Dedicated to Professor Philippe Ciarlet on the occasion of his 80th birthday

ON OLDROYD MODELS WITHOUT EXPLICIT DISSIPATION

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We review an argument of Renardy proving existence and regularity for a subset of a class of models of non-Newtonian fluids suggested by Oldroyd, including the upper-convected and lower-convected Maxwellian models. We suggest an effective method for solving these models, including a variational formulation suitable for finite element computation.

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1. INTRODUCTION

We consider some model equations proposed for non-Newtonian fluids that are a subset of the Oldroyd models [15]. This includes the upper-convected and lower-convected Maxwellian models. Our objective is to extend the existence proof of Renardy [16] for these equations in various ways. In particular, we show that a variant of his proof can be the basis for an effective solution algorithm. The subset of the Oldroyd models that we study involves three parameters, the fluid kinematic viscosity η and two rheological parameters λ_1 and μ_1. We will refer to this subset as the "three-parameter" subset. An extended version of these results appeared in [11] and were announced in [10].

1.1. Notation

Let d denote the space dimension. We assume that the fluid domain $\mathcal{D} \subset \mathbb{R}^d$ is connected and has a boundary $\partial\mathcal{D}$ with different degrees of regularity for different results. For simplicity, we assume that the boundary conditions on the fluid velocity are Dirichlet: $\mathbf{u} = \mathbf{0}$ on $\partial\mathcal{D}$, although these can be relaxed to allow $\mathbf{u} = \mathbf{g}$ on $\partial\mathcal{D}$ provided $\mathbf{g}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{D}$. We utilize standard Sobolev spaces $W_q^s(\mathcal{D})$ for nonnegative integers s and $1 \le q \le \infty$, consisting of functions whose derivatives of order s or less are in the Lebesgue space $L_q(\mathcal{D})$ [1,4,6]. For vector-valued functions $\mathbf{v}$ and matrix-valued functions $\mathbf{T}$, we will write $\mathbf{v} \in W_q^s(\mathcal{D})^d$ or $\mathbf{T} \in W_q^s(\mathcal{D})^{d^2}$ to indicate that each component of $\mathbf{v}$ or $\mathbf{T}$ is in $W_q^s(\mathcal{D})$. For tensor-valued functions of tensor order larger than 2, we will use analogous notation. The highest order of tensors considered here is 3, but we develop some identities in Section 9 for general tensor-valued functions.

We will also write the corresponding norms with the understanding that the norms for vector-valued and tensor-valued functions are evaluated appropriately. More precisely, we define

$$\| \mathbf{T} \|_{W_q^s(\mathcal{D})} = \sum_{m=0}^{s} \| \, |\nabla^m \mathbf{T}| \, \|_{L_q(\mathcal{D})},$$

where, for example, $|\mathbf{T}(x)|$ is the Frobenius norm of $\mathbf{T}(x)$ in the case when $\mathbf{T}(x)$ is a matrix and the Euclidean norm in the case when $\mathbf{T}(x)$ is a vector. We give details about generalizations to arbitrary tensors in Section 9. For simplicity, we do not use bold face to indicate points in $\mathbb{R}^d$.

We collect here our assumptions regarding the regularity of the domain boundary. We will always assume that $\mathcal{D}$ is bounded and $\partial\mathcal{D}$ is Lipschitz, but in addition we make the following assumptions. Consider the elliptic equations

$$\begin{aligned} v - \Delta v &= f \quad \text{in } \mathcal{D} \\ \nabla v \cdot \mathbf{n} &= 0 \quad \text{on } \partial\mathcal{D}, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} -\Delta v &= f \quad \text{in } \mathcal{D} \\ v &= 0 \quad \text{on } \partial\mathcal{D}. \end{aligned} \tag{1.2}$$

We introduce the following condition: suppose that the domain $\mathcal{D}$ has the property that there is a constant C such that each problem (1.1) and (1.2) has a unique solution $v \in H^2(\mathcal{D})$ for all $f \in L_2(\mathcal{D})$ satisfying

$$\| v \|_{H^2(\mathcal{D})} \le C \| f \|_{L_2(\mathcal{D})}. \tag{1.3}$$

Similarly, we consider a Stokes system,

$$\begin{aligned} -\Delta \mathbf{v} + \nabla p &= \mathbf{f} \text{ in } \mathcal{D} \\ \nabla\cdot \mathbf{v} &= 0 \text{ in } \mathcal{D}, \quad \mathbf{v} = \mathbf{0} \text{ on } \partial\mathcal{D}. \end{aligned} \tag{1.4}$$

We introduce the following condition: suppose that, for some $q > 1$, the domain $\mathcal{D}$ has the property that there is a constant $C_{q,\mathcal{D}}$ such that for all $\mathbf{f} \in L_q(\mathcal{D})^d$

there is a unique pair $\mathbf{v} \in W_q^2(\mathcal{D})^d$ and $p \in W_q^1(\mathcal{D})/\mathbb{R}$ solving (1.4) such that

$$\|\mathbf{v}\|_{W_q^2(\mathcal{D})} + \|p\|_{W_q^1(\mathcal{D})/\mathbb{R}} \le C_{q,\mathcal{D}}\|\mathbf{f}\|_{L_q(\mathcal{D})} \text{ for all } \mathbf{f} \in L_q(\mathcal{D})^d. \tag{1.5}$$

We assume this holds for all $q \le q_0$ where $q_0 > 1$. Ultimately, many of the results will be restricted to the case $q_0 > d$, where d is the dimension of $\mathcal{D}$.

We will utilize Sobolev's inequality, which says that for $q > d$, functions in $W_q^{s+1}(\mathcal{D})$ may be viewed as being in $C^s(\mathcal{D})$. We will in particular use the case $s = 0$ frequently, and we introduce the corresponding Sobolev constant σ_q which is the smallest real number such that

$$\|v\|_{L_\infty(\mathcal{D})} \le \sigma_q\|v\|_{W_q^1(\mathcal{D})} \text{ for all } v \in W_q^1(\mathcal{D}). \tag{1.6}$$

We will be interested in the cases $d = 2$ and $d = 3$, and our estimates will always be restricted to the case $q < \infty$. The constant σ_q depends on d and the domain $\mathcal{D}$, but we will suppress this dependence in what follows.

Another type of Sobolev inequality is

$$\|v\|_{L_{2q/(q-2)}(\mathcal{D})} \le \sigma_q\|v\|_{H^1(\mathcal{D})} \text{ for all } v \in H^1(\mathcal{D}), \tag{1.7}$$

provided that $q > 2$ for $d = 2$ and $q \ge d$ for $d \ge 3$. Although the constant σ_q may be different from the one in (1.6), we will use the same notation for both, that is, we will assume that σ_q is the maximum of the two constants.

2. THREE-PARAMETER OLDROYD MODELS

In all (time-independent) models of fluids, the basic equation can be written as

$$\mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = \nabla\cdot \mathbf{T} + \mathbf{f}, \tag{2.1}$$

where $\mathbf{T}$ is called the extra (also called deviatoric) stress and $\mathbf{f}$ represents externally given data. The models only differ according to the dependence of the stress on the velocity $\mathbf{u}$.

In the case of a Newtonian fluid

$$\mathbf{T} = \eta(\nabla\mathbf{u} + \nabla\mathbf{u}^t)\,.$$

Thus, when $\nabla\cdot \mathbf{u} = 0$, it follows that $\nabla\cdot \mathbf{T} = \eta\Delta\mathbf{u}$, and we obtain the well known Navier-Stokes equations for Newtonian flow, where η is the kinematic viscosity [13].

We now describe the particular family of non-Newtonian models on which we focus here.

A three parameter subset of the eight parameter model of Oldroyd [15] for the extra stress takes the form

$$\mathbf{T} + \lambda_1(\mathbf{u} \cdot \nabla\mathbf{T} + \mathbf{R}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{R}^t) - \mu_1(\mathbf{E}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{E}) = 2\eta\mathbf{E},$$

where the five parameters λ_2, μ_2, μ_0, ν_0, and ν_1 in [15] are set to zero, and

$$\mathbf{R} = \tfrac{1}{2}(\nabla\mathbf{u}^t - \nabla\mathbf{u}) \;\text{ and }\; \mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t).$$

Note that $\mathbf{E}^t = \mathbf{E}$, $\mathbf{R}^t = -\mathbf{R}$, $\mathbf{R} + \mathbf{E} = \nabla\mathbf{u}^t$, and $\mathbf{R} - \mathbf{E} = -\nabla\mathbf{u}$.

We can write the full model in the steady case as

$$\begin{aligned} \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \nabla\cdot\mathbf{T} + \mathbf{f} \;\text{ in } \mathcal{D}, \\ \nabla\cdot\mathbf{u} = 0 \;\text{ in } \mathcal{D}, &\quad \mathbf{u} = \mathbf{0} \;\text{ on } \partial\mathcal{D}, \end{aligned} \tag{2.2}$$

$$\mathbf{T} + \lambda_1(\mathbf{u}\cdot\nabla\mathbf{T} + \mathbf{R}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{R}^t) - \mu_1(\mathbf{E}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{E}) = 2\eta\mathbf{E} \;\text{ in } \mathcal{D}. \tag{2.3}$$

By combining $\mathbf{R}$ and $\mathbf{E}$, formula (2.3) has the equivalent expression

$$\mathbf{T} + \lambda_1(\mathbf{u}\cdot\nabla\mathbf{T} - (\nabla\mathbf{u})\circ\mathbf{T} - \mathbf{T}\circ(\nabla\mathbf{u}^t)) + (\lambda_1 - \mu_1)(\mathbf{E}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{E}) = 2\eta\mathbf{E}. \tag{2.4}$$

There are physical reasons to assume that $\lambda_1 > 0$, but we will allow $\lambda_1 < 0$ as well. The case $\lambda_1 = 0$ and $\mu_1 = 0$ which corresponds to the Navier-Stokes equations, has not been considered here, but it can be treated similarly and is essentially trivial by comparison. Therefore, from now on, we assume that $\lambda_1 \neq 0$.

3. ALTERNATIVE FORMULATION

The difficulty with the simple formulation (2.2–2.3) is that there is no obvious smoothing for $\mathbf{u}$, *i.e.*, there is no explicit dissipation in the basic equation (2.1). In Section 8, we describe a technique proposed by Renardy in [16] that addresses this issue by making a substitution based on (2.2). Of course, this is not the only option. Following the work of Fernandez-Cara *et al* in [5], we develop a modified version of the Renardy formulation that uses a more selective substitution. This formulation is simplified in several terms and may be more effective both analytically and numerically. Renardy suggested writing (2.2) as

$$\nabla\cdot\mathbf{T} = \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{f}, \tag{3.1}$$

and then inserting this expression for $\nabla\cdot\mathbf{T}$ into the divergence of (2.3), or equivalently (2.4). We can use the expression (3.1) for $\nabla\cdot\mathbf{T}$ selectively in (2.3) to get different formulations with different properties. In order to do so, we need to use some identities, which we now develop.

3.1. Some identities

The reader will find in Section 9 the general definitions of the operators used here. Let us now compute the divergence of the left-hand side of (2.4).

We compute the divergence of $\mathbf{u}\cdot\nabla\mathbf{T}$ as follows:

$$\begin{aligned}(\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{T}))_i &= \sum_j (\mathbf{u}\cdot\nabla\mathbf{T})_{ij,j} = \sum_j (\mathbf{u}\cdot\nabla T_{ij})_{,j} = \sum_{jk}\big(u_k T_{ij,k}\big)_{,j} \\ &= \sum_{jk}\big(u_k T_{ij,kj} + u_{k,j}T_{ij,k}\big) = (\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{T}))_i + \sum_{jk} u_{k,j}T_{ij,k}.\end{aligned}$$

We compute the divergence of $\mathbf{T}\circ\nabla\mathbf{u}^t$ as follows:

$$\begin{aligned}\big(\nabla\cdot\big(\mathbf{T}\circ(\nabla\mathbf{u})^t\big)\big)_i &= \sum_j \big(\mathbf{T}\circ(\nabla\mathbf{u})^t\big)_{ij,j} = \sum_{jk}(T_{ik}(\nabla\mathbf{u})_{jk})_{,j} = \sum_{jk}(T_{ik}u_{j,k})_{,j} \\ &= \sum_{jk}\big(T_{ik,j}u_{j,k} + T_{ik}u_{j,kj}\big) = \sum_{jk}T_{ik,j}u_{j,k} + \sum_k T_{ik}(\nabla\cdot\mathbf{u})_{,k} \\ &= \sum_{jk}T_{ik,j}u_{j,k} = \sum_{jk}T_{ij,k}u_{k,j} = \sum_{jk}u_{k,j}T_{ij,k},\end{aligned}$$

provided that $\nabla\cdot\mathbf{u}=0$. Therefore we have proved the following identity:

$$\nabla\cdot\big(\mathbf{u}\cdot\nabla\mathbf{T} - \mathbf{T}\circ(\nabla\mathbf{u})^t\big) = \mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{T}), \tag{3.2}$$

valid in the sense of distributions for all sufficiently regular functions and tensors. For instance, it holds when the left and right sides of equation (3.2) define elements of $H^{-1}(\mathcal{D})$, e.g., if the components of $\mathbf{u}$ and $\mathbf{T}$ belong to $W^1_q(\mathcal{D})$ for $q>d$; then

$$\begin{aligned}\langle\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{T}),\boldsymbol{\phi}\rangle &= -\int_{\mathcal{D}}(\mathbf{u}\cdot\nabla\mathbf{T}):\nabla\boldsymbol{\phi}\,\mathrm{d}x \\ \langle\nabla\cdot(\mathbf{T}\circ(\nabla\mathbf{u})^t),\boldsymbol{\phi}\rangle &= -\int_{\mathcal{D}}(\mathbf{T}\circ(\nabla\mathbf{u})^t):\nabla\boldsymbol{\phi}\,\mathrm{d}x\end{aligned}$$

for all $\boldsymbol{\phi}\in H^1_0(\mathcal{D})^d$. If moreover $\nabla\cdot\mathbf{u}=0$, then

$$\begin{aligned}\langle\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{T}),\boldsymbol{\phi}\rangle &= \langle\sum_i u_i\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{T}),\boldsymbol{\phi}\rangle = \sum_i\langle\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{T}),u_i\boldsymbol{\phi}\rangle \\ &= -\langle\nabla\cdot\mathbf{T},\sum_i\frac{\partial}{\partial x_i}(u_i\boldsymbol{\phi})\rangle = -\langle\nabla\cdot\mathbf{T},\sum_i u_i\frac{\partial}{\partial x_i}\boldsymbol{\phi}\rangle \\ &= -\int_{\mathcal{D}}(\nabla\cdot\mathbf{T})\cdot(\mathbf{u}\cdot\nabla\boldsymbol{\phi})\,\mathrm{d}x \quad \forall\boldsymbol{\phi}\in H^1_0(\mathcal{D})^d.\end{aligned} \tag{3.3}$$

The main point of (3.2) is that the expression on the left, which involves second derivatives of $\mathbf{T}$, has the property that all such second derivatives can be written as a first-order derivative of $\nabla\cdot\mathbf{T}$.

If $\nabla\cdot \mathbf{u} = 0$, then we can establish another identity:

(3.4)
$$\begin{aligned}
\nabla\cdot((\nabla\mathbf{u})\circ\mathbf{v}) &= \sum_\ell \frac{\partial}{\partial x_\ell}((\nabla\mathbf{u})\circ\mathbf{v})_\ell = \sum_\ell \frac{\partial}{\partial x_\ell}\Big(\sum_k (\nabla\mathbf{u})_{\ell,k} v_k\Big)\\
&= \sum_\ell \frac{\partial}{\partial x_\ell}\sum_k u_{\ell,k}v_k = \sum_\ell\sum_k \frac{\partial}{\partial x_\ell}(u_{\ell,k}v_k)\\
&= \sum_\ell\sum_k \big(u_{\ell,k\ell}v_k + u_{\ell,k}v_{k,\ell}\big) = \sum_k\sum_\ell u_{\ell,k\ell}v_k + \sum_\ell\sum_k u_{\ell,k}v_{k,\ell}\\
&= \sum_k \frac{\partial}{\partial x_k}\Big(\sum_\ell u_{\ell,\ell}\Big)v_k + \nabla\mathbf{u}^t : \nabla\mathbf{v} = \nabla\mathbf{u}^t : \nabla\mathbf{v}.
\end{aligned}$$

3.2. **Applying the identities**

For example, using (3.2), we get

$$\nabla\cdot\big(\mathbf{u}\cdot\nabla\mathbf{T} - (\nabla\mathbf{u})\circ\mathbf{T} - \mathbf{T}\circ(\nabla\mathbf{u})^t\big) = \mathbf{u}\cdot\nabla\,(\nabla\cdot\mathbf{T}) - \nabla\cdot\,((\nabla\mathbf{u})\circ\mathbf{T})\,. \tag{3.5}$$

Thus the divergence of (2.4) becomes

$$\nabla\cdot\mathbf{T} + \lambda_1(\mathbf{u}\cdot\nabla\,(\nabla\cdot\mathbf{T}) - \nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})) + (\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E}) = \eta\Delta\mathbf{u}. \tag{3.6}$$

The only troublesome term in (3.6) is $\mathbf{u}\cdot\nabla\,(\nabla\cdot\mathbf{T})$. Although we have bounds for this term, we cannot show that it is suitably smooth in the relevant spaces required for a proof of existence. Thus we eliminate it by using (3.1). Inserting the expression (3.1) for $\nabla\cdot\mathbf{T}$ into (3.6) gives

$$\begin{aligned}
\eta\Delta\mathbf{u} &= \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{f} + \lambda_1\mathbf{u}\cdot\nabla\,(\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{f})\\
&\quad - \lambda_1\nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T}) + (\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p + \lambda_1\mathbf{u}\cdot\nabla(\nabla p) = \mathbf{f} + \lambda_1\mathbf{u}\cdot\nabla\mathbf{f}\\
&\quad - \lambda_1\Big(\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})\Big) - (\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E}).
\end{aligned} \tag{3.7}$$

Remark 3.1. If we consider an Oldroyd model with additional parameters, other than λ_1, μ_1, and η, for instance the five-parameter model with λ_2 and μ_2, then the right-hand side of formula (3.7) has an additional term, say $\nabla\cdot\mathcal{T}(\nabla\mathbf{u},\lambda_2,\mu_2)$, where $\mathcal{T}$ is some function, which is much more problematic, since it involves third derivatives of $\mathbf{u}$. This is consistent with the fact that certain Oldroyd models are asymptotically equivalent to a grade-two model [17]. This is the reason why we focus only on the equation (2.3).

3.3. **Pressure equation**

Define an auxiliary pressure function π by

$$\pi = p + \lambda_1 \mathbf{u} \cdot \nabla p. \tag{3.8}$$

Then

(3.9)
$$\begin{aligned}\nabla\pi &= \nabla p + \lambda_1 \nabla(\mathbf{u}\cdot\nabla p) = \nabla p + \lambda_1 \nabla\Big(\sum_i u_i p_{,i}\Big) \\ &= \nabla p + \lambda_1 \sum_i \big((\nabla u_i) p_{,i} + u_i \nabla p_{,i}\big) = \nabla p + \lambda_1 \left((\nabla \mathbf{u})^t \nabla p + \mathbf{u}\cdot\nabla(\nabla p)\right),\end{aligned}$$

which agrees with (9.6) in this case. Substituting (3.9) in (3.7) yields

$$\begin{aligned} -\eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi - \lambda_1(\nabla\mathbf{u})^t\nabla p &= \mathbf{f} + \lambda_1\mathbf{u}\cdot\nabla\mathbf{f} \\ - \lambda_1\Big(\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})\Big) &- (\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{E}).\end{aligned} \tag{3.10}$$

We can think of (3.8) as determining p from π. This is exactly the problem addressed in [12] as described subsequently in Lemma 4.2. Thus the following result can be proved; for the proof see [12] or the proof of Lemma 4.2.

LEMMA 3.2. *Suppose that $2 \le d \le 4$, $q > d$, $\mathcal{D} \subset \mathbb{R}^d$ is a bounded, Lipschitz domain, and $\mathbf{u} \in W^1_\infty(\mathcal{D})^d$ with $\nabla\cdot\mathbf{u} = 0$ in $\mathcal{D}$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$. Define $\mathcal{U} = \|\nabla\mathbf{u}\|_{L_\infty(\mathcal{D})}$ and suppose that $\mathcal{U} < |\lambda_1|^{-1}$. Let p be determined from π via (3.8). Then*

$$\|p\|_{W^1_q(\mathcal{D})} \le \frac{1}{1-|\lambda_1|\mathcal{U}} \|\pi\|_{W^1_q(\mathcal{D})}.$$

3.4. **A Navier-Stokes system**

Re-phrasing (3.10), we find

$$-\eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \mathcal{F}(\mathbf{f},\mathbf{u},p,\mathbf{T}), \tag{3.11}$$

where $\mathcal{F}$ is defined by

(3.12)
$$\begin{aligned}\mathcal{F}(\mathbf{f},\mathbf{u},p,\mathbf{T}) = \mathbf{f} + \lambda_1\mathbf{u}\cdot\nabla\mathbf{f} + \lambda_1(\nabla\mathbf{u})^t\nabla p - \lambda_1\Big(\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})\Big) \\ - (\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{E}).\end{aligned}$$

Lemma 3.3. *Suppose that* $q>d$, $\mathbf{v}\in W_q^2(\mathcal{D})^d$, $\mathbf{T}\in W_q^1(\mathcal{D})^{d^2}$, $\mathbf{f}\in W_q^1(\mathcal{D})^d$, *and* $p\in W_q^1(\mathcal{D})$. *Then*

(3.13)
$$\|\mathcal{F}(\mathbf{f},\mathbf{v},p,\mathbf{T})\|_{L_q(\mathcal{D})}\le\|\mathbf{f}\|_{L_q(\mathcal{D})}+\sigma_q|\lambda_1|\,\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\Big(\|\mathbf{f}\|_{W_q^1(\mathcal{D})}+\|p\|_{W_q^1(\mathcal{D})}$$
$$+2\sigma_q\|\mathbf{v}\|^2_{W_q^2(\mathcal{D})}+\|\mathbf{T}\|_{W_q^1(\mathcal{D})}\Big)+4\sigma_q|\lambda_1-\mu_1|\,\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\|\mathbf{T}\|_{W_q^1(\mathcal{D})},$$

where σ_q *is the Sobolev constant* (1.6).

