Dedicated to Philippe G. Ciarlet on his 80th birthday, in friendship and admiration

LOCAL EXISTENCE IN FREE INTERFACE PROBLEMS WITH UNDERLYING SECOND-ORDER STEFAN CONDITION

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In this survey we consider free interface problems that do not fall within the class of Stefan problems, as there is no specific condition on the velocity of the interface. At least near some equilibrium, we are able to associate the velocity with a combination of spatial derivatives up to the second order that we define as a *second-order Stefan condition*. Then, we may reformulate the system as a fully nonlinear problem, for which it holds local in time existence and uniqueness.

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1. INTRODUCTION

In the physical literature, free interface evolution problems are often called "*Stefan problems*", with reference to the work of Joseph Stefan who introduced the general class of moving boundary problems around 1890, in relation to the melting of polar ice cap (see, *e.g.*, [15, Section 1.1]). In dimension one, a simple one-phase problem for the temperature distribution in the water and the position of the melting interface reads in non-dimensional variables

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \qquad t>0,\ 0<x<\xi(t), \tag{1.1}$$

subject to a fixed boundary condition at $x=0$. At the free interface $x=\xi(t)$ it holds for $t>0$:

$$u(t,\xi(t)) = 0, \tag{1.2}$$

$$\frac{\partial u}{\partial x}(t,\xi(t)) = -\xi'(t). \tag{1.3}$$

The second condition (1.3) is usually called a *Stefan condition*; we will refer to as a *first-order Stefan condition*, because it involves a first-order spatial derivative of u at the interface. The classical Stefan problem (1.1)–(1.3) has multiple generalizations: multiphase, nonlinear Stefan problems (see [14, 16]), two and three space dimensions, etc. There exists a vast literature on the matter, which among many other relevant references includes the books by Crank [15], Gupta [17], Meïrmanov [28], Rodrigues [29], Rubenšteĭn [30].

However, several free interface evolution problems in physics, in particular in combustion theory, do not a priori belong to the class of Stefan problems, for the simple reason that there is no Stefan condition in the formulation. More specifically, the velocity of the interface is not explicit. In the case of thin flames the temperature's gradient is discontinuous at the free interface (the flame front), which characterizes "combustion type" free boundary conditions (see Caffarelli-Vázquez [13]).

The purpose of this survey is to discuss local in time existence and uniqueness of a solution of some typical free interface problems in which there is no prima facie velocity of the free interface, at least near some equilibrium. We will revisit two models which stem from combustion theory: a simple one-dimensional, one-phase problem [10], and the Near-Equidiffusional flames (NEF) system in the whole space ([7], see also [23–25] for a comprehensive study). Then, we study a general overdetermined problem in a domain Ω_t of $\mathbb{R}^N, N \geq 1$, whose boundary $\partial\Omega_t$ is now the free interface (see [6]). In all these cases, further analysis enables us to relate the interface's velocity to a combination of spatial derivatives up to the second-order. As we called (1.3) a *first-order Stefan condition*, it sounds natural to call *second-order Stefan condition* a similar condition involving second-order spatial derivatives (see, *e.g.*, [4]).

We want to emphasize that first-order and second-order Stefan conditions lead to different mathematical problems. For simplicity, consider the following one-dimensional free interface problem:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \qquad t > 0, \; -\infty < x < \xi(t). \tag{1.4}$$

At the free interface $x = \xi(t)$, we may consider, either a *first-order* (*nonlinear*) *Stefan condition* such as

$$\xi'(t) = f_1\left(\frac{\partial u}{\partial x}(t,\xi(t))\right), \tag{1.5}$$

where $f_1 : \mathbb{R} \to \mathbb{R}$ is a smooth function, or a *second-order* (*nonlinear*) *Stefan condition* of the form:

$$\xi'(t) = f_2\left(\frac{\partial u}{\partial x}(t,\xi(t)), \frac{\partial^2 u}{\partial x^2}(t,\xi(t))\right), \tag{1.6}$$

where f_2 is a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$. Each condition, (1.5) or (1.6), is supplemented by another condition at the free interface, such as $u(t, \xi(t))$ given.

Next, in the coordinates $t' = t$, $x' = x - \xi(t)$, the interface is fixed at $x' = 0$. Equation (1.4) reads:

$$\frac{\partial u}{\partial t'}(t', x') - \frac{d\xi}{dt'}(t')\frac{\partial u}{\partial x'}(t', x') = \frac{\partial^2 u}{\partial x' \partial x'}(t', x'), \quad x' \in (-\infty, 0).$$

Replacing $-\frac{d\xi}{dt'}(t')$ by (1.5), we get

$$\frac{\partial u}{\partial t'}(t', x') = f_1\left(\frac{\partial u}{\partial x}(t', 0)\right)\frac{\partial u}{\partial x'}(t', x') + \frac{\partial^2 u}{\partial x' \partial x'}(t', x'),$$

which is a quasilinear parabolic equation with a nonlocal term; replacing it by (1.6), now it follows that

$$\frac{\partial u}{\partial t'}(t', x') = f_2\left(\frac{\partial u}{\partial x}(t', 0), \frac{\partial^2 u}{\partial x^2}(t', 0)\right)\frac{\partial u}{\partial x'}(t', x') + \frac{\partial^2 u}{\partial x' \partial x'}(t', x'),$$

which is a *fully nonlinear* parabolic equation with a nonlocal term.

Therefore, in contrast to the classical first-order situation which leads to quasilinear problems, system containing a *second-order Stefan condition* may be reformulated as *fully nonlinear* problems of the form:

$$\text{(1.7)} \qquad \begin{cases} \dfrac{\partial w}{\partial t} = \mathcal{L}w + \mathcal{F}(w), \\ \mathcal{B}w = \mathcal{G}(w), \end{cases}$$

supplemented by an ad hoc initial condition at time $t = 0$. Here, $\mathcal{L}$ and $\mathcal{B}$ are linear differential operators, whereas $\mathcal{F}$ and $\mathcal{G}$ are nonlinear functions, see Section 5 for a general framework.

Let us conclude this introduction with two comments.

(i) It is worthwhile noting that the above approach via *second-order Stefan condition* and *fully nonlinear* problems has been also successful in interface problems with no jumps at the free interface. For example, in the case of thick flames, the temperature remains continuously differentiable. In thermo-diffusive model of flame propagation with stepwise temperature kinetics and zero-order reaction (see [2]), the main qualitative feature is that it has two interfaces: the ignition interface where the ignition temperature is attained and the trailing interface where the concentration of deficient reactant reaches zero. For this model, underlying *second-order Stefan conditions* appear naturally at both fronts (see [1, 3]). When converted into free interface problems, similar features may hold in parabolic problems with ignition temperature or with discontinuous nonlinearities (see [1] and the references therein).

(ii) Local existence near equilibria has been initially motivated by stability issues. However, stability in *fully nonlinear* problems goes beyond the scope of this survey. We refer the interested reader to [1], [3–10], [19], [25] and to *e.g.*, [26] for a comprehensive abstract analysis.

