cases} 1 & \textit{if } d = s = 1 \textit{ or} \\ & 1 \le d < s \le \infty \\ (\ln \frac{H}{h} + 1)^{\frac{d-1}{d}} & \textit{if } 1 < d = s < \infty \\ (\frac{H}{h})^{\frac{d-s}{s}} & \textit{if } 1 \le s < d < \infty, \end{cases} \tag{5.12}$$

To prove that Assumption 2.2 holds, we associate to the decomposition (5.1) of Ω some functions $\theta_i \in C(\bar{\Omega}_i)$, $\theta_i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$, such that

$$\begin{aligned} &0 \le \theta_i \le 1 \text{ on } \Omega,\ \theta_i = 0 \text{ on } \cup_{j=i+1}^m \Omega_j \backslash \Omega_i \text{ and} \\ &\theta_i = 1 \text{ on } \Omega_i \backslash \cup_{j=i+1}^m \Omega_j. \end{aligned} \tag{5.13}$$

Such functions θ_i with the above properties have been introduced in [1] and they are constructed using unity partitions of the domains $\cup_{j=i}^m \Omega_j$, $i = 1, \ldots, m$. Using these functions we define

$$\begin{aligned} &u_{11} = L_h(\theta_1 u_1 + (1 - \theta_1) w_{11}) \text{ and} \\ &u_{1i} = L_h(\theta_i (u_1 - \textstyle\sum_{j=1}^{i-1} u_{1j}) + (1 - \theta_i) w_{1i}), \quad i = 2, \ldots, m, \end{aligned} \tag{5.14}$$

L_h being the P_1-Lagrangian interpolation. Also, to prove that Assumption 2.3 holds, we associate to the decomposition (5.1), a unity partition $\{\theta_i\}_{1 \le i \le m}$, with $\theta_i \in C^0(\bar{\Omega})$, $\theta_i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$,

$$0 \le \theta_i \le 1 \text{ on } \Omega,\ \text{supp } \theta_i \subset \overline{\Omega}_i \text{ and } \textstyle\sum_{i=1}^m \theta_i = 1 \tag{5.15}$$

and write

$$u_{1i} = L_h(\theta^i u_1), \quad i = 1, \ldots, m. \tag{5.16}$$

Since the overlapping size of the domain decomposition is δ, the functions θ_i in (5.13) and (5.15) can be chosen to satisfy

$$|\partial_{x_k}\theta_i| \leq C/\delta, \text{ a.e. in } \Omega, \text{ for any } k = 1, \ldots, d \tag{5.17}$$

As in (5.17), we denote in the following by C a generic constant which does not depend on either the mesh or the decomposition of the domain.

Using the u_0 in (5.9) and u_{1i}, $i = 1, \ldots, m$ in (5.14) or in (5.16) we can prove the following proposition which shows that the convergence rate of the algorithms depends very weakly (through constant C_0 in Assumptions 2.2 and 2.3) on the mesh and domain decomposition parameters and is independent of them if H/δ and H/h are kept constant when $h \to 0$. The result concerning Assumption 2.2 is a particular case of Proposition 3.4 in [6] and the proof for Assumption 2.3 is very similar. Also, the proof of conditions (3.2) and (4.3), for the multiplicative algorithms, is almost identical with that in Proposition 2 in [5]. In the case of the additive algorithms, the proof of conditions (3.3) and (4.4) uses the same techniques and is similar.

PROPOSITION 5.2. *Assumptions 2.2 and 2.3 holds for the convex sets K_1 and K_0 defined in (5.8) with the constant C_0 written as*

$$C_0 = C(m+1)C_{d,s}(H,h)[1 + (m-1)\tfrac{H}{\delta}] \tag{5.18}$$

where C is independent of the mesh and domain decomposition parameters, and $C_{d,s}(H,h)$ is given in (5.12). *Also, conditions (3.2) and (3.3), for functionals φ of the form (5.6), and (4.3) and (4.4), for the functionals φ in (5.7), are satisfied.*

Remark 5.1. In this Section 5, we have assumed that, in the case of the quasi-variational inequalities, the functional φ is of the form (5.7). We notice that the proofs of Proposition 5.2 also hold, if we replace the functional $\varphi(u,v)$ in (5.7) with

$$\varphi(u,v) = \sum_{k\in\mathcal{N}_h} s_k(h)\phi(u(x_k), v(x_k)) \tag{5.19}$$

where $s_k(h) \geq 0$, and $\phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous and convex in the second variable. In general, (5.6), (5.7) or (5.19) represent numerical approximations of some integrals.

The results have referred to problems in $W^{1,s}$ with Dirichlet boundary conditions. We point out that similar results can be obtained for problems in $(W^{1,s})^d$ or problems with mixed boundary conditions.