Proof. We use some relations in Section 9. From (9.7), we have
$$\|\mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla\mathbf{v})\|_{L_q(\mathcal{D})}\le 2\|\mathbf{v}\|_{L_\infty(\mathcal{D})}\|\mathbf{v}\|_{W^1_\infty(\mathcal{D})}\|\mathbf{v}\|_{W_q^2(\mathcal{D})}.$$
From (9.5), we have
$$\|\nabla\cdot((\nabla\mathbf{v})\circ\mathbf{T})\|_{L_q(\mathcal{D})}\le\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\|\mathbf{T}\|_{L_\infty(\mathcal{D})}+\|\mathbf{v}\|_{W^1_\infty(\mathcal{D})}\|\mathbf{T}\|_{W_q^1(\mathcal{D})},$$
$$\|\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E})\|_{L_q(\mathcal{D})}\le 2\big(\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\|\mathbf{T}\|_{L_\infty(\mathcal{D})}+\|\mathbf{v}\|_{W^1_\infty(\mathcal{D})}\|\mathbf{T}\|_{W_q^1(\mathcal{D})}\big).$$
The remaining terms are simpler. Thus
$$\begin{aligned}\|\mathcal{F}(\mathbf{f},\mathbf{v},p,\mathbf{T})\|_{L_q(\mathcal{D})}&\le\|\mathbf{f}\|_{L_q(\mathcal{D})}+|\lambda_1|\,\|\mathbf{v}\|_{W^1_\infty(\mathcal{D})}\big(\|\mathbf{f}\|_{W_q^1(\mathcal{D})}+\|p\|_{W_q^1(\mathcal{D})}\\&\quad+2\|\mathbf{v}\|_{L_\infty(\mathcal{D})}\|\mathbf{v}\|_{W_q^2(\mathcal{D})}+\|\mathbf{T}\|_{W_q^1(\mathcal{D})}\big)+|\lambda_1|\,\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\|\mathbf{T}\|_{L_\infty(\mathcal{D})}\\&\quad+2|\lambda_1-\mu_1|\big(\|\mathbf{v}\|_{W^1_\infty(\mathcal{D})}\|\mathbf{T}\|_{W_q^1(\mathcal{D})}+\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\|\mathbf{T}\|_{L_\infty(\mathcal{D})}\big)\\&\le\|\mathbf{f}\|_{L_q(\mathcal{D})}+\sigma_q|\lambda_1|\,\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\big(\|\mathbf{f}\|_{W_q^1(\mathcal{D})}+\|p\|_{W_q^1(\mathcal{D})}+2\sigma_q\|\mathbf{v}\|^2_{W_q^2(\mathcal{D})}\\&\quad+2\|\mathbf{T}\|_{W_q^1(\mathcal{D})}\big)+4\sigma_q|\lambda_1-\mu_1|\,\|\mathbf{v}\|_{W_q^2(\mathcal{D})}\|\mathbf{T}\|_{W_q^1(\mathcal{D})}.\quad\square\end{aligned}$$

3.5. The new system

We can now state the alternative system. It involves (2.4) to define $\mathbf{T}$ in terms of $\mathbf{u}$, the Navier-Stokes system (3.11), and the pressure transport equation (3.8):

(3.14)
$$\begin{gathered}-\eta\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=\mathcal{F}(\mathbf{f},\mathbf{u},p,\mathbf{T})\\ \nabla\cdot\mathbf{u}=0\text{ in }\mathcal{D}\text{ and }\mathbf{u}=\mathbf{0}\text{ on }\partial\mathcal{D}\\ p+\lambda_1\mathbf{u}\cdot\nabla p=\pi\\ \mathbf{T}+\lambda_1(\mathbf{u}\cdot\nabla\mathbf{T}-(\nabla\mathbf{u})\circ\mathbf{T}-\mathbf{T}\circ(\nabla\mathbf{u}^t))+(\lambda_1-\mu_1)(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E})=2\eta\mathbf{E},\end{gathered}$$

where $\mathcal{F}$ is defined by (3.12) and $\mathbf{E}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^t)$.

We have the following equivalence theorem. Its proof is not straightforward and is developed below in several steps.

THEOREM 3.4. *The formulations* (2.2)–(2.3) *and* (3.14) *are equivalent. More precisely, let $q > d$. If $\mathbf{u} \in W_q^2(\mathcal{D})^d$, $\mathbf{T} \in W_q^1(\mathcal{D})^{d^2}$, and $p \in W_q^1(\mathcal{D})/\mathbb{R}$ satisfy one of them, then they satisfy the other.*

In our derivation of (3.14), we assumed we had a solution of (2.2)–(2.3) with the stated regularity. Thus we have proved one direction of the equivalence. To prove the other direction, we must deal with the issue that we have created a new system by differentiation. Thus we need a way to be sure that we can go back to the original system and still have a solution. To do so, we will make use of the following result.

LEMMA 3.5. *Suppose that $\mathbf{v} \in W_q^2(\mathcal{D})^d$ with $\nabla\cdot\mathbf{v} = 0$ in $\mathcal{D}$ and $\mathbf{v} = \mathbf{0}$ on $\partial\mathcal{D}$, that $\mathbf{z} \in L_q(\mathcal{D})^m$, and that*

$$\mathbf{z} + \mathbf{v}\cdot\nabla\mathbf{z} = \mathbf{0}, \tag{3.15}$$

where we interpret $\mathbf{v}\cdot\nabla\mathbf{z} \in H^{-1}(\mathcal{D})^m$ as in (3.3). *Then $\mathbf{z} = \mathbf{0}$.*

Proof. The equation (3.15) implies that $\mathbf{v}\cdot\nabla z_i = -z_i \in L_q(\mathcal{D})$ for $i = 1,\ldots,m$. Thus the uniqueness results in [9] imply $\mathbf{z} = \mathbf{0}$. $\square$

Remark. What makes the uniqueness result of Lemma 3.5 so much simpler than the results of [9] is the extra regularity we are assuming on $\mathbf{v}$. Thus the product of $\mathbf{v} \in W_q^2(\mathcal{D})^d$ and $\nabla\mathbf{z}$ is well defined in $H^{-1}(\mathcal{D})^{dm}$, whereas if we only assume that $\mathbf{v} \in H^1(\mathcal{D})^d$ as in [9], such a product is defined only in a weaker space than $H^{-1}(\mathcal{D})^{dm}$.

We now return to the proof of Theorem 3.4. Recall that (2.3) and (2.4) are equivalent algebraic restatements of the last equation in (3.14). So we need to verify only the first line of (2.2), which is equivalent to (3.1). Let us verify that (3.1) holds provided that (3.14) holds. Define

$$\mathbf{w} = \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{f} \in L_q(\mathcal{D})^d. \tag{3.16}$$

To prove (3.1), we have to show that $\nabla\cdot\mathbf{T} = \mathbf{w}$. With the definition (3.16) of $\mathbf{w}$, we have

$$\begin{aligned}
-\eta\Delta\mathbf{u} + \mathbf{w} &= -\eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{f}\\
&= -\eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi + \nabla(p-\pi) - \mathbf{f}\\
&= \mathcal{F}(\mathbf{f},\mathbf{u},p,\mathbf{T}) + \nabla(p-\pi) - \mathbf{f}\\
&= \nabla(p-\pi) + \lambda_1\mathbf{u}\cdot\nabla\mathbf{f} + \lambda_1(\nabla\mathbf{u})^t\nabla p\\
-\lambda_1\big(\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) &- \nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})\big) - (\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{E}),
\end{aligned}$$

using (3.11) and (3.12). Now using (3.9), which is the gradient of the third equation in (3.14), we find

$$\begin{aligned}
-\eta\Delta\mathbf{u}+\mathbf{w} &= -\lambda_1\big((\nabla\mathbf{u})^t\nabla p+\mathbf{u}\cdot\nabla(\nabla p)\big)+\lambda_1\mathbf{u}\cdot\nabla\mathbf{f}+\lambda_1(\nabla\mathbf{u})^t\nabla p\\
&\quad -\lambda_1\big(\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u})-\nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})\big)-(\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E})\\
&= -\lambda_1\left(\mathbf{u}\cdot\nabla(\nabla p)\right)+\lambda_1\mathbf{u}\cdot\nabla\mathbf{f}\\
&\quad -\lambda_1\big(\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u})-\nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})\big)-(\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E})\\
&= -\lambda_1\mathbf{u}\cdot\nabla\mathbf{w}+\lambda_1\nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})-(\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E}).
\end{aligned}$$

Therefore

(3.17)
$$\eta\Delta\mathbf{u}=\mathbf{w}+\lambda_1\mathbf{u}\cdot\nabla\mathbf{w}-\lambda_1\nabla\cdot((\nabla\mathbf{u})\circ\mathbf{T})+(\lambda_1-\mu_1)\nabla\cdot(\mathbf{E}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{E}).$$

Note that (3.6) is just the divergence of the last equation in (3.14), in view of (3.5). Subtracting (3.17) from (3.6), we find

$$\nabla\cdot\mathbf{T}+\lambda_1\mathbf{u}\cdot\nabla\left(\nabla\cdot\mathbf{T}\right)=\mathbf{w}+\lambda_1\mathbf{u}\cdot\nabla\mathbf{w}.$$

By the uniqueness result in Lemma 3.5, we conclude that $\nabla\cdot\mathbf{T}=\mathbf{w}$.

This completes the proof of Theorem 3.4. □

The next three sections are devoted to showing that the system (3.14) has a solution $\mathbf{u}\in W^2_q(\mathcal{D})^d$, $\mathbf{T}\in W^1_q(\mathcal{D})^{d^2}$, and $p\in W^1_q(\mathcal{D})$ for $q>d$. This will be done in three steps, first establishing regularity of solutions of (2.3) given smooth $\mathbf{u}$ in Section 4. The reversed roles, showing $\mathbf{u}$ is smooth given smooth $\mathbf{T}$ is standard Navier-Stokes theory, which we address in Section 5. We then show how, by an iterative scheme, we can combine the two together in Section 6.

4. REGULARITY FOR T

We now consider the question of determining the regularity of the solution $\mathbf{T}$ of (2.3), or equivalently (2.4), in terms of corresponding regularity of $\mathbf{u}$. We will later return to the Navier-Stokes type equation (3.11) to close the loop, deriving regularity of $\mathbf{u}$ in terms of $\mathbf{T}$.

The tensor $\mathbf{T}$ can be viewed as a type of projection of the symmetric gradient $\mathbf{E}$ of $\mathbf{u}$. For tensor quantities $\mathbf{T}$ of any order $r\geq 1$, we denote by $|\mathbf{T}|$ the Euclidean norm of $\mathbf{T}$ when viewed as a vector of dimension d^r. We can simplify (2.4) by defining $\mathbf{v}=\lambda_1\mathbf{u}$, and it becomes

$$\mathbf{T}+(\mathbf{v}\cdot\nabla\mathbf{T}-(\nabla\mathbf{v})\circ\mathbf{T}-\mathbf{T}\circ(\nabla\mathbf{v}^t))+(1-\mu_1/\lambda_1)(\widetilde{\mathbf{E}}\circ\mathbf{T}+\mathbf{T}\circ\widetilde{\mathbf{E}})=2\eta\mathbf{E},$$

where $\widetilde{\mathbf{E}}=\lambda_1\mathbf{E}=\frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^t)$.

4.1. **Bounds for T in L_q**

The following result can be derived from [3, 12].

LEMMA 4.1. *Suppose that $2 \le d \le 4$, $\tilde\mu \in \mathbb{R}$, $q \ge 2$, $\mathcal{D} \subset \mathbb{R}^d$ is bounded and Lipschitz, and $\mathbf{v} \in W^1_\infty(\mathcal{D})^d$, with $\nabla\cdot \mathbf{v} = 0$ in $\mathcal{D}$, $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$ and*

$$\|\nabla \mathbf{v}\|_{L_\infty(\mathcal{D})} = \|\,|\nabla \mathbf{v}|\,\|_{L_\infty(\mathcal{D})} \le \frac{(1-c_0)}{|1+\tilde\mu| + |1-\tilde\mu|}, \quad \text{where } 0 < c_0 < 1. \tag{4.1}$$

Then for each $\mathbf{g} \in L_q(\mathcal{D})^{d^2}$, there is a unique solution $\mathbf{T} \in L_q(\mathcal{D})^{d^2}$ of the equation

$$\mathbf{T} + \mathbf{v}\cdot\nabla\mathbf{T} + \widetilde{\mathbf{R}}\circ\mathbf{T} + \mathbf{T}\circ\widetilde{\mathbf{R}}^t - \tilde\mu(\widetilde{\mathbf{E}}\circ\mathbf{T} + \mathbf{T}\circ\widetilde{\mathbf{E}}) = \mathbf{g}, \tag{4.2}$$

satisfying

$$\|\mathbf{T}\|_{L_q(\mathcal{D})} \le \frac{1}{c_0}\|\mathbf{g}\|_{L_q(\mathcal{D})}. \tag{4.3}$$

Here $\widetilde{\mathbf{R}} = \frac{1}{2}(\nabla\mathbf{v}^t - \nabla\mathbf{v})$ and $\widetilde{\mathbf{E}} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^t)$. Furthermore,

$$\|\mathbf{v}\cdot\nabla\mathbf{T}\|_{L_q(\mathcal{D})} \le \frac{3}{c_0}\|\mathbf{g}\|_{L_q(\mathcal{D})}. \tag{4.4}$$

The proof of this result will assume $q < \infty$, but once it is proved for arbitrary $q < \infty$, the case $q = \infty$ immediately follows by taking limits on both sides of (4.3) and (4.4) as $q \to \infty$.

Proof. The estimate (4.4) follows from (4.3) by using the equation (4.2) as follows:

$$\begin{aligned}
\|\mathbf{v}\cdot\nabla\mathbf{T}\|_{L_q(\mathcal{D})} &\le \|\mathbf{T}\|_{L_q(\mathcal{D})} + \|\widetilde{\mathbf{R}}\circ\mathbf{T} + \mathbf{T}\circ\widetilde{\mathbf{R}}^t\|_{L_q(\mathcal{D})} \\
&\qquad + |\tilde\mu|\,\|\widetilde{\mathbf{E}}\circ\mathbf{T} + \mathbf{T}\circ\widetilde{\mathbf{E}}\|_{L_q(\mathcal{D})} + \|\mathbf{g}\|_{L_q(\mathcal{D})} \\
&\le \|\mathbf{T}\|_{L_q(\mathcal{D})}\big(1 + 2(1+|\tilde\mu|)\|\nabla\mathbf{v}\|_{L^\infty(\mathcal{D})}\big) + \|\mathbf{g}\|_{L_q(\mathcal{D})} \\
&\le \Big(1 + \frac{1+2(1+|\tilde\mu|)}{c_0}\|\nabla\mathbf{v}\|_{L^\infty(\mathcal{D})}\Big)\|\mathbf{g}\|_{L_q(\mathcal{D})} \\
&= \frac{1+c_0+2(1+|\tilde\mu|)}{c_0}\|\nabla\mathbf{v}\|_{L^\infty(\mathcal{D})}\|\mathbf{g}\|_{L_q(\mathcal{D})} \\
&\le \frac{(1+c_0+2(1+|\tilde\mu|))(1-c_0)}{c_0(|1+\tilde\mu|+|1-\tilde\mu|)}\|\mathbf{g}\|_{L_q(\mathcal{D})}.
\end{aligned}$$

But

$$\begin{aligned}
\frac{(1+c_0+2(1+|\tilde\mu|))(1-c_0)}{|1+\tilde\mu|+|1-\tilde\mu|} &= \frac{1-c_0^2+2(1-c_0)(1+|\tilde\mu|)}{|1+\tilde\mu|+|1-\tilde\mu|} \\
&\le \frac{1+2(1+|\tilde\mu|)}{|1+\tilde\mu|+|1-\tilde\mu|} \le 3.
\end{aligned}$$

Thus we only have to prove the well-posedness of (4.2) and establish the bound (4.3). Let us make the change of variable

$$\mathbf{S} = -\tfrac{1}{2}\tilde{\mu}(\nabla\mathbf{v}^t + \nabla\mathbf{v}) + \widetilde{\mathbf{R}} = -\tilde{\mu}\widetilde{\mathbf{E}} + \widetilde{\mathbf{R}} = \tfrac{1}{2}(1-\tilde{\mu})\nabla\mathbf{v}^t - \tfrac{1}{2}(1+\tilde{\mu})\nabla\mathbf{v}. \tag{4.5}$$

Then

$$\mathbf{S}\circ\mathbf{T}+\mathbf{T}\circ\mathbf{S}^t = (-\tilde{\mu}\widetilde{\mathbf{E}}+\widetilde{\mathbf{R}})\circ\mathbf{T}+\mathbf{T}\circ(-\tilde{\mu}\widetilde{\mathbf{E}}+\widetilde{\mathbf{R}}^t) = \widetilde{\mathbf{R}}\circ\mathbf{T}+\mathbf{T}\circ\widetilde{\mathbf{R}}^t - \tilde{\mu}(\widetilde{\mathbf{E}}\circ\mathbf{T}+\mathbf{T}\circ\widetilde{\mathbf{E}}).$$

Thus (4.2) becomes

$$\mathbf{T} + \mathbf{v}\cdot\nabla\mathbf{T} + \mathbf{S}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{S}^t = \mathbf{g}. \tag{4.6}$$

To fit into the framework of [12], we view $\mathbf{T}$ as a function whose values are vectors of dimension d^2, and we use the Frobenius product " : " as the inner-product on such vectors, with norm $|\mathbf{T}(x)| = \sqrt{\mathbf{T}(x) : \mathbf{T}(x)}$. In particular, [12, (4)] and [12, Theorem 3] can be phrased as follows.

Lemma 4.2. *Suppose that $2 \le d \le 4$, $q \ge 2$, $\mathcal{D} \subset \mathbb{R}^d$ is a bounded, Lipschitz domain, and $\mathbf{v} \in H^1(\mathcal{D})^d$ with $\nabla\cdot\mathbf{v} = 0$ in $\mathcal{D}$ and $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$. Suppose further that $\mathbf{C}$ is an $m\times m$ matrix valued function such that for some constant $c_0 > 0$*

$$\big(\mathbf{C}(x)\boldsymbol{\xi}\big)\cdot\boldsymbol{\xi} \ge c_0|\boldsymbol{\xi}|^2 \;\forall\boldsymbol{\xi}\in\mathbb{R}^m$$

for almost all $x \in \mathcal{D}$. Then for all $\mathbf{g} \in L_q(\mathcal{D})^m$, there is a unique solution $\mathbf{T} \in L_q(\mathcal{D})^m$ to

$$\mathbf{v}\cdot\nabla\mathbf{T} + \mathbf{C}\circ\mathbf{T} = \mathbf{g},$$

satisfying

$$\|\,\mathbf{T}\,\|_{L_q(\mathcal{D})} \le \frac{1}{c_0}\|\,\mathbf{g}\,\|_{L_q(\mathcal{D})}. \tag{4.7}$$

We note that the results in [12] were stated for the special case when the size of the vector m was the same as the dimension of the domain d (*i.e.*, $m = d$), but it can be easily checked that the result holds for vectors of arbitrary length $m \ge 1$. As stated after Lemma 4.1, (4.7) is also valid for $q = \infty$.

Since the mapping

$$\mathbf{T} \mapsto \mathbf{S}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{S}^t$$

is linear, there is a matrix-valued function $\mathcal{M}$ such that

$$\mathcal{M}\circ\mathbf{T} = \mathbf{S}\circ\mathbf{T} + \mathbf{T}\circ\mathbf{S}^t. \tag{4.8}$$

Therefore (4.2) takes the form

$$\mathbf{T} + \mathbf{v}\cdot\nabla\mathbf{T} + \mathcal{M}\circ\mathbf{T} = \mathbf{g}, \tag{4.9}$$

which corresponds to the equation in [12, (2)] with $\mathbf{C} = \mathbf{I} + \mathcal{M}$. Thus we need to show that $\mathbf{I}+\mathcal{M}$ can be bounded below appropriately (that is, it is coercive). For almost all $x \in \mathcal{D}$,

$$\begin{aligned}(\mathbf{T}(x) + \mathcal{M}(x)\circ\mathbf{T}(x)) : \mathbf{T}(x) &= |\mathbf{T}(x)|^2 + \mathcal{M}(x)\circ\mathbf{T}(x) : \mathbf{T}(x)\\ &\ge |\mathbf{T}(x)|^2 - |\mathcal{M}(x)\circ\mathbf{T}(x) : \mathbf{T}(x)|.\end{aligned} \tag{4.10}$$

By the definitions of $\mathbf{S}$ and $\mathcal{M}$, we have by the multiplicative property of the Frobenius norm that, for any tensor $\mathbf{U}$ of order 2,

$$\begin{aligned} |\mathcal{M}(x)\circ\mathbf{T}(x):\mathbf{U}| &= |\mathbf{S}(x)\circ\mathbf{T}(x):\mathbf{U}+\mathbf{T}(x)\circ\mathbf{S}(x)^t:\mathbf{U}| \\ &\le |\mathbf{S}(x)\circ\mathbf{T}(x):\mathbf{U}|+|\mathbf{T}(x)\circ\mathbf{S}(x)^t:\mathbf{U}| \\ &\le |\mathbf{S}(x)\circ\mathbf{T}(x)|\,|\mathbf{U}|+|\mathbf{T}(x)\circ\mathbf{S}(x)^t|\,|\mathbf{U}| \\ &\le |\mathbf{S}(x)|\,|\mathbf{T}(x)|\,|\mathbf{U}|+|\mathbf{T}(x)|\,|\mathbf{S}(x)^t|\,|\mathbf{U}| \\ &= 2|\mathbf{S}(x)|\,|\mathbf{T}(x)|\,|\mathbf{U}|. \end{aligned} \tag{4.11}$$

Recalling the definition of $\mathbf{S}$ in (4.5), we have

$$\begin{aligned} |\mathbf{S}(x)| &= \tfrac{1}{2}|(1-\tilde{\mu})\nabla\mathbf{v}(x)^t-(1+\tilde{\mu})\nabla\mathbf{v}(x)| \\ &\le \tfrac{1}{2}(|1-\tilde{\mu}|+|1+\tilde{\mu}|)|\nabla\mathbf{v}(x)|. \end{aligned} \tag{4.12}$$

Therefore (4.11) and (4.12) imply

$$|\mathcal{M}(x)\circ\mathbf{T}(x):\mathbf{T}(x)| \le (|1-\tilde{\mu}|+|1+\tilde{\mu}|)|\nabla\mathbf{v}(x)|\,|\mathbf{T}(x)|^2 \le (1-c_0)|\mathbf{T}(x)|^2, \tag{4.13}$$

where c_0 is given by (4.1). Thus (4.10) and (4.13) imply that

$$(\mathbf{T}(x)+\mathcal{M}(x)\circ\mathbf{T}(x)):\mathbf{T}(x) \ge c_0|\mathbf{T}(x)|^2. \tag{4.14}$$

This gives the required coercivity to use the results of [12]. In particular, (4.3) follows from [12, Theorem 3]. □

Writing $\mathbf{v}=\lambda_1\mathbf{u}$, and picking $\tilde{\mu}=\mu_1/\lambda_1$, Lemma 4.1 implies the following.

Lemma 4.3. *Suppose that $\mathcal{D}$ and q satisfy the conditions of Lemma 4.1 and that $\mathbf{u}\in W^1_\infty(\mathcal{D})^d$, with $\nabla\cdot\mathbf{u}=0$ in $\mathcal{D}$ and $\mathbf{u}=\mathbf{0}$ on $\partial\mathcal{D}$. Define*

$$\nu=|\lambda_1+\mu_1|+|\lambda_1-\mu_1|.$$

Suppose

$$\mathcal{U}=\|\nabla\mathbf{u}\|_{L_\infty(\mathcal{D})}=\|\,|\nabla\mathbf{u}|\,\|_{L_\infty(\mathcal{D})}<\frac{1}{\nu}.$$

Then there is a unique solution $\mathbf{T}\in L_q(\mathcal{D})^{d^2}$ to (2.3) such that

$$\max\left\{\|\mathbf{T}\|_{L_q(\mathcal{D})},\ \frac{|\lambda_1|}{3}\|\mathbf{u}\cdot\nabla\mathbf{T}\|_{L_q(\mathcal{D})}\right\}\le\frac{2\eta}{1-\nu\mathcal{U}}\|\nabla\mathbf{u}\|_{L_q(\mathcal{D})}.$$

The proof follows from Lemma 4.1, by taking $c_0=1-\nu\mathcal{U}$ and $\mathbf{g}=2\eta\mathbf{E}$ and applying (4.3) and (4.4).