2. A ONE-PHASE, ONE-DIMENSIONAL PROBLEM ([10])

As a first step, let us consider a simple, one-phase, one-dimensional problem on the real line, which stems from combustion theory (see [10]). For a two-phase, one-dimensional problem, see [8] (and [9] in dimension two). It reads for $t > 0$:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \qquad -\infty < x < \xi(t), \tag{2.1}$$

subject to conditions at the free interface $x = \xi(t)$. To fix ideas, for $t > 0$, the (normalized) temperature at the flame front is set to 1 and the temperature gradient is equal to $a > 0$:

$$u(t,\xi(t)) = 1, \tag{2.2}$$

$$\frac{\partial u}{\partial x}(t,\xi(t)^-) = a. \tag{2.3}$$

It is easy to see that problem (2.1)–(2.3) admits a traveling wave solution $U(z) = \exp(cz)$, $c = a$, $z < 0$, which travels at velocity $-c$. It is convenient to set $x' = x - \xi(t) = x + ct - s(t)$, $t' = t$, where s is a perturbation of the traveling wave interface $-ct$. In the coordinates (t', x'), the system (2.1)–(2.3) reads:

$$\begin{cases} \dfrac{\partial u}{\partial t'}(t',x') + (c - s'(t'))\dfrac{\partial u}{\partial x'}(t',x') = \dfrac{\partial^2 u}{\partial x'\partial x'}(t',x'), & t' > 0, \quad x' \in (-\infty, 0), \\ u(t',0) = 1, & t > 0, \\ \dfrac{\partial u}{\partial x'}(t,0^-) = a, & t > 0. \end{cases}$$

Omitting the primes, we may also write $u(t,x) = U(x) + v(t,x)$, where v is a perturbation of the traveling wave U. It comes for $t > 0$ and $x < 0$:

$$\frac{\partial v}{\partial t}(t,x) + c\frac{\partial v}{\partial x}(t,x) - \frac{\partial^2 v}{\partial x^2}(t,x) - s'(t)U'(x) = s'(t)\frac{\partial v}{\partial x}(t,x), \tag{2.4}$$

$$v(t,0) = \frac{\partial v}{\partial x}(t,0^-) = 0. \tag{2.5}$$

The next step is the ansatz:

$$v(t,x) = s(t)U'(x) + w(t,x). \tag{2.6}$$

First, it allows the elimination of $-s(t)U'$ in the left-hand side of (2.4). Keeping v for convenience in the right-hand side, the equation for w easily reads:

$$\frac{\partial w}{\partial t}(t,x)+c\frac{\partial w}{\partial x}(t,x)-\frac{\partial^2 w}{\partial x^2}(t,x)=s'(t)\frac{\partial v}{\partial x}(t,x),\qquad x<0. \tag{2.7}$$

Second, the ansatz (2.6) enables to express the (perturbation of the) interface s thanks to the conditions (2.5):

$$s(t)=-\frac{1}{c}w(t,0^-)=-\frac{1}{c^2}\frac{\partial w}{\partial x}(t,0^-),\qquad s'(t)=-\frac{1}{c}\frac{\partial w}{\partial t}(t,0^-). \tag{2.8}$$

Therefore, the "natural" boundary condition associated with w at $x=0^-$ is $\frac{\partial w}{\partial x}(t,0^-)-cw(t,0^-)=0$. The information about $\frac{\partial w}{\partial t}(t,0^-)$ is missing, however the latter can be easily retrieved by evaluating both sides of (2.7) at $x=0^-$:

$$\frac{\partial w}{\partial t}(t,0^-)=-c\frac{\partial w}{\partial x}(t,0^-)+\frac{\partial^2 w}{\partial x^2}(t,0^-), \tag{2.9}$$

taking advantage of the second condition in (2.5).

Summarizing, it follows from (2.8) and (2.9) the underlying *second-order Stefan condition*:

$$s'(t)=\frac{\partial w}{\partial x}(t,0^-)-\frac{1}{c}\frac{\partial^2 w}{\partial x^2}(t,0^-),\qquad t>0.$$

The problem for w can finally be formulated as a *fully nonlinear* parabolic boundary value problem of the form (1.7) where $\mathcal{L}w=\frac{\partial^2 w}{\partial x^2}-c\frac{\partial w}{\partial x}$, $\mathcal{B}w=\frac{\partial w}{\partial x}(0^-)-cw(0^-)$, $\mathcal{G}(w)=0$ and

$$\begin{aligned}\mathcal{F}(w)&=s'\left(sU''+\frac{\partial w}{\partial x}\right)\\&=\left(\frac{\partial w}{\partial x}(\cdot,0^-)-\frac{1}{c}\frac{\partial^2 w}{\partial x^2}(\cdot,0^-)\right)\left(-w(\cdot,0^-)U'+\frac{\partial w}{\partial x}\right).\end{aligned}$$

For such a problem, the following local in time existence and uniqueness result can be proved.

THEOREM 2.1. *Fix any $T>0$ and $\alpha\in(0,1)$. There exists $\rho>0$ such that for every $u_0\in C_b^{2+\alpha}((-\infty,0))$ with $\|u_0\|_{C_b^{2+\alpha}((-\infty,0))}\le\rho$ and satisfying the compatibility conditions*

$$\mathcal{B}(u_0)=\mathcal{G}(u_0)=0,$$

problem (1.7), with $\mathcal{L}$, $\mathcal{B}$, $\mathcal{F}$ and $\mathcal{G}$ as above, admits a unique solution $u\in C_b^{1+\alpha/2,2+\alpha}((0,T)\times(-\infty,0))$ such that $u(0,\cdot)=u_0$. Moreover,

$$\|u\|_{C_b^{1+\alpha/2,2+\alpha}((0,T)\times(-\infty,0))}\le c\|u_0\|_{C_b^{2+\alpha}((-\infty,0))}.$$

In the previous theorem, $C_b^{2+\alpha}((-\infty,0))$ is the space of all twice continuously differentiable functions $v : (-\infty, 0] \to \mathbb{R}$, which are bounded together with their first- and second-order derivatives and with second-order derivative which is α-Hölder continuous in $(-\infty, 0]$. It is normed by setting

$$\|v\|_{C_b^{2+\alpha}((-\infty,0))} = \|v\|_\infty + \|v'\|_\infty + \|v''\|_\infty + [v'']_{C_b^\alpha((-\infty,0))},$$

where $[v'']_{C_b^\alpha((-\infty,0))}$ denotes the classical Hölder seminorm of v''. Similarly, $C_b^{1+\alpha/2,2+\alpha}((0,T) \times (-\infty,0))$ is the parabolic Hölder space of functions $w : [0,T] \times (-\infty, 0] \to \mathbb{R}$ which are once continuously differentiable with respect to the time variable and twice-continuously differentiable with respect to the spatial variable. Moreover, w and its derivatives are bounded, and the time derivative and the second-order spatial derivative are α-Hölder continuous in $[0,T] \times (-\infty, 0]$ with respect to the parabolic distance $d((t,x),(s,y))=\sqrt{|t-s|+|x-y|^2}$. It is normed by setting

$$\begin{aligned}
&\|v\|_{C_b^{1+\alpha/2,2+\alpha}((0,T)\times(-\infty,0))}\\
=&\|w\|_\infty + \left\|\frac{\partial w}{\partial t}\right\|_\infty + \left\|\frac{\partial w}{\partial x}\right\|_\infty + \left\|\frac{\partial^2 w}{\partial x^2}\right\|_\infty\\
&+\left[\frac{\partial w}{\partial t}\right]_{C_b^{\alpha/2,\alpha}((0,T)\times(-\infty,0))} + \left[\frac{\partial^2 w}{\partial x^2}\right]_{C_b^{\alpha/2,\alpha}((0,T)\times(-\infty,0))}
\end{aligned}$$

where $[\cdot]_{C_b^{\alpha/2,\alpha}((0,T)\times(-\infty,0))}$ denotes the Hölder-seminorm with respect to the parabolic distance.

3. THE "NEF" SYSTEM IN $\mathbb{R}^2$ ([7])

A paradigm in premixed flame combustion is the two-dimensional thermo-diffusive model, a simplified combustion model that involves two equations: the heat equation for the system's temperature and the diffusion equation for the deficient reactant's concentration (see, *e.g.*, [11]):

$$\frac{\partial T}{\partial t} = \Delta T + \omega(Y,T), \qquad \frac{\partial Y}{\partial t} = \mathrm{Le}^{-1}\Delta Y - \omega(Y,T). \tag{3.1}$$

The parameter Le is the Lewis number, the reaction rate $\omega(Y,T)$ is given by the Arrhenius law

$$\omega = BY\exp(-E/RT), \tag{3.2}$$

E and R being, respectively, the activation energy and the gas constant. The conventional high activation energy limit converts the reaction rate term into a localized source distributed over a free-interface, $x = \xi(t,y)$, the flame front (see [11, p. 218]).

The Near-Equidiffusional Flame (NEF) model (see Matkowsky-Sivashinsky [27]) combines the limit of large normalized activation energy β with the limit of Lewis number near unity. The NEF theory is characterized ([12]) by the requirements:

(i) $\mathrm{Le}^{-1} = 1 - \beta^{-1}\ell$, where $\ell = O(1)$ is the reduced Lewis number;

(ii) $H = H_f + O(\beta^{-1})$, where $H = Y + T$ is the enthalpy and H_f is its limit as x tends to $-\infty$.

