To Philippe Ciarlet and Moshe Marcus,
with friendship and admiration on their 80th birthdays

THE ROLE OF THE HARDY TYPE INEQUALITIES IN THE THEORY OF FUNCTION SPACES

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We illustrate the crucial importance of the Hardy type inequalities in the study of function spaces, especially of fractional regularity. Immediate applications include Sobolev and Morrey type embeddings, and properties of the superposition operator $f \mapsto \Phi \circ f$. Another fundamental consequence is the trace theory of weighted Sobolev spaces. In turn, weighted Sobolev spaces are useful in the regularity theory of the superposition operators. More involved applications, that we present in the final section, are related to Sobolev spaces of maps with values into manifolds.

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1. INTRODUCTION

Fractional regularity function spaces, in particular Sobolev spaces $W^{s,p}$ with non integer s, have attracted considerable interest in the latest years, for example in connection with fractional processes and operators. Typical and classical questions related to these spaces are their embeddings, the properties of the superposition operators $f \mapsto \Phi \circ f$, or the possibility of giving a meaning to the pullback $f^\sharp \omega$ when ω is an alternate object, *e.g.* a form.

One of our purposes is to present a user-friendly introduction to fractional Sobolev spaces and their analysis. This text is an elementary and, to a significant extent, self-contained presentation of these topics. The main thread is the effectiveness of the Hardy type inequalities in the study of the aforementioned properties. Fractional Sobolev spaces are at the intersection of two important classes of function spaces: Besov spaces and Triebel-Lizorkin spaces; see *e.g.*

Triebel [57, Chapters 2–4] for an overview of the theory of these function spaces. In order to establish the properties of these general classes, one usually has to use relatively advanced tools in analysis (linear or nonlinear interpolation theory, Littlewood-Paley theory, etc.). As we will see below, completely elementary arguments, many of them based on Hardy type inequalities, suffice in the case of fractional Sobolev spaces. (Number of these proofs can be adapted to Besov spaces, but I took the party of not working with these spaces in this text.) This is not the first relatively elementary introduction to fractional Sobolev spaces; see *e.g.* Leoni [33, Chapter 14] or Di Nezza, Palatucci and Valdinoci [22]. However, we think that the systematic use of the Hardy type inequalities provides the basis for a unified approach that may be of interest even to the expert reader.

The results in Sections 2, 3 and in the first part of Section 5 are well-known since the 60's. Few proofs in these sections are either classical or possibly known to experts, but we also present a significant number of new proofs. We gave references whenever we were aware of the use of similar arguments in the literature. In the other sections, we present more recent results, some of them with new proofs.

In order to keep the reading as smooth as possible, a final appendix gathers some calculations which, though essential in the proofs, are not in line with the main type of arguments we present here.

This text is not a survey of the subject; the references list is very limited. The interested reader may google the keywords and find the huge literature existing on these topics.

Notation

1. All the functions we consider in $\mathbb{R}^n$ are implicitly assumed to be Borel measurable.
2. $x \vee y := \max\{x, y\}$, $x \wedge y := \min\{x, y\}$. (Warning: "$\wedge$" will also be used for the exterior product vector of vectors in $\mathbb{R}^2$, see item 19. below.)
3. When $x \in \mathbb{R}^n$, $|x|$ stands for the (standard) Euclidean norm of x. The standard scalar product is denoted $\langle x, y\rangle$, $x, y \in \mathbb{R}^n$.
4. $B_r(x)$ is the Euclidean open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$.
5. When $\rho : \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon > 0$, we set $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$, $\forall\, x \in \mathbb{R}^n$.
6. A standard mollifier is a function $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\rho \geq 0$ and $\int_{\mathbb{R}^n} \rho = 1$. We often assume in addition that $\operatorname{supp} \rho \subset B_1(0)$.
7. $|A|$ is the Lebesgue measure of a Borel set $A \subset \mathbb{R}^n$.
8. $\fint_A f$ stands for the average of f on A. Typically, $A \subset \mathbb{R}^n$, and then

$$\fint_A f(x)\,\mathrm{d}x := \frac{1}{|A|}\int_A f(x)\,\mathrm{d}x.$$

9. Almost everywhere (a.e.) for some function f is understood with respect to the Lebesgue (or Hausdorff) measure of the underlying space.
10. When $f : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{M}f(x)$ is the (standard uncentered) maximal function of f at x, *i.e.*,

$$\mathcal{M}f(x) := \sup\left\{\fint_B |f(y)|\,\mathrm{d}y;\ B \text{ ball in } \mathbb{R}^n \text{ such that } x \in \overline{B}\right\}.$$

When f is defined on $(0,\infty) \subset \mathbb{R}$, we consider only balls (=intervals) contained in $(0,\infty)$.
11. "$\hookrightarrow$" stands for continuous embeddings of Banach spaces X and Y: $X \hookrightarrow Y$ indicates that X is continuously embedded into Y.
12. In many estimates, it is crucial to indicate the dependence of constants on various parameters. The notation we use is explained in Remark 2.
13. We use several notation for partial derivatives of a function f. The "abstract" one is $\partial^\alpha f$, with $\alpha \in \mathbb{N}^n$. (We denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$ the total number of derivatives). In concrete cases, we rather write $\partial_1\partial_2 f$ for the second order partial derivative, once with respect to x_1, once with respect to x_2, etc.
14. If $\Omega \subset \mathbb{R}^n$ is an open set, $m \geq 1$ is an integer and $1 \leq p < \infty$, we let

$$W^{m,p}(\Omega) := \{f \in L^p(\Omega);\ \partial^\alpha f \in L^p(\Omega),\ \forall\, \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq m\},$$
$$W^{m,p}_{loc}(\Omega) := \{f \in L^p_{loc}(\Omega);\ \partial^\alpha f \in L^p_{loc}(\Omega),\ \forall\, \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq m\}.$$

15. For $f \in W^{m,p}_{loc}(\Omega)$, we let $|D^m u| := \sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|=m}} |\partial^\alpha u|$.
16. We set $\mathbb{R}^{n+1}_{+,*} := \mathbb{R}^n \times (0,\infty)$ and $\mathbb{R}^{n+1}_{+} := \mathbb{R}^n \times [0,\infty)$.
17. $\lceil s\rceil$ denotes the smallest integer $k \geq s$.
18. $\mathbb{S}^k$ is the unit Euclidean sphere in $\mathbb{R}^{k+1}$.
19. $a \wedge b$ stands for the vector product of vectors $a, b \in \mathbb{R}^2$:

$$(a_1, a_2) \wedge (b_1, b_2) := a_1\, b_2 - a_2\, b_1 \in \mathbb{R}.$$

More generally, if $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^m \times \mathbb{R}^m$, we set

$$(a_1, a_2) \wedge (b_1, b_2) := a_1\, b_2 - a_2\, b_1 \in \mathbb{R}^m.$$

2. HARDY INEQUALITIES

2.1. Integer order inequalities

The "standard" Hardy inequality asserts that for every $1 < p < \infty$ and every $f \in W^{1,p}((0,\infty))$ such that $f(0) = 0$ we have

$$\int_0^\infty \frac{|f(x)|^p}{x^p}\,\mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f'(x)|^p\,\mathrm{d}x. \tag{2.1}$$

If we set, with f as above, $g := |f'| : (0,\infty) \to [0,\infty]$, then (2.1) follows from

$$\int_0^\infty x^{-p}\left(\int_0^x g(u)\,\mathrm{d}u\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty (g(u))^p\,\mathrm{d}u. \tag{2.2}$$

In turn, (2.2) is a member of the following family of estimates, commonly referred to as the "Hardy inequalities"; see *e.g.* Hardy, Littlewood and Pólya [29, Section 9.9].