4.2. **Smoothness of T**

LEMMA 4.4. *Suppose that the conditions of Lemma 4.1 hold, that condition* (1.3) *holds, and that* $\mathbf{g} \in H^1(\mathcal{D})^d$. *Suppose moreover that* $\mathbf{v} \in W_q^2(\mathcal{D})^d$ *for some* $q > d$ *and*

$$\| \nabla \mathbf{v} \|_{L_\infty(\mathcal{D})} \le \frac{(1-c_1)}{1+|1+\tilde{\mu}|+|1-\tilde{\mu}|}, \tag{4.15}$$

where $0 < c_1 < 1$. *Then the solution* $\mathbf{T}$ *to* (4.2) *satisfies* $\mathbf{T} \in H^1(\mathcal{D})^{d^2}$.

Proof. We recall that (4.2) is equivalent to (4.6). Following [2], we introduce a regularized problem: find $\mathbf{T}^\epsilon \in H^1(\mathcal{D})^{d^2}$ such that

$$-\epsilon\Delta\mathbf{T}^\epsilon + \mathbf{T}^\epsilon + \mathbf{v}\cdot\nabla\mathbf{T}^\epsilon + \mathbf{S}\circ\mathbf{T}^\epsilon + \mathbf{T}^\epsilon\circ\mathbf{S}^t = \mathbf{g} \quad \text{in } \mathcal{D}, \tag{4.16}$$

where $\mathbf{S}$ is defined in (4.5), with natural boundary conditions as in (1.1), that is,

$$\nabla(\mathbf{T}^\epsilon)_{ij}\cdot\mathbf{n} = 0 \quad \text{on } \partial\mathcal{D}, \quad \text{for } i,j = 1,\dots,d.$$

Multiplying (4.16) by $\mathbf{T}^\epsilon$, integrating over $\mathcal{D}$, and integrating by parts, we find

$$\begin{aligned}\epsilon\int_{\mathcal{D}} |\nabla\mathbf{T}^\epsilon|^2\,\mathrm{d}x + \int_{\mathcal{D}} |\mathbf{T}^\epsilon|^2\,\mathrm{d}x + \int_{\mathcal{D}} (\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon) : \mathbf{T}^\epsilon\,\mathrm{d}x \\ + \int_{\mathcal{D}} (\mathbf{S}\circ\mathbf{T}^\epsilon + \mathbf{T}^\epsilon\circ\mathbf{S}^t) : \mathbf{T}^\epsilon\,\mathrm{d}x = \int_{\mathcal{D}} \mathbf{g} : \mathbf{T}^\epsilon\,\mathrm{d}x.\end{aligned} \tag{4.17}$$

We have

$$\int_{\mathcal{D}} (\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon) : \mathbf{T}^\epsilon\,\mathrm{d}x = \sum_{ij}\int_{\mathcal{D}} (\mathbf{v}\cdot\nabla T_{ij}^\epsilon) T_{ij}^\epsilon\,\mathrm{d}x = 0,$$

since $\nabla\cdot\mathbf{v} = 0$ in $\mathcal{D}$ and $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$. From (9.14), (4.12), and (4.1), we have

$$\begin{aligned}\left|\int_{\mathcal{D}} (\mathbf{S}\circ\mathbf{T}^\epsilon + \mathbf{T}^\epsilon\circ\mathbf{S}^t) : \mathbf{T}^\epsilon\,\mathrm{d}x\right| &\le 2\|\mathbf{S}\|_{L_\infty(\mathcal{D})}\|\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}^2 \\ &\le (|1+\tilde{\mu}|+|1-\tilde{\mu}|)\|\nabla\mathbf{v}\|_{L_\infty(\mathcal{D})}\|\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}^2 \\ &\le (1-c_0)\|\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}^2.\end{aligned}$$

Applying these estimates to (4.17), we obtain

$$\begin{aligned}\epsilon\int_{\mathcal{D}} |\nabla\mathbf{T}^\epsilon|^2\,\mathrm{d}x + c_0\int_{\mathcal{D}} |\mathbf{T}^\epsilon|^2\,\mathrm{d}x \le \left|\int_{\mathcal{D}} \mathbf{g} : \mathbf{T}^\epsilon\,\mathrm{d}x\right| &\le \|\mathbf{g}\|_{L_2(\mathcal{D})}\|\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})} \\ &\le \frac{1}{2c_0}\|\mathbf{g}\|_{L_2(\mathcal{D})}^2 + \tfrac{1}{2}c_0\|\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}^2.\end{aligned}$$

In particular, we obtain

$$\|\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})} \le \frac{1}{c_0}\|\mathbf{g}\|_{L_2(\mathcal{D})}, \tag{4.18}$$

consistent with (4.3).

To proceed, we want to take the $L^2(\mathcal{D})^{d^2}$ inner-product of the terms on both sides of (4.16) with $-\Delta\mathbf{T}^\epsilon$ and integrate by parts. Formally, this gives

$$\begin{aligned}\epsilon\int_{\mathcal{D}}|\Delta\mathbf{T}^\epsilon|^2\,\mathrm{d}x+\int_{\mathcal{D}}|\nabla\mathbf{T}^\epsilon|^2\,\mathrm{d}x+\int_{\mathcal{D}}\nabla(\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x\\+\int_{\mathcal{D}}\nabla(\mathbf{S}\circ\mathbf{T}^\epsilon+\mathbf{T}^\epsilon\circ\mathbf{S}^t):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x=\int_{\mathcal{D}}\nabla\mathbf{g}:\nabla\mathbf{T}^\epsilon\,\mathrm{d}x.\end{aligned}\tag{4.19}$$

On the one hand, since $\mathbf{T}^\epsilon\in H^1(\mathcal{D})^{d^2}$, we infer from (4.16) that $-\Delta\mathbf{T}^\epsilon\in L^2(\mathcal{D})^{d^2}$ and thus the scalar products leading to (4.19) are well defined. But on the other hand, integration by parts requires that $\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon$ be in $H^1(\mathcal{D})^{d^2}$. This follows from the regularity assumption (1.3). In particular, (9.6) implies

$$\nabla(\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon)=\nabla\mathbf{T}^\epsilon\circ\nabla\mathbf{v}+\mathbf{v}\cdot\nabla(\nabla\mathbf{T}^\epsilon)=\nabla\mathbf{T}^\epsilon\circ\nabla\mathbf{v}+(\nabla^2\mathbf{T}^\epsilon)\circ\mathbf{v},$$

and we see that it is necessary that $\nabla^2\mathbf{T}^\epsilon$ be integrable to some degree to justify (4.19). We can expand the corresponding term as follows:

$$\begin{aligned}\int_{\mathcal{D}}\nabla(\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x&=\int_{\mathcal{D}}(\nabla\mathbf{T}^\epsilon\circ\nabla\mathbf{v}+\mathbf{v}\cdot\nabla(\nabla\mathbf{T}^\epsilon)):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x\\&=\int_{\mathcal{D}}(\nabla\mathbf{T}^\epsilon\circ\nabla\mathbf{v}):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x.\end{aligned}$$

Thus we have the bound

$$\left|\int_{\mathcal{D}}\nabla(\mathbf{v}\cdot\nabla\mathbf{T}^\epsilon):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x\right|\le\|\,\nabla\mathbf{T}^\epsilon\,\|^2_{L_2(\mathcal{D})}\|\,\nabla\mathbf{v}\,\|_{L_\infty(\mathcal{D})}.\tag{4.20}$$

Next, by (9.12), we have

$$\begin{aligned}&\int_{\mathcal{D}}\nabla(\mathbf{S}\circ\mathbf{T}^\epsilon+\mathbf{T}^\epsilon\circ\mathbf{S}^t):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x\\&=\int_{\mathcal{D}}\big(\mathbf{S}\circ\nabla\mathbf{T}^\epsilon+\mathcal{B}(\nabla\mathbf{S},\mathbf{T}^\epsilon)+\mathbf{T}^\epsilon\circ\nabla\mathbf{S}^t+\mathcal{B}(\nabla\mathbf{T}^\epsilon,\mathbf{S}^t)\big):\nabla\mathbf{T}^\epsilon\,\mathrm{d}x,\end{aligned}\tag{4.21}$$

where $\mathcal{B}$ is a bilinear mapping on tensors defined by (9.10), which here reduces to

$$(\mathcal{B}(\mathbf{W},\mathbf{U}))_{ijk}=\sum_{\ell=1}^{d}(\mathbf{W})_{i\ell k}U_{\ell j}\,.\tag{4.22}$$

In what follows, we will frequently make use of two estimates. The first is

$$\|\,u\,v\,\|_{L_2(\mathcal{D})}\le\|\,u\,\|_{L^q(\mathcal{D})}\|\,v\,\|_{L^{\frac{2q}{q-2}}(\mathcal{D})},\tag{4.23}$$

valid provided $q>2$. To prove this, we use Hölder's inequality to get

$$\|\,u\,v\,\|^2_{L_2(\mathcal{D})}=\int_{\mathcal{D}}u^2v^2\,\mathrm{d}x\le\|\,u^2\,\|_{L^s(\mathcal{D})}\|\,v^2\,\|_{L^{s'}(\mathcal{D})}=\|\,u\,\|^2_{L^q(\mathcal{D})}\|\,v\,\|^2_{L^{2s'}(\mathcal{D})},$$

where $s = q/2$ and $s' = s/(s-1) = q/(q-2)$. This proves (4.23). The second inequality, which follows from (4.23) and the Sobolev inequality (1.7), is

$$\begin{aligned} \Big| \int_{\mathcal{D}} u(x)v(x)w(x)\,\mathrm{d}x \Big| &\le \| u\,v \|_{L_2(\mathcal{D})} \| w \|_{L_2(\mathcal{D})} \\ &\le \| u \|_{L_q(\mathcal{D})} \| v \|_{L_{\frac{2q}{q-2}}(\mathcal{D})} \| w \|_{L_2(\mathcal{D})} \\ &\le \sigma_q \| u \|_{L_q(\mathcal{D})} \| v \|_{H^1(\mathcal{D})} \| w \|_{L_2(\mathcal{D})}, \end{aligned} \tag{4.24}$$

valid provided $q > d$ for $d = 2$ or $q \ge d$ for $d \ge 3$.

From (9.13), $|\mathcal{B}(\nabla \mathbf{V}(x), \mathbf{U}(x))| \le |\nabla \mathbf{V}(x)|\,|\mathbf{U}(x)|$, so (4.21) and (4.23) imply

$$\begin{aligned} \Big| \int_{\mathcal{D}} \nabla(\mathbf{S} \circ \mathbf{T}^\epsilon + \mathbf{T}^\epsilon \circ \mathbf{S}^t) : \nabla \mathbf{T}^\epsilon \,\mathrm{d}x \Big| &\le 2 \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})}^2 \| \mathbf{S} \|_{L_\infty(\mathcal{D})} \\ &\quad + 2 \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})} \| \mathbf{T}^\epsilon \|_{L_{\frac{2q}{q-2}}(\mathcal{D})} \| \nabla \mathbf{S} \|_{L_q(\mathcal{D})}. \end{aligned} \tag{4.25}$$

By the Gagliardo-Nirenberg inequality [4, 6] and (4.18), there is a constant $c_q < \infty$ such that

$$\begin{aligned} \| \mathbf{T}^\epsilon \|_{L_{\frac{2q}{q-2}}(\mathcal{D})} &\le c_q \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})}^{d/q} \| \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})}^{1-d/q} \\ &\le \frac{c_q}{c_0^{1-d/q}} \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})}^{d/q} \| \mathbf{g} \|_{L_2(\mathcal{D})}^{1-d/q}, \end{aligned} \tag{4.26}$$

provided $q > d$. We need to use the elementary inequality

$$a^\theta b^{1-\theta} \le \theta a + (1-\theta) b, \tag{4.27}$$

valid for $a, b \ge 0$ and $0 \le \theta \le 1$, which is a consequence of the concavity of the logarithm function. As a consequence of (4.27), we have

$$AB \le \frac{1}{r} A^r + \frac{1}{r'} B^{r'} \text{ for } A, B \ge 0, \quad \frac{1}{r} + \frac{1}{r'} = 1 \quad (1 < r < \infty), \tag{4.28}$$

by choosing $r = 1/\theta$, $A = a^\theta$, and $B = b^{1-\theta}$. From (4.26) and (4.27), we have

$$\| \mathbf{T}^\epsilon \|_{L_{\frac{2q}{q-2}}(\mathcal{D})} \le \delta \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})} + C_\delta \| \mathbf{g} \|_{L_2(\mathcal{D})},$$

where $\delta > 0$ is arbitrary. Thus the estimate (4.25) becomes

(4.29)

$$\begin{aligned} \Big| \int_{\mathcal{D}} \nabla(\mathbf{S} \circ \mathbf{T}^\epsilon + \mathbf{T}^\epsilon \circ \mathbf{S}^t) : \nabla \mathbf{T}^\epsilon \,\mathrm{d}x \Big| &\le 2 \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})}^2 \big(\| \mathbf{S} \|_{L_\infty(\mathcal{D})} + \delta \| \nabla \mathbf{S} \|_{L_q(\mathcal{D})} \big) \\ &\quad + 2 C_\delta \| \mathbf{g} \|_{L_2(\mathcal{D})} \| \nabla \mathbf{T}^\epsilon \|_{L_2(\mathcal{D})} \| \nabla \mathbf{S} \|_{L_q(\mathcal{D})}. \end{aligned}$$

Combining (4.20) and (4.29), and using (4.15) and (4.12), (4.19) becomes

(4.30)
$$
\begin{aligned}
\epsilon\int_{\mathcal{D}}|\Delta\mathbf{T}^\epsilon|^2\,\mathrm{d}x + \int_{\mathcal{D}}|\nabla\mathbf{T}^\epsilon|^2\,\mathrm{d}x \le{}& \|\nabla\mathbf{g}\|_{L_2(\mathcal{D})}\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}\\
&+\|\nabla\mathbf{T}^\epsilon\|^2_{L_2(\mathcal{D})}\Big(\|\nabla\mathbf{v}\|_{L_\infty(\mathcal{D})}+2\|\mathbf{S}\|_{L_\infty(\mathcal{D})}+\delta\|\nabla\mathbf{S}\|_{L_q(\mathcal{D})}\Big)\\
&+2C_\delta\|\mathbf{g}\|_{L_2(\mathcal{D})}\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}\|\nabla\mathbf{S}\|_{L_q(\mathcal{D})}\\
\le{}& \|\nabla\mathbf{g}\|_{L_2(\mathcal{D})}\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}+\|\nabla\mathbf{T}^\epsilon\|^2_{L_2(\mathcal{D})}\Big((1-c_1)+\delta\|\nabla\mathbf{S}\|_{L_q(\mathcal{D})}\Big)\\
&+2C_\delta\|\mathbf{g}\|_{L_2(\mathcal{D})}\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}\|\nabla\mathbf{S}\|_{L_q(\mathcal{D})}.
\end{aligned}
$$

If we choose $\delta>0$ so that $\delta\|\nabla\mathbf{S}\|_{L_q(\mathcal{D})}\le\frac{1}{2}c_1$, then (4.30) implies

$$
\begin{aligned}
\epsilon\int_{\mathcal{D}}|\Delta\mathbf{T}^\epsilon|^2\,\mathrm{d}x + \tfrac{1}{2}c_1\int_{\mathcal{D}}|\nabla\mathbf{T}^\epsilon|^2\,\mathrm{d}x \le{}& \|\nabla\mathbf{g}\|_{L_2(\mathcal{D})}\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}\\
&+C\|\mathbf{g}\|_{L_2(\mathcal{D})}\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}\|\nabla\mathbf{S}\|_{L_q(\mathcal{D})}.
\end{aligned}
\tag{4.31}
$$

Dividing by $\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}$, we see that $\|\nabla\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}$ is bounded independently of ϵ. Using (4.18), we conclude that $\|\mathbf{T}^\epsilon\|_{H^1(\mathcal{D})}$ is also bounded independently of ϵ. Thus there is a subsequence ϵ_j such that $\mathbf{T}^{\epsilon_j}$ converges weakly to $\widetilde{\mathbf{T}}\in H^1(\mathcal{D})^{d^2}$. The estimate (4.31) also shows that

$$\epsilon\int_{\mathcal{D}}|\Delta\mathbf{T}^\epsilon|^2\,\mathrm{d}x\le C$$

for some constant C independent of ϵ, and thus

$$\|\epsilon\Delta\mathbf{T}^\epsilon\|_{L_2(\mathcal{D})}\le\sqrt{C\epsilon}$$

for all ϵ. Taking the weak limit $\epsilon_j\to0$ in (4.16) shows that $\widetilde{\mathbf{T}}\in H^1(\mathcal{D})^{d^2}$ is a solution to (4.2), and by uniqueness of such solutions, we conclude that the original solution $\mathbf{T}$ must be in $H^1(\mathcal{D})^{d^2}$. □

4.3. Bounds for $\nabla\mathbf{T}$

LEMMA 4.5. *Under the conditions of Lemma 4.4, for each* $\mathbf{g}\in W^1_q(\mathcal{D})^d$, $q>d$, *there is a unique solution* $\mathbf{T}\in W^1_q(\mathcal{D})^{d^2}$ *to* (4.2) *such that*

$$\|\nabla\mathbf{T}\|_{L_q(\mathcal{D})}\le\frac{1}{c_1}\Big(\|\nabla\mathbf{g}\|_{L_q(\mathcal{D})}+\frac{|1-\tilde\mu|+|1+\tilde\mu|}{c_0}\|\nabla^2\mathbf{v}\|_{L_q(\mathcal{D})}\|\mathbf{g}\|_{L_\infty(\mathcal{D})}\Big).$$

Recall that we have already obtained a bound for $\|\mathbf{v}\cdot\nabla\mathbf{T}\|_{L_q(\mathcal{D})}$ in (4.4).

Proof. To estimate $\nabla \mathbf{T}$ in $L^q(\mathcal{D})^{d^3}$, we write $\mathbf{W} = \nabla \mathbf{T}$. In view of our previous arguments, if $\mathbf{g} \in H^1(\mathcal{D})^d$, then $\mathbf{T} \in H^1(\mathcal{D})^{d^2}$. Applying ∇ to (4.6), which is equivalent to (4.2), we see that $\mathbf{W} \in L_2(\mathcal{D})^{d^2}$ solves

$$\mathbf{W} + \mathbf{v} \cdot \nabla \mathbf{W} + \mathbf{W} \circ \nabla \mathbf{v} + \mathbf{S} \circ \mathbf{W} + \mathcal{B}(\mathbf{W}, \mathbf{S}^t) = \nabla \mathbf{g} - \mathcal{B}(\nabla \mathbf{S}, \mathbf{T}) - \mathbf{T} \circ \nabla \mathbf{S}^t, \tag{4.32}$$

where $\mathbf{S}$ is defined in (4.5), $\mathcal{B}$ is defined in (4.22), and we have used the tensor identities (9.6) and (9.12). Thus we seek coercivity for the linear map $\mathbf{C}$ where

$$\mathbf{C} \circ \mathbf{W} = \mathbf{W} + \mathbf{W} \circ \nabla \mathbf{v} + \mathbf{S} \circ \mathbf{W} + \mathcal{B}(\mathbf{W}, \mathbf{S}^t).$$

By analogy with (4.8), we can write $\mathbf{C} = \mathbf{I} + \mathcal{M}$ where

$$\mathcal{M}\mathbf{W} = \mathbf{W} \circ \nabla \mathbf{v} + \mathbf{S} \circ \mathbf{W} + \mathcal{B}(\mathbf{W}, \mathbf{S}^t). \tag{4.33}$$

However, $\mathbf{W}$ is a tensor of order 3, so we view it as a vector of dimension d^3, and we need to recapitulate the previous arguments. Let us introduce the notation $\mathcal{T}_r$ for the set of tensors in d dimensions of order r. We can extend the concept of the Frobenius product of matrices to $\mathcal{T}_r$, because it corresponds simply to the ℓ_2 inner-product of vectors of dimension d^r. Thus (4.10) becomes

$$\begin{aligned}(\mathbf{W}(x) + \mathcal{M}(x)\mathbf{W}(x)) : \mathbf{W}(x) &= |\mathbf{W}(x)|^2 + \mathcal{M}(x)\mathbf{W}(x) : \mathbf{W}(x) \\ &\geq |\mathbf{W}(x)|^2 - |\mathcal{M}(x)\mathbf{W}(x) : \mathbf{W}(x)|.\end{aligned}$$

Here $\mathcal{M}$ is simply a linear operator mapping $\mathcal{T}_3$ to $\mathcal{T}_3$ given by (4.33). This requires the interpretation that the tensor contractions $\mathbf{W} \mapsto \mathbf{W} \circ \nabla \mathbf{v}$ and $\mathbf{W} \mapsto \mathbf{S} \circ \mathbf{W}$, and the map $\mathbf{W} \mapsto \mathcal{B}(\mathbf{W}, \mathbf{S}^t)$, give linear operators on $\mathcal{T}_3$. Indeed,

$$(\mathbf{S} \circ \mathbf{W})_{ijk} = \sum_{l=1}^{d} (\mathbf{S})_{il} (\mathbf{W})_{ljk},$$

with a similar interpretation derived from (4.22). With the interpretation that " : " is the usual ℓ_2 inner-product on vectors of dimension d^3, and that $|\mathbf{W}|$ denotes the corresponding norm, (4.11) remains valid in this context, as a consequence of (9.14):

$$|\mathbf{S}(x) \circ \mathbf{W}(x) : \mathbf{W}(x)| \leq |\mathbf{S}(x)|\, |\mathbf{W}(x)|^2.$$

Similarly, (9.14) implies

$$|\mathbf{W}(x) \circ \nabla \mathbf{v}(x) : \mathbf{W}(x)| \leq |\nabla \mathbf{v}(x)|\, |\mathbf{W}(x)|^2.$$

Applying (9.13) with $\mathbf{U} = \mathbf{S}^t$, we find

$$|\mathcal{B}(\mathbf{W}(x), \mathbf{S}^t(x)) : \mathbf{W}(x)| \leq |\mathcal{B}(\mathbf{W}(x), \mathbf{S}^t(x))|\, |\mathbf{W}(x)| \leq |\mathbf{W}(x)|^2 |\mathbf{S}(x)|.$$

Thus the following analog of (4.13) holds:

$$\begin{aligned}|\mathcal{M}(x)\mathbf{W}(x) : \mathbf{W}(x)| &\leq (2|\mathbf{S}(x)| + |\nabla \mathbf{v}(x)|)\, |\mathbf{W}(x)|^2 \\ &\leq (1 + |1 - \tilde{\mu}| + |1 + \tilde{\mu}|) |\nabla \mathbf{v}(x)|\, |\mathbf{W}(x)|^2 \leq (1 - c_1) |\mathbf{W}(x)|^2.\end{aligned}$$

Therefore

$$(\mathbf{C}(x)\circ\mathbf{W}(x)) : \mathbf{W}(x) = |\mathbf{W}(x)|^2 + \mathcal{M}(x)\mathbf{W}(x) : \mathbf{W}(x) \geq c_1|\mathbf{W}(x)|^2,$$

which provides an analog of (4.14). So applying Lemma 4.2 to (4.32) yields

$$\| \mathbf{W} \|_{L_q(\mathcal{D})} \leq \frac{1}{c_1} \| \nabla\mathbf{g} - \mathcal{B}(\nabla\mathbf{S}, \mathbf{T}) - \mathbf{T}\circ\nabla\mathbf{S}^t \|_{L_q(\mathcal{D})}. \tag{4.34}$$

Applying (9.13), we have

$$|\mathcal{B}(\nabla\mathbf{S}, \mathbf{T}) + \mathbf{T}\circ\nabla\mathbf{S}^t| \leq 2|\nabla\mathbf{S}|\,|\mathbf{T}|.$$

Recall from (4.5) that $2\mathbf{S} = (1-\tilde{\mu})\nabla\mathbf{v}^t - (1+\tilde{\mu})\nabla\mathbf{v}$. Hence

$$|\mathcal{B}(\nabla\mathbf{S}(x), \mathbf{T}(x)) + \mathbf{T}(x)\circ\nabla\mathbf{S}(x)^t| \leq (|1-\tilde{\mu}| + |1+\tilde{\mu}|)|\nabla^2\mathbf{v}(x)|\,|\mathbf{T}(x)|,$$

for almost all $x \in \mathcal{D}$. Therefore (4.34) becomes, in view of (4.7) applied with $q = \infty$,

$$\begin{aligned} \| \mathbf{W} \|_{L_q(\mathcal{D})} &\leq \frac{1}{c_1}\left(\| \nabla\mathbf{g} \|_{L_q(\mathcal{D})} + (|1-\tilde{\mu}| + |1+\tilde{\mu}|)\| \nabla^2\mathbf{v}(x) \|_{L_q(\mathcal{D})}\| \mathbf{T} \|_{L_\infty(\mathcal{D})}\right) \\ &\leq \frac{1}{c_1}\left(\| \nabla\mathbf{g} \|_{L_q(\mathcal{D})} + \frac{|1-\tilde{\mu}| + |1+\tilde{\mu}|}{c_0}\| \nabla^2\mathbf{v}(x) \|_{L_q(\mathcal{D})}\| \mathbf{g} \|_{L_\infty(\mathcal{D})}\right). \end{aligned}$$

Recalling that $\mathbf{W} = \nabla\mathbf{T}$ completes the proof. □

Writing $\mathbf{v} = \lambda_1\mathbf{u}$, and picking $\tilde{\mu} = \mu_1/\lambda_1$, Lemmas 4.1 and 4.5 combine to yield the following.