In particular (ii) corresponds to expand T and Y as follows:

$$T = T_0 + \beta^{-1}T_1 + \cdots, \qquad Y = (H_f - T_0) + \beta^{-1}(H_1 - T_1) + \cdots .$$

Under the above assumptions, model (3.1)-(3.2) yields to a free-interface problem for T_0 and H_1. Writing θ and S instead of T_0 and $H_1/2$, $\ell = -2\lambda$, to be consistent with the notation of [31], the NEF system for θ, S and the flame front $x = \xi(t, y)$ reads:

$$\begin{cases} \dfrac{\partial\theta}{\partial t}(t,x,y) = \Delta\theta(t,x,y), & t>0, \quad x<\xi(t,y), \quad y\in\mathbb{R}, \\ \theta(t,x,y) = 1, & t>0, \quad x\geq\xi(t,y), \quad y\in\mathbb{R}, \\ \dfrac{\partial S}{\partial t}(t,x,y) = \Delta S(t,x,y) - \lambda\Delta\theta(t,x,y), & t>0, \quad x\neq\xi(t,y), \quad y\in\mathbb{R}. \end{cases}$$

The functions θ and S are continuous at the front, whereas their normal derivatives satisfy the following jump conditions at the interface:[1]

$$\left[\frac{\partial\theta}{\partial n}\right] = -\exp(S), \qquad \left[\frac{\partial S}{\partial n}\right] = \lambda\left[\frac{\partial\theta}{\partial n}\right]. \tag{3.3}$$

Further, as x tends to $\pm\infty$, the following conditions are prescribed

$$\theta(t,-\infty,y) = S(t,-\infty,y) = S(t,+\infty,y) = 0. \tag{3.4}$$

As it is easily verified, this system admits a planar traveling wave solution, with velocity -1, which reads in the coordinate $z = x + t$:

$$\Theta^0 = e^z, \quad S^0 = \lambda z e^z, \quad z\leq 0, \qquad \Theta^0 = 1, \quad S^0 = 0, \quad z>0.$$

It is standard to fix the interface at the origin by setting $\xi(t,y) = -t + s(t,y)$, $x' = x - \xi(t,y) = z - s(t,y)$. In this new framework:

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + \left(1 - \dfrac{\partial s}{\partial t}\right)\dfrac{\partial\theta}{\partial x'} = \Delta_s\theta, & \text{in } (0,+\infty)\times(-\infty,0)\times\mathbb{R}, \\ \theta = 1, & \text{in } (0,+\infty)\times(0,+\infty)\times\mathbb{R}, \\ \dfrac{\partial S}{\partial t} + \left(1 - \dfrac{\partial s}{\partial t}\right)\dfrac{\partial S}{\partial x'} = \Delta_s S - \lambda\Delta_s\theta, & \text{in } (0,+\infty)\times\mathbb{R}\setminus\{0\}\times\mathbb{R}, \end{cases} \tag{3.5}$$

[1] Here, $[v](t,y) := \lim_{x\to\xi(t,y)^+} v(t,x,y) - \lim_{x\to\xi(t,y)^-} v(t,x,y)$ for a given function v.

where

$$\Delta_s = \left[1 + \left(\frac{\partial s}{\partial y}\right)^2\right]\frac{\partial^2}{\partial x'\partial x'} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2 s}{\partial y^2}\frac{\partial}{\partial x'} - 2\frac{\partial s}{\partial y}\frac{\partial^2}{\partial x'\partial y}.$$

The jump conditions (computed at $x' = 0$) are $[\theta] = [S] = 0$ and

$$\sqrt{1 + \left(\frac{\partial s}{\partial y}\right)^2}\left[\frac{\partial \theta}{\partial x'}\right] = -\exp(S), \qquad \left[\frac{\partial S}{\partial x'}\right] = \lambda\left[\frac{\partial \theta}{\partial x'}\right], \tag{3.6}$$

which follow from (3.3). Omitting the primes, the main step now is the ansatz,

$$\theta = \Theta^0 + s\frac{d\Theta^0}{dx} + v, \qquad S = S^0 + s\frac{dS^0}{dx} + w, \tag{3.7}$$

which, taking advantage of the boundary conditions

$$[\theta] = [\Theta^0] = 0, \qquad \left[\frac{\partial \theta^0}{\partial x}\right] = \left[\frac{d\Theta^0}{dx}\right] = -1,$$

enables us to express the interface s in terms of the trace of v at $x = 0^-$:

$$s(t, y) = [v] = -v(t, 0^-, y). \tag{3.8}$$

Replacing (3.8) in (3.5) and (3.6), we obtain a system in the only unknowns v, w. However, it is convenient to rewrite it in the standard form of a system in $\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2 : \ x < 0\}$, setting $\mathbf{u} = (v, w, h)$ where $h(t, x, y) = w(t, -x, y)$ for $x < 0$ and $y \in \mathbb{R}$. We get

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = \mathcal{L}\mathbf{u} + \mathcal{F}_0(\mathbf{u}) - \dfrac{\partial v}{\partial t}(\cdot, 0, \cdot)\Psi(\mathbf{u}), & \text{in } (0, \infty) \times \mathbb{R}^2_-, \\ \mathcal{B}\mathbf{u} = \mathcal{G}(\mathbf{u}), & \text{in } (0, \infty) \times \mathbb{R}, \end{cases} \tag{3.9}$$

where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L}\mathbf{u} = \mathcal{L}(v, w, h) = \left(\Delta v - \frac{\partial v}{\partial x}, \, \Delta w - \frac{\partial w}{\partial x} - \lambda \Delta v, \, \Delta h + \frac{\partial h}{\partial x}\right),$$

the linear boundary operator $\mathcal{B}$ has three components $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$, defined by

$$\begin{cases} \mathcal{B}_1\mathbf{u} = \lambda v(0, \cdot) - w(0, \cdot) + h(0, \cdot), \\ \mathcal{B}_2\mathbf{u} = \lambda v(0, \cdot) + \lambda\dfrac{\partial v}{\partial x}(0, \cdot) - \dfrac{\partial w}{\partial x}(0, \cdot) - \dfrac{\partial h}{\partial x}(0, \cdot), \\ \mathcal{B}_3\mathbf{u} = v(0, \cdot) + h(0, \cdot) - \dfrac{\partial v}{\partial x}(0, \cdot), \end{cases} \tag{3.10}$$

$\mathcal{F}_0(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u}))$ with

$$f_1(\mathbf{u}) = \left(\frac{\partial v}{\partial y}(0, \cdot)\right)^2\left(\frac{d^2\Theta^0}{dx^2} - v(0, \cdot)\frac{d^3\Theta^0}{dx^3} + \frac{\partial^2 v}{\partial x^2}\right)$$

$$
\begin{aligned}
&+2\frac{\partial v}{\partial y}(0,\cdot)\Big(-\frac{\partial v}{\partial y}(0,\cdot)\frac{d^2\Theta^0}{dx^2}+\frac{\partial^2 v}{\partial x\partial y}\Big)\\
&+\frac{\partial^2 v}{\partial y^2}(0,\cdot)\Big(-v(0,\cdot)\frac{d^2\Theta^0}{dx^2}+\frac{\partial v}{\partial x}\Big),
\end{aligned}
$$

$$
\begin{aligned}
f_2(\mathbf{u}) =&\Big(\frac{\partial v}{\partial y}(0,\cdot)\Big)^2\Big(\frac{d^2 S^0}{dx^2}-v(0,\cdot)\frac{d^3 S^0}{dx^3}+\frac{\partial^2 w}{\partial x^2}\Big)\\
&+2\frac{\partial v}{\partial y}(0,\cdot)\Big(-\frac{\partial v}{\partial y}(0,\cdot)\frac{d^2 S^0}{dx^2}+\frac{\partial^2 w}{\partial x\partial y}\Big)\\
&+\frac{\partial^2 v}{\partial y^2}(0,\cdot)\Big(-v(0,\cdot)\frac{d^2 S^0}{dx^2}+\frac{\partial w}{\partial x}\Big)-\lambda f_1(\mathbf{u}),
\end{aligned}
$$

$$
f_3(\mathbf{u}) =\Big(\frac{\partial v}{\partial y}(0,\cdot)\Big)^2\frac{\partial^2 h}{\partial x^2}-2\frac{\partial v}{\partial y}(0,\cdot)\frac{\partial^2 h}{\partial x\partial y}-\frac{\partial^2 v}{\partial y^2}(0,\cdot)\frac{\partial h}{\partial x}.
$$