LEMMA 1. *Let $1 \le q < \infty$, $0 < r < \infty$ and let g be a nonnegative measurable function on $(0,\infty)$. Then we have "Hardy's inequality at 0"*

$$\int_0^\infty x^{-r-1}\left(\int_0^x g(u)\,\mathrm{d}u\right)^q \mathrm{d}x \le \left(\frac{q}{r}\right)^q \int_0^\infty u^{-r+q-1}(g(u))^q\,\mathrm{d}u \tag{2.3}$$

and "Hardy's inequality at ∞"

$$\int_0^\infty x^{r-1}\left(\int_x^\infty g(u)\,\mathrm{d}u\right)^q \mathrm{d}x \le \left(\frac{q}{r}\right)^q \int_0^\infty u^{r+q-1}(g(u))^q\mathrm{d}u. \tag{2.4}$$

Let us recall, following Stein and Weiss [53, Lemma 3.14, pp. 196–197], a proof of the above inequalities.

Proof. We rely on Jensen + Fubini. More specifically, for $x > 0$ the measure $\mu_x := u^{r/q-1}\,\mathrm{d}u/C_x$, with $C_x := (q/r)\,x^{r/q}$, is a probability on $(0,x)$. Jensen's inequality applied on $(0,x)$ to the convex function $\Phi(s) := s^q$, $s \ge 0$, and to the probability measure μ_x yields

$$\begin{aligned}\left(\int_0^x g(u)\,\mathrm{d}u\right)^q &= (C_x)^q\left(\int_0^x g(u)\,u^{1-r/q}\,\mathrm{d}\mu_x\right)^q\\ &\le (C_x)^q \int_0^x (g(u))^q\,u^{q-r}\,\mathrm{d}\mu_x\\ &= (C_x)^{q-1}\int_0^x (g(u))^q\,u^{q-r+r/q-1}\,\mathrm{d}u.\end{aligned} \tag{2.5}$$

Multiplying (2.5) by x^{-r-1}, integrating over x and using Fubini's theorem, we find that the left-hand side of (2.3) does not exceed

$$\begin{aligned}
&\left(\frac{q}{r}\right)^{q-1} \int_0^\infty \int_u^\infty x^{-1-r/q}\,\mathrm{d}x\,(g(u))^q\,u^{q-r+r/q-1}\,\mathrm{d}u \\
&= \left(\frac{q}{r}\right)^{q} \int_0^\infty (g(u))^q\,u^{-r+q-1}\,\mathrm{d}u,
\end{aligned}$$

and therefore (2.3) holds.

In order to obtain (2.4), we proceed as above, starting from the probability measure $\nu_x := u^{-r/q-1}\,\mathrm{d}u/D_x$ on (x,∞), with $D_x := (q/r)\,x^{-r/q}$. □

Remark 1. Far-reaching extensions of Lemma 1 yield necessary and sufficient conditions for the validity of estimates of the form

$$\int_0^\infty \left(\int_0^x g(u)\,\mathrm{d}u\right)^q \mathrm{d}\mu(x) \le C\left(\int_0^\infty (g(u))^p\,\mathrm{d}\nu(u)\right)^{q/p}, \tag{2.6}$$

for $1 \le p, q < \infty$, $g : (0,\infty) \to [0,\infty)$ and ν, μ Radon measures on $(0,\infty)$, as well as the value of the best constant C in (2.6). See the exposition of this subject by Maz'ya [36, Sections 1.3.2–1.3.3], and the historical comments there [36, p. 63].

2.2. **Three basic fractional order inequalities**

The above Hardy inequalities involve f and its derivative f'. Fractional order versions of these inequalities involve f and the average rate of change $(f(x) - f(y)/(x-y)$ (in place of $f'(x)$). We present here three basic lemmas, that we will interpret later in terms of fractional Sobolev spaces $W^{s,p}$.

LEMMA 2 (Fractional Hardy inequality). *Let* $1 \le p < \infty$, $0 < \lambda < \infty$, $\lambda \ne 1$, *and* $f : (0,\infty) \to \mathbb{R}$.

Assume that

$$\int_0^\infty \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x < \infty. \tag{2.7}$$

Then, for some finite constant $C = C_{p,\lambda}$, *we have*

$$\int_0^\infty \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x \le C \int_0^\infty \int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y. \tag{2.8}$$

In particular, (2.7)–(2.8) *hold when* $f \in C_c^\infty([0,\infty))$ *and* $0 < \lambda < 1$, *respectively when* $f \in C_c^\infty([0,\infty))$, $f(0) = 0$ *and* $1 < \lambda < p+1$.

Remark 2. In the above and in what follows, C denotes a *generic finite positive constant* independent of f or other relevant objects, whose value may change with different occurrences. If we want to specify what C depends on, we use subscript indices; *e.g.*, in the above $C = C_{p,\lambda}$ indicates that C depends on p and λ (but not on f).

We also write "$A \lesssim B$" instead of "$A \le C\,B$", provided the constant C does not depend on f or other relevant objects. The notation "$A \approx B$" indicates that $A \lesssim B \lesssim A$.

The proof of Lemma 2 we present below is inspired by [14, Proof of Lemma F.2]. It only uses the triangle inequality!

Proof. We have

$$|f(x)|^p \le 2^{p-1}|f(y)|^p + 2^{p-1}|f(x)-f(y)|^p. \tag{2.9}$$

We divide (2.9) by $\alpha\, x^{1+\lambda}$, and integrate over $x > 0$ and $\alpha\, x < y < 2\alpha\, x$. Here, the constant $\alpha > 0$ will be chosen later. Using Fubini's theorem, we find that

$$\begin{aligned}\int_0^\infty \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x \le & \frac{2^{p-1}\,(2^\lambda-1)\,\alpha^{\lambda-1}}{\lambda}\int_0^\infty \frac{|f(y)|^p}{y^\lambda}\,\mathrm{d}y \\ & + \frac{2^{p-1}}{\alpha}\int_0^\infty\int_{\alpha x}^{2\alpha x}\frac{|f(x)-f(y)|^p}{x^{1+\lambda}}\,\mathrm{d}y\mathrm{d}x \\ \le & \frac{2^{p-1}\,(2^\lambda-1)\,\alpha^{\lambda-1}}{\lambda}\int_0^\infty \frac{|f(y)|^p}{y^\lambda}\,\mathrm{d}y \\ & + C_{\alpha,p,\lambda}\int_0^\infty\int_0^\infty\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.10}$$

Here,

$$\begin{aligned}C_{\alpha,p,\lambda} := & \frac{2^{p-1}}{\alpha}\sup_{0<\alpha x<y<2\alpha x}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}} \\ = & \frac{2^{p-1}}{\alpha}\left(|1-\alpha| \vee |2\alpha-1|\right)^{1+\lambda} < \infty.\end{aligned} \tag{2.11}$$

We now pick α such that $\dfrac{2^{p-1}\,(2^\lambda-1)\,\alpha^{\lambda-1}}{\lambda} \le \dfrac{1}{2}$. Since $\lambda \ne 1$, this is possible provided $\alpha > 0$ is: sufficiently large when $\lambda < 1$, respectively sufficiently small when $\lambda > 1$. For such α, (2.10) yields

$$\int_0^\infty \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x \le \frac{1}{2}\int_0^\infty \frac{|f(y)|^p}{y^\lambda}\,\mathrm{d}y + C_{\alpha,p,\lambda}\int_0^\infty\int_0^\infty\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y,$$

and thus (2.8) holds with $C := 2\,C_{\alpha,p,\lambda}$, thanks to the assumption (2.7). $\square$

Remark 3. Let us note that when $\lambda > 1$ we may choose $\alpha < 1/2$. This implies that, when we estimate, by the above procedure, the integral $\int_0^a \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x$ in terms of an integral involving the quotient $\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}$ (where this time a is *finite*), it suffices to consider, in (2.9), only values of y in the interval $(0,a)$. Therefore, the proof of Lemma 2 (but not its statement) leads to the following version of Lemma 2.