Lemma 4.6. *Suppose that $\mathcal{D}$ satisfies the condition (1.3), $q > d$, and $\mathbf{u} \in W_q^2(\mathcal{D})^d$, with $\nabla\cdot\mathbf{u} = 0$ in $\mathcal{D}$ and $\mathbf{u} = \mathbf{0}$ on $\partial\mathcal{D}$. Define*

$$\nu = |\lambda_1 + \mu_1| + |\lambda_1 - \mu_1|.$$

Suppose

$$\mathcal{U} = \| \nabla\mathbf{u} \|_{L_\infty(\mathcal{D})} = \| \,|\nabla\mathbf{u}|\, \|_{L_\infty(\mathcal{D})} < \frac{1}{|\lambda_1| + \nu}. \tag{4.35}$$

Then there is a unique solution $\mathbf{T} \in W_q^1(\mathcal{D})^{d^2}$ to (2.3) such that

$$\begin{aligned} &\| \mathbf{T} \|_{L_q(\mathcal{D})} + \| \nabla\mathbf{T} \|_{L_q(\mathcal{D})} \\ &\qquad \leq \left(\frac{2\eta}{1-\mathcal{U}\,\nu}\right)\left(\| \nabla\mathbf{u} \|_{L_q(\mathcal{D})} + \frac{1}{1-\mathcal{U}(|\lambda_1|+\nu)}\| \nabla^2\mathbf{u} \|_{L_q(\mathcal{D})}\right). \end{aligned} \tag{4.36}$$

In particular, if we assume that

$$|\lambda_1| \leq \lambda_0\eta, \quad |\mu_1| \leq \mu_0|\lambda_1|, \tag{4.37}$$

and

$$\eta\mathcal{U} \leq \frac{1}{6\lambda_0(1+\frac{2}{3}\mu_0)}, \tag{4.38}$$

then

$$\| \mathbf{T} \|_{L_q(\mathcal{D})} + \| \nabla \mathbf{T} \|_{L_q(\mathcal{D})} \le 4\eta \| \nabla \mathbf{u} \|_{L_q(\mathcal{D})} + 8\eta \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})}. \tag{4.39}$$

Similarly,

$$|\lambda_1| \, \| \mathbf{u} \cdot \nabla \mathbf{T} \|_{L_q(\mathcal{D})} \le \frac{6\eta}{1 - \mathcal{U}\,\nu} \| \nabla \mathbf{u} \|_{L_q(\mathcal{D})}.$$

Proof. We can choose $c_0 = 1 - \nu\mathcal{U}$ in Lemma 4.1 and pick $\mathbf{g} = 2\eta\mathbf{E}$, and this shows that

$$\max \left\{ \| \mathbf{T} \|_{L_q(\mathcal{D})}, \frac{|\lambda_1|}{3} \| \mathbf{u} \cdot \nabla \mathbf{T} \|_{L_q(\mathcal{D})} \right\} \le \frac{1}{1 - \nu\mathcal{U}} \| \mathbf{g} \|_{L_q(\mathcal{D})} = \frac{2\eta}{1 - \nu\mathcal{U}} \| \mathbf{E} \|_{L_q(\mathcal{D})}.$$

Similarly, we can choose $c_1 = 1 - (|\lambda_1| + \nu)\mathcal{U} \le c_0$ in Lemma 4.5, and this yields

$$\begin{aligned}
\| \nabla \mathbf{T} \|_{L_q(\mathcal{D})} &\le \frac{2\eta}{1 - (|\lambda_1| + \nu)\mathcal{U}} \| \nabla \mathbf{E} \|_{L_q(\mathcal{D})} + \frac{2\nu\,\eta}{c_0\, c_1} \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})} \| \mathbf{E} \|_{L_\infty(\mathcal{D})} \\
&\le \frac{2\eta}{1 - (|\lambda_1| + \nu)\mathcal{U}} \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})} + \frac{2\nu\,\eta\mathcal{U}}{c_0\, c_1} \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})} \\
&= \frac{2\eta}{c_1} \left(1 + \frac{\nu\mathcal{U}}{1 - \nu\mathcal{U}} \right) \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})} = \frac{2\eta}{c_1} \left(\frac{1}{1 - \nu\mathcal{U}} \right) \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})},
\end{aligned}$$

where we recall that $\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$. Summing these completes the proof of (4.36).

If (4.37) holds, then

$$\nu = |\lambda_1| \Big(\big|1 + \frac{\mu_1}{\lambda_1}\big| + \big|1 - \frac{\mu_1}{\lambda_1}\big| \Big) \le 2|\lambda_1|(1 + \mu_0),$$

and so

$$|\lambda_1| + \nu \le |\lambda_1| \big(1 + 2(1 + \mu_0)\big) \le \eta\, \lambda_0 \big(1 + 2(1 + \mu_0)\big) = 3\eta\, \lambda_0 (1 + \tfrac{2}{3}\mu_0).$$

So the assumption (4.38) implies that

$$\mathcal{U} \le \frac{1}{6\eta\lambda_0(1 + \frac{2}{3}\mu_0)} \le \frac{1}{2(|\lambda_1| + \nu)} \le \frac{1}{2\nu},$$

and hence $\mathcal{U}\nu \le \mathcal{U}(|\lambda_1| + \nu) \le 1/2$. Thus (4.36) implies

$$\| \mathbf{T} \|_{L_q(\mathcal{D})} + \| \nabla \mathbf{T} \|_{L_q(\mathcal{D})} \le 4\eta \Big(\| \nabla \mathbf{u} \|_{L_q(\mathcal{D})} + 2 \| \nabla^2 \mathbf{u} \|_{L_q(\mathcal{D})} \Big),$$

which completes the proof of (4.39). $\square$

Based on Lemma 4.6, we can think of (2.3) as defining a mapping $\mathbf{u} \mapsto \mathbf{T}$ such that, for $q > d$,

$$\| \mathbf{T}(\mathbf{u}) \|_{W^1_q(\mathcal{D})} \le C_1 \eta \| \mathbf{u} \|_{W^2_q(\mathcal{D})}, \tag{4.40}$$

provided $\| \mathbf{u} \|_{W^2_q(\mathcal{D})} \le C_2$, $\eta \ge \eta_0$, $|\lambda_1| \le \lambda_0 \eta_0$, and $|\mu_1| \le \mu_0 |\lambda_1|$, where C_1 and C_2 depend only on q, $\mathcal{D}$, $\eta_0 > 0$, $\lambda_0 < \infty$, and $\mu_0 < \infty$.

5. REGULARITY FOR u

We consider the system

$$-\eta\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f}\text{ in }\mathcal{D},$$
$$\nabla\cdot\mathbf{u}=0\text{ in }\mathcal{D},\quad \mathbf{u}=\mathbf{0}\text{ on }\partial\mathcal{D}. \tag{5.1}$$

It is well known [14] that, via a variational formulation, (5.1) always has a solution $\mathbf{u}\in H^1(\mathcal{D})^d$ even for $\mathbf{f}\in H^{-1}(\mathcal{D})^d$, and that all such solutions satisfy

$$\|\mathbf{u}\|_{H^1(\mathcal{D})}\le C\eta^{-1}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}. \tag{5.2}$$

From (1.5), we have

$$\eta\|\mathbf{u}\|_{W_q^2(\mathcal{D})}+\|p\|_{W_q^1(\mathcal{D})/\mathbb{R}}\le C\left(\|\mathbf{f}\|_{L_q(\mathcal{D})}+\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})}\right), \tag{5.3}$$

although so far the last term might be infinite. But if $\mathbf{u}\in H^1(\mathcal{D})^d$, then $\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})}<\infty$ for some $q>1$ sufficiently small, and that allows us to bootstrap with respect to q to conclude that (5.3) holds for the desired value of q, as follows. By Hölder's inequality, and Sobolev's inequality, we have, for $1<q\le 3/2$, when $d=3$,

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})}\le\|\mathbf{u}\|_{L_{\frac{2q}{2-q}}(\mathcal{D})}\|\nabla\mathbf{u}\|_{L^2(\mathcal{D})}\le C\|\mathbf{u}\|^2_{H^1(\mathcal{D})}. \tag{5.4}$$

When $d=2$, (5.4) can be extended to hold for $1<q<2$, but with a constant $C=C_q\to\infty$ as $q\to 2$. Define $q_d=\frac{1}{2}(6-d)$, that is, $q_2=2$ and $q_3=3/2$, the limiting Lebesgue indices for the validity of (5.4). With these results, we easily prove the following lemma.

LEMMA 5.1. *Suppose that $d=2$ or 3 and define $q_d=\frac{1}{2}(6-d)$. Suppose further that $\mathbf{f}\in H^{-1}(\mathcal{D})^d$ and that $\mathbf{u}\in H^1(\mathcal{D})^d$ solves (5.1) in the sense of distributions. Suppose finally that (1.5) holds for some q satisfying $1<q<q_d$. Then there is a constant $C_{q,\mathcal{D}}<\infty$ such that for all $\mathbf{f}\in L_q(\mathcal{D})^d\cap H^{-1}(\mathcal{D})^d$, we have*

$$\eta\|\mathbf{u}\|_{W_q^2(\mathcal{D})}+\|p\|_{W_q^1(\mathcal{D})/\mathbb{R}}\le C_{q,\mathcal{D}}\big(\|\mathbf{f}\|_{L_q(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big), \tag{5.5}$$

where $C_{q,\mathcal{D}}$ remains bounded independently of q for $d=3$ as $q\to q_3=3/2$, and moreover (5.5) holds for $q=3/2$ as well for $d=3$.

Note that we must assume that $\mathbf{f}\in L_q(\mathcal{D})^d\cap H^{-1}(\mathcal{D})^d$ since the assumption $\mathbf{f}\in L_q(\mathcal{D})^d$ alone does not imply that $\mathbf{f}\in H^{-1}(\mathcal{D})^d$ for $d=3$. The next lemma gives a preliminary range of q that will be sharpened further on. Now we use bootstrapping to increase the range of q for which bounds can be proved.

LEMMA 5.2. *Suppose that $q>1$, that (1.5) holds, that $\mathbf{f}\in L_q(\mathcal{D})^d\cap H^{-1}(\mathcal{D})^d$, and that $\mathbf{u}\in H^1(\mathcal{D})^d$ solves (5.1) in the sense of distributions. Define*

$$f_{-1}=\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\text{ and }f_q=\|\mathbf{f}\|_{L_q(\mathcal{D})}. \tag{5.6}$$

Then for $d = 2$,

$$\eta\| \mathbf{u} \|_{W_q^2(\mathcal{D})} + \| p \|_{W_q^1(\mathcal{D})/\mathbb{R}} \le C_{q,\mathcal{D}} \Bigg(f_q + \begin{cases} \eta^{-2} f_{-1}^2 & 1 < q < 2 \\ \eta^{-2} f_{-1}\big(f_2 + \eta^{-2} f_{-1}^2\big) & 2 \le q < \infty \end{cases} \Bigg). \tag{5.7}$$

For $d = 3$,

$$\eta\| \mathbf{u} \|_{W_q^2(\mathcal{D})} + \| p \|_{W_q^1(\mathcal{D})/\mathbb{R}} \le C_{q,\mathcal{D}} \Bigg(f_q + \begin{cases} \eta^{-2} f_{-1}^2 & 1 < q \le 3/2 \\ \eta^{-2} f_{-1}\big(f_{3/2} + \eta^{-2} f_{-1}^2\big) & 3/2 < q \le 2 \\ \eta^{-2} f_{-1}\big(f_2 + \eta^{-2} f_{-1}(f_{3/2} + \eta^{-2} f_{-1}^2)\big) & 2 < q \le 3 \\ \eta^{-2}\Big(f_3 + \eta^{-2} f_{-1}\big(f_2 + \eta^{-2} f_{-1}(f_{3/2} + \eta^{-2} f_{-1}^2)\big)\Big)^2 & 3 < q < \infty \end{cases} \Bigg). \tag{5.8}$$

Proof. Here C denotes various constants which may be different but are independent of η.

Let us begin with the case $d = 2$. From Lemma 5.1, we have

$$\begin{aligned} \eta\| \mathbf{u} \|_{W_q^2(\mathcal{D})} + \| p \|_{W_q^1(\mathcal{D})/\mathbb{R}} &\le C\big(\| \mathbf{f} \|_{L_q(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^2\big) \\ &\le C\big(\| \mathbf{f} \|_{L_2(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^2\big), \end{aligned} \tag{5.9}$$

for $1 < q < 2$. By Sobolev's inequality, we conclude $\mathbf{u} \in W_r^1(\mathcal{D})^2$ for any $r \le 2q/(2-q)$, and so we find from (5.2) and (5.9), for any $1 < q < \infty$, that

$$\begin{aligned} \| \mathbf{u} \cdot \nabla \mathbf{u} \|_{L_q(\mathcal{D})} &\le \| \mathbf{u} \|_{L_{2q}(\mathcal{D})} \| \mathbf{u} \|_{W_{2q}^1(\mathcal{D})} \le C\| \mathbf{u} \|_{H^1(\mathcal{D})} \| \mathbf{u} \|_{W_{2q/(q+1)}^2(\mathcal{D})} \\ &\le C\eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}\big(\| \mathbf{f} \|_{L_2(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^2\big), \end{aligned} \tag{5.10}$$

since $2q/(q+1) < q$ for all $q > 1$. From (5.3) and (5.10), we conclude that, for any $1 < q < \infty$,

$$\begin{aligned} &\eta\| \mathbf{u} \|_{W_q^2(\mathcal{D})} + \| p \|_{W_q^1(\mathcal{D})/\mathbb{R}} \\ &\qquad \le C\Big(\| \mathbf{f} \|_{L_q(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}\big(\| \mathbf{f} \|_{L_2(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^2\big)\Big). \end{aligned}$$

Now suppose $d = 3$. From Lemma 5.1, we have

$$\eta\| \mathbf{u} \|_{W_{3/2}^2(\mathcal{D})} + \| p \|_{W_{3/2}^1(\mathcal{D})/\mathbb{R}} \le C\big(\| \mathbf{f} \|_{L_{3/2}(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^2\big).$$

By Sobolev's inequality, we have $\mathbf{u} \in W_3^1(\mathcal{D})^3$, with the bound

$$\eta\| \mathbf{u} \|_{W_3^1(\mathcal{D})} \le C\eta\| \mathbf{u} \|_{W_{3/2}^2(\mathcal{D})} \le C\big(\| \mathbf{f} \|_{L_{3/2}(\mathcal{D})} + \eta^{-2}\| \mathbf{f} \|_{H^{-1}(\mathcal{D})}^2\big).$$

Therefore Hölder's and Sobolev's inequalities give

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_2(\mathcal{D})} &\le \|\mathbf{u}\|_{L_6(\mathcal{D})}\|\nabla\mathbf{u}\|_{L_3(\mathcal{D})} \le C\|\mathbf{u}\|_{H^1(\mathcal{D})}\|\mathbf{u}\|_{W_3^1(\mathcal{D})}\\ &\le C\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\big(\|\mathbf{f}\|_{L_{3/2}(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big).\end{aligned}$$

Thus

$$\begin{aligned}\eta\|\mathbf{u}\|_{H^2(\mathcal{D})}+\|p\|_{H^1(\mathcal{D})/\mathbb{R}} &\le C\Big(\|\mathbf{f}\|_{L_2(\mathcal{D})}\\ &\quad+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\big(\|\mathbf{f}\|_{L_{3/2}(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big)\Big).\end{aligned}\tag{5.11}$$

Next, Schwarz's and Sobolev's inequalities show, using (5.11), that

$$\begin{aligned}&\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_3(\mathcal{D})} \le \|\mathbf{u}\|_{L_6(\mathcal{D})}\|\nabla\mathbf{u}\|_{L_6(\mathcal{D})} \le C\|\mathbf{u}\|_{H^1(\mathcal{D})}\|\mathbf{u}\|_{H^2(\mathcal{D})}\\ &\le C\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\Big(\|\mathbf{f}\|_{L_2(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\big(\|\mathbf{f}\|_{L_{3/2}(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big)\Big),\end{aligned}$$

and so (5.3) yields

$$\begin{aligned}\eta\|\mathbf{u}\|_{W_3^2(\mathcal{D})}+\|p\|_{W_3^1(\mathcal{D})/\mathbb{R}} &\le C\bigg(\|\mathbf{f}\|_{L_3(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\Big(\|\mathbf{f}\|_{L_2(\mathcal{D})}\\ &\quad+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\big(\|\mathbf{f}\|_{L_{3/2}(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big)\Big)\bigg).\end{aligned}$$

Finally, Sobolev's inequality shows that, for any $q<\infty$,

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})} &\le C\|\mathbf{u}\|^2_{W_3^2(\mathcal{D})} \le C\eta^{-2}\bigg(\|\mathbf{f}\|_{L_3(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\Big(\|\mathbf{f}\|_{L_2(\mathcal{D})}\\ &\quad+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\big(\|\mathbf{f}\|_{L_{3/2}(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big)\Big)\bigg)^2,\end{aligned}$$

and so (5.3) yields, for any $q<\infty$,

$$\begin{aligned}&\eta\|\mathbf{u}\|_{W_q^2(\mathcal{D})}+\|p\|_{W_q^1(\mathcal{D})/\mathbb{R}} \le C\Bigg(\|\mathbf{f}\|_{L_q(\mathcal{D})}+\eta^{-2}\bigg(\|\mathbf{f}\|_{L_3(\mathcal{D})}\\ &+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\Big(\|\mathbf{f}\|_{L_2(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|_{H^{-1}(\mathcal{D})}\big(\|\mathbf{f}\|_{L_{3/2}(\mathcal{D})}+\eta^{-2}\|\mathbf{f}\|^2_{H^{-1}(\mathcal{D})}\big)\Big)\bigg)^2\Bigg),\end{aligned}$$

and this completes the proof. □

Although the above result is sufficient for some purposes, it suggests that the dependence of $\mathbf{u}$ and p on $\mathbf{f}$ is discontinuous with respect to q. We can smooth out this dependence in the following.

We need to estimate the nonlinear term in (5.3) for $q>q_d$. By Hölder's inequality, we have, for any t satisfying $1<t<\infty$,

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})} \le C\|\mathbf{u}\|_{L_{qt}(\mathcal{D})}\|\mathbf{u}\|_{W^1_{t'q}(\mathcal{D})},\tag{5.12}$$

where $t' = t/(t-1)$. By the Gagliardo-Nirenberg inequality [6, page 24, Theorem 9.3], we have

$$\|\mathbf{u}\|_{W^1_{t'q}(\mathcal{D})} \le C\|\mathbf{u}\|^{\theta}_{H^1(\mathcal{D})}\|\mathbf{u}\|^{1-\theta}_{W^2_q(\mathcal{D})}, \tag{5.13}$$

where $\theta(t')$ is determined from

$$-1+\frac{d}{t'q} = \theta(t')\Big(-1+\frac{d}{2}\Big)+(1-\theta(t'))\Big(-2+\frac{d}{q}\Big) = \theta(t')(1+\kappa)+\Big(-2+\frac{d}{q}\Big),$$

where $\kappa = (d/2)-(d/q)$, so that

$$\theta(t') = \frac{1}{1+\kappa}\Big(1+\frac{d}{t'q}-\frac{d}{q}\Big) = \frac{1}{1+\kappa}\Big(1-\frac{d}{tq}\Big). \tag{5.14}$$

The estimate (5.13) is valid only for $\theta \in]0,1[$.

For $d=2$, we have by Sobolev's inequality that

$$\|\mathbf{u}\|_{L_{qt}(\mathcal{D})} \le C\|\mathbf{u}\|_{H^1(\mathcal{D})}$$

for all $1<t<\infty$. Thus (5.12) and (5.13) imply that

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})} \le C\|\mathbf{u}\|^{1+\theta}_{H^1(\mathcal{D})}\|\mathbf{u}\|^{1-\theta}_{W^2_q(\mathcal{D})},$$

where θ is given in (5.14), but the constant C depends on the choice of t. Applying (4.28), we have for any $\delta>0$,

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L_q(\mathcal{D})} \le C\Big(\theta\delta^{(\theta-1)/\theta}\|\mathbf{u}\|^{(1+\theta)/\theta}_{H^1(\mathcal{D})}+\delta(1-\theta)\|\mathbf{u}\|_{W^2_q(\mathcal{D})}\Big). \tag{5.15}$$

By taking $\delta = c\eta$ with an appropriate choice of c, we find from (5.3), (5.15), and (5.2) that

$$\begin{aligned}\tfrac12\eta\|\mathbf{u}\|_{W^2_q(\mathcal{D})}+\|p\|_{W^1_q(\mathcal{D})/\mathbb{R}} &\le C\Big(\|\mathbf{f}\|_{L_q(\mathcal{D})}+\eta^{(\theta-1)/\theta}\|\mathbf{u}\|^{(1+\theta)/\theta}_{H^1(\mathcal{D})}\Big)\\ &\le C\Big(\|\mathbf{f}\|_{L_q(\mathcal{D})}+\eta^{-2/\theta}\|\mathbf{f}\|^{1+(1/\theta)}_{H^{-1}(\mathcal{D})}\Big).\end{aligned}$$

Note that $1+\kappa = 2-(2/q) = 2/q'$. Thus for any $\epsilon>0$, we can choose $t<\infty$ such that $\theta = \frac12 q'-\epsilon$, and we have proved the following.