6. CONCLUSIONS

We have introduced and analyzed two-level methods, one of multiplicative type and another one of additive type, for variational inequalities of the second kind and quasi-variational inequalities whose convex set is of two-obstacle type. First, the methods are introduced as subspace correction methods in a general framework of a reflexive Banach space. In this context, in order to ensure an optimal computing complexity of the iterations, an assumption on the construction of the level convex sets where we look for the corrections is introduced. Also, we introduce two hypotheses, one for the multiplicative algorithm and the other one for the additive algorithm, which refer to the decomposition of the elements of the convex set. These two assumptions contain a constant C_0 which will play an important role in the writing of the convergence rate. We prove that, under these hypotheses, the introduced algorithms are globally convergent and estimate their convergence rates.

In the case of the finite element spaces, the introduced abstract algorithms become two-level Schwarz methods. For these spaces, we show that the previous assumptions hold for two-obstacle convex sets and explicitly write the constant C_0 depending on the mesh and domain decomposition parameters. In this way, we get that convergence rates of the two-level methods we have introduced are almost independent of these parameters. In comparison with the two-level methods introduced in [5], the convergence condition of the new algorithms for quasi-variational inequalities is less restrictive and does not depend on the number of the subdomains in the decomposition of the domain. This fact together with the optimal computing complexity of the algorithms represent the main contribution of the paper. In a forthcoming paper we intend to compare the numerical results obtained with the methods introduced in this paper with those obtained with the methods in [5] for a frictional contact problem.

Acknowledgements. The author acknowledges the partial support of the network GDRI ECO Math for this paper.

REFERENCES

[1] L. Badea, *On the Schwarz alternating method with more than two subdomains for nonlinear monotone problems.* SIAM J. Numer. Anal. **28** (1991), *1*, 179–204.

[2] L. Badea, *Convergence rate of a Schwarz multilevel method for the constrained minimization of non-quadratic functionals.* SIAM J. Numer. Anal. **44** (2006), *2*, 449–477.

[3] L. Badea, *Schwarz methods for inequalities with contraction operators.* J. Comput. Appl. Math. **215** (2008), *1*, 196–219.

[4] L. Badea, *Additive Schwarz method for the constrained minimization of functionals in reflexive Banach spaces.* In: U. Langer et al. (Eds.), *Domain decomposition methods in*

science and engineering XVII, LNSE 60, pp. 427–434, Springer, 2008.

[5] L. Badea and R. Krause, *One- and two-level Schwarz methods for inequalities of the second kind and their application to frictional contact problems.* Numer. Math. **120** (2012), *4*, 573–599.

[6] L. Badea, *Global convergence rate of a standard multigrid method for variational inequalities.* IMA J. Numer. Anal. **34** (2014), 197–216.

[7] L. Badea, *Convergence rate of some hybrid multigrid methods for variational inequalities.* J. Numer. Math. **23** (2015), *3*, 195–210.

[8] L. Badea, *Globally convergent multigrid method for variational inequalities with a nonlinear term.* In: T. Dickopf et al. (Eds.), *Domain Decomposition Methods in Science and Engineering XXII*, LNCSE 104, pp. 427–435, Springer, 2016.

[9] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[10] V. Dolean, P. Jolivet and F. Nataf, *An introduction to domain decomposition methods: algorithms, theory, and parallel implementation.* SIAM, 2015.

[11] I. Ekeland and R. Temam, *Convex analysis and variational problems.* North-Holland, Amsterdam, 1976.

[12] E. Gelman and J. Mandel, *On multilevel iterative method for optimization problems.* Math. Program. **48** (1990), 1–17.

[13] R. Glowinski, *Numerical methods for nonlinear variational problems.* Series in Computational Physics, New York, 1984.

[14] R. Glowinski, G.H. Golub, G.A. Meurant and J. Périeux (Eds.), *First Int. Symp. on Domain Decomposition Methods.* SIAM, Philadelphia, 1988.

[15] R. Glowinski, J.L. Lions and R. Trémolières, *Analyse numérique des inéquations variationnelles.* Dunod, 1976.

[16] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities I.* Numer. Math. **69** (1994), 167–184.

[17] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities II.* Numer. Math. **72** (1996), 481–499.

[18] R. Kornhuber, *Adaptive monotone multigrid methods for nonlinear variational problems.* Teubner-Verlag, 1997.

[19] J. Mandel, *A multilevel iterative method for symmetric, positive definite linear complementary problems.* Appl. Math. Optim. **11** (1984), 77–95.

[20] J. Mandel, *Etude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre.* C. R. Acad. Sci. Ser. I **298** (1984), 469–472.

[21] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations.* Oxford Science Publications, 1999.

[22] B.F. Smith, P.E. Bjørstad and William Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Differential Equations.* Cambridge Univ. Press, 1996.

[23] A. Toselli and O. Widlund, *Domains Decomposition Methods – Algorithms and Theory.* Springer-Verlag, Berlin, 2005.

Received 2 April 2018

Institute of Mathematics
of the Romanian Academy
and the Francophone Center for Mathematics
in Bucharest,
P.O. Box 1-764, 014700 Bucharest, Romania
lori.badea@imar.ro