COROLLARY 1. *Let* $1 \le p < \infty$, $1 < \lambda < \infty$, $0 < a \le \infty$, *and* $f : (0,a) \to \mathbb{R}$.

Assume that

$$\int_0^a \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x < \infty. \tag{2.12}$$

Then, for some finite constant $C = C_{p,\lambda}$ *(independent of a!), we have*

$$\int_0^a \frac{|f(x)|^p}{x^\lambda}\,\mathrm{d}x \le C \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y. \tag{2.13}$$

LEMMA 3 (Hardy implies Morrey). *Let* $1 \le p < \infty$, $1 < \lambda < p+1$ *and let* $I \subset \mathbb{R}$ *be an interval. Assume that* $f : I \to \mathbb{R}$ *satisfies*

$$\int_I \int_I \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y < \infty. \tag{2.14}$$

Then f *equals a.e. some continuous function* g.

Assuming that f *itself is continuous, we have, for every* $a, b \in I$ *such that* $a < b$,

$$|f(b)-f(a)|^p \le C_{p,\lambda}(b-a)^{\lambda-1} \int_a^b \int_a^b \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y. \tag{2.15}$$

Proof. Assume first that f is smooth on $[a,b]$. By Corollary 1, we have

$$\int_a^b \frac{|f(x)-f(a)|^p}{(x-a)^\lambda}\,\mathrm{d}x \le C_{p,\lambda} \int_a^b \int_a^b \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y, \tag{2.16}$$

and similarly

$$\int_a^b \frac{|f(x)-f(b)|^p}{(b-x)^\lambda}\,\mathrm{d}x \le C_{p,\lambda} \int_a^b \int_a^b \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y. \tag{2.17}$$

Let $J := ((2a+b)/3, (a+2b)/3)$. When $x \in J$, we have

$$\begin{aligned} \frac{|f(b)-f(a)|^p}{(b-a)^\lambda} &\le 2^{p-1}\left(\frac{|f(x)-f(a)|^p}{(b-a)^\lambda} + \frac{|f(x)-f(b)|^p}{(b-a)^\lambda}\right) \\ &\le 2^{p-1}\left(\frac{|f(x)-f(a)|^p}{(x-a)^\lambda} + \frac{|f(x)-f(b)|^p}{(b-x)^\lambda}\right). \end{aligned} \tag{2.18}$$

Using (2.16)–(2.18), we find that

$$\begin{aligned}\frac{|f(b)-f(a)|^p}{(b-a)^{\lambda-1}} =&3\int_J \frac{|f(b)-f(a)|^p}{(b-a)^{\lambda}}\,\mathrm{d}x\\ \le& C_{p,\lambda}\int_J\left(\frac{|f(x)-f(a)|^p}{(x-a)^{\lambda}}+\frac{|f(x)-f(b)|^p}{(b-x)^{\lambda}}\right)\mathrm{d}x\\ \le& C_{p,\lambda}\int_a^b\int_a^b\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

whence (2.15) for smooth f.

We next remove the smoothness assumption. We note that (2.14) implies that

$$\int_I \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x<\infty \text{ for some } y\in I,$$

so that $f\in L^p_{loc}(\overline{I})$. Fix some compact interval $K\subset I$ and set $\varepsilon_0:=\mathrm{dist}(K,\partial I)/2$. Consider a standard mollifier $\rho\in C_c^\infty((-1,1))$ and set, for $0<\varepsilon<\varepsilon_0$, $f_\varepsilon(x):=f*\rho_\varepsilon(x)$, $\forall\, x\in K$. Then f_ε is smooth in K. By the first part of the proof, for every $a,b\in K$ such that $a<b$ we have

$$|f_\varepsilon(b)-f_\varepsilon(a)|^p\le C_{p,\lambda}(b-a)^{\lambda-1}\int_a^b\int_a^b\frac{|f_\varepsilon(x)-f_\varepsilon(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y. \tag{2.19}$$

We claim that

$$\int_a^b\int_a^b\frac{|f_\varepsilon(x)-f_\varepsilon(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y\le\int_{a-\varepsilon}^{b+\varepsilon}\int_{a-\varepsilon}^{b+\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y. \tag{2.20}$$

Indeed, let, for $g:(A,B)\to\mathbb{R}$,

$$\Delta_h^1 g(x):=g(x+h)-g(x),\ \forall\, h\in(0,B-A),\ \forall\, x\in(A,B-h). \tag{2.21}$$

Then

$$\int_A^B\int_A^B\frac{|g(x)-g(y)|^p}{|x-y|^{1+\lambda}}\,\mathrm{d}x\mathrm{d}y=2\int_0^{B-A}\frac{\|\Delta_h^1 g\|^p_{L^p((A,B-h))}}{h^{1+\lambda}}\,\mathrm{d}h. \tag{2.22}$$

Next, we have

$$\Delta_h^1 f_\varepsilon(x)=\Delta_h^1\left(\int_{-\varepsilon}^{\varepsilon} f(\cdot-y)\,\rho_\varepsilon(y)\,\mathrm{d}y\right)(x)=\int_{-\varepsilon}^{\varepsilon}\Delta_h^1 f(x-y)\,\rho_\varepsilon(y)\,\mathrm{d}y,$$

and thus, for $0<h<b-a$, we have

$$\begin{aligned}\|\Delta_h^1 f_\varepsilon\|_{L^p((a,b-h))}\le&\int_{-\varepsilon}^{\varepsilon}\|\Delta_h^1 f(\cdot-y)\|_{L^p((a,b-h))}\,\rho_\varepsilon(y)\,\mathrm{d}y\\ =&\int_{-\varepsilon}^{\varepsilon}\|\Delta_h^1 f\|_{L^p((a-y,b-h-y))}\,\rho_\varepsilon(y)\,\mathrm{d}y\\ \le&\int_{-\varepsilon}^{\varepsilon}\|\Delta_h^1 f\|_{L^p((a-\varepsilon,b-h+\varepsilon))}\,\rho_\varepsilon(y)\,\mathrm{d}y\\ =&\|\Delta_h^1 f\|_{L^p((a-\varepsilon,b-h+\varepsilon))}.\end{aligned} \tag{2.23}$$

We obtain (2.20) from (2.22) and (2.23).

We conclude as follows. From (2.19) and (2.20), we have $|f_\varepsilon(b) - f_\varepsilon(a)| \le C(b-a)^\alpha$, with $\alpha := (\lambda-1)/p > 0$ and C independent of a, b, ε. We find that f_ε satisfies a uniform Hölder estimate on K, and thus converges when $\varepsilon \to 0$, up to a subsequence and an additive constant, to some Hölder continuous function g. Since, on the other hand, we have $f_\varepsilon \to f$ in $L^p_{loc}(I)$ as $\varepsilon \to 0$, we find that $f = g$ a.e. Assuming that $f = g$, we obtain (2.15) by passing to the limits in (2.19) and using (2.20). $\square$

LEMMA 4 (Hardy implies Sobolev). *Let* $1 \le p < \infty$, $0 < \lambda < 1$, *and* $f : (0, \infty) \to \mathbb{R}$. *Let* $q := p/(1-\lambda) \in (p, \infty)$.