LEMMA 5.3. *Suppose that $d=2$, that $2<q<\infty$, that (1.5) holds, that $\mathbf{f}\in L_q(\mathcal{D})^2$, and that $\mathbf{u}\in H^1(\mathcal{D})^2$ solves (5.1) in the sense of distributions. Then there is a constant $C<\infty$ such that*

$$\tfrac12\eta\|\mathbf{u}\|_{W^2_q(\mathcal{D})}+\|p\|_{W^1_q(\mathcal{D})/\mathbb{R}} \le C\Big(\|\mathbf{f}\|_{L_q(\mathcal{D})}+\eta^{-2/\theta}\|\mathbf{f}\|^{1+(1/\theta)}_{H^{-1}(\mathcal{D})}\Big), \tag{5.16}$$

for any $\theta<\frac12 q'$, where $q'=q/(q-1)$, and C depends on θ and q but is independent of $\mathbf{f}$, $\mathbf{u}$, and η.

The right-hand side of estimate (5.16) is arbitrarily close to

$$\|\mathbf{f}\|_{L_q(\mathcal{D})}+\eta^{-4(1-1/q)}\|\mathbf{f}\|^{3-(2/q)}_{H^{-1}(\mathcal{D})},$$

which interpolates the extremes in (5.7). Now we consider the case $d=3$.

LEMMA 5.4. *Let $d = 3$ and suppose that $q > 3/2$. Define $q' = q/(q-1)$, so that $q' < 3$. Define*

$$\theta = \frac{1}{1 - q'/6}. \tag{5.17}$$

Then there is a constant C such that, for all $\mathbf{v} \in W_q^2(\mathcal{D})^3$,

$$\| \mathbf{v} \cdot \nabla \mathbf{v} \|_{L_q(\mathcal{D})} \le C \| \mathbf{v} \|^{\theta}_{H^1(\mathcal{D})} \| \mathbf{v} \|^{2-\theta}_{W_q^2(\mathcal{D})}. \tag{5.18}$$

The conditions of Lemma 5.4 imply that $\frac{6}{5} < \theta < 2$.

Proof. For the moment, let us consider a general dimension d. By the Gagliardo-Nirenberg inequality [6, page 24, Theorem 9.3], for $s = 0, 1$ we have

$$\| \mathbf{u} \|_{W^s_{qt}(\mathcal{D})} \le C \| \mathbf{u} \|^{\theta_s}_{H^1(\mathcal{D})} \| \mathbf{u} \|^{1-\theta_s}_{W_q^2(\mathcal{D})}, \tag{5.19}$$

where $1 < t < \infty$, $t' = t/(t-1)$, and $\theta_s(t)$ is determined from

$$-s + \frac{d}{qt} = \theta_s(t)\Big(-1 + \frac{d}{2}\Big) + (1 - \theta_s(t))\Big(-2 + \frac{d}{q}\Big) = \theta_s(t)(1+\kappa) + \Big(-2 + \frac{d}{q}\Big),$$

where $\kappa = (d/2) - (d/q)$ as in (5.14), so that

$$\theta_s(t) = \theta_s(t; q) = \frac{1}{1+\kappa}\Big(2 - s + \frac{d}{tq} - \frac{d}{q}\Big) = \frac{1}{1+\kappa}\Big(2 - s - \frac{d}{t'q}\Big).$$

The estimate (5.19) is valid only for $\theta_s \in]0, 1[$. Assuming for the moment that it is possible to find a value of t such that this holds for $\theta_0(t)$ and $\theta_1(t')$, we conclude from (5.12) that (5.18) holds for $\theta = \theta_0(t) + \theta_1(t')$. Since $(t')' = t$, we find

$$\begin{aligned} \theta = \theta_0(t) + \theta_1(t') &= \frac{1}{1+\kappa}\Big(2 - \frac{d}{t'q}\Big) + \frac{1}{1+\kappa}\Big(1 - \frac{d}{tq}\Big) \\ &= \frac{1}{1+\kappa}\Big(3 - \frac{d}{q}\Big) = \frac{1}{1+\kappa}\Big(3 - d + \frac{d}{q'}\Big). \end{aligned}$$

Choosing $d = 3$ yields $\theta = (1+\kappa)^{-1}(3/q')$ and θ verifies (5.17).

It remains to prove that, for all $q > 3/2$, there is a t such that $0 < \theta_0(t) < 1$ and $0 < \theta_1(t') < 1$. Let $d = 3$ and consider the choice $t = 4$ for $s = 0$. (In Section 5.2, we will see why we cannot have $d = 2$ and the reasoning behind the choice $t = 4$.) In this case

$$\theta_0(4) = h_0(q) := \frac{1}{1+\kappa}\Big(2 + \frac{3}{4q} - \frac{3}{q}\Big) = \frac{1}{5/2 - 3/q}\Big(2 - \frac{9}{4q}\Big) = \frac{4}{5} + \frac{3}{50}\Big(q - \frac{6}{5}\Big)^{-1}.$$

Then $h_0(3/2) = 1$, h_0 is strictly decreasing for $q > 6/5$, and $h_0(q) \to 4/5$ as $q \to \infty$. Thus for $3/2 < q < \infty$, $4/5 < h_0(q) < 1$, and thus $4/5 < \theta_0(4) < 1$ as well.

Let $s=1$. Since $t=4$, then $t'=4/3$, and

$$\theta_1(4/3)=h_1(q):=\frac{1}{5/2-3/q}\Big(1-\frac{3}{4q}\Big)=\frac{2}{5}+\frac{9}{50}\Big(q-\frac{6}{5}\Big)^{-1}.$$

Then $h_1(3/2)=1$, h_1 is strictly decreasing for $q>6/5$, and $h_1(q)\to 2/5$ as $q\to\infty$. Thus for $3/2<q<\infty$, $2/5<h_1(q)<1$, and thus $2/5<\theta_1(4/3)<1$ as well.

The result now follows from (5.12), with $t=4$ (and $t'=4/3$). $\square$

The following is an immediate consequence of (5.3) and (5.18):

$$\eta\|\,\mathbf{u}\,\|_{W^2_q(\mathcal{D})}+\|\,p\,\|_{W^1_q(\mathcal{D})/\mathbb{R}}\le C_{q,\mathcal{D}}\Big(\|\,\mathbf{f}\,\|_{L_q(\mathcal{D})}+\|\,\mathbf{u}\,\|^{\theta}_{H^1(\mathcal{D})}\|\,\mathbf{u}\,\|^{2-\theta}_{W^2_q(\mathcal{D})}\Big), \tag{5.20}$$

where θ is defined in (5.17) and satisfies $6/5<\theta<2$. Thus $2-\theta\in\,]0,4/5[$. Applying (4.28) with $1/r'=2-\theta$ (and $1/r=\theta-1\in\,]1/5,1[$), we have for any $\delta>0$

$$\begin{aligned}\|\,\mathbf{u}\,\|^{\theta}_{H^1(\mathcal{D})}\|\,\mathbf{u}\,\|^{2-\theta}_{W^2_q(\mathcal{D})}&=\delta^{\theta-2}\|\,\mathbf{u}\,\|^{\theta}_{H^1(\mathcal{D})}\big(\delta\|\,\mathbf{u}\,\|_{W^2_q(\mathcal{D})}\big)^{2-\theta}\\&\le(\theta-1)\big(\delta^{\theta-2}\|\,\mathbf{u}\,\|^{\theta}_{H^1(\mathcal{D})}\big)^{1/(\theta-1)}+(2-\theta)\delta\|\,\mathbf{u}\,\|_{W^2_q(\mathcal{D})}.\end{aligned}\tag{5.21}$$

By choosing $\delta=c\eta$ with an appropriate c, (5.20) and (5.21) combine to yield

$$\begin{aligned}\tfrac12\eta\|\,\mathbf{u}\,\|_{W^2_q(\mathcal{D})}+\|\,p\,\|_{W^1_q(\mathcal{D})/\mathbb{R}}&\le C_{q,\mathcal{D}}\Big(\|\,\mathbf{f}\,\|_{L_q(\mathcal{D})}+\eta^{(\theta-2)/(\theta-1)}\|\,\mathbf{u}\,\|^{\theta/(\theta-1)}_{H^1(\mathcal{D})}\Big)\\&\le C_{q,\mathcal{D}}\Big(\|\,\mathbf{f}\,\|_{L_q(\mathcal{D})}+\eta^{-2/(\theta-1)}\|\,\mathbf{f}\,\|^{\theta/(\theta-1)}_{H^{-1}(\mathcal{D})}\Big).\end{aligned}$$

Recall that $\theta=1/(1-q'/6)$, so

$$\theta-1=\frac{q'}{6-q'},\quad(\theta-1)^{-1}=\frac{6}{q'}-1,\quad\text{and}\quad\frac{\theta}{\theta-1}=\frac{6}{q'}$$

Thus we have proved the following.

Lemma 5.5. *Suppose that $d=3$, that $3/2<q<\infty$, that (1.5) holds, that $\mathbf{f}\in L_q(\mathcal{D})^d$, and that $\mathbf{u}\in H^1(\mathcal{D})^d$ solves (5.1) in the sense of distributions. Let $q'=q/(q-1)\in\,]1,3[$. Then*

$$\tfrac12\eta\|\,\mathbf{u}\,\|_{W^2_q(\mathcal{D})}+\|\,p\,\|_{W^1_q(\mathcal{D})/\mathbb{R}}\le C_{q,\mathcal{D}}\big(\|\,\mathbf{f}\,\|_{L_q(\mathcal{D})}+\eta^{2-(12/q')}\|\,\mathbf{f}\,\|^{6/q'}_{H^{-1}(\mathcal{D})}\big), \tag{5.22}$$

where $C_{q,\mathcal{D}}$ is independent of $\mathbf{f}$, $\mathbf{u}$, *and* η.

5.1. Some corollaries

First we give an example that clarifies the meaning of Lemmas 5.3 and 5.5, especially in contrast with Lemma 5.2. Let $\mathcal{D}=]0,1[^d$ and suppose that we define $\mathbf{f}_h$ via

$$\mathbf{f}_h(x)=h^{-1}\big(\sin(x_1/h),0,\ldots,0\big),$$

where x_1 is the first coordinate of x. Then $\| \mathbf{f}_h \|_{H^{-1}(\mathcal{D})} \leq C_1$ where C_1 is independent of h, but $\| \mathbf{f}_h \|_{L_q(\mathcal{D})} \geq C_2/h$ where $C_2 > 0$ is also independent of h. Thus Lemmas 5.3 and 5.5 show that the corresponding solution $\mathbf{u}_h$ satisfies $\| \mathbf{u}_h \|_{W^2_q(\mathcal{D})} \leq Ch^{-1}$ with C independent of h, whereas Lemma 5.2 would only guarantee that $\| \mathbf{u}_h \|_{W^2_q(\mathcal{D})} \leq Ch^{-2}$ with C independent of h with $d = 3$ and $q > 3$.

As a corollary of Lemmas 5.1, 5.2, 5.3, and 5.5, we have the following.

LEMMA 5.6. *Suppose that $q > 1$ for $d = 2$ and $q \geq 6/5$ for $d = 3$, that* (1.5) *holds, M is any positive real number, and $\eta \geq \eta_0 > 0$. Then for $d = 2$ and $d = 3$, there is a constant $C_{q,\mathcal{D},\eta_0,M}$ such that for all $\mathbf{f} \in H^{-1}(\mathcal{D})^d$ satisfying $\| \mathbf{f} \|_{H_{-1}(\mathcal{D})} \leq M$ and for all $\mathbf{u} \in H^1(\mathcal{D})^d$ solving* (5.1) *in the sense of distributions, we have*

$$\tfrac{1}{2}\eta \| \mathbf{u} \|_{W^2_q(\mathcal{D})} + \| p \|_{W^1_q(\mathcal{D})/\mathbb{R}} \leq C_{q,\mathcal{D},\eta_0,M} \| \mathbf{f} \|_{L_q(\mathcal{D})}. \tag{5.23}$$

Proof. Since $\| \mathbf{f} \|_{H^{-1}(\mathcal{D})} \leq C \| \mathbf{f} \|_{L_q(\mathcal{D})}$, we have for $s \geq 0$ and $t \geq 1$,

$$\eta^{-s} \| \mathbf{f} \|^t_{H^{-1}(\mathcal{D})} \leq C\eta_0^{-s} M^{t-1} \| \mathbf{f} \|_{L_q(\mathcal{D})}.$$

Thus (5.23) follows from (5.5), (5.16), and (5.22), except that for $d = 2$ we require Lemma 5.2 for the case $q = 2$. $\square$

As another corollary of Lemma 5.5, we have the following.

COROLLARY 5.7. *Suppose that the conditions of Lemma 5.5 hold with two data functions $\mathbf{f}_1$, $\mathbf{f}_2$, and that there are two solutions $(\mathbf{u}_1, \pi_1)$, $(\mathbf{u}_2, \pi_2)$ to* (5.1), *that is,*

$$\begin{aligned} -\eta\Delta \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \nabla \pi_i &= \mathbf{f}_i \ \textit{in } \mathcal{D} \\ \nabla\cdot \mathbf{u}_i = 0 \ \textit{in } \mathcal{D}, \quad \mathbf{u}_i &= \mathbf{z} \ \textit{on } \partial\mathcal{D}, \quad \text{for } i = 1, 2. \end{aligned} \tag{5.24}$$

Then there is an $\epsilon > 0$ such that, provided $\max_{i=1,2} \| \mathbf{f}_i \|_{H^{-1}(\mathcal{D})} \leq \epsilon\eta^2$,

$$\eta \| \mathbf{u}_1 - \mathbf{u}_2 \|_{H^1(\mathcal{D})} + \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} \leq C_{\mathcal{D},\epsilon} \| \mathbf{f}_1 - \mathbf{f}_2 \|_{H^{-1}(\mathcal{D})},$$

for both $d = 2$ and $d = 3$.

Proof. The proof is straightforward, see for example [8], but we present it here for the reader's convenience. From (5.2), we have, for $i = 1, 2$,

$$\eta \| \mathbf{u}_i \|_{H^1(\mathcal{D})} \leq C_{\mathcal{D}} \| \mathbf{f}_i \|_{H^{-1}(\mathcal{D})}. \tag{5.25}$$

Now we multiply (5.24) by $\mathbf{u}_1 - \mathbf{u}_2$ for each i, integrate over $\mathcal{D}$, integrate by parts, and then subtract to get

$$\begin{aligned}
&\eta \int_{\mathcal{D}} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \,\mathrm{d}x + \int_{\mathcal{D}} (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \,\mathrm{d}x \\
&= \int_{\mathcal{D}} (\mathbf{f}_1 - \mathbf{f}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \,\mathrm{d}x \\
&\le C'_{\mathcal{D}} \eta^{-1} \| \mathbf{f}_1 - \mathbf{f}_2 \|^2_{H^{-1}(\mathcal{D})} + \tfrac{1}{2}\eta \int_{\mathcal{D}} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \,\mathrm{d}x.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tfrac{1}{2}\eta^2 \int_{\mathcal{D}} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \,\mathrm{d}x &\le C'_{\mathcal{D}} \| \mathbf{f}_1 - \mathbf{f}_2 \|^2_{H^{-1}(\mathcal{D})} \\
&\quad + \eta \Big| \int_{\mathcal{D}} (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \,\mathrm{d}x \Big|.
\end{aligned} \tag{5.26}$$

Adding and subtracting, we find from (5.25) and Green's formula that

$$\begin{aligned}
&\Big| \int_{\mathcal{D}} (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \mathrm{d}x \Big| \\
&\qquad = \Big| \int_{\mathcal{D}} ((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{u}_1 + \mathbf{u}_2 \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \,\mathrm{d}x \Big| \\
&\qquad \le \| \mathbf{u}_1 - \mathbf{u}_2 \|_{L_4(\mathcal{D})} \| \mathbf{u}_1 \|_{H^1(\mathcal{D})} \| \mathbf{u}_1 - \mathbf{u}_2 \|_{L_4(\mathcal{D})} \\
&\qquad \le C''_{\mathcal{D}} \| \mathbf{u}_1 - \mathbf{u}_2 \|^2_{H^1(\mathcal{D})} \| \mathbf{u}_1 \|_{H^1(\mathcal{D})} \\
&\qquad \le C''_{\mathcal{D}} \big(\max_{i=1,2} \| \mathbf{u}_i \|_{H^1(\mathcal{D})} \big) \| \mathbf{u}_1 - \mathbf{u}_2 \|^2_{H^1(\mathcal{D})} \\
&\qquad \le C'''_{\mathcal{D}}\, \eta^{-1} \big(\max_{i=1,2} \| \mathbf{f}_i \|_{H^{-1}(\mathcal{D})} \big) \| \mathbf{u}_1 - \mathbf{u}_2 \|^2_{H^1(\mathcal{D})}.
\end{aligned} \tag{5.27}$$

By combining (5.26) and (5.27), we find

$$\begin{aligned}
\tfrac{1}{2}\eta^2 \int_{\mathcal{D}} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \,\mathrm{d}x &\le C'_{\mathcal{D}} \| \mathbf{f}_1 - \mathbf{f}_2 \|^2_{H^{-1}(\mathcal{D})} \\
&\quad + C'''_{\mathcal{D}} \big(\max_{i=1,2} \| \mathbf{f}_i \|_{H^{-1}(\mathcal{D})} \big) \| \mathbf{u}_1 - \mathbf{u}_2 \|^2_{H^1(\mathcal{D})} \\
&\le C'_{\mathcal{D}} \| \mathbf{f}_1 - \mathbf{f}_2 \|^2_{H^{-1}(\mathcal{D})} + C'''_{\mathcal{D}} \epsilon \eta^2 \| \mathbf{u}_1 - \mathbf{u}_2 \|^2_{H^1(\mathcal{D})}.
\end{aligned}$$

Choosing $\epsilon = (4C'''_{\mathcal{D}})^{-1}$, we find

$$\eta^2 \int_{\mathcal{D}} |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \,\mathrm{d}x \le 4C'_{\mathcal{D}} \| \mathbf{f}_1 - \mathbf{f}_2 \|^2_{H^{-1}(\mathcal{D})}. \tag{5.28}$$

To estimate the pressure terms, let V be the subspace of divergence-free functions of $H^1_0(\mathcal{D})^d$, and $V^\perp$ its orthogonal in $H^1_0(\mathcal{D})^d$ for the scalar product $(\nabla \mathbf{u}, \nabla \mathbf{v})$. We multiply (5.24) by $\mathbf{v} \in V^\perp$, integrate over $\mathcal{D}$, integrate by parts, subtract, and use the orthogonality of $V^\perp$, to get

$$\int_{\mathcal{D}} (\pi_1 - \pi_2)\nabla\cdot \mathbf{v}\, dx = -\int_{\mathcal{D}} (\mathbf{f}_1 - \mathbf{f}_2)\cdot \mathbf{v}\, dx + \int_{\mathcal{D}} (\mathbf{u}_1\cdot\nabla\mathbf{u}_1 - \mathbf{u}_2\cdot\nabla\mathbf{u}_2)\cdot\mathbf{v}\, dx.$$

The same argument used in deriving (5.27) gives for the nonlinear term,

$$\Big|\int_{\mathcal{D}} \big(\mathbf{u}_1\cdot\nabla\mathbf{u}_1 - \mathbf{u}_2\cdot\nabla\mathbf{u}_2\big)\cdot\mathbf{v}\, dx\Big| \\ \le C'''_{\mathcal{D}}\,\eta^{-1}\big(\max_{i=1,2}\|\,\mathbf{f}_i\,\|_{H^{-1}(\mathcal{D})}\big)\|\,\mathbf{u}_1 - \mathbf{u}_2\,\|_{H^1(\mathcal{D})}\|\,\mathbf{v}\,\|_{H^1(\mathcal{D})},$$

and from (5.28), we conclude that

$$\Big|\int_{\mathcal{D}} (\pi_1 - \pi_2)\nabla\cdot\mathbf{v}\, dx\Big| \le C^{(4)}_{\mathcal{D}}\|\,\mathbf{f}_1 - \mathbf{f}_2\,\|_{H^{-1}(\mathcal{D})}\|\,\mathbf{v}\,\|_{H^1(\mathcal{D})}.$$

Then we complete the proof by applying Ladyzhenskaya's Lemma [7]. □

The equations (3.11), (3.8), and (2.3) provide an alternative formulation of the 3-parameter Oldroyd model (2.2)–(2.3). Using this formulation, we shall prove the following in Section 6.

THEOREM 5.8. *Suppose that $q > d$, that* (1.3) *and* (1.5) *hold, that the coefficients λ_1 and μ_1 satisfy*

$$|\lambda_1| \le \lambda_0\eta, \quad |\mu_1| \le \mu_0|\lambda_1|, \quad \text{and} \quad \eta \ge \eta_0. \tag{5.29}$$

Then there are constants $C < \infty$ and $\widetilde{C} > 0$, depending only on q, $\mathcal{D}$, λ_0, μ_0, and η_0, such that the 3-parameter Oldroyd system (2.2)–(2.3) *has solutions satisfying*

$$\eta\|\,\mathbf{u}\,\|_{W^2_q(\mathcal{D})} + \|\,\mathbf{T}\,\|_{W^1_q(\mathcal{D})} + \|\,p\,\|_{W^1_q(\mathcal{D})/\mathbb{R}} \le C\|\,\mathbf{f}\,\|_{W^1_q(\mathcal{D})}, \tag{5.30}$$

provided that $\|\,\mathbf{f}\,\|_{W^1_q(\mathcal{D})} \le \widetilde{C}$.

Note that this is suboptimal in terms of the relation between the regularity of $\mathbf{f}$ and $\mathbf{u}$, but the term $\mathbf{u}\cdot\nabla\mathbf{f}$ in (3.12) appears to require this in the case of the estimate (5.30).

The parameter λ in [16] corresponds to λ_1^{-1} here, and thus the auxiliary pressure function q in [16] corresponds to $\lambda_1^{-1}\pi$. However, there appears to be a discrepancy with equations (2.5-6) in [16] with regard to the scaling of the pressure function q.

5.2. The choice of t

We now return to the proof of Lemma 5.4 to understand the choice of t and the restriction $d \ne 2$. Define $Q = d/q$ and $T = 1/t'$. Then

$$\theta_s = \frac{2 - s - QT}{1 + \frac{1}{2}d - Q}.$$

Thus for $d = 2$ and $s = 0$,

$$\theta_0 = \frac{2 - QT}{2 - Q} > 1,$$

and so the inequality (5.19) is not valid. This is the reason why we restrict to the case $d = 3$, as we do from now on.