Finally,

$$
\Psi(\mathbf{u}) = \Big(-v(0,\cdot)\frac{d^2\Theta^0}{dx^2}+\frac{\partial v}{\partial x},\ -v(0,\cdot)\frac{d^2 S^0}{dx^2}+\frac{\partial w}{\partial x},\ -\frac{\partial h}{\partial x}\Big),
$$

and

$$
\mathcal{G}(\mathbf{u}) = (0,0,g(\mathbf{u})), \qquad g(\mathbf{u}) = 1+h(0,\cdot)-\Big[1+\Big(\frac{\partial v}{\partial y}(0,\cdot)^2\Big)\Big]^{-\frac{1}{2}}e^{h(0,\cdot)}.
$$

However, the differential system in (3.9) contains $\frac{\partial v}{\partial t}(t,0,y)$ in the right-hand side. The main point is that Equation (3.8) yields $\frac{\partial v}{\partial t}(t,0,y) = -\frac{\partial s}{\partial t}(t,y)$. The first equation in (3.9) reads for v and $\frac{\partial v}{\partial x}$ small enough:

$$
\begin{aligned}
\frac{\partial v}{\partial t}(t,x,y) =&\Delta v(t,x,y)-\frac{\partial v}{\partial x}(t,x,y)+(f_1(\mathbf{u}(t\cdot,\cdot))(x,y)\\
&-\frac{\partial v}{\partial t}(t,0,y)\Big(-v(t,0,y)e^x+\frac{\partial v}{\partial x}(t,x,y)\Big),
\end{aligned}
$$

so that if we evaluate it at $x=0$ then we get the formula:

$$
\frac{\partial s}{\partial t}(t,y) = -\frac{\Delta v(t,0,y)-\frac{\partial v}{\partial x}(t,0,y)+(f_1(\mathbf{u}(t,\cdot,\cdot)))(0,y)}{1-v(t,0,y)+\frac{\partial v}{\partial x}(t,0,y)}. \tag{3.11}
$$

Therefore, the velocity of the interface s is expressed in terms of the trace of first- and second-order derivatives of $\mathbf{u}$ at the interface itself. The relation (3.11) is the underlying *second-order Stefan condition.*

Plugging (3.11) in (3.9), we get the fully nonlinear parabolic problem for $\mathbf{u}$ which is of the form (1.7):

(3.12)
$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(t,x,y) = \mathcal{L}\mathbf{u}(t,x,y) + (\mathcal{F}(\mathbf{u}(t,\cdot,\cdot)))(x,y), & t \geq 0, \quad x<0, \quad y\in\mathbb{R}, \\ (\mathcal{B}\mathbf{u}(t,\cdot))(y) = \mathcal{G}(\mathbf{u}(t,\cdot))(y), & t\geq 0, \qquad\qquad y\in\mathbb{R}, \end{cases}$$

with

$$\mathcal{F}(\mathbf{u}) = \mathcal{F}_0(\mathbf{u}) - \frac{\Delta v(0,\cdot) - \frac{\partial v}{\partial x}(0,\cdot) + f_1(\mathbf{u})(0,\cdot)}{1 - v(0,\cdot) + \frac{\partial v}{\partial x}(0,\cdot)}\Psi(\mathbf{u}).$$

We shall set problem (3.12) in spaces of Hölder continuous functions. So, for $\alpha\in(0,1)$, we define

$$X_\alpha = \{\mathbf{u}\in C^\alpha(\mathbb{R}^2_-;\mathbb{R}^3)\cap L^\infty(\mathbb{R}^2_-;\mathbb{R}^3) : \lim_{x\to-\infty}|\mathbf{u}(x,y)| = 0 \text{ for all } y\in\mathbb{R}\},$$

where $C^\alpha(\mathbb{R}^2_-;\mathbb{R}^3)$ is the set of functions $\mathbf{u}:\overline{\mathbb{R}^2_-}\to\mathbb{R}^3$ such that

$$[\mathbf{u}]_{C^\alpha(\mathbb{R}^2_-)} = \sup_{\substack{x,y\in\mathbb{R}^2_- \\ x\neq y}} \frac{|\mathbf{u}(x)-\mathbf{u}(y)|}{|x-y|^\alpha} < +\infty\,.$$

It is endowed with the norm $\|\mathbf{u}\|_{X_\alpha} = \|\mathbf{u}\|_\infty + [\mathbf{u}]_{C^\alpha(\mathbb{R}^2_-)}$.

Similarly, $X_{2+\alpha}$ is the set of all twice continuously differentiable functions $\mathbf{u}:\overline{\mathbb{R}^2_-}\to\mathbb{R}^3$ such that $\dfrac{\partial^2\mathbf{u}}{\partial x^{\gamma_1}\partial y^{\gamma_2}}\in C^\alpha(\mathbb{R}^2_-)$ for every $\gamma_1+\gamma_2=2$ and $\mathbf{u}$ together with its first- and second-order derivatives vanishes as x tends to $-\infty$ for each $y\in\mathbb{R}$. The norm is, as usual,

$$\|\mathbf{u}\|_{X_{2+\alpha}} = \sum_{\gamma_1+\gamma_2=0}^{2}\left\|\frac{\partial^{\gamma_1+\gamma_2}\mathbf{u}}{\partial x^{\gamma_1}\partial y^{\gamma_2}}\right\|_\infty + \sum_{\gamma_1+\gamma_2=2}\left[\frac{\partial^2\mathbf{u}}{\partial x^{\gamma_1}\partial y^{\gamma_2}}\right]_{C^\alpha(\mathbb{R}^2_-)}.$$

For $T>0$ we also introduce the parabolic Hölder spaces $\mathcal{X}_{\alpha/2,\alpha}(0,T)$ and $\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)$ defined by

$$\mathcal{X}_{\alpha/2,\alpha}(0,T) = \Big\{\mathbf{u}:\ \mathbf{u}(t,\cdot)\in X_\alpha \text{ for all } t\in[0,T],\ \sup_{0<t<T}\|\mathbf{u}(t,\cdot)\|_{X_\alpha}<\infty,$$
$$\mathbf{u}(\cdot,x,y)\in C^{\alpha/2}([0,T];\mathbb{R}^3) \text{ for all } x<0,\ y\in\mathbb{R},$$
$$\sup_{x<0,\, y\in\mathbb{R}}\|\mathbf{u}(\cdot,x,y)\|_{C^{\alpha/2}([0,T];\mathbb{R}^3)}<\infty\Big\},$$

$$\|\mathbf{u}\|_{\mathcal{X}_{\alpha/2,\alpha}(0,T)} = \sup_{0<t<T}\|\mathbf{u}(t,\cdot)\|_{X_\alpha} + \sup_{x<0,\ y\in\mathbb{R}}[\mathbf{u}(\cdot,x,y)]_{C^{\alpha/2}([0,T];\mathbb{R}^3)},$$

$$\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T) = \left\{ \mathbf{u} : \frac{\partial^{\gamma_1+\gamma_2+\gamma_3}\mathbf{u}}{\partial t^{\gamma_1}\partial x^{\gamma_2}\partial y^{\gamma_3}} \in \mathcal{X}_{\alpha/2,\alpha}(0,T) \text{ for } 2\gamma_1+\gamma_2+\gamma_3 \le 2 \right\},$$

$$\|\mathbf{u}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)} = \sum_{2\gamma_1+\gamma_2+\gamma_3\le 2} \left\| \frac{\partial^{\gamma_1+\gamma_2+\gamma_3}\mathbf{u}}{\partial t^{\gamma_1}\partial x^{\gamma_2}\partial y^{\gamma_3}} \right\|_{\mathcal{X}_{\alpha/2,\alpha}(0,T)}.$$

We finally state a local in time existence and uniqueness theorem:

THEOREM 3.1. *Fix any $T>0$ and $\alpha\in(0,1)$. There exist ρ, $\rho_0>0$ such that for every $\mathbf{u_0}\in X_{2+\alpha}$ with $\|\mathbf{u_0}\|_{X_{2+\alpha}}\le\rho_0$ and satisfying the compatibility conditions*

$$\mathcal{B}_1\mathbf{u_0} = \mathcal{B}_2\mathbf{u_0} = 0, \qquad \mathcal{B}_3\mathbf{u_0} = g(\mathbf{u_0}), \qquad \mathcal{B}_1(\mathcal{L}\mathbf{u_0} + \mathcal{F}(\mathbf{u_0})) = 0,$$

problem (3.12) *admits a unique solution $\mathbf{u}\in\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)$ such that $\mathbf{u}(0)=\mathbf{u_0}$ and $\|\mathbf{u}\|_{\mathcal{X}_{1+\alpha/2,2+\alpha}(0,T)}\le\rho$.*

4. OVERDETERMINED PARABOLIC PROBLEMS IN $\mathbb{R}^N$ ([6])

In this section, we are going to present a general approach in some more abstract setting. We assume that Ω_t is a bounded domain in $\mathbb{R}^N$, $N\ge1$, with moving boundary $\partial\Omega_t$, $\mathcal{L}$ is a time independent uniformly elliptic operator with smooth coefficients, f and g are given smooth functions defined on the whole of $\mathbb{R}^N$. We consider the problem:

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x) + f(t,x), \qquad t>0,\ x\in\Omega_t, \tag{4.1}$$

with free boundary conditions on $\partial\Omega_t$:

$$u = g_1 \quad \text{and} \quad \frac{\partial u}{\partial n} = g_2. \tag{4.2}$$

We assume that there exists a pair (Ω, U), with Ω bounded, $\partial\Omega$ and U smooth, which is an equilibrium for problem (4.1)–(4.2), *i.e.*,

$$\mathcal{L}U + f = 0 \text{ in } \Omega \text{ with } U = g_1 \text{ and } \frac{\partial U}{\partial\nu} = g_2 \text{ on } \partial\Omega.$$

The main hypothesis is that the functions g_1 and g_2 satisfy a non-degeneracy (or transversality) condition

$$\frac{\partial g_1}{\partial n} - g_2 \ne 0 \text{ at } \partial\Omega. \tag{4.3}$$

As it is usual, we transform the problem on the variable domain Ω_t to a problem on the fixed domain Ω and we denote by y and ν, respectively, the spatial variable and the outward unit normal vector to $\partial\Omega$.

Since we are interested in solutions close to the equilibrium (Ω, U), the first main idea is to look for Ω_t in the form

$$\partial\Omega_t = \{x = y' + s(t, y')\nu(y'),\ y' \in \partial\Omega\}, \tag{4.4}$$

where $s : I \times \partial\Omega \to [-\delta, \delta]$ is a smooth function corresponding to a free interface, I is a suitable interval, containing 0, and $\delta > 0$ is chosen sufficiently small such that the function $\Psi : [-\delta, \delta] \times \partial\Omega \to \mathbb{R}^N$, defined by $\Psi(r, y') = x + r\nu(y')$ for any $r \in [-\delta, \delta]$ and $y' \in \partial\Omega$, is bijective from $I \times \partial\Omega$ to a small neighborhood $\mathcal{O}$ of $\partial\Omega$. Thus, Ω_t lies inside $\mathcal{O}$ for any $t \in I$.

For $y' \in \partial\Omega$ we write

$$\xi(t, y') = s(t, y')\nu(y')$$

and we localize the field ξ near $\partial\Omega$ thanks to a mollifier,[2] as usual, and define the (bijective) coordinate transformation:

$$t = \tau, \qquad x = y + \xi(\tau, y).$$

Then, the time derivative and the spatial gradient transform as

$$\nabla_x = (I + J_\xi^\intercal)^{-1}\nabla_y, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{\partial \xi}{\partial \tau} \cdot (I + J_\xi^\intercal)^{-1}\nabla_y,$$

where $J_\xi^\intercal$ is the transposed Jacobian matrix of ξ, just with respect to the spatial variable.

The transformation of Ω_t to Ω also acts on the equilibrium U itself. Computing the Taylor expansion of U at y, we get

$$U(y + \xi(\tau, y)) = U(y) + (\nabla_y U(y)) \cdot \xi(\tau, y) + R(y, \xi(\tau, y)),$$

where R is a (smooth) remainder,[3] which is quadratic in $\xi(\tau, y)$. This expansion suggests the following splitting (the ansatz) for the unknown function u in the new variables τ and y (which we call $\hat{u}$):

$$\hat{u}(\tau, y) = U(y) + (\nabla_y U(y)) \cdot \xi(\tau, y) + w(\tau, y). \tag{4.5}$$

Using (4.5) to compute $\mathcal{L}u$, we get

$$(\mathcal{L}u)(t, x) = (\mathcal{L}\hat{u})(\tau, y) + (\mathcal{L}_1\hat{u})(\tau, y) + (\mathcal{L}_2\hat{u})(\tau, y), \tag{4.6}$$

where $\mathcal{L}_1$ is an operator whose coefficients depend linearly on ξ and its spatial derivatives up to the second-order, and the operator $\mathcal{L}_2$ has coefficients which can be bounded by suitable multiples of $|\xi|^2 + |D_y\xi|^2 + |D_y^2\xi|^2$.

[2] *i.e.*, we set $\xi(t, y) = \alpha(y)s(t, y')\nu(y')$ for any $t \in I$ and $y \in \mathbb{R}^N$, where $\alpha \in C_c^\infty(\mathcal{O})$ and y' denotes the orthogonal projection of y on $\partial\Omega$.

[3] Throughout the section, we denote still by R (possibly) different remainders which are quadratic in the unknown functions.

Similarly, computing the time derivative of u, using the expansion (4.5), we get

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) =& \frac{\partial \hat{u}}{\partial \tau}(\tau,y) - \frac{\partial \xi}{\partial \tau}(\tau,y)\cdot\nabla_y \hat{u}(\tau,y)\\ &+ \frac{\partial \xi}{\partial \tau}(\tau,y)\cdot J_\xi^{\intercal}(\tau,y)(I+J_\xi^{\intercal}(\tau,y))^{-1}\nabla_y \hat{u}(\tau,y).\end{aligned} \tag{4.7}$$

Replacing (4.6) and (4.7) in (4.1), yields to the following equation for w:

$$\frac{\partial w}{\partial \tau} = \mathcal{L}w + \mathcal{F}_1(y,w,\nabla w, D^2 w, \xi, D\xi, D^2\xi) + \frac{\partial \xi}{\partial \tau}\cdot \mathcal{F}_2(y,\nabla w,\xi,D\xi), \tag{4.8}$$

where

$$\begin{aligned}\mathcal{F}_1(y,w,\nabla w, D^2 w,\xi,D\xi,D^2\xi) &= \mathcal{L}_1 w + \mathcal{L}_2 w + \mathcal{L}_1(\xi\cdot\nabla_y U)\\ &\qquad + \mathcal{L}_2(U+\xi\cdot\nabla_y U) + R(\cdot,\xi(\cdot,\cdot)),\\ \mathcal{F}_2(y,\nabla w,\xi,D\xi) &= \nabla_y w - J_\xi^{\intercal}(I+J_\xi^{\intercal})^{-1}\nabla_y w + \nabla_y(\xi\cdot\nabla_y U)\\ &\qquad - J_\xi^{\intercal}(I+J_\xi^{\intercal})^{-1}\nabla_y(U+\xi\cdot\nabla_y U).\end{aligned}$$

Next, we transform the free boundary conditions in (4.2). At the boundary, formula (4.5) gives

$$g_1(y+s(\tau,y)\nu(y)) = g_1(y) + s(\tau,y)g_2(y) + w(\tau,y), \qquad y\in\partial\Omega.$$

Computing the Taylor expansion centered at y of the left-hand side of the previous formula, we get the equation

$$s(\tau,y)\left(\frac{\partial g_1}{\partial \nu}(y) - g_2(y)\right) + R(y,s(\tau,y)) = w(\tau,y). \tag{4.9}$$