Assume that

$$\int_0^\infty \frac{|f(x)|^p}{x^\lambda} \, \mathrm{d}x < \infty. \tag{2.24}$$

Then

$$\int_0^\infty |f(x)|^q \, \mathrm{d}x \le C_{p,\lambda} \left(\int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x-y|^{1+\lambda}} \, \mathrm{d}x\mathrm{d}y \right)^{q/p}. \tag{2.25}$$

The proof we present below relies on a decomposition method that goes back to Hedberg [30], and has been widely used since then. This kind of technique is consubstantial with the interpolation theory.

Proof. We may assume that the right-hand side of (2.25) is finite. Set

$$G(x) := \int_0^\infty \frac{|f(x) - f(y)|^p}{|x-y|^{1+\lambda}} \, \mathrm{d}y \text{ and } M := \int_0^\infty G(x) \, \mathrm{d}x. \tag{2.26}$$

We will establish the following point estimate

$$|f(x)| \le C_{p,\lambda} M^{\lambda/p} G(x)^{(1-\lambda)/p}, \ \forall\, x > 0, \tag{2.27}$$

which clearly implies (2.25).

We may assume that x satisfies $G(x) < \infty$. We first prove that we have

$$\int_x^\infty \frac{|f(y)|^p}{(y-x)^\lambda} \, \mathrm{d}y < \infty \tag{2.28}$$

and thus (by Lemma 2)

$$\int_x^\infty \frac{|f(y)|^p}{(y-x)^\lambda} \, \mathrm{d}y \le C_{p,\lambda} \, M. \tag{2.29}$$

Indeed, let us note that, by (2.24), we have

$$\int_{x+1}^\infty \frac{|f(y)|^p}{(y-x)^\lambda} \, \mathrm{d}y < \infty.$$

On the other hand, if $G(x)<\infty$ then $\displaystyle\int_x^{x+1}\frac{|f(x)-f(y)|^p}{(y-x)^{1+\lambda}}\,\mathrm{d}y<\infty$, and thus for any such x we have

$$\int_x^{x+1}\frac{|f(y)|^p}{(y-x)^{\lambda}}\,\mathrm{d}y\le 2^{p-1}\int_x^{x+1}\left(\frac{|f(x)-f(y)|^p}{(y-x)^{1+\lambda}}+\frac{|f(x)|^p}{(y-x)^{\lambda}}\right)\mathrm{d}y<\infty;$$

here, we use the assumption $\lambda<1$.

Therefore, (2.28) and (2.29) hold for any x such that $G(x)<\infty$, as claimed.

Let $\varepsilon>0$ and set

$$f_\varepsilon(x):=\fint_x^{x+\varepsilon}f(y)\,\mathrm{d}y=\frac{1}{\varepsilon}\int_x^{x+\varepsilon}f(y)\,\mathrm{d}y,\ \forall\, x>0,\forall\,\varepsilon>0.$$

On the one hand, we have (using (2.29))

$$\begin{aligned}|f_\varepsilon(x)|^p\le&\varepsilon^{-p}\left(\int_x^{x+\varepsilon}|f(y)|\,\mathrm{d}y\right)^p\le\varepsilon^{\lambda-p}\left(\int_x^{x+\varepsilon}\frac{|f(y)|}{(y-x)^{\lambda/p}}\,\mathrm{d}y\right)^p\\ \le&\varepsilon^{\lambda-p}\int_x^{x+\varepsilon}\frac{|f(y)|^p}{(y-x)^{\lambda}}\,\mathrm{d}y\left(\int_x^{x+\varepsilon}\mathrm{d}y\right)^{p-1}\\ \le&C_{p,\lambda}\varepsilon^{\lambda-1}\int_x^{x+\varepsilon}\frac{|f(y)|^p}{(y-x)^{\lambda}}\,\mathrm{d}y\le C_{p,\lambda}\varepsilon^{\lambda-1}M.\end{aligned}\tag{2.30}$$

Similarly, we have

$$\begin{aligned}|f(x)-f_\varepsilon(x)|^p\le&\varepsilon^{-p}\left(\int_x^{x+\varepsilon}|f(x)-f(y)|\,\mathrm{d}y\right)^p\\ \le&\varepsilon^{1+\lambda-p}\left(\int_x^{x+\varepsilon}\frac{|f(x)-f(y)|}{(y-x)^{(1+\lambda)/p}}\,\mathrm{d}y\right)^p\\ \le&\varepsilon^{1+\lambda-p}\int_x^{x+\varepsilon}\frac{|f(x)-f(y)|^p}{(y-x)^{1+\lambda}}\,\mathrm{d}y\left(\int_x^{x+\varepsilon}\mathrm{d}y\right)^{p-1}\\ \le&\varepsilon^{\lambda}G(x).\end{aligned}\tag{2.31}$$

By (2.30) and (2.31), we find that

$$|f(x)|\le C_{p,\lambda}\left(\varepsilon^{(\lambda-1)/p}M^{1/p}+\varepsilon^{\lambda/p}(G(x))^{1/p}\right).\tag{2.32}$$

We next “optimize” (2.32) by choosing $\varepsilon:=M/G(x)$ and obtain (2.27). $\square$

2.3. **Further developments**

Fact 1. In the previous section, one can clearly work in $\mathbb{R}$ instead of $(0,\infty)$.

Fact 2. The extensions of the results in the previous section to $\mathbb{R}^n$ with arbitrary $n \geq 1$ are obtained starting from the following version of Lemma 2.

LEMMA 5. *Let* $1 \leq p < \infty$, $0 < \lambda < \infty$, $\lambda \neq n$, *and* $f : \mathbb{R}^n \to \mathbb{R}$. *Assume that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^\lambda}\, \mathrm{d}x < \infty. \tag{2.33}$$

Then, for some finite constant $C = C_{p,\lambda,n}$, *we have*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^\lambda}\, \mathrm{d}x \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y. \tag{2.34}$$

In order to prove (2.34), one divides (2.9) by $|x|^{n+\lambda}$ and integrates over $x \in \mathbb{R}^n$ and $y \in B_{\alpha\,|x|/2}(3\alpha\, x/2)$, for appropriate $\alpha \ll 1$ (when $\lambda > n$), respectively $\alpha \gg 1$ (when $\lambda < n$).

Fact 3. When $\lambda > n$, we may replace in (2.34) $\mathbb{R}^n$ with suitable subsets of $\mathbb{R}^n$; this is similar to Remark 3 and Corollary 1. More specifically, fix some constants $k > 0$ and $\alpha_0 < 1/2$. Assume that $\Omega \subset \mathbb{R}^n$ is a set such that for every $x \in \Omega$ and $0 < \alpha < \alpha_0$ we have

$$|B_{\alpha\,|x|/2}(3\alpha\, x/2) \cap \Omega| \geq k\, \alpha^n,\ \forall\, x \in \Omega,\ \forall\, 0 < \alpha < \alpha_0. \tag{2.35}$$

Then we may reproduce the proof of (2.34) (explained above) and obtain the following local version of Lemma 5.