Although we have seen that the choice of $t = 4$ in (5.19) works, it may be of interest to see how we arrived at this unique choice. The condition $0 < \theta_s < 1$ translates to

$$0 < 2 - s - QT < \frac{5}{2} - Q,$$

which we can write as two inequalities:

$$QT < 2 - s \qquad \text{and} \qquad \frac{Q}{t} = Q(1 - T) < s + \frac{1}{2}. \tag{5.31}$$

Since $q > 3/2$, $Q < 2$. Since $1 < t < \infty$, $0 < T < 1$. Thus the first of the inequalities in (5.31) holds automatically for $s = 0$. The second of the inequalities in (5.31) for $s = 0$ translates to $t > 2Q = 6/q$ or

$$t > t_0(q) = \max\{1, 6/q\} = \begin{cases} 6/q & q \le 6 \\ 1 & q \ge 6 \end{cases}.$$

Note that $\max\{t_0(q) \mid q > 3/2\} = t_0(3/2) = 4$. Thus we can say that (5.19) holds for all $t > t_0(q)$ for the case $s = 0$.

The first of the inequalities in (5.31) for $s = 1$ is equivalent to $QT < 1$, which means $t' > Q = 3/q$, or

$$t' > t'_1(q) = \max\{1, 3/q\} = \begin{cases} 3/q & q \le 3 \\ 1 & q \ge 3 \end{cases}.$$

Note that $\max\{t'_1(q) \mid q > 3/2\} = t'_1(3/2) = 2$. The second of the inequalities in (5.31) for $s = 1$ translates to $t > \frac{2}{3}Q = \frac{2}{q}$ or

$$t > t_1(q) = \max\{1, 2/q\} = \begin{cases} 2/q & q \le 2 \\ 1 & q \ge 2 \end{cases}.$$

Note that $\max\{t_1(q) \mid q > 3/2\} = t_1(3/2) = 4/3$. Thus we can say that (5.19) holds, in the case $s = 1$, for all $t > t_1(q)$ and $t' > t'_1(q)$. We need to translate this to a bound on t' only, and the former inequality can be written $1/t < 1/t_1(q)$ and hence $1/t' = 1 - 1/t > 1 - 1/t_1(q)$. Thus our conditions on t' for the case $s = 1$ are

$$t'_1(q) < t' \quad \text{and} \quad t' < \frac{t_1(q)}{t_1(q) - 1}. \tag{5.32}$$

Here the singularity in the denominator in the right-hand inequality in (5.32) simply translates to $t' < \infty$, so it provides no extra condition. We can make the constraints (5.32) explicit in the case $s = 1$ for various ranges of q as follows:

$$\begin{aligned} \frac{3}{q} < t' < \frac{2}{2-q} &= \gamma_1(q) \text{ for } (3/2) < q < 2 \\ \frac{3}{q} &< t' < \infty \text{ for } 2 \le q \le 3 \\ 1 &< t' < \infty \text{ for } 3 < q < \infty, \end{aligned} \tag{5.33}$$

where the constraint function $\gamma_1(q) = 2/(2-q)$. Note that the first line of (5.33) is the most restrictive of the three and any $t' \in [2, 4]$ satisfies all three for all $3/2 < q < \infty$. For the case $s = 0$, the constraints can be made explicit via

$$\begin{aligned} \gamma_0(q) = \frac{6}{q} &< t < \infty \text{ for } 3/2 < q \le 6 \\ 1 &< t < \infty \text{ for } 6 < q < \infty, \end{aligned} \tag{5.34}$$

where the constraint function $\gamma_0(q) = 6/q$. Note that the critical constraint functions satisfy $\gamma_1(3/2) = 4 = \gamma_0(3/2)$. However, γ_1 is strictly increasing on $[3/2, 2[$, and γ_0 is strictly decreasing on $[3/2, 2[$. Thus for $3/2 < q < 2$,

$$\gamma_0(q) < 4 < \gamma_1(q).$$

Thus $t' = 4$ satisfies the constraints (5.33) for all $(3/2) < q < \infty$ and $t = 4$ satisfies the constraints (5.34) for all $(3/2) < q < \infty$. Moreover, for all $(3/2) < q < \infty$, there is an open interval of values of t such that the constraints are satisfied, and $t = 4$ is in the interior of this interval.

6. SOLUTION ALGORITHM

In this section, we present the proof of Theorem 5.8. The following algorithm is a modification of the iteration proposed by Renardy to demonstrate existence. Given $\mathbf{u}^{n-1}$, $\mathbf{T}^{n-1}$, p^{n-1}, we define $\mathbf{u}^n$, $\mathbf{T}^n$, p^n as follows. First we solve

$$\begin{gathered} -\eta\Delta\mathbf{u}^n + \mathbf{u}^n \cdot \nabla\mathbf{u}^n + \nabla\pi^n = \mathcal{F}(\mathbf{f}, \mathbf{u}^{n-1}, p^{n-1}, \mathbf{T}^{n-1}) \text{ in } \mathcal{D}, \\ \nabla\cdot\mathbf{u}^n = 0 \text{ in } \mathcal{D}, \quad \mathbf{u}^n = \mathbf{0} \text{ on } \partial\mathcal{D} \end{gathered} \tag{6.1}$$

to determine $\mathbf{u}^n$ and π^n, where $\mathcal{F}$ was defined in (3.12). Then we solve

$$p^n + \lambda_1 \mathbf{u}^n \cdot \nabla p^n = \pi^n \tag{6.2}$$

to determine p^n. We recall the notation

$$\mathbf{E}^n = \tfrac{1}{2}\big(\nabla\mathbf{u}^n + (\nabla\mathbf{u}^n)^t\big) \text{ and } \mathbf{R}^n = \tfrac{1}{2}\big(-\nabla\mathbf{u}^n + (\nabla\mathbf{u}^n)^t\big).$$

Finally, we solve

$$\begin{aligned}\mathbf{T}^n + \lambda_1 \big(\mathbf{u}^n \cdot \nabla \mathbf{T}^n - (\nabla \mathbf{u}^n)\circ \mathbf{T}^n - \mathbf{T}^n \circ (\nabla \mathbf{u}^n)^t\big) \\ + (\lambda_1 - \mu_1)(\mathbf{E}^n \circ \mathbf{T}^n + \mathbf{T}^n \circ \mathbf{E}^n) = 2\eta \mathbf{E}^n\end{aligned} \tag{6.3}$$

for $\mathbf{T}^n$. Recall that (6.3) is equivalent to

$$\begin{aligned}\mathbf{T}^n + \lambda_1(\mathbf{u}^n \cdot \nabla \mathbf{T}^n + \mathbf{R}^n \circ \mathbf{T}^n + \mathbf{T}^n \circ (\mathbf{R}^n)^t) \\ - \mu_1(\mathbf{E}^n \circ \mathbf{T}^n + \mathbf{T}^n \circ \mathbf{E}^n) = 2\eta \mathbf{E}^n.\end{aligned} \tag{6.4}$$

More precisely, we first solve the Navier-Stokes equations (6.1) for $\mathbf{u}^n \in W_q^2(\mathcal{D})^d$ and $\pi^n \in W_q^1(\mathcal{D})$. Then we solve the scalar transport equation (6.2) for $p^n \in W_q^1(\mathcal{D})$. Finally, we solve either (6.3) or (6.4) for $\mathbf{T}^n \in W_q^1(\mathcal{D})^{d^2}$. We begin the iteration with $\mathbf{u}^0 = \mathbf{0}$, $p^0 = 0$ and $\mathbf{T}^0 = \mathbf{0}$.

The following lemma gives bounds on p^n and $\mathbf{T}^n$ in terms of $\mathbf{u}^n$, collecting the results of Lemmas 4.6 and 3.2.

Lemma 6.1. *Suppose that $\mathcal{D}$ satisfies the condition* (1.3) *and $q > d$. Assume that* (4.37) *holds. Let $\sigma_q > 0$ be the constant in the Sobolev inequality* (1.6).

Then there is a constant $\hat\sigma < \infty$, depending only on λ_0, μ_0, q and $\mathcal{D}$, such that if

$$\| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} \le \frac{1}{2\sigma_q(|\lambda_1| + |\lambda_1 + \mu_1| + |\lambda_1 - \mu_1|)},$$

there is a unique solution $\mathbf{T}^n \in W_q^1(\mathcal{D})^{d^2}$ to (6.3) *such that*

$$\| \mathbf{T}^n \|_{W_q^1(\mathcal{D})} \le \hat\sigma \eta \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} \tag{6.5}$$

and a unique solution $p^n \in W_q^1(\mathcal{D})$ to (6.2) *such that*

$$\| p^n \|_{W_q^1(\mathcal{D})} \le \hat\sigma \| \pi^n \|_{W_q^1(\mathcal{D})}. \tag{6.6}$$

6.1. **Bounds for the iterates**

Let us prove (by induction) that, for some $\gamma > 0$, the following holds for $n \ge 0$:

$$\eta \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} + \| \pi^n \|_{W_q^1(\mathcal{D})} \le \gamma. \tag{6.7}$$

For $n = 0$, this holds for any $\gamma > 0$. Suppose that $\gamma > 0$ has been chosen small enough so that

$$\gamma \le \frac{\eta}{2\sigma_q(|\lambda_1| + |\lambda_1 + \mu_1| + |\lambda_1 - \mu_1|)}, \tag{6.8}$$

where σ_q is the Sobolev constant in (1.6). In particular, this implies that

$$\| \nabla \mathbf{u}^n \|_{L_\infty(\mathcal{D})} \le \sigma_q \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} \le \gamma\sigma_q/\eta \le \frac{1}{2(|\lambda_1| + |\lambda_1+\mu_1| + |\lambda_1-\mu_1|)}.$$

In this case, we can apply Lemma 6.1. Note that (6.7) and (4.37) imply that

$$|\lambda_1| \, \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} \le \lambda_0\gamma. \tag{6.9}$$

Let $\varphi > 0$ and assume that $\| \mathbf{f} \|_{W_q^1(\mathcal{D})} \le \varphi$. In view of (3.13), (5.29), (6.5), (6.6), and (6.9), we have

$$\begin{aligned}
&\| \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \|_{L_q(\mathcal{D})} \le \| \mathbf{f} \|_{L_q(\mathcal{D})} + \sigma_q\Big(|\lambda_1| \, \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} \big(\| \mathbf{f} \|_{W_q^1(\mathcal{D})} \\
&\quad + \hat\sigma \| \pi^n \|_{W_q^1(\mathcal{D})} + 2\sigma_q \| \mathbf{u}^n \|^2_{W_q^2(\mathcal{D})} + \hat\sigma\eta \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} \big) + 4\hat\sigma|\lambda_1-\mu_1|\eta \| \mathbf{u}^n \|^2_{W_q^2(\mathcal{D})} \Big) \\
&\quad \le \varphi + \sigma_q\Big(\lambda_0\gamma\big(\varphi + \hat\sigma\gamma + 2\sigma_q(\gamma/\eta)^2 + \hat\sigma\gamma\big) + 4\hat\sigma\lambda_0|1-\mu_1/\lambda_1|\gamma^2 \Big) \\
&\quad \le \varphi + \sigma_q\Big(\lambda_0\gamma\big(\varphi + 2\hat\sigma\gamma + 2\sigma_q(\gamma/\eta)^2\big) + 4\hat\sigma\lambda_0(1+\mu_0)\gamma^2 \Big) \\
&\quad \le \varphi(1+\sigma_q\lambda_0\gamma) + 2\sigma_q\lambda_0\gamma\big(\sigma_q(\gamma/\eta_0)^2 + (3\hat\sigma + 2\hat\sigma\mu_0)\gamma\big) \\
&\quad = \big(1 + C\gamma\big)\varphi + C'\gamma^2 + C''\gamma^3,
\end{aligned}$$

where $C = \sigma_q\lambda_0$, $C' = 2C(3\hat\sigma + 2\hat\sigma\mu_0)$, and $C'' = 2C\sigma_q/\eta_0^2$. By taking φ and γ small enough, we can guarantee that

$$\| \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \|_{H^{-1}(\mathcal{D})} \le c_{q,\mathcal{D}} \| \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \|_{L_q(\mathcal{D})} \le 1. \tag{6.10}$$

Thus we can apply (5.23) with $M = 1$ to get

$$\begin{aligned}
\eta \| \mathbf{u}^{n+1} \|_{W_q^2(\mathcal{D})} + \| \pi^{n+1} \|_{W_q^1(\mathcal{D})} &\le C_{q,\mathcal{D},\eta_0,1} \| \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \|_{L_q(\mathcal{D})} \\
&\le C_{q,\mathcal{D},\eta_0,1}\Big(\big(1 + C\gamma\big)\varphi + C'\gamma^2 + C''\gamma^3 \Big) \le \gamma,
\end{aligned}$$

provided that φ and γ are small enough. We thus ensure (by induction) that

$$\eta \| \mathbf{u}^n \|_{W_q^2(\mathcal{D})} + \| \pi^n \|_{W_q^1(\mathcal{D})} \le \gamma$$

for all n. Note that by (6.5) and (6.6), we also have

$$\| p^n \|_{W_q^1(\mathcal{D})} + \| \mathbf{T}^n \|_{W_q^1(\mathcal{D})} \le \hat\sigma\gamma. \tag{6.11}$$

We collect the constraints required by γ and φ, with the constants defined above:

$$\begin{gathered}
\gamma \le \frac{\eta}{2\sigma_q(|\lambda_1| + |\lambda_1+\mu_1| + |\lambda_1-\mu_1|)}, \\
\big(1 + C\gamma\big)\varphi + C'\gamma^2 + C''\gamma^3 \le \min\left\{ \frac{1}{c_{q,\mathcal{D}}}, \frac{\gamma}{C_{q,\mathcal{D},\eta_0,1}} \right\}.
\end{gathered} \tag{6.12}$$

The first condition in (6.12) is satisfied if we assume

$$\gamma \le \frac{1}{2\sigma_q \lambda_0 (1 + 2(1+\mu_0))}.$$

All constraints can be satisfied independently of $\mathbf{f}$ provided that $\| \mathbf{f} \|_{W^1_q(\mathcal{D})} \le \varphi$.

6.2. **Convergence of the iterates**

To prove convergence of the iterates, we use the bounds in Section 6.1. Thus we assume that the parameters $\gamma > 0$ and $\varphi > 0$ have been chosen small enough so that all of the iterates remain bounded independently of n. More precisely, we will assume that we have iterates satisfying

$$\begin{aligned} \eta \| \mathbf{u}^n \|_{W^2_q(\mathcal{D})} + \| \pi^n \|_{W^1_q(\mathcal{D})} &\le \gamma \\ \| \mathbf{T}^n \|_{W^1_q(\mathcal{D})} + \| p^n \|_{W^1_q(\mathcal{D})} &\le \hat{\sigma}\gamma, \end{aligned} \tag{6.13}$$

where γ has been chosen to satisfy (6.12) and $\hat{\sigma}$ is given in Lemma 6.1.

To show convergence, we will demonstrate Lipschitz continuity of the solution operator for (3.11) and also for the mapping $\mathbf{T}(\mathbf{u})$, cf. (4.40). Thus we will assume that we have $\mathbf{v}_i$ satisfying the bound (6.7). We will apply this in the specific case where $\mathbf{v}_1 = \mathbf{u}^n$ and $\mathbf{v}_2 = \mathbf{u}^{n-1}$.

The system (2.3) can be written as in (4.9) via

$$\mathbf{T} + \mathcal{M}(\hat{\mathbf{v}})\mathbf{T} + \hat{\mathbf{v}} \cdot \nabla \mathbf{T} = \eta(\nabla \mathbf{v} + \nabla \mathbf{v}^t) \ \text{ in } \mathcal{D},$$

where $\hat{\mathbf{v}} = \lambda_1 \mathbf{v}$ and $\mathcal{M}(\hat{\mathbf{v}})\mathbf{T}$ is defined by

$$\mathcal{M}(\hat{\mathbf{v}})\mathbf{T} = \widetilde{\mathbf{R}} \circ \mathbf{T} + \mathbf{T} \circ \widetilde{\mathbf{R}}^t - \tilde{\mu}(\widetilde{\mathbf{E}} \circ \mathbf{T} + \mathbf{T} \circ \widetilde{\mathbf{E}}), \tag{6.14}$$

where $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{R}}$ are defined by

$$\widetilde{\mathbf{E}} = \tfrac{1}{2}\lambda_1(\nabla \mathbf{v} + \nabla \mathbf{v}^t) = \tfrac{1}{2}(\nabla \hat{\mathbf{v}} + \nabla \hat{\mathbf{v}}^t) \ \text{ and } \ \widetilde{\mathbf{R}} = \tfrac{1}{2}\lambda_1(\nabla \mathbf{v}^t - \nabla \mathbf{v}) = \tfrac{1}{2}(\nabla \hat{\mathbf{v}}^t - \nabla \hat{\mathbf{v}})$$

and $\tilde{\mu} = \mu_1/\lambda_1$. We want to show that the mapping $\mathbf{v} \mapsto \mathbf{T} = \mathbf{T}(\mathbf{v})$ is Lipschitz continuous. Let $\mathbf{g}_i = \eta(\nabla \mathbf{v}_i + \nabla \mathbf{v}_i^t)$ and consider the problems

$$\mathbf{T}_i + \mathcal{M}(\hat{\mathbf{v}}_i)\mathbf{T}_i + \hat{\mathbf{v}}_i \cdot \nabla \mathbf{T}_i = \mathbf{g}_i \ \text{ in } \mathcal{D},$$

for $i = 1, 2$. Define $\mathbf{U} = \mathbf{T}_1 - \mathbf{T}_2$ and $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2$. Let $\mathbf{G} = \mathbf{g}_1 - \mathbf{g}_2 = \eta(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$. Then

$$\begin{aligned} \mathbf{U} + \mathcal{M}(\hat{\mathbf{v}}_1)\mathbf{U} + \hat{\mathbf{v}}_1 \cdot \nabla \mathbf{U} &= \big(\mathbf{I} + \mathcal{M}(\hat{\mathbf{v}}_1)\big)(\mathbf{T}_1 - \mathbf{T}_2) + \hat{\mathbf{v}}_1 \cdot \nabla(\mathbf{T}_1 - \mathbf{T}_2) \\ &= \mathbf{G} - \hat{\mathbf{u}} \cdot \nabla \mathbf{T}_2 + \big(\mathcal{M}(\hat{\mathbf{v}}_2) - \mathcal{M}(\hat{\mathbf{v}}_1)\big)\mathbf{T}_2\,. \end{aligned} \tag{6.15}$$

Applying Lemma 4.1 with $q = 2$ to (6.15), we find

$$(6.16)\quad \| \mathbf{T}_1 - \mathbf{T}_2 \|_{L_2(\mathcal{D})} = \| \mathbf{U} \|_{L_2(\mathcal{D})} \leq \frac{1}{c_0} \| \mathbf{G} - \lambda_1 \big(\mathbf{u} \cdot \nabla \mathbf{T}_2 + \big(\mathcal{M}(\mathbf{v}_2) - \mathcal{M}(\mathbf{v}_1) \big) \mathbf{T}_2 \big) \|_{L_2(\mathcal{D})},$$

where we can define c_0 via

$$(6.17)\quad c_0 = 1 - \big(|1+\tilde{\mu}| + |1-\tilde{\mu}| \big) \| \nabla \hat{\mathbf{v}}_1 \|_{L_\infty(\mathcal{D})} = 1 - \big(|\lambda_1+\mu_1| + |\lambda_1-\mu_1| \big) \| \nabla \mathbf{v}_1 \|_{L_\infty(\mathcal{D})},$$

provided that the formula (6.17) yields $c_0 > 0$. But our assumptions (6.13) and (6.8) on γ imply that

$$\big(|\lambda_1+\mu_1| + |\lambda_1-\mu_1| \big) \| \nabla \mathbf{v}_1 \|_{L_\infty(\mathcal{D})} \leq \big(|\lambda_1+\mu_1| + |\lambda_1-\mu_1| \big) \sigma_q \| \nabla \mathbf{v}_1 \|_{W_q^2(\mathcal{D})} \leq \big(|\lambda_1+\mu_1| + |\lambda_1-\mu_1| \big) (\gamma \sigma_q / \eta) \leq \tfrac{1}{2},$$

so that $c_0 \geq \frac{1}{2}$. Thus we can prove the following lemma.

Lemma 6.2. *Suppose that the conditions of Lemma 4.6 hold for* $\mathbf{u} = \mathbf{v}_i \in W_q^2(\mathcal{D})$, $i = 1, 2$, *so that* $\nabla \cdot \mathbf{v}_i = 0$ *in* $\mathcal{D}$ *and* $\mathbf{v}_i = 0$ *on* $\partial \mathcal{D}$, *and the bound* (4.35) *holds for both* $\mathbf{v}_1$ *and* $\mathbf{v}_2$. *Let* $\mathbf{T}_i$ *solve*

$$\mathbf{T}_i + \mathcal{M}(\hat{\mathbf{v}}_i) \mathbf{T}_i + \hat{\mathbf{v}}_i \cdot \nabla \mathbf{T}_i = \mathbf{g}_i \quad \text{in } \mathcal{D},$$

for $\mathbf{g}_i = \eta \big(\nabla \mathbf{v}_i + (\nabla \mathbf{v}_i)^t \big)$, *where* $\mathcal{M}$ *is defined in* (6.14) *and* $\hat{\mathbf{v}}_i = \lambda_1 \mathbf{v}_i$, $i = 1, 2$. *Then*

$$(6.18)\quad \| \mathbf{T}_1 - \mathbf{T}_2 \|_{L_2(\mathcal{D})} \leq 2(2\eta + C) \| \mathbf{T}_2 \|_{W_q^1(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})},$$

where $C = |\lambda_1| \sigma_q (3 + 2|\tilde{\mu}|$.