The non-degeneracy assumption (4.3) allows us to make s explicit in terms of w, at least for w small enough, and we obtain

$$s(\tau,y) = \frac{w(\tau,y)}{\frac{\partial g_1}{\partial \nu}(y) - g_2(y)} + R(y,w(\tau,y)). \tag{4.10}$$

Let us now consider the second free boundary condition in (4.2), which involves the normal derivative of u at $\partial\Omega_t$. Since

$$n = \frac{(I+J_\xi^{\intercal})^{-1}\nu}{|(I+J_\xi^{\intercal})^{-1}\nu|}$$

and

$$(I+J_\xi^{\intercal})^{-1} = I - J_\xi^{\intercal} + (J_\xi^{\intercal})^2(I+J_\xi^{\intercal})^{-1},$$

we can expand

$$n = \nu - \nabla_y s + R(\cdot,s,\nabla_y s), \tag{4.11}$$

Thus, using (4.5), (4.11) and computing the Taylor expansion of g_2 as it has been done for g_1, in the end, after some computations, we can write

$$\frac{\partial w}{\partial \nu}+s\left(\frac{\partial^2 U}{\partial \nu^2}-\frac{\partial g_2}{\partial \nu}\right)-\nabla^{tang}s\cdot\nabla^{tang}g_1 = B(y,s,\nabla s)\nabla w+R(y,s,\nabla s), \tag{4.12}$$

which is reminiscent of Hadamard's work [18]. Here, B is a smooth matrix-valued function, whose entries can be bounded (in moduli) in terms of $s^2 + |\nabla_y s|^2$.

We now have in mind to derive the *second-order Stefan condition* and to eliminate the dependence of the right-hand side of (4.8) from ξ and its derivatives. Due to the form of ξ, we need to express s and its derivatives in terms of w. First, we evaluate both the sides of Equation (4.8) on the boundary $\partial\Omega$. Since $\xi(\tau,y)=s(\tau,y)\nu(y)$ for $y\in\partial\Omega$, we get

$$\frac{\partial w}{\partial \tau} = \mathcal{L}w + \tilde{\mathcal{F}}_1(y,w,\nabla w,D^2w,s,\nabla s,D^2 s) + \frac{\partial s}{\partial \tau}\tilde{\mathcal{F}}_2(y,\nabla w,s,\nabla s). \tag{4.13}$$

Removing the dependence on s from $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ is easy: it suffices to use formula (4.10). To eliminate the τ-derivative of s from the right-hand side of (4.13), we first differentiate (4.9) with respect to τ to obtain

$$\frac{\partial w}{\partial \tau} - \left(\frac{\partial g_1}{\partial \nu} - g_2\right)\frac{\partial s}{\partial \tau} = \frac{\partial R}{\partial s}(y,s)\frac{\partial s}{\partial \tau} = R_*(y,w)\frac{\partial s}{\partial \tau}. \tag{4.14}$$

Note that $R_*(\cdot,w)$ is smooth as well and bounded by a multiple of w. From (4.13) and (4.14), we obtain

$$\left(\frac{\partial g_1}{\partial \nu}-g_2+R_*(y,w)\right)\frac{\partial s}{\partial \tau}-\mathcal{L}w = \overline{\mathcal{F}}_1(y,w,\nabla w,D^2w)+\frac{\partial s}{\partial \tau}\overline{\mathcal{F}}_2(y,w,\nabla w,D^2w),$$

where we have also used (4.10) to absorb the s-dependence in w-dependence.

Summing up, the underlying *second-order Stefan condition* reads for y at the boundary $\partial\Omega$:

$$\frac{\partial s}{\partial \tau} = \frac{\mathcal{L}w + \overline{\mathcal{F}}_1(\cdot,w,\nabla w,D^2w)}{\frac{\partial g_1}{\partial \nu} - g_2 + R_*(\cdot,w) - \overline{\mathcal{F}}_2(\cdot,w,\nabla w,D^2w)},$$

which holds provided w, ∇w and D^2w are small. Taking into account that the function Ψ is localized in the neighborhood $\mathcal{O}$ of $\partial\Omega$, thanks to the above formulas we can write the final equation for w as follows:

$$\frac{\partial w}{\partial \tau} = \mathcal{L}w + \mathcal{F}(y,w(\tau,y),\nabla w(\tau,y),D^2w(\tau,y),w(y',\tau),\nabla w(y',\tau),D^2w(y',\tau)) \tag{4.15}$$

for $\tau\geq 0$ and $y\in\overline{\Omega}\cap\mathcal{O}$, and

$$\frac{\partial w}{\partial \tau} = \mathcal{L}w \tag{4.16}$$

for $\tau \geq 0$ and $y \in \overline{\Omega} \setminus \mathcal{O}$. We recall that y' is the projection on $\partial\Omega$ of $y \in \mathcal{O}$. The boundary condition to be satisfied by w follows directly from (4.9), (4.10) and (4.12). It comes out as

(4.17)

$$\mathcal{B}w = \frac{\partial w}{\partial \nu} + \frac{w}{\frac{\partial g_1}{\partial \nu} - g_2}\left(\frac{\partial^2 U}{\partial \nu^2} - \frac{\partial g_2}{\partial \nu}\right) - \nabla^{tang}\left(\frac{w}{\frac{\partial g_1}{\partial \nu} - g_2}\right) \cdot \nabla^{tang} g_1 = \mathcal{G}(y, w, \nabla w),$$

where $\mathcal{G}$ is sufficiently smooth and quadratic in w.

We have obtained a fully nonlinear problem of the form (1.7) for w, namely (4.15)–(4.17), with initial datum $w(0,\cdot) = w_0$ determined by Ω_0 and u_0 via (4.4) and (4.5). Since u_0 is assumed to satisfy the boundary conditions at $t = 0$, it follows that w_0 satisfies $\mathcal{B}w_0 = \mathcal{G}(\cdot, w_0, \nabla w_0)$ at $\partial\Omega$. We may solve this initial boundary value problem for w provided w_0 is sufficiently small, by means of the general result in the next section and we get:

THEOREM 4.1. *For each $T > 0$ there exist r and ρ positive such that problem* (4.15)–(4.17) *admits a solution $w \in C^{1+\alpha/2, 2+\alpha}((0,T) \times \Omega)$ if the $C^{2+\alpha}$-norm of the datum w_0 does not exceed ρ. Moreover, w is the unique solution to that problem in the ball $B(0,r)$ of $C^{1+\alpha/2,2+\alpha}((0,T) \times \Omega)$.*

5. FUNCTIONAL ANALYTIC TOOLS

In this last section, we recall some basic features about *fully nonlinear problems* of parabolic type in a sufficiently smooth (not necessarily bounded) domain of $\mathbb{R}^N, N \geq 1$. We refer the reader to *e.g.*, [20–22, 26] for a comprehensive analysis of partial differential equations of parabolic type.

Consider a *fully nonlinear* problem for the unknown $\mathbf{u} = (u_1, \ldots, u_d)$:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(t,x) = \mathcal{L}\mathbf{u}(t,x) + \mathcal{F}(\mathbf{u}(t,\cdot))(x), & t > 0, \quad x \in \overline{\Omega}, \\ \mathcal{B}\mathbf{u}(t,x) = \mathcal{G}(\mathbf{u}(t,\cdot))(x), & t > 0, \quad x \in \partial\Omega, \end{cases} \tag{5.1}$$

supplemented by the initial condition

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad x \in \overline{\Omega}. \tag{5.2}$$