LEMMA 6. *Let* $\Omega \subset \mathbb{R}^n$ *satisfy* (2.35) *for some constants* $k > 0$ *and* α_0. *Let* $1 \leq p < \infty$ *and* $\lambda > n$. *Let* $f : \Omega \to \mathbb{R}$. *Assume that*

$$\int_{\Omega} \frac{|f(x)|^p}{|x|^\lambda}\, \mathrm{d}x < \infty. \tag{2.36}$$

Then, for some finite constant $C = C_{p,\lambda,n,k,\alpha_0}$, *we have*

$$\int_{\Omega} \frac{|f(x)|^p}{|x|^\lambda}\, \mathrm{d}x \leq C \int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y. \tag{2.37}$$

In particular, if Ω *is a ball having* 0 *on its boundary, then we may choose* k *and* α_0 *independent of* Ω, *and thus* (2.37) *holds with a constant* $C = C_{p,\lambda,n}$.

Fact 4. By straightforward adaptations of the proofs of Lemmas 3 and 4, and using Lemma 6 in a ball, we obtain the following

LEMMA 7. *Let* $1 \leq p < \infty$ *and* $n < \lambda < \infty$ *be such that* $\lambda < p+1$. *Assume that* $f : \mathbb{R}^n \to \mathbb{R}$ *satisfies*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y < \infty. \tag{2.38}$$

Then f equals a.e. some continuous function g.

Assume that f itself is continuous. For every $a, b \in \mathbb{R}^n$, let $c := (a+b)/2$ and $r := |a-b|/2$. Then

$$|f(b)-f(a)|^p \le C_{p,\lambda,n}(b-a)^{\lambda-n} \int_{B_r(c)} \int_{B_r(c)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y. \tag{2.39}$$

More generally, let p and λ as above and let $\Omega \subset \mathbb{R}$ be an open set. If $f : \Omega \to \mathbb{R}$ satisfies

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y < \infty, \tag{2.40}$$

then f equals a.e. some continuous function g.

Assuming f continuous, let $a, b \in \Omega$ be such that $\overline{B}_r(c) \subset \Omega$ (with c and r as above). Then (2.39) *holds.*

The higher dimensional analogue of Lemma 4 is

Lemma 8. *Let $1 \le p < \infty$, $0 < \lambda < n$, and $f : \mathbb{R}^n \to \mathbb{R}$. Let $q := (np)/(n-\lambda) \in (p, \infty)$.*

Assume that

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^\lambda}\, \mathrm{d}x < \infty. \tag{2.41}$$

Then

$$\int_{\mathbb{R}^n} |f(x)|^q\, \mathrm{d}x \le C_{p,\lambda,n} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y \right)^{q/p}. \tag{2.42}$$

Fact 5. Let $\lambda \ge p$. Assuming that g is a smooth function on an interval $I \subset \mathbb{R}$ and that x is a point in I, we have (by Taylor's formula at x)

$$\int_I \frac{|g(x)-g(y)|^p}{|x-y|^{1+\lambda}}\, \mathrm{d}y = \infty \text{ possibly unless } g'(x) = 0,$$

and therefore, for smooth g, we have

$$\int_I \int_I \frac{|g(x)-g(y)|^p}{|x-y|^{1+\lambda}}\, \mathrm{d}x\mathrm{d}y = \infty \text{ unless } g \text{ is constant.} \tag{2.43}$$

Comparing (2.43) (with $g := f_\varepsilon$) with (2.20), we obtain the following result, stated below in dimension n; this was obtained with different arguments in [5] (see Corollaries 4 and 5 there).

Lemma 9. *Let $1 \le p < \infty$ and $p \le \lambda < \infty$. Let $f : \mathbb{R}^n \to \mathbb{R}$ satisfy*

$$\int_{(0,1)^n} \int_{(0,1)^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\lambda}}\, \mathrm{d}x\mathrm{d}y < \infty.$$

Then f is constant a.e.

Therefore, although Lemmas 2 and 3 (respectively 5 and 7) are stated for larger ranges of λ, we may always assume that $\lambda < p$, for otherwise the hypotheses of the lemmas are fulfilled only by $f \equiv 0$. For example, in Lemma 7 the relevant range is $n < p < \infty$ and $n < \lambda < p$.

Fact 6. Let λ, p and q be as in Lemma 4. Assume that we know in advance that $f \in L^q$. Then it is possible to obtain (2.25) without using Hardy's inequality. I know the beautiful argument below from Brezis [8]; it holds in any dimension, but I present it only in $\mathbb{R}$.

LEMMA 10 ([8]). *Let* $1 \le p < \infty$, $0 < \lambda < 1$, *and* $f : (0,\infty) \to \mathbb{R}$. *Let* $q := p/(1-\lambda) \in (p,\infty)$.

Assume that $f \in L^q((0,\infty))$. *Then*

$$\int_0^\infty |f(x)|^q \, \mathrm{d}x \le C_{p,\lambda} \left(\int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x-y|^{1+\lambda}} \, \mathrm{d}x \mathrm{d}y \right)^{q/p}. \tag{2.44}$$

Proof. Let $G(x)$ be as in (2.26) and set $N := \|f\|_{L^q} < \infty$. We will establish the following point estimate

$$|f(x)| \le 2\, N^\lambda G(x)^{(1-\lambda)/p}, \ \forall\, x > 0, \tag{2.45}$$

which implies (2.44).

With f_ε as in the proof of Lemma 4, we have

$$\begin{aligned} |f_\varepsilon(x)| &\le \varepsilon^{-1} \int_x^{x+\varepsilon} |f(y)| \, \mathrm{d}y \le \varepsilon^{-1/q} \left(\int_x^{x+\varepsilon} |f(y)|^q \, \mathrm{d}y \right)^{1/q} \\ &\le \varepsilon^{(\lambda-1)/p} N. \end{aligned} \tag{2.46}$$

By (2.31) and (2.46), we find that

$$|f(x)| \le \varepsilon^{(\lambda-1)/p}\, N + \varepsilon^{\lambda/p} (G(x))^{1/p}. \tag{2.47}$$

Choosing $\varepsilon := N^p/G(x)$ in (2.47), we obtain (2.45). □

Fact 7. In Lemmas 2 and 5, we have assumed that $\lambda \ne n$. If we are in the range $\lambda < p$ (for otherwise these results are empty, by Lemma 9), then the condition $\lambda \ne n$ is necessary for the validity of Lemmas 2 and 5. In order to prove this fact *e.g.* when $n = 1$ (and thus $1 < p < \infty$) we will construct a family $(f^\varepsilon)_{0<\varepsilon<1}$ such that

$$\int_0^\infty \int_0^\infty \frac{|f^\varepsilon(x) - f^\varepsilon(y)|^p}{|x-y|^2} \, \mathrm{d}x\mathrm{d}y \le C, \ \forall\, \varepsilon \in (0,1), \tag{2.48}$$

and

$$\int_0^\infty \frac{|f^\varepsilon(x)|^p}{x} \, \mathrm{d}x \to \infty \text{ as } \varepsilon \to 0. \tag{2.49}$$

The existence of such a family implies that the conclusion of Lemma 2 does not hold when $\lambda = 1$ and $1 < p < \infty$.