Proof. Estimates (6.16), (4.23) and (1.7) imply

$$\| \mathbf{T}_1 - \mathbf{T}_2 \|_{L_2(\mathcal{D})} \leq 2 \big(\| \mathbf{G} \|_{L_2(\mathcal{D})} + |\lambda_1| \big(\| \mathbf{u} \|_{L_{\frac{2q}{q-2}}(\mathcal{D})} \| \mathbf{T}_2 \|_{W_q^1(\mathcal{D})} + \| \mathcal{M}(\mathbf{u}) \mathbf{T}_2 \|_{L_2(\mathcal{D})} \big) \big) \leq 2 \big(2\eta + |\lambda_1| (\sigma_q + c_M) \| \mathbf{T}_2 \|_{W_q^1(\mathcal{D})} \big) \| \mathbf{u} \|_{H^1(\mathcal{D})},$$

where the constant c_M is the smallest real number such that

$$\| \mathcal{M}(\mathbf{u}) \mathbf{T} \|_{L_2(\mathcal{D})} \leq c_M \| \mathbf{T} \|_{W_q^1(\mathcal{D})} \| \mathbf{u} \|_{H^1(\mathcal{D})} \ \forall \mathbf{u} \in H^1(\mathcal{D})^d, \ \mathbf{T} \in W_q^1(\mathcal{D})^{d^2}.$$

We estimate c_M as follows. From the definition (6.14), we see that

$$\| \mathcal{M}(\mathbf{u}) \mathbf{T} \|_{L_2(\mathcal{D})} \leq 2(1 + |\tilde{\mu}|) \| \mathbf{T} \|_{L_\infty(\mathcal{D})} \| \mathbf{u} \|_{H^1(\mathcal{D})} \leq 2\sigma_q (1 + |\tilde{\mu}|) \| \mathbf{T} \|_{W_q^1(\mathcal{D})} \| \mathbf{u} \|_{H^1(\mathcal{D})},$$

where σ_q is the constant in Sobolev's inequality (1.6) and $\tilde{\mu} = \mu_1/\lambda_1$, so we can be assured that $c_M \le 2\sigma_q(1 + |\tilde{\mu}|)$. $\square$

Using (6.13), (6.18) becomes

(6.19)
$$\begin{aligned} \| \mathbf{T}_1 - \mathbf{T}_2 \|_{L_2(\mathcal{D})} &\le 2\big(2\eta + |\lambda_1|\sigma_q(3 + 2|\tilde{\mu}|)(\hat{\sigma}\gamma)\big)\| \mathbf{u} \|_{H^1(\mathcal{D})} \\ &= C_T \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})}, \end{aligned}$$

where $\hat{\sigma}$ is the constant in (6.5) and (6.11), γ is the constant in the bound (6.13), and

(6.20)
$$C_T = 2\big(2\eta + \lambda_0\eta\sigma_q(3 + 2|\tilde{\mu}|)(\hat{\sigma}\gamma)\big).$$

Thus we conclude that the mapping $\mathbf{v} \mapsto \mathbf{T}(\mathbf{v})$ is Lipschitz continuous $H^1(\mathcal{D}) \to L_2(\mathcal{D})$, but only on bounded sets in $W_q^2(\mathcal{D})$. Moreover, we note that the Lipschitz constant C_T is not particularly small in this case.

In a similar way, we can provide a Lipschitz bound for the pressure terms. Suppose that

$$p_i + \lambda_1 \mathbf{v}_i \cdot \nabla p_i = \pi_i.$$

Then

$$p_1 - p_2 + \lambda_1 \mathbf{v}_1 \cdot \nabla(p_1 - p_2) = \pi_1 - \pi_2 + \lambda_1(\mathbf{v}_2 - \mathbf{v}_1) \cdot \nabla p_2.$$

Using [9], (4.23), and (1.6), we find

(6.21)
$$\begin{aligned} \| p_1 - p_2 \|_{L_2(\mathcal{D})} &\le \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} + |\lambda_1| \, \| (\mathbf{v}_2 - \mathbf{v}_1) \cdot \nabla p_2 \|_{L_2(\mathcal{D})} \\ &\le \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} + |\lambda_1| \, \| \mathbf{v}_2 - \mathbf{v}_1 \|_{L_{2q/(q-2)}(\mathcal{D})} \| p_2 \|_{W_q^1(\mathcal{D})} \\ &\le \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} + \sigma_q|\lambda_1| \, \| \mathbf{v}_2 - \mathbf{v}_1 \|_{H^1(\mathcal{D})} \| p_2 \|_{W_q^1(\mathcal{D})} \\ &\le \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} + \sigma_q\lambda_0\eta\hat{\sigma}\gamma \| \mathbf{v}_2 - \mathbf{v}_1 \|_{H^1(\mathcal{D})}. \end{aligned}$$

Next, we estimate

$$\| \mathcal{F}(\mathbf{f}, \mathbf{v}_1, p_1, \mathbf{T}(\mathbf{v}_1)) - \mathcal{F}(\mathbf{f}, \mathbf{v}_2, p_2, \mathbf{T}(\mathbf{v}_2)) \|_{H^{-1}(\mathcal{D})}.$$

It helps to split

$$\mathcal{F}(\mathbf{f}, \mathbf{v}, p, \mathbf{T}) = \mathcal{F}_1(\mathbf{f}, \mathbf{v}) + \lambda_1\mathcal{F}_2(\mathbf{v}, p) - \lambda_1\mathcal{F}_3(\mathbf{v}) + \lambda_1\mathcal{F}_4(\mathbf{v}) - (\lambda_1 - \mu_1)\mathcal{F}_5(\mathbf{v}),$$

where

$$\begin{aligned} \mathcal{F}_1(\mathbf{f}, \mathbf{v}) &= \mathbf{f} + \lambda_1 \mathbf{v} \cdot \nabla \mathbf{f} \\ \mathcal{F}_2(\mathbf{v}, p) &= (\nabla \mathbf{v})^t \nabla p \\ \mathcal{F}_3(\mathbf{v}) &= \mathbf{v} \cdot \nabla(\mathbf{v} \cdot \nabla \mathbf{v}) \\ \mathcal{F}_4(\mathbf{v}) &= \nabla \cdot ((\nabla \mathbf{v}) \circ \mathbf{T}(\mathbf{v})) \\ \mathcal{F}_5(\mathbf{v}) &= \nabla \cdot (\mathbf{E}(\mathbf{v}) \circ \mathbf{T}(\mathbf{v}) + \mathbf{T}(\mathbf{v}) \circ \mathbf{E}(\mathbf{v})) \end{aligned}$$

and $\mathbf{E}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$.

To begin with, we have the simple estimate

$$\| \mathcal{F}_1(\mathbf{f},\mathbf{v}_1) - \mathcal{F}_1(\mathbf{f},\mathbf{v}_2) \|_{H^{-1}(\mathcal{D})} \le |\lambda_1| \, \| (\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla\mathbf{f} \|_{H^{-1}(\mathcal{D})}.$$

For $\boldsymbol{\phi} \in H^1(\mathcal{D})^d$ we have by (4.24)

$$\begin{aligned} |\langle(\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla\mathbf{f}, \boldsymbol{\phi}\rangle| &= |\langle \mathbf{f}, (\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla\boldsymbol{\phi}\rangle| \\ &\le \sigma_q \| \mathbf{f} \|_{L_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}. \end{aligned} \tag{6.22}$$

Here and below, we are able to derive an estimate for $\boldsymbol{\phi} \in H^1(\mathcal{D})^d$ even though we only need it for $\boldsymbol{\phi} \in H^1_0(\mathcal{D})^d$. When the restriction to the smaller space is needed for the derivation of the inequality, we will note it. Returning to (6.22), we see that it implies

$$\begin{aligned} &\| \mathcal{F}_1(\mathbf{f},\mathbf{v}_1) - \mathcal{F}_1(\mathbf{f},\mathbf{v}_2) \|_{H^{-1}(\mathcal{D})} \le \sigma_q |\lambda_1| \, \| \mathbf{f} \|_{L_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \\ &\quad \le \sigma_q |\lambda_1| \, \varphi \, \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \le \sigma_q \lambda_0 \eta\varphi \, \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} = c_1 \varphi\eta \, \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})}, \end{aligned}$$

where $c_1 = \sigma_q \lambda_0$ and we recall that $\varphi \ge \| \mathbf{f} \|_{L_q(\mathcal{D})}$ and that λ_1 satisfies the bound (5.29).

For the next term, we find

$$\mathcal{F}_2(\mathbf{v}_1, p_1) - \mathcal{F}_2(\mathbf{v}_2, p_2) = \nabla(\mathbf{v}_1 - \mathbf{v}_2)^t \mathrm{S} \nabla p_1 - (\nabla\mathbf{v}_2)^t \nabla(p_2 - p_1). \tag{6.23}$$

For $\boldsymbol{\phi} \in H^1_0(\mathcal{D})^d$, we have

$$\begin{aligned} |\langle(\nabla\mathbf{v}_2)^t \nabla(p_2 - p_1), \boldsymbol{\phi}\rangle| &= |\langle \nabla(p_2 - p_1), (\nabla\mathbf{v}_2)\boldsymbol{\phi}\rangle| = |\langle p_2 - p_1, \nabla\cdot\big((\nabla\mathbf{v}_2)\boldsymbol{\phi}\big)\rangle| \\ &= |\langle p_2 - p_1, (\nabla\mathbf{v}_2)^t : \nabla\boldsymbol{\phi}\rangle|, \end{aligned}$$

where we have used (3.4) at the last step. Thus

$$|\langle(\nabla\mathbf{v}_2)^t \nabla(p_2 - p_1), \boldsymbol{\phi}\rangle| \le \sigma_q \| p_2 - p_1 \|_{L^2(\mathcal{D})} \| \mathbf{v}_2 \|_{W^2_q(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})},$$

using (4.24). Thus (6.21) implies

$$\begin{aligned} \| (\nabla\mathbf{v}_2)^t \nabla(p_2 - p_1) \|_{H^{-1}(\mathcal{D})} &\le \sigma_q \| p_2 - p_1 \|_{L^2(\mathcal{D})} \| \mathbf{v}_2 \|_{W^2_q(\mathcal{D})} \\ &\le \sigma_q \gamma\eta^{-1} \big(\| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} + \sigma_q \lambda_0 \eta\hat{\sigma}\gamma \| \mathbf{v}_2 - \mathbf{v}_1 \|_{H^1(\mathcal{D})} \big). \end{aligned} \tag{6.24}$$

For $\boldsymbol{\phi} \in H^1_0(\mathcal{D})^d$, we also have

$$\begin{aligned} |\langle \nabla(\mathbf{v}_1 - \mathbf{v}_2)^t \nabla p_1, \boldsymbol{\phi}\rangle| &= |\langle \nabla p_1, (\nabla(\mathbf{v}_1 - \mathbf{v}_2))\boldsymbol{\phi}\rangle| \\ &\le \sigma_q \| p_1 \|_{W^1_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}, \end{aligned}$$

again using (4.24). Thus

$$\begin{aligned} \| \nabla(\mathbf{v}_1 - \mathbf{v}_2)^t \nabla p_1 \|_{H^{-1}(\mathcal{D})} &\le \sigma_q \| p_1 \|_{W^1_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \\ &\le \sigma_q \hat{\sigma}\gamma \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})}. \end{aligned} \tag{6.25}$$

Combining (6.23), (6.24), and (6.25) we obtain

$$\| \mathcal{F}_2(\mathbf{v}_1, p_1) - \mathcal{F}_2(\mathbf{v}_2, p_2) \|_{H^{-1}(\mathcal{D})} \le c_2 \gamma \big(\| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})} + \| \mathbf{v}_2 - \mathbf{v}_1 \|_{H^1(\mathcal{D})} \big),$$

where $c_2 = \sigma_q \max\{\eta^{-1}, \hat{\sigma}(\lambda_0\gamma\sigma_q + 1)\}$. Moving along, we expand

$$\begin{aligned}\mathcal{F}_3(\mathbf{v}_1) - \mathcal{F}_3(\mathbf{v}_2) &= \mathbf{v}_1 \cdot \nabla(\mathbf{v}_1 \cdot \nabla\mathbf{v}_1) - \mathbf{v}_2 \cdot \nabla(\mathbf{v}_2 \cdot \nabla\mathbf{v}_2)\\ &= (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla(\mathbf{v}_1 \cdot \nabla\mathbf{v}_1) - \mathbf{v}_2 \cdot \nabla(\mathbf{v}_2 \cdot \nabla\mathbf{v}_2 - \mathbf{v}_1 \cdot \nabla\mathbf{v}_1)\\ &= (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla(\mathbf{v}_1 \cdot \nabla\mathbf{v}_1) - \mathbf{v}_2 \cdot \nabla((\mathbf{v}_2 - \mathbf{v}_1) \cdot \nabla\mathbf{v}_2)\\ &\qquad + \mathbf{v}_2 \cdot \nabla(\mathbf{v}_1 \cdot \nabla(\mathbf{v}_1 - \mathbf{v}_2)).\end{aligned}$$

We estimate the first of these three terms using (4.24):

$$\begin{aligned}|\langle(\mathbf{v}_1 - \mathbf{v}_2)\cdot\nabla(\mathbf{v}_1 \cdot \nabla\mathbf{v}_1), \boldsymbol{\phi}\rangle| &= |\langle(\mathbf{v}_1 \cdot \nabla\mathbf{v}_1), (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla\boldsymbol{\phi}\rangle|\\ &\le \sigma_q \| \mathbf{v}_1 \cdot \nabla\mathbf{v}_1 \|_{L_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}\\ &\le \sigma_q \| \mathbf{v}_1 \|_{L_\infty(\mathcal{D})} \| \nabla\mathbf{v}_1 \|_{L_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}\\ &\le \sigma_q^2 \| \mathbf{v}_1 \|_{W_q^1(\mathcal{D})}^2 \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}.\end{aligned}$$

Similarly, the second of the three terms is estimated using (4.24) by

$$\begin{aligned}|\langle \mathbf{v}_2 \cdot \nabla((\mathbf{v}_2 - \mathbf{v}_1) \cdot \nabla\mathbf{v}_2), \boldsymbol{\phi}\rangle| &= |\langle(\mathbf{v}_2 - \mathbf{v}_1) \cdot \nabla\mathbf{v}_2, \mathbf{v}_2 \cdot \nabla\boldsymbol{\phi}\rangle|\\ &\le \sigma_q^2 \| \mathbf{v}_2 \|_{W_q^1(\mathcal{D})}^2 \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}.\end{aligned}$$

Finally, Hölder's and Sobolev's inequalities give

$$\begin{aligned}|\langle \mathbf{v}_2 \cdot \nabla(\mathbf{v}_1 \cdot \nabla(\mathbf{v}_1 - \mathbf{v}_2)), \boldsymbol{\phi}\rangle| &= |\langle \mathbf{v}_1 \cdot \nabla(\mathbf{v}_1 - \mathbf{v}_2), \mathbf{v}_2 \cdot \nabla\boldsymbol{\phi}\rangle|\\ &\le \| \mathbf{v}_1 \|_{L_\infty(\mathcal{D})} \| \mathbf{v}_2 \|_{L_\infty(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}\\ &\le \sigma_q^2 \| \mathbf{v}_1 \|_{W_q^1(\mathcal{D})} \| \mathbf{v}_2 \|_{W_q^1(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}.\end{aligned}$$

Thus (6.7) implies

$$\begin{aligned}\| \mathcal{F}_3(\mathbf{v}_1) - \mathcal{F}_3(\mathbf{v}_2) \|_{H^{-1}(\mathcal{D})} &\le \sigma_q^2 \big(\| \mathbf{v}_1 \|_{W_q^1(\mathcal{D})}^2 + \| \mathbf{v}_2 \|_{W_q^1(\mathcal{D})}^2\\ &\qquad + \| \mathbf{v}_1 \|_{W_q^1(\mathcal{D})} \| \mathbf{v}_2 \|_{W_q^1(\mathcal{D})} \big) \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})}\\ &\le 3\sigma_q^2 \gamma^2 \eta^{-2} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} = c_3\gamma \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})},\end{aligned}$$

where $c_3 = 3\sigma_q^2\gamma\eta^{-2}$. For the next term, we have

(6.26)
$$\begin{aligned}\mathcal{F}_4(\mathbf{v}_1) - \mathcal{F}_4(\mathbf{v}_2) &= \nabla\cdot((\nabla\mathbf{v}_1)\circ\mathbf{T}(\mathbf{v}_1)) - \nabla\cdot((\nabla\mathbf{v}_2)\circ\mathbf{T}(\mathbf{v}_2))\\ &= \nabla\cdot((\nabla(\mathbf{v}_1 - \mathbf{v}_2))\circ\mathbf{T}(\mathbf{v}_1)) - \nabla\cdot((\nabla\mathbf{v}_2)\circ(\mathbf{T}(\mathbf{v}_2) - \mathbf{T}(\mathbf{v}_1)).\end{aligned}$$

For the first of these terms, we have, for $\boldsymbol{\phi} \in H_0^1(\mathcal{D})^d$,

$$\begin{aligned}|\langle\nabla\cdot((\nabla(\mathbf{v}_1 - \mathbf{v}_2))\circ\mathbf{T}(\mathbf{v}_1)), \boldsymbol{\phi}\rangle| &= |\langle(\nabla(\mathbf{v}_1 - \mathbf{v}_2))\circ\mathbf{T}(\mathbf{v}_1), \nabla\boldsymbol{\phi}\rangle|\\ &\le \| \mathbf{T}(\mathbf{v}_1) \|_{L_\infty(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}\\ &\le \sigma_q \| \mathbf{T}(\mathbf{v}_1) \|_{W_q^1(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}.\end{aligned} \tag{6.27}$$

Then (6.27) and (6.13) imply

$$\begin{aligned}\| \nabla\cdot ((\nabla(\mathbf{v}_1 - \mathbf{v}_2))\circ\mathbf{T}(\mathbf{v}_1)) \|_{H^{-1}(\mathcal{D})} &\le \sigma_q \| \mathbf{T}(\mathbf{v}_1) \|_{W^1_q(\mathcal{D})} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \\ &\le \sigma_q \hat{\sigma}\gamma \mathrm{S} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})}.\end{aligned} \tag{6.28}$$

For the second of the two terms in (6.26), we have, for $\boldsymbol{\phi} \in H^1_0(\mathcal{D})^d$,

$$\begin{aligned}|\langle \nabla\cdot ((\nabla\mathbf{v}_2)\circ(\mathbf{T}(\mathbf{v}_2) - \mathbf{T}(\mathbf{v}_1)), \boldsymbol{\phi}\rangle| &= |\langle (\nabla\mathbf{v}_2)\circ(\mathbf{T}(\mathbf{v}_2) - \mathbf{T}(\mathbf{v}_1)), \nabla\boldsymbol{\phi}\rangle| \\ &\le \| \mathbf{v}_2 \|_{W^1_\infty(\mathcal{D})} \| \mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2) \|_{L_2(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})} \\ &\le \sigma_q \| \mathbf{v}_2 \|_{W^2_q(\mathcal{D})} \| \mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2) \|_{L_2(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})}.\end{aligned} \tag{6.29}$$

Then (6.29) and (6.19) imply

$$\begin{aligned}\| \nabla\cdot (\nabla\mathbf{v}_2)\circ(\mathbf{T}(\mathbf{v}_2) - \mathbf{T}(\mathbf{v}_1)) \|_{H^{-1}(\mathcal{D})} &\le \sigma_q \| \mathbf{v}_2 \|_{W^2_q(\mathcal{D})} \| \mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2) \|_{L_2(\mathcal{D})} \\ &\le \sigma_q (\gamma/\eta) C_T \| \mathbf{v}_1 - \mathbf{v}_2 \|_{L_2(\mathcal{D})},\end{aligned} \tag{6.30}$$

where C_T is defined in (6.20). Estimates (6.28) and (6.30) combine to yield

$$\begin{aligned}&\| \mathcal{F}_4(\mathbf{v}_1) - \mathcal{F}_4(\mathbf{v}_2) \|_{H^{-1}(\mathcal{D})} \\ &\qquad \le \sigma_q (\gamma/\eta)\big(C_T + \hat{\sigma}\eta\big) \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} \le c_4 \gamma \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})},\end{aligned}$$

where $c_4 = \sigma_q \eta^{-1}\big(C_T + \hat{\sigma}\eta\big)$. Last and least, we examine $\mathcal{F}_5$. Note first that, for any $\boldsymbol{\phi} \in H^1_0(\mathcal{D})^d$ and $\mathbf{T} \in L_2(\mathcal{D})^{d^2}$,

$$|\langle \nabla\cdot \mathbf{T}, \boldsymbol{\phi}\rangle| = |\langle \mathbf{T}, \nabla\boldsymbol{\phi}\rangle| \le \| \mathbf{T} \|_{L_2(\mathcal{D})} \| \boldsymbol{\phi} \|_{H^1(\mathcal{D})},$$

so that for all $\mathbf{T} \in L_2(\mathcal{D})^{d^2}$,

$$\| \nabla\cdot \mathbf{T} \|_{H^{-1}(\mathcal{D})} \le \| \mathbf{T} \|_{L_2(\mathcal{D})}. \tag{6.31}$$

Expanding, we have

$$\begin{aligned}\mathcal{F}_5(\mathbf{v}_1) - \mathcal{F}_5(\mathbf{v}_2) &= \nabla\cdot \big(\mathbf{E}(\mathbf{v}_1)\circ\mathbf{T}(\mathbf{v}_1) - \mathbf{E}(\mathbf{v}_2)\circ\mathbf{T}(\mathbf{v}_2)\big) \\ &\quad + \nabla\cdot \big(\mathbf{T}(\mathbf{v}_1)\circ\mathbf{E}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2)\circ\mathbf{E}(\mathbf{v}_2)\big) \\ &= \nabla\cdot \big(\mathbf{E}(\mathbf{v}_1 - \mathbf{v}_2)\circ\mathbf{T}(\mathbf{v}_1) - \mathbf{E}(\mathbf{v}_2)\circ(\mathbf{T}(\mathbf{v}_2) - \mathbf{T}(\mathbf{v}_1))\big) \\ &\quad + \nabla\cdot \big((\mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2))\circ\mathbf{E}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2)\circ\mathbf{E}(\mathbf{v}_2 - \mathbf{v}_1)\big).\end{aligned}$$

Thus (6.31) implies

(6.32)

$$\begin{aligned}
\| \mathcal{F}_5(\mathbf{v}_1) - \mathcal{F}_5(\mathbf{v}_2) \|_{H^{-1}(\mathcal{D})} &\le \big(\| \mathbf{E}(\mathbf{v}_1 - \mathbf{v}_2) \circ \mathbf{T}(\mathbf{v}_1) \|_{L_2(\mathcal{D})} \\
&\quad + \| \mathbf{E}(\mathbf{v}_2) \circ (\mathbf{T}(\mathbf{v}_2) - \mathbf{T}(\mathbf{v}_1)) \|_{L_2(\mathcal{D})} + \| (\mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2)) \circ \mathbf{E}(\mathbf{v}_1) \|_{L_2(\mathcal{D})} \\
&\quad + \| \mathbf{T}(\mathbf{v}_2) \circ \mathbf{E}(\mathbf{v}_2 - \mathbf{v}_1) \|_{L_2(\mathcal{D})}\big) \\
&\le \Big(\| \mathbf{E}(\mathbf{v}_1 - \mathbf{v}_2) \|_{L_2(\mathcal{D})} \big(\| \mathbf{T}(\mathbf{v}_1) \|_{L_\infty(\mathcal{D})} + \| \mathbf{T}(\mathbf{v}_2) \|_{L_\infty(\mathcal{D})}\big) \\
&\quad + \| \mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2) \|_{L_2(\mathcal{D})} \big(\| \mathbf{E}(\mathbf{v}_1) \|_{L_\infty(\mathcal{D})} + \| \mathbf{E}(\mathbf{v}_2) \|_{L_\infty(\mathcal{D})}\big)\Big) \\
&\le \sigma_q \big(\| \mathbf{E}(\mathbf{v}_1 - \mathbf{v}_2) \|_{L_2(\mathcal{D})} + \| \mathbf{T}(\mathbf{v}_1) - \mathbf{T}(\mathbf{v}_2) \|_{L_2(\mathcal{D})}\big) \\
&\quad \times \sum_{i=1,2} \big(\| \mathbf{T}(\mathbf{v}_i) \|_{W^1_q(\mathcal{D})} + \| \mathbf{E}(\mathbf{v}_i) \|_{W^1_q(\mathcal{D})}\big).
\end{aligned}$$

Applying (6.19) and (6.13) to (6.32), we find

$$\begin{aligned}
\| \mathcal{F}_5(\mathbf{v}_1) - \mathcal{F}_5(\mathbf{v}_2) \|_{H^{-1}(\mathcal{D})} &\le \sigma_q (1 + C_T) \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} 2(\hat{\sigma}\eta + 1)(\gamma/\eta) \\
&\le c_5 \gamma \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})},
\end{aligned}$$

where $c_5 = \sigma_q (1 + C_T) 2(\hat{\sigma}\eta + 1)\eta^{-1}$.