Here, $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$ and

$$\mathcal{L}_k \mathbf{u} = \sum_{i,j=1}^{N} a_{ij}^k D_{ij} u_k + \sum_{i=1}^{N} \sum_{j=1}^{d} b_{i,j}^k D_i u_j + \sum_{j=1}^{d} c_{kj} u_j, \qquad k = 1, \ldots, d,$$

is a uniformly elliptic operator (with sufficiently smooth coefficients), $\mathcal{F}$ and $\mathcal{G}$ are smooth enough functions defined in a neighborhood of 0 in $C_b^2(\overline{\Omega})$ with values in $C_b(\overline{\Omega})$ and $C_b^1(\partial\Omega)$ respectively and $\mathcal{L}$ and $\mathcal{B}$ are linear differential operators with regular coefficients. Moreover,

$$\mathcal{F}(\mathbf{0})=\mathbf{0},\ \mathcal{F}'(\mathbf{0})=\mathbf{0},\qquad \mathcal{G}(\mathbf{0})=\mathbf{0},\ \mathcal{G}'(\mathbf{0})=\mathbf{0},$$

so that $\mathbf{u}\equiv\mathbf{0}$ is a solution of problem (5.1) and the linearization of (5.1) around the null solution is

$$(5.3)\qquad \begin{cases} \dfrac{\partial\mathbf{v}}{\partial t}(t,x)=\mathcal{L}\mathbf{v}(t,x)+\mathbf{f}(t,x), & t>0,\quad x\in\Omega,\\ \mathcal{B}\mathbf{v}(t,x)=\mathbf{g}(t,x), & t>0,\quad x\in\partial\Omega,\end{cases}$$

where $\mathbf{f}\in C^{\alpha/2,\alpha}((0,T)\times\Omega;\mathbb{R}^d)$ and $\mathbf{g}\in C^{(1+\alpha)/2,1+\alpha}([0,T]\times\partial\Omega;\mathbb{R}^d)$, for some $\alpha\in(0,1)$, are given functions. Of course, if $\mathcal{G}(\mathbf{u})$ has some null components, then we can limit ourselves to considering $\mathbf{g}$ with nontrivial components only in correspondence of nontrivial components of $\mathcal{G}(\mathbf{u})$.

If one, as in this paper, is interested in the Cauchy problem (5.3) for initial data $\mathbf{u}_0$ close to $\mathbf{0}$, the strategy is to consider first the linear problem (5.3) and then apply a fixed point argument to solve the original problem (5.1). As the examples of the previous sections show, the nonlinear term $\mathcal{F}$ could depend also on the trace on $\partial\Omega$ of the second-order spatial derivatives of the solution $\mathbf{u}$. This causes some additional difficulties and optimal regularity results are required already for the Cauchy problem (5.3).

The picture is well-understood in the classical case (considered in Section 4) when $d=1$, Ω is bounded with a boundary of class $C^{2+\alpha}$ and

$$\mathcal{B}=\sum_{i=1}^N\beta_iD_i+\gamma$$

is a nontangential operator, *i.e.*, $\sum_{i=1}^N\beta_i\nu_i$ never vanishes on $\partial\Omega$, where $\nu(x)$ denotes the outward unit normal vector to $\partial\Omega$ at x. In this case, under the assumptions

H1 $\mathcal{F}:B(0,R)\subset C^2(\overline{\Omega})\to C(\overline{\Omega})$ is continuously differentiable with Lipschitz continuous derivative, $\mathcal{F}(0)=0$, $\mathcal{F}'(0)=0$ and the restriction of $\mathcal{F}$ to $B(0,R)\subset C^{2+\alpha}(\Omega)$ takes values in $C^\alpha(\Omega)$ and is continuously differentiable;

H2 $\mathcal{G}:B(0,R)\subset C^1(\overline{\Omega})\to C(\partial\Omega)$ is continuously differentiable with Lipschitz continuous derivative, $\mathcal{G}(0)=0$, $\mathcal{G}'(0)=0$ and the restriction of $\mathcal{G}$ to $B(0,R)\subset C^{2+\alpha}(\Omega)$ takes values in $C^{1+\alpha}(\partial\Omega)$ and is continuously differentiable too;

H3 $u_0\in C^{2+\alpha}(\overline{\Omega})$ satisfies[4] the compatibility condition $\mathcal{B}u_0=\mathcal{G}(u_0)$ in $\partial\Omega$;

[4] as usual, when dealing with real-valued functions, we do not use bold style.

H4 the coefficients of the operator $\mathcal{L} = \mathcal{L}_1$ belong to $C^\alpha(\Omega)$, whereas β_i ($i = 1, \ldots, N$) and γ belongs to $C^{1+\alpha}(\partial\Omega)$,

a local in time existence and uniqueness result for problem (5.1), subject to the initial condition $u(0,\cdot) = u_0$, can be proved and it reads:

THEOREM 5.1. *Under the above assumptions, for every $T > 0$ there exist r, $\rho > 0$ such that problem* (5.1), (5.2) *admits a solution $u \in C^{1+\alpha/2,2+\alpha}((0,T)\times \Omega)$ if $\|u_0\|_{C^{2+\alpha}(\Omega)} \le \rho$. Moreover u is the unique solution in $B(0,r) \subset C^{1+\alpha/2,2+\alpha}((0,T) \times \Omega)$.*

The main core is the proof of the existence (and uniqueness) of a classical smooth solution to problem (5.3), subject to the initial condition $v(0,\cdot) = v_0 \in C^{2+\alpha}(\Omega)$, where $\mathcal{B}v_0 = g(0,\cdot)$. It is based on decoupling that problem introducing a suitable lifting operator $\mathcal{N}$, to lift-up the boundary term g, with the property $\mathcal{B}\mathcal{N}g = g$ for each g as in problem (5.3). More precisely, the smoothness of $\partial\Omega$ allows to prove (via local-charts) the existence of a bounded operator $\mathcal{N} \in L(C^1(\partial\Omega), C^2(\overline{\Omega})) \cap L(C^{1+\alpha}(\partial\Omega), C^{2+\alpha}(\Omega))$ such that $\mathcal{B}\mathcal{N}g = g$ on functions g as in problem (5.3). Thus, one first solves the Cauchy problem

$$
(5.4) \qquad \begin{cases} \dfrac{\partial v_1}{\partial t}(t,x) = \mathcal{L}v_1(t,x) + f(t,x) + \mathcal{L}\mathcal{N}g(t,x), & t>0, \quad x \in \Omega, \\ \mathcal{B}v_1(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ v_1(0,x) = v_0(x), & \qquad\quad\; x \in \Omega. \end{cases}
$$

Since $f + \mathcal{N}g$ belongs to $C^{\alpha/2,\alpha}((0,T) \times \Omega)$ and $v_0 \in C^{2+\alpha}(\Omega)$, classical results for parabolic equations (see *e.g.*, [22]) show that the above problem admits a unique solution $v_1 \in C^{1+\alpha/2,2+\alpha}((0,T) \times \Omega)$. Moreover,

$$
\begin{aligned}
&\|v_1\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)} \\
\le& c(\|u_0\|_{C^{2+\alpha}(\Omega)} + \|f\|_{C^{\alpha/2,\alpha}((0,T)\times\Omega)} + \|g\|_{C^{(1+\alpha)/2,1+\alpha}([0,T]\times\partial\Omega)}),
\end{aligned}
$$

for some positive constant c, independent of v_1 and the data.

Next one considers the Cauchy problem

$$
\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \mathcal{L}w(t,x) + \mathcal{N}\psi(t,x) - \mathcal{N}\psi(0,x), & t > 0, \quad x \in \Omega, \\ \mathcal{B}w(t,x) = 0, & t>0, \quad x \in \partial\Omega, \\ w(0,x) = 0, & \qquad\quad\; x \in \Omega. \end{cases}
$$

For the same reasons as above, this problem admits a (unique) solution $w \in C^{1+\alpha/2,2+\alpha}((0,T) \times \Omega)$, which satisfies the estimate

$$
\|w\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)} \le c\|g\|_{C^{(1+\alpha)/2,1+\alpha}([0,T]\times\partial\Omega)}.
$$

Actually, due to the null initial condition, w is smoother. More precisely, the function $\mathcal{L}w$ lies in $C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)$ and it satisfies an estimate similar to that satisfied by w. Hence, the function $v_2 = \mathcal{L}w + \mathcal{N}\psi - \mathcal{N}\psi(0,\cdot)$ belongs to $C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)$ and

$$\|v_2\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times\Omega)} \le c\|g\|_{C^{(1+\alpha)/2,1+\alpha}([0,T]\times\partial\Omega)}.$$

Moreover, simple computations show that

$$\text{(5.5)}\qquad \begin{cases} \dfrac{\partial v_2}{\partial t}(t,x) = \mathcal{L}v_2(t,x) - \mathcal{L}\mathcal{N}g(t,x), & t>0, \quad x\in\Omega,\\ \mathcal{B}v_2(t,x) = g(t,x), & t>0, \quad x\in\partial\Omega,\\ v_2(0,x) = 0, & \qquad\quad x\in\Omega. \end{cases}$$

As a byproduct, the function $v = v_1 + v_2$ solves the Cauchy problem (5.3) and $w(0,\cdot) = u_0$. Finally, the uniqueness of such a solution follows straightforwardly from the classical maximum principle.