In order to define f^ε, we start from a (fixed) function $f \in C^1([0,\infty))$ such that $f(x) \equiv 1$ on $[0,1]$ and $f(x) \equiv 0$ when $x \geq 2$. We then set

$$f^\varepsilon(x) := \begin{cases} x/\varepsilon, & \text{if } 0 \leq x \leq \varepsilon \\ f(x), & \text{if } x > \varepsilon \end{cases}, \ \forall\, 0 < \varepsilon < 1.$$

Since f is Lipschitz and bounded, we have

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \lesssim & \int_0^2 \int_3^\infty \frac{1}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \\ & + \int_0^2 \int_0^3 \frac{|x-y|^p}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y := K < \infty. \end{aligned} \tag{2.50}$$

Using the fact that $f^\varepsilon = f$ on $[\varepsilon,\infty)$, we find that

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{|f^\varepsilon(x)-f^\varepsilon(y)|^p}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \lesssim & K + \int_0^\varepsilon \int_0^\infty \frac{|f^\varepsilon(x)-f^\varepsilon(y)|^p}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \\ \lesssim & K + \int_0^\varepsilon \int_{2\varepsilon}^\infty \frac{1}{|y-x|^2}\,\mathrm{d}x\mathrm{d}y \\ & + \int_0^\varepsilon \int_0^\varepsilon \frac{|x/\varepsilon - y/\varepsilon|^p}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y \\ & + \int_0^\varepsilon \int_\varepsilon^{2\varepsilon} \frac{|1-y/\varepsilon|^p}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y := C < \infty; \end{aligned} \tag{2.51}$$

here, we use the convergence and the scale invariance of the last three integrals in (2.51). It follows that (2.48) holds.

On the other hand, by monotone convergence we find that

$$\lim_{\varepsilon\to 0} \int_0^\infty \frac{|f^\varepsilon(x)|^p}{x}\,\mathrm{d}x = \int_0^\infty \frac{|f(x)|^p}{x}\,\mathrm{d}x = \infty,$$

so that (2.49) holds.

Fact 8. The method presented in the proof of Lemma 2 allows to obtain a weak form of the standard Hardy inequality, more specifically the existence, for $1 < p < \infty$, of some C_p such that for every $f \in W^{1,p}((0,\infty))$ satisfying $f(0) = 0$ we have

$$\int_0^\infty \frac{|f(x)|^p}{x^p}\,\mathrm{d}x \leq C_p \int_0^\infty |f'(x)|^p\,\mathrm{d}x. \tag{2.52}$$

This time, we use, in addition to the triangle inequality, the Hardy-Littlewood maximal function theorem, asserting that for $1 < p < \infty$ and

$g \in L^p((0,\infty))$ the (uncentered) maximal function $\mathcal{M}g$ of g satisfies

$$\|\mathcal{M}g\|_{L^p} \le C_p \|g\|_{L^p}. \tag{2.53}$$

(The idea of the use of the maximal inequalities in similar contexts goes back to Hedberg [30].)

In order to obtain (2.52), we let $0 < \alpha < 1$ to be determined later and start from

$$\begin{aligned} |f(x)|^p \le & 2^{p-1}|f(\alpha x)|^p + 2^{p-1}|f(\alpha x) - f(x)|^p \\ \le & 2^{p-1}|f(\alpha x)|^p + 2^{p-1}\left(\int_{\alpha x}^{x} |f'(y)|\,\mathrm{d}y\right)^p \\ \le & 2^{p-1}|f(\alpha x)|^p + 2^{p-1}(1-\alpha)^p x^p \left(\mathcal{M}f'(x)\right)^p. \end{aligned} \tag{2.54}$$

Dividing (2.54) by x^p and integrating over x, we find that

$$\int_0^\infty \frac{|f(x)|^p}{x^p}\,\mathrm{d}x \le (2\alpha)^{p-1}\int_0^\infty \frac{|f(y)|^p}{y^p}\,\mathrm{d}y + 2^{p-1}(1-\alpha)^p \|\mathcal{M}f'\|_{L^p}^p. \tag{2.55}$$

If we let $\alpha < 1/2$ in (2.55) and use (2.53) with $g := f'$, we obtain (2.52), at least when $f \in C_c^\infty([0,\infty))$. The general case follows from the density of $C_c^\infty([0,\infty))$ into $W_0^{1,p}((0,\infty))$.

Fact 9. We present here a variant of (2.52). Let $I \subset \mathbb{R}$ be an open interval and let $f \in W^{1,p}(I)$. Assume that f vanishes at each finite endpoint of I. Then

$$\int_I \frac{|f(x)|^p}{[\mathrm{dist}(x,\partial I)]^p}\,\mathrm{d}x \le C_p \int_I |f'(x)|^p\,\mathrm{d}x. \tag{2.56}$$

Indeed, if $I = \mathbb{R}$ there is nothing to prove. If I is a half-line, then (2.56) is equivalent to (2.52). Finally, assume that $I = (a,b)$, with $a, b \in \mathbb{R}$. Arguing as in Remark 3, the proof of (2.52) (but not the inequality (2.52) itself) leads to

$$\int_a^b \frac{|f(x)|^p}{(x-a)^p}\,\mathrm{d}x \le C_p \int_a^b |f'(x)|^p\,\mathrm{d}x, \tag{2.57}$$

$$\int_a^b \frac{|f(x)|^p}{(b-x)^p}\,\mathrm{d}x \le C_p \int_a^b |f'(x)|^p\,\mathrm{d}x. \tag{2.58}$$

We obtain (2.56) using (2.57), (2.58) and the fact that

$$\mathrm{dist}(x,\partial I) = (x-a) \wedge (b-x),\ \forall\, x \in (a,b).$$

3. FRACTIONAL SOBOLEV SPACES

3.1. One dimensional spaces and embeddings

As in Section 2, we first focus on the one dimensional setting. When $0 < s < 1$, $1 \le p < \infty$ and $f : \mathbb{R} \to \mathbb{R}$, the $W^{s,p}$-semi-norm of f is

$$|f|_{W^{s,p}} = |f|_{W^{s,p}(\mathbb{R})} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^p}{|x - y|^{1+sp}} \, \mathrm{d}x \mathrm{d}y \right)^{1/p}. \tag{3.1}$$

Similarly, we set, for every open interval $I \subset \mathbb{R}$,

$$|f|_{W^{s,p}(I)} := \left(\int_{I} \int_{I} \frac{|f(x) - f(y)|^p}{|x - y|^{1+sp}} \, \mathrm{d}x \mathrm{d}y \right)^{1/p}. \tag{3.2}$$

One then defines

$$W^{s,p}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R};\ f \in L^p(\mathbb{R}) \text{ and } |f|_{W^{s,p}} < \infty \}, \tag{3.3}$$

equipped with the "natural" norm

$$\|f\|^p_{W^{s,p}} := \|f\|^p_{L^p} + |f|^p_{W^{s,p}}; \tag{3.4}$$

the definition of $W^{s,p}(I)$ is similar.

$W^{s,p}(\mathbb{R})$ is a "fractional Sobolev" or Slobodeskii space.

We now interpret the results in Section 2.2 in terms of fractional Sobolev spaces.