For any $\alpha > 0$, we can choose γ and φ sufficiently small so that

(6.33)
$$\begin{aligned}
&\| \mathcal{F}(\mathbf{f}, \mathbf{v}_1, p_1, \mathbf{T}_1) - \mathcal{F}(\mathbf{f}, \mathbf{v}_2, p_2, \mathbf{T}_2) \|_{H^{-1}(\mathcal{D})} \\
&\qquad\qquad \le \alpha \big(\eta \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^1(\mathcal{D})} + \| \pi_1 - \pi_2 \|_{L_2(\mathcal{D})}\big).
\end{aligned}$$

Choosing $\alpha > 0$ appropriately, we find

$$\begin{aligned}
&\eta \| \mathbf{u}^{n+1} - \mathbf{u}^n \|_{H^1(\mathcal{D})} + \| \pi^{n+1} - \pi^n \|_{L_2(\mathcal{D})} \\
&\le \tfrac{1}{2} \big(\eta \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_{H^1(\mathcal{D})} + \| \pi^n - \pi^{n-1} \|_{L_2(\mathcal{D})}\big).
\end{aligned}$$

Here we used Corollary 5.7 and (6.10). This proves that the sequence $(\mathbf{u}^n, \pi^n)$ converges geometrically in $H^1(\mathcal{D})^d \times L^2(\mathcal{D})$, and (6.19) and (6.21) prove that the full sequence $(\mathbf{u}^n, \pi^n, p^n, \mathbf{T}^n)$ converges geometrically to a limit $(\mathbf{u}, \pi, p, \mathbf{T}) \in H^1(\mathcal{D})^d \times L^2(\mathcal{D}) \times L^2(\mathcal{D}) \times L^2(\mathcal{D})^{d^2}$.

To show that this gives a solution of the 3-parameter Oldroyd system, we need to show that the limit satisfies the Navier-Stokes system (3.10). To show convergence in the Navier-Stokes system, we need to study the convergence of $\mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n)$ to $\mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T})$. From (6.33), we conclude that

(6.34)
$$\mathcal{F}^n := \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \to \mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T})$$

strongly in $H^{-1}(\mathcal{D})^d$ as $n \to \infty$. This implies that (3.11) holds via the following standard variational argument. We can express (3.11) in variational form as

$$\eta \int_{\mathcal{D}} \nabla \mathbf{u}^n : \nabla \mathbf{v} \, \mathrm{d}x - \int_{\mathcal{D}} (\mathbf{u}^n \cdot \nabla \mathbf{v}) \cdot \mathbf{u}^n \, \mathrm{d}x - \int_{\mathcal{D}} \pi^n \, \nabla \cdot \mathbf{v} \, \mathrm{d}x = \langle \mathcal{F}^n, \mathbf{v} \rangle$$

for all $\mathbf{v} \in H_0^1(\mathcal{D})^d$. Given the strong convergence of $\mathbf{u}^n \to \mathbf{u}$ in $H^1(\mathcal{D})^d$ and $\pi^n \to \pi$ in $L_2(\mathcal{D})$, together with (6.34), we conclude that

$$\eta \int_{\mathcal{D}} \nabla \mathbf{u} : \nabla \mathbf{v} \, \mathrm{d}x - \int_{\mathcal{D}} (\mathbf{u} \cdot \nabla \mathbf{v}) \cdot \mathbf{u} \, \mathrm{d}x - \int_{\mathcal{D}} \pi \, \nabla \cdot \mathbf{v} \, \mathrm{d}x = \big\langle \mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T}), \mathbf{v} \big\rangle$$

for all $\mathbf{v} \in H_0^1(\mathcal{D})^d$, confirming (3.11), and equivalently (3.10).

7. VARIATIONAL FORMULATION

The variational formulation is based on a standard one for Navier-Stokes:

$$\begin{aligned} \eta \int_{\mathcal{D}} \nabla \mathbf{u}^{n+1} : \nabla \mathbf{v} \, \mathrm{d}x + \int_{\mathcal{D}} (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}) \cdot \mathbf{v} \, \mathrm{d}x - \int_{\mathcal{D}} \pi^{n+1} \nabla \cdot \mathbf{v} \, \mathrm{d}x & \\ = \int_{\mathcal{D}} \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \cdot \mathbf{v} \, \mathrm{d}x, & \end{aligned}$$

where we recall that

$$\begin{aligned} \mathcal{F}(\mathbf{f}, \mathbf{u}, p, \mathbf{T}) = \mathbf{f} + \lambda_1 \mathbf{u} \cdot \nabla \mathbf{f} + \lambda_1 (\nabla \mathbf{u})^t \nabla p - \lambda_1 \big(\mathbf{u} \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) - \nabla \cdot ((\nabla \mathbf{u}) \circ \mathbf{T}) \big) & \\ - (\lambda_1 - \mu_1) \nabla \cdot (\mathbf{E}(\mathbf{u}) \circ \mathbf{T} + \mathbf{T} \circ \mathbf{E}(\mathbf{u})). & \end{aligned}$$

We develop some identities that are useful for simplifying the terms involving $\mathcal{F}$. For any tensor function $\mathbf{T}$ of arity 2 (that is, a matrix function) and any vector function $\mathbf{v} \in H_0^1(\mathcal{D})^d$,

(7.1)

$$\int_{\mathcal{D}} (\nabla \cdot \mathbf{T}) \cdot \mathbf{v} \, \mathrm{d}x = \int_{\mathcal{D}} \sum_{ij} T_{ij,j} \, v_i \, \mathrm{d}x = - \int_{\mathcal{D}} \sum_{ij} T_{ij} \, v_{i,j} \, \mathrm{d}x = - \int_{\mathcal{D}} \mathbf{T} : \nabla \mathbf{v} \, \mathrm{d}x.$$

Note that, if $\nabla \cdot \mathbf{u} = 0$ in $\mathcal{D}$ and $\mathbf{v} = \mathbf{0}$ on $\partial \mathcal{D}$,

(7.2)

$$\begin{aligned} \int_{\mathcal{D}} \big((\nabla \mathbf{u})^t \cdot \nabla p \big) \cdot \mathbf{v} \, \mathrm{d}x &= \int_{\mathcal{D}} \sum_{ij} u_{j,i} \, p_{,j} \, v_i \, \mathrm{d}x = - \int_{\mathcal{D}} \sum_{ij} \big(u_{j,i} \, v_i \big)_{,j} \, p \, \mathrm{d}x \\ &= - \int_{\mathcal{D}} \Big(\sum_{ij} u_{j,ij} \, v_i + u_{j,i} \, v_{i,j} \Big) \, p \, \mathrm{d}x \\ &= - \int_{\mathcal{D}} \sum_i ((\nabla \cdot \mathbf{u})_{,i}) v_i \, p \, \mathrm{d}x - \int_{\mathcal{D}} \big((\nabla \mathbf{u})^t : (\nabla \mathbf{v}) \big) \, p \, \mathrm{d}x \\ &= - \int_{\mathcal{D}} \big((\nabla \mathbf{u})^t : (\nabla \mathbf{v}) \big) \, p \, \mathrm{d}x \, ; \end{aligned}$$

compare with (3.4). Similarly, if $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{v} = \mathbf{0}$ on $\partial \mathcal{D}$, for any $\mathbf{w} \in H^1(\mathcal{D})^d$ we have

$$\int_{\mathcal{D}} (\mathbf{u} \cdot \nabla \mathbf{w}) \cdot \mathbf{v} \, \mathrm{d}x = - \int_{\mathcal{D}} \mathbf{w} \cdot (\mathbf{u} \cdot \nabla \mathbf{v}) \, \mathrm{d}x. \tag{7.3}$$

Using (7.1), (7.2) and (7.3), we find

$$
\begin{aligned}
&\int_{\mathcal{D}} \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \cdot \mathbf{v} \, \mathrm{d}x \\
&\qquad = \int_{\mathcal{D}} \mathbf{f} \cdot (\mathbf{v} - \lambda_1 \mathbf{u}^n \cdot \nabla \mathbf{v}) \, \mathrm{d}x - \lambda_1 \Big(\int_{\mathcal{D}} p^n (\nabla \mathbf{u}^n)^t : \nabla \mathbf{v} \, \mathrm{d}x \\
&\qquad - \int_{\mathcal{D}} (\mathbf{u}^n \cdot \nabla \mathbf{u}^n) \cdot (\mathbf{u}^n \cdot \nabla \mathbf{v}) \, \mathrm{d}x + \int_{\mathcal{D}} (\nabla \mathbf{u}^n \circ \mathbf{T}^n) : \nabla \mathbf{v} \, \mathrm{d}x \Big) \\
&\qquad + (\lambda_1 - \mu_1) \int_{\mathcal{D}} (\mathbf{E}(\mathbf{u}^n) \circ \mathbf{T}^n + \mathbf{T}^n \circ \mathbf{E}(\mathbf{u}^n)) : \nabla \mathbf{v} \, \mathrm{d}x.
\end{aligned} \tag{7.4}
$$

Thus a variational form for the algorithm (6.1) is as follows. First, knowing $\mathbf{u}^n$, p^n, and $\mathbf{T}^n$, we find $\mathbf{u}^{n+1} \in V$ and $\pi^{n+1} \in \Pi$ such that

$$
\begin{aligned}
&\eta \int_{\mathcal{D}} \nabla \mathbf{u}^{n+1} : \nabla \mathbf{v} \, \mathrm{d}x + \int_{\mathcal{D}} (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}) \cdot \mathbf{v} \, \mathrm{d}x - \int_{\mathcal{D}} \pi^{n+1} \nabla \cdot \mathbf{v} \, \mathrm{d}x \\
&\quad = \int_{\mathcal{D}} \mathcal{F}(\mathbf{f}, \mathbf{u}^n, p^n, \mathbf{T}^n) \cdot \mathbf{v} \, \mathrm{d}x = \int_{\mathcal{D}} \mathbf{f} \cdot (\mathbf{v} - \lambda_1 \mathbf{u}^n \cdot \nabla \mathbf{v}) \, \mathrm{d}x \\
&\quad - \lambda_1 \Big(\int_{\mathcal{D}} p^n (\nabla \mathbf{u}^n)^t : \nabla \mathbf{v} \, \mathrm{d}x - \int_{\mathcal{D}} (\mathbf{u}^n \cdot \nabla \mathbf{u}^n) \cdot (\mathbf{u}^n \cdot \nabla \mathbf{v}) \, \mathrm{d}x \\
&\qquad\qquad + \int_{\mathcal{D}} (\nabla \mathbf{u}^n \circ \mathbf{T}^n) : \nabla \mathbf{v} \, \mathrm{d}x \Big) \\
&\qquad\qquad + (\lambda_1 - \mu_1) \int_{\mathcal{D}} (\mathbf{E}(\mathbf{u}^n) \circ \mathbf{T}^n + \mathbf{T}^n \circ \mathbf{E}(\mathbf{u}^n)) : \nabla \mathbf{v} \, \mathrm{d}x,
\end{aligned}
$$

for all $\mathbf{v}$ in a suitable space V, where as usual, $\mathbf{E}(\mathbf{w}) = \frac{1}{2}\big(\nabla \mathbf{w} + (\nabla \mathbf{w})^t\big)$. We omit the details for solving for π^{n+1}; in the discrete case, this will depend on the particular implementation of the velocity and pressure spaces. Next, we solve the transport problem for p^{n+1} via two possible formulations: find $p^{n+1} \in \widehat{\Pi}$ such that

$$
\int_{\mathcal{D}} p^{n+1} (v - \lambda_1 \mathbf{u}^{n+1} \cdot \nabla v) \, \mathrm{d}x = \int_{\mathcal{D}} \pi^{n+1} v \, \mathrm{d}x \quad \forall v \in \widehat{\Pi}
$$

or

$$
\int_{\mathcal{D}} (p^{n+1} + \lambda_1 \mathbf{u}^{n+1} \cdot \nabla p^{n+1}) v \, \mathrm{d}x = \int_{\mathcal{D}} \pi^{n+1} v \, \mathrm{d}x \quad \forall v \in \widehat{\Pi}
$$

for a suitable space $\widehat{\Pi}$. Finally, we solve the transport problem (6.3) or (6.4) for $\mathbf{T}^{n+1}$ in a similar fashion. For example, one option would be

$$
\begin{aligned}
&\int_{\mathcal{D}} \mathbf{T}^{n+1} : (\mathbf{U} - \lambda_1 \mathbf{u}^{n+1} \cdot \nabla \mathbf{U}) \, \mathrm{d}x + \int_{\mathcal{D}} \mathcal{M}(\mathbf{u}^{n+1}) \mathbf{T}^{n+1} : \mathbf{U} \, \mathrm{d}x \\
&\qquad\qquad = 2\eta \int_{\mathcal{D}} \mathbf{E}(\mathbf{u}^{n+1}) : \mathbf{U} \, \mathrm{d}x \quad \forall \mathbf{U} \in \widetilde{\Pi}^{d^2}
\end{aligned}
$$

for a suitable space $\widetilde{\Pi}$, where $\hat{\mathbf{v}} = \lambda_1 \mathbf{v}$ and $\mathcal{M}(\hat{\mathbf{v}})\mathbf{T}$ is defined by (6.14).

8. RENARDY'S ORIGINAL PROOF

Define the operator $\mathbf{T} : \partial^2$ as follows:

$$\left(\mathbf{T} : \partial^2 \mathbf{u}\right)_i = \sum_{jk} \mathbf{T}_{jk} u_{i,jk}.$$

We compute the divergence of $(\nabla \mathbf{u}) \circ \mathbf{T}$ as follows:

$$\begin{aligned}
(\nabla \cdot ((\nabla \mathbf{u}) \circ \mathbf{T}))_i &= \sum_j ((\nabla \mathbf{u}) \circ \mathbf{T})_{ij,j} \\
&= \sum_{jk} ((\nabla \mathbf{u})_{ik} T_{kj})_{,j} \\
&= \sum_{jk} (u_{i,k} T_{kj})_{,j} = \sum_{jk} \left(u_{i,jk} T_{kj} + u_{i,k} T_{kj,j}\right) \\
&= \left(\mathbf{T} : \partial^2 \mathbf{u}\right)_i + \sum_{jk} u_{i,k} T_{kj,j} = \left(\mathbf{T} : \partial^2 \mathbf{u}\right)_i + \sum_k u_{i,k} \Big(\sum_j T_{kj,j}\Big) \\
&= \left(\mathbf{T} : \partial^2 \mathbf{u}\right)_i + \sum_k u_{i,k} \left(\nabla \cdot \mathbf{T}\right)_k = \left(\mathbf{T} : \partial^2 \mathbf{u}\right)_i + (\nabla u_i) \cdot (\nabla \cdot \mathbf{T}).
\end{aligned}$$

Therefore

$$\nabla \cdot ((\nabla \mathbf{u}) \circ \mathbf{T}) = \mathbf{T} : \partial^2 \mathbf{u} + (\nabla \mathbf{u}) \circ (\nabla \cdot \mathbf{T}). \tag{8.1}$$

Thus (3.2) and (8.1) imply that

$$\begin{aligned}
\nabla \cdot \left(\mathbf{u} \cdot \nabla \mathbf{T} - \mathbf{T} \circ (\nabla \mathbf{u})^t - (\nabla \mathbf{u}) \circ \mathbf{T}\right) &= \mathbf{u} \cdot \nabla \left(\nabla \cdot \mathbf{T}\right) - \mathbf{T} : \partial^2 \mathbf{u} - (\nabla \mathbf{u}) \circ \nabla \cdot \mathbf{T} \\
&= \mathcal{R}(\mathbf{u}) \left(\nabla \cdot \mathbf{T}\right) - \mathbf{T} : \partial^2 \mathbf{u},
\end{aligned}$$

where we define the operator $\mathcal{R}(\mathbf{u})$ by

$$\mathcal{R}(\mathbf{u}) \mathbf{v} = \mathbf{u} \cdot \nabla \mathbf{v} - (\nabla \mathbf{u}) \circ \mathbf{v}.$$

Renardy used (2.1) to replace $\nabla \cdot \mathbf{T}$ in the divergence of (2.4) to get

$$\begin{aligned}
\eta \Delta \mathbf{u} + \lambda_1 \mathbf{T} : \partial^2 \mathbf{u} &= \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f} \\
&\quad + \lambda_1 \mathcal{R}(\mathbf{u}) \left(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}\right) + (\lambda_1 - \mu_1) \nabla \cdot \left(\mathbf{E} \circ \mathbf{T} + \mathbf{T} \circ \mathbf{E}\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
-\eta \Delta \mathbf{u} - \lambda_1 \mathbf{T} : \partial^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f} \\
&\quad - \lambda_1 \mathcal{R}(\mathbf{u}) \left(\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{f}\right) - (\lambda_1 - \mu_1) \nabla \cdot \left(\mathbf{E} \circ \mathbf{T} + \mathbf{T} \circ \mathbf{E}\right).
\end{aligned} \tag{8.2}$$

Renardy used the modified Stokes operator on the left-hand side of (8.2) as the basis of his existence proof. For the regularity results, this requires verifying the appropriate coercivity and regularity results for variable-coefficient, Stokes-like equations. Details were omitted from [16].

9. TENSOR CALCULUS

Here we collect some tensor identities from [11].

For $\mathbf{T}, \mathbf{U} \in \mathcal{T}_r$, we define the contraction $\mathbf{T} : \mathbf{U}$ via

$$\mathbf{T} : \mathbf{U} = \sum_{i_1,\ldots,i_r} T_{i_1,\ldots,i_r} U_{i_1,\ldots,i_r}. \tag{9.1}$$

Another tensor contraction formula is

$$(\mathbf{T} \circ \mathbf{U})_{i_1\ldots i_{r-1} j_2 \ldots j_{r'}} = \sum_{\ell=1}^{d} T_{i_1\ldots i_{r-1}\ell} U_{\ell j_2\ldots j_{r'}}, \tag{9.2}$$

where $\mathbf{T} \in \mathcal{T}_r$ and $\mathbf{U} \in \mathcal{T}_{r'}$, and this defines $\mathbf{T} \circ \mathbf{U} \in \mathcal{T}_{r+r'-2}$. We have the following identities:

$$\mathbf{v} \cdot \nabla \mathbf{T} = (\nabla \mathbf{T}) \circ \mathbf{v}. \tag{9.3}$$

$$(\mathbf{W} : \mathbf{U})_i = \sum_{j,k=1}^{d} W_{ijk} U_{jk} \text{ for } \mathbf{W} \in \mathcal{T}_3,\ \mathbf{U} \in \mathcal{T}_2. \tag{9.4}$$

$$\nabla \cdot (\mathbf{T} \circ \mathbf{U}) = (\nabla \mathbf{T}) : \mathbf{U} + \mathbf{T} \circ (\nabla \cdot \mathbf{U}). \tag{9.5}$$

Note that the operator " $\circ$ " in $\mathbf{T} \circ (\nabla \cdot \mathbf{U})$ denotes an ordinary matrix-vector product also

$$\nabla(\mathbf{v} \cdot \nabla \mathbf{T}) = \nabla \mathbf{T} \circ \nabla \mathbf{v} + \mathbf{v} \cdot \nabla(\nabla \mathbf{T}) = \nabla \mathbf{T} \circ \nabla \mathbf{v} + (\nabla^2 \mathbf{T}) \circ \mathbf{v}. \tag{9.6}$$

We can combine (9.3) and (9.6) to compute

$$\mathbf{v} \cdot \nabla(\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla(\mathbf{v} \cdot \nabla \mathbf{v}) \circ \mathbf{v} = \big(\nabla \mathbf{v} \circ \nabla \mathbf{v} + (\nabla^2 \mathbf{v}) \circ \mathbf{v}\big) \circ \mathbf{v}. \tag{9.7}$$

When $\mathbf{T}$ is a scalar-valued function, (9.6) can be written alternatively as

$$\nabla(\mathbf{v} \cdot \nabla f) = \nabla \mathbf{v}^t \nabla f + \mathbf{v} \cdot \nabla(\nabla f), \tag{9.8}$$

since

$$(\nabla f \circ \nabla \mathbf{v})_k = \sum_j f_{,j} v_{j,k} = \sum_j (\nabla \mathbf{v}^t)_{k,j} f_j.$$

Based on the tensor contraction formula (9.2), we compute the derivative of a product:

$$(\nabla(\mathbf{T} \circ \mathbf{U}) - \mathbf{T} \circ \nabla \mathbf{U})_{i_1\ldots i_{r-1} j_2 \ldots j_{r'} k} = \sum_{\ell=1}^{d} (\nabla \mathbf{T})_{i_1\ldots i_{r-1}\ell k} U_{\ell j_2\ldots j_{r'}}, \tag{9.9}$$

but the term on the right-hand side is not an obvious product. Define a bilinear mapping $\mathcal{B} : \mathcal{T}_{r+1} \times \mathcal{T}_{r'} \to \mathcal{T}_{r+r'-1}$ by

$$(\mathcal{B}(\mathbf{W}, \mathbf{U}))_{i_1 \dots i_{r-1} j_2 \dots j_{r'} k} = \sum_{\ell=1}^{d} (\mathbf{W})_{i_1 \dots i_{r-1} \ell k} U_{\ell j_2 \dots j_{r'}}. \tag{9.10}$$

Then (9.9) becomes

$$\nabla(\mathbf{T} \circ \mathbf{U}) = \mathbf{T} \circ \nabla \mathbf{U} + \mathcal{B}(\nabla \mathbf{T}, \mathbf{U}). \tag{9.11}$$

In particular,

$$\nabla(\mathbf{S} \circ \mathbf{T} + \mathbf{T} \circ \mathbf{S}^t) = \mathbf{S} \circ \nabla \mathbf{T} + \mathcal{B}(\nabla \mathbf{S}, \mathbf{T}) + \mathbf{T} \circ \nabla \mathbf{S}^t + \mathcal{B}(\nabla \mathbf{T}, \mathbf{S}^t). \tag{9.12}$$

From the definition (9.10), we have

$$|\mathcal{B}(\mathbf{W}, \mathbf{U}))| \le |\mathbf{W}|\, |\mathbf{U}|. \tag{9.13}$$

There is a useful inequality involving three tensors. Suppose that $\mathbf{T} \in \mathcal{T}_2$ and $\mathbf{W}, \mathbf{U} \in \mathcal{T}_r$ where $r \ge 1$. Note that $\mathbf{T} \circ \mathbf{W} \in \mathcal{T}_r$ and $\mathbf{T} \circ \mathbf{U} \in \mathcal{T}_r$. We can interpret the contraction " : " as the usual ℓ_2 inner-product on vectors of dimension d^r, and $|\mathbf{W}|$ as the corresponding norm. Then we claim that

$$|\mathbf{T} \circ \mathbf{W} : \mathbf{U}| \le |\mathbf{T}|\, |\mathbf{W}|\, |\mathbf{U}|, \tag{9.14}$$

which is a generalization of (4.11).

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