Things are less trivial when $d > 1$, the boundary conditions are of jump type and the domain is unbounded, as in the relevant cases of halfplane, half-space and strips. In these situations the existence of a solution to problems (5.4) and (5.5) is not for free and a deeper analysis is required. One possible way to "attack the problem" is via the theory of analytic semigroups, which is a powerful tool in the theory of PDEs (of parabolic type). Some steps are required, which we briefly describe here below.

(i) One needs to show that the realization[5] L of the operator $\mathcal{L}$ in $X = C_b(\Omega;\mathbb{R}^d)$ (or even[6] in $X = \{\mathbf{u}\in C_b(\Omega;\mathbb{R}^d) : \mathbf{u}$ vanishes at infinity (along suitable directions)$\}$) generates an analytic semigroup. This can be done looking at the so-called *resolvent equation*, *i.e.*, the equation $\lambda\mathbf{u} - L\mathbf{u} = \mathbf{f}\in X$ with $\lambda\in\mathbb{C}$. If one proves that the above equation admits a unique solution $\mathbf{u} := R(\lambda,L)\mathbf{f}$ in $D(L)$ for λ in a suitable right-halfplane and, in such halfplane, $|\lambda|^{-1}\|R(\lambda,L)\mathbf{f}\|_X \le c\|\mathbf{f}\|_X$, with c independent of λ and $\mathbf{f}$, then one concludes that the operator L is *sectorial* and, hence, it generates an analytic semigroup in X.

(ii) Next, one needs to characterize the so-called interpolation spaces, which roughly speaking, are the subsets of maximal regularity for the equation

[5]Here, by realization of $\mathcal{L}$ in X we mean the operator $L : D(L) = \{\mathbf{u}\in X : \mathcal{L}\mathbf{u}\in X\}\to X$, defined by $L\mathbf{u} = \mathcal{L}\mathbf{u}$ for any $\mathbf{u}\in D(L)$, where $\mathcal{L}\mathbf{u}$ is meant in the sense of distributions.

[6]For instance, for the problem considered in Section 2, $C_b(\Omega)$ is a suitable choice. On the other hand, for the "NEF" system in Section 3 $X = C_b(\Omega,\mathbb{R}^3)$ is not a suitable choice since it does not take the conditions at infinity (3.4) into due account. Indeed, a straightforward computation reveals that, if θ and S are as in the ansatz (3.7), then conditions (3.4) result in the following conditions on v and w: $v(t,-\infty,y) = 0$ and $w(t,\pm\infty,y) = 0$ for any t and y.

$\lambda\mathbf{u} - \mathcal{L}\mathbf{u} = \mathbf{f}$. In much more precise terms, for any $\alpha \in (0,1)$ the interpolation space $D_L(\alpha/2,\infty)$ is the set of all $\mathbf{f} \in X$ such that $\sup_{\lambda\in(0,1)} \lambda^{1-\alpha/2}\|LR(\lambda,L)\mathbf{f}\|_X < +\infty$, whereas $D_L(1+\alpha/2,\infty)$ is the set of all $\mathbf{u} \in D(L)$ such that $L\mathbf{u} \in D_L(\alpha/2,\infty)$. To characterize these interpolation spaces a representation formula for the operator $R(\lambda, L)$ (which in many concrete cases can be obtained, *e.g.* via Fourier transform) and a much more precise characterization of $D(L)$ are of much help. One expects that

$$D_L(\alpha/2,\infty) = C^\alpha(\Omega;\mathbb{R}^d) \cap X,$$

if the operators $\mathcal{B}_1,\ldots,\mathcal{B}_d$ are all of the first-order, whereas

$$D_L(\alpha/2,\infty) = \{\mathbf{u} \in C^\alpha(\Omega;\mathbb{R}^d) \cap X : \mathcal{B}_j\mathbf{u} = \mathbf{0},\ j \in J\},$$

otherwise, where $\mathcal{B}\mathbf{u} = (\mathcal{B}_1\mathbf{u},\ldots,\mathcal{B}_d\mathbf{u})$ and $\mathcal{B}_j$ $(j \in J)$ are the zeroth-order operators in the definition of $\mathcal{B}$. Similarly,

$$D_L(1+\alpha/2,\infty) = \{\mathbf{u} \in C_b^{2+\alpha}(\Omega;\mathbb{R}^d) \cap X : \mathcal{L}\mathbf{u} \in X,\ \mathcal{B}\mathbf{u} = \mathbf{0}\},$$

if the (jump condition in the) boundary operator $\mathcal{B}$ are all of the first-order, whereas

$$\begin{aligned} &D_L(1+\alpha/2,\infty) \\ =&\{\mathbf{u} \in C_b^{2+\alpha}(\Omega;\mathbb{R}^d) : \mathcal{L}\mathbf{u} \in X,\ \mathcal{B}\mathbf{u} = \mathbf{0},\ \mathcal{B}_j\mathcal{L}\mathbf{u} = 0,\ j \in J\}, \end{aligned}$$

otherwise.[7] The previous (topological equality) can be proved under very reasonable assumptions on the operators $\mathcal{L}$ and $\mathcal{B}$. Then, one needs also to prove that $\{\mathbf{u} \in C_b^1(\overline{\Omega};\mathbb{R}^d) \cap X : \mathcal{B}_j\mathbf{u} = 0,\ j \in J\}$ is continuously embedded into $D_L(1/2,\infty)$.

(iii) Finally, one needs to define a lifting operator $\mathcal{N}$ mapping $C^1(\partial\Omega;\mathbb{R}^d)$ into $C_b^2(\overline{\Omega};\mathbb{R}^d)$ and $C^{1+\alpha}(\partial\Omega;\mathbb{R}^d)$ into $C_b^{2+\alpha}(\Omega;\mathbb{R}^d)$, and such that $\mathcal{B}\mathcal{N}\mathbf{g} = \mathbf{g}$ for each function $\mathbf{g}$ as in problem (5.3). Typically, these operators are defined via an appropriate integral operator. For instance, if $\mathcal{B}$ is as in Section 2, then a possible choice of $\mathcal{N}$ is the following: $\mathcal{N}\mathbf{g} = (-N_1g_3, -\lambda N_1g_3, 0)$ for any $\mathbf{g} = (g_1, g_2, g_3)$, where

$$(N_1g_3)(x,y) = \eta(x)x\int_{\mathbb{R}} \varphi(\xi)g_3(y+\xi x)d\xi, \qquad x \le 0,\ y \in \mathbb{R}.$$

Here, η is a smooth function vanishing on $(-\infty,-2]$ and identically equal to one in $[-1,0]$, whereas φ is smooth in $\mathbb{R}$, with compact support and such that $\|\varphi\|_{L^1(\mathbb{R})} = 1$.

[7]For instance, if $\mathcal{B}$ is the operator in Section 3 (see (3.10)), then $D_L(\alpha/2,\infty) = \{\mathbf{u} \in X_\alpha : \lambda v(0,\cdot) - w(0,\cdot) + h(0,\cdot) = 0\}$ and $D_L(1+\alpha/2,\infty) = \{\mathbf{u} \in X_{2+\alpha} : \mathcal{B}\mathbf{u} = \mathbf{0}$ and $\lambda(\mathcal{L}\mathbf{u})_1(0,\cdot) - (\mathcal{L}\mathbf{u})_2(0,\cdot) + (\mathcal{L}\mathbf{u})_3(0,\cdot) = 0\}$.

Having all these tools at hand, one can apply time and spatial regularity results for abstract Cauchy problems associated with sectorial operator (see *e.g.* [26, Theorems 4.3.1, 4.3.8, 4.3.16], to prove Theorem 5.1, assuming the vector-valued counterpart of Hypotheses **H1**–**H3** and the first part of Hypothesis **H4**.

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