Lemma 11. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < 1$. Then*

$$\int_{\mathbb{R}} \frac{|f(x)|^p}{|x|^{sp}} \, \mathrm{d}x \le C_{s,p} \, |f|^p_{W^{s,p}}, \ \forall f \in W^{s,p}(\mathbb{R}). \tag{3.5}$$

Proof. Let $f \in W^{s,p}(\mathbb{R})$. Since $|f|_{W^{s,p}} < \infty$, for a.e. $z \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \frac{|f(x) - f(z)|^p}{|x - z|^{1+sp}} \, \mathrm{d}x < \infty. \tag{3.6}$$

Set $A := \{ z \in \mathbb{R};\ (3.6) \text{ holds} \}$. We note that A is dense in $\mathbb{R}$ (since it is a full measure set). By the proof of (2.28) and the fact that $f \in L^p(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \frac{|f(x)|^p}{|x - z|^{sp}} \, \mathrm{d}x < \infty, \ \forall z \in A. \tag{3.7}$$

By Lemma 2 and Fact 1, we obtain

$$\int_{\mathbb{R}} \frac{|f(x)|^p}{|x - z|^{sp}} \, \mathrm{d}x \le C_{s,p} \, |f|^p_{W^{s,p}}, \ \forall z \in A. \tag{3.8}$$

Consider now a sequence $(z_k) \subset A$ such that $z_k \to 0$. Applying (3.8) with $z = z_k$, letting $k \to \infty$ and using Fatou's lemma, we find that (3.5) holds. $\square$

From Lemmas 4 and 11 and Fact 1, we derive the following

Corollary 2. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < 1$. Set $q := p/(1 - sp) \in (p, \infty)$. Then $W^{s,p}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$. More specifically, we have*

$$\|f\|_{L^q} \le C_{s,p}|f|_{W^{s,p}}, \ \forall\, f \in W^{s,p}(\mathbb{R}). \tag{3.9}$$

By Lemma 9, our next result is equivalent to Lemma 3.

Corollary 3. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp > 1$. Assume that $f : I \to \mathbb{R}$ satisfies $|f|_{W^{s,p}(I)} < \infty$. Then f equals a.e. some continuous function g.*

Assuming that f itself is continuous, we have, for every $a, b \in I$ such that $a < b$,

$$|f(b) - f(a)|^p \le C_{p,\lambda}(b-a)^{\lambda-1}|f|^p_{W^{s,p}((a,b))}. \tag{3.10}$$

3.2. **Higher dimensional spaces and embeddings**

When $0 < s < 1$, $1 \le p < \infty$ and $f : \mathbb{R}^n \to \mathbb{R}$, the $W^{s,p}$-semi-norm of f is

$$\begin{aligned} |f|_{W^{s,p}} = |f|_{W^{s,p}(\mathbb{R})} &:= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x+h)-f(x)|^p}{|h|^{n+sp}}\,\mathrm{d}x\mathrm{d}h\right)^{1/p}. \end{aligned} \tag{3.11}$$

Similarly, we set, for every open set Ω having "some smoothness" (*e.g.* bounded Lipschitz domain, or a convex set)

$$|f|_{W^{s,p}(\Omega)} := \left(\int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{1/p}. \tag{3.12}$$

One then defines

$$W^{s,p} = W^{s,p}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R};\ f \in L^p(\mathbb{R}^n) \text{ and } |f|_{W^{s,p}} < \infty\}, \tag{3.13}$$

equipped with

$$\|f\|^p_{W^{s,p}} := \|f\|^p_{L^p} + |f|^p_{W^{s,p}}; \tag{3.14}$$

the definition of $W^{s,p}(\Omega)$ is similar.

Remark 4. A warning. One can use (3.12) to define $W^{s,p}(\Omega)$ for any Ω. The drawback of this is that the definition will coincide with other reasonable possible definitions of $W^{s,p}(\Omega)$ only when Ω is sufficiently smooth (in particular bounded Lipschitz, or convex). We will not discuss this point here. However, we call the attention of the reader to the fact that whenever we consider the semi-norm $|\ |_{W^{s,p}(\Omega)}$, we implicitly assume that either Ω is $\mathbb{R}^n$ (and then we simply write $|\ |_{W^{s,p}}$), or Ω is bounded Lipschitz, or convex.

As in Section 3.1, we obtain the following.

LEMMA 12. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < n$. Then*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{sp}}\, \mathrm{d}x \le C_{s,p}\, |f|^p_{W^{s,p}}, \ \forall\, f \in W^{s,p}(\mathbb{R}^n). \tag{3.15}$$

COROLLARY 4. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < n$. Set $q := (np)/(n - sp) \in (p, \infty)$. Then $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$. More specifically, we have*

$$\|f\|_{L^q} \le C_{s,p} |f|_{W^{s,p}}, \ \forall\, f \in W^{s,p}(\mathbb{R}^n). \tag{3.16}$$

COROLLARY 5. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp > n$. Assume that $f : \Omega \to \mathbb{R}$ satisfies $|f|_{W^{s,p}(\Omega)} < \infty$. Then f equals a.e. some continuous function g.*

Assuming that f itself is continuous, set, for $a, b \in \Omega$, $c := (a + b)/2$ and $r := |a - b|/2$. If $\overline{B}_r(c) \subset \Omega$, then

$$|f(b) - f(a)|^p \le C_{p,\lambda} (b - a)^{\lambda - n} |f|^p_{W^{s,p}(B_r(c))}. \tag{3.17}$$

3.3. **An elementary embedding**

One should see $W^{s,p}$ as a space of functions "having up to s derivatives in L^p". With this interpretation in mind, it is reasonable to expect the validity of the following result.

LEMMA 13. *Let $0 < s_1 < s_2 < 1$ and $1 \le p < \infty$. Then we have*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow W^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{s_1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n). \tag{3.18}$$

Proof. The last embedding is clear. The embedding $W^{s_2,p} \hookrightarrow W^{s_1,p}$ follows from

$$\begin{aligned}
|f|^p_{W^{s_1,p}} &= \int_{|h|<1} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+s_1 p}}\, \mathrm{d}x \mathrm{d}h \\
&\quad + \int_{|h|\ge 1} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+s_1 p}}\, \mathrm{d}x \mathrm{d}h \\
&\le \int_{|h|<1} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+s_2 p}}\, \mathrm{d}x \mathrm{d}h \\
&\quad + 2^{p-1} \int_{|h|\ge 1} \frac{\|f\|^p_{L^p}}{|h|^{n+s_1 p}}\, \mathrm{d}h \lesssim |f|^p_{W^{s_2,p}} + \|f\|^p_{L^p}.
\end{aligned}$$

Finally, we prove that $W^{1,p} \hookrightarrow W^{s_2,p}$. If $f \in W^{1,p}$, then

$$
\begin{aligned}
\|f(\cdot + h) - f\|_{L^p} &= \left\| [f(\cdot + t\,h)]_{t=0}^{t=1} \right\|_{L^p} \\
&= \left\| \int_0^1 \nabla f(\cdot + t\,h) \cdot h \,\mathrm{d}t \right\|_{L^p} \\
&\leq \int_0^1 \|\nabla f(\cdot + t\,h)\|_{L^p} \,\mathrm{d}t\,|h| \\
&= \|\nabla f\|_{L^p}\,|h|, \forall\, h \in \mathbb{R}^n.
\end{aligned}
\tag{3.19}
$$

Using (3.19), we find that

$$
\begin{aligned}
|f|^p_{W^{s_2,p}} &= \int_{|h|<1} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+s_2 p}} \,\mathrm{d}x\mathrm{d}h \\
&\quad + \int_{|h|\geq 1} \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|^p}{|h|^{n+s_2 p}} \,\mathrm{d}x\mathrm{d}h \\
&\leq \int_{|h|<1} \frac{\|\nabla f\|^p_{L^p}\,|h|^p}{|h|^{n+s_2 p}} \,\mathrm{d}h \\
&\quad + 2^{p-1} \int_{|h|\geq 1} \frac{\|f\|^p_{L^p}}{|h|^{n+s_1 p}} \,\mathrm{d}x\mathrm{d}h \lesssim \|\nabla f\|^p_{L^p} + \|f\|^p_{L^p}.
\end{aligned}
$$

This completes the proof of Lemma 13. □

3.4. **Homogeneous spaces**

Sobolev spaces are often used in connection with optimal Sobolev and Morrey embeddings. In this perspective, it is convenient to consider larger spaces, that contain the original Sobolev ones, and satisfy the same embedding properties. In order to motivate what follows, let us briefly recall what happens in the context of Sobolev spaces $W^{1,p} = W^{1,p}(\mathbb{R}^n)$. When $p > n$, we see that the Morrey estimate

$$
|f(x) - f(y)| \leq C\,|x - y|^{1-n/p}\,\|\nabla f\|_{L^p},\ \forall\, f \in W^{1,p},\ \forall\, x, y \in \mathbb{R}^n \tag{3.20}
$$

involves only $\|\nabla f\|_{L^p}$, and an inspection of its proof shows that the estimate holds for f in the larger space $\{f : \mathbb{R}^n \to \mathbb{R};\ \|\nabla f\|_{L^p} < \infty\}$. (Strictly speaking, in (3.20) we have to replace f by its continuous representative.)

When $p = n$, there is no “optimal embedding” to look at.

When $1 \leq p < n$, the optimal Sobolev embedding

$$
\|f\|_{L^{(np)/(n-p)}} \leq C\,\|\nabla f\|_{L^p},\ \forall\, f \in W^{1,p}, \tag{3.21}
$$

does not hold solely under the assumption $\nabla f \in L^p$. Indeed, it suffices to see that $f \equiv 1$ does not satisfy (3.21). However, the conclusion (3.21) holds if we require that "f is small at infinity" in an appropriate sense. There are several possible definitions of the smallness, and they all yield the same "homogeneous space" $\dot{W}^{1,p} = \dot{W}^{1,p}(\mathbb{R}^n)$.

LEMMA 14. *Let $1 \le p < n$ and let $q := (np)/(n-p)$. Set*

$$X_1 := \text{the closure of } C_c^\infty(\mathbb{R}^n) \text{ equipped with the norm } f \mapsto \|\nabla f\|_{L^p}, \tag{3.22}$$

$$X_2 := \left\{ f : \mathbb{R}^n \to \mathbb{R};\ \nabla f \in L^p(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\, \mathrm{d}x < \infty \right\}, \tag{3.23}$$

$$X_3 := \{ f : \mathbb{R}^n \to \mathbb{R};\ \nabla f \in L^p(\mathbb{R}^n) \text{ and } f \in L^q \}, \tag{3.24}$$

$$X_4 := \left\{ f : \mathbb{R}^n \to \mathbb{R};\ \nabla f \in L^p(\mathbb{R}^n) \text{ and } \lim_{R\to\infty} \fint_{B_R(0)} f = 0 \right\}. \tag{3.25}$$

Then $X_1 = X_2 = X_3 = X_4$. Moreover, if we endow X_j, $j = 1, \ldots, 4$, with its "natural" norm

$$\|f\|_{X_1}^p := \|\nabla f\|_{L^p}^p,\ \|f\|_{X_2}^p := \|\nabla f\|_{L^p}^p + \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\, \mathrm{d}x,$$
$$\|f\|_{X_3}^p := \|\nabla f\|_{L^p}^p + \|f\|_{L^q}^p,\ \|f\|_{X_4}^p := \|\nabla f\|_{L^p}^p,$$

then these norms are equivalent. In particular, each X_j is complete.

We denote $\dot{W}^{1,p}$ one of the spaces X_j, $j = 1, \ldots, 4$, with its natural norm.

Proof. When $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$f(r\,\omega) = -\left[f(t\,\omega)\right]_{t=r}^{t=\infty} = -\int_r^\infty [\nabla f(t\,\omega)] \cdot \omega \, \mathrm{d}t,\ \forall\, r > 0,\ \forall\, \omega \in \mathbb{S}^{n-1},$$

and thus

$$|f(r\,\omega)| \le \int_r^\infty |\nabla f(t\,\omega)|\, \mathrm{d}t, \forall\, r > 0,\ \forall\, \omega \in \mathbb{S}^{n-1}. \tag{3.26}$$

Using (3.26) and Hardy's inequality at infinity (2.4) (with $r := n - p$, $q := p$ and $g(u) := |\nabla f(u\,\omega)|$), we find that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\, \mathrm{d}x &= \int_{\mathbb{S}^{n-1}} \int_0^\infty r^{n-p-1} |f(r\,\omega)|^p \, \mathrm{d}r \mathrm{d}s_\omega \\ &\lesssim \int_{\mathbb{S}^{n-1}} \int_0^\infty r^{n-1} |\nabla f(r\,\omega)|^p \, \mathrm{d}r \mathrm{d}s_\omega = \int_{\mathbb{R}^n} |\nabla f(x)|^p \, \mathrm{d}x. \end{aligned} \tag{3.27}$$

We find that (3.27) holds for every $f \in X_1$, and thus $X_1 \hookrightarrow X_2$ with norm equivalence.

If $f \in L^q(\mathbb{R}^n)$, then $\lim_{R\to\infty} \fint_{B_R(0)} f = 0$, by Hölder's inequality applied to f in $B_R(0)$. We find that $X_3 \hookrightarrow X_4$. By a similar argument, we have $X_2 \hookrightarrow X_4$.

Assume now that $f \in X_4$. Since $\nabla f \in L^p$, we have $f \in L^q_{loc}(\mathbb{R}^n)$. Set $f_R := \fint_{B_R(0)} f$. Since $\int_{B_R(0)} (f - f_R) = 0$, we have, by the local Sobolev embedding,

$$\int_{B_R(0)} |f(x) - f_R|^q \, \mathrm{d}x \leq C \left(\int_{B_R(0))} |\nabla f(x)|^p \, \mathrm{d}x \right)^{q/p}. \tag{3.28}$$

Note that $C = C_{p,n}$ does not depend on R, by the scale invariance of (3.28). Letting $R \to \infty$ in (3.28) and using Fatou's lemma, we find that $f \in X_3$ and that $X_4 \hookrightarrow X_3$.

In order to complete the proof of the lemma, it suffices to prove that $X_3 \hookrightarrow X_1$. Let $f \in X_3$. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier; thus $\rho \geq 0$ and $\int_{\mathbb{R}^n} \rho = 1$. Set $f_\varepsilon := f * \rho_\varepsilon$. Then

$$\|f_\varepsilon\|_{L^p} = \|f * \rho_\varepsilon\|_{L^p} \leq \|f\|_{L^p} \|\rho_\varepsilon\|_{L^1} = \|f\|_{L^p}$$

and similarly $\|\nabla f_\varepsilon\|_{L^p} \leq \|\nabla f\|_{L^p}$.

Consider now some $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi = 1$ in $B(0,1)$ and $\operatorname{supp} \psi \subset B(0,2)$. Set

$$\psi^\varepsilon(x) := \psi(\varepsilon x) \text{ and } g_\varepsilon := \psi^\varepsilon f_\varepsilon = \psi^\varepsilon (f * \rho_\varepsilon). \tag{3.29}$$

Then $g_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. We claim that $g_\varepsilon \to f$ in X_3 as $\varepsilon \to 0$. (This implies that $f \in X_1$ and that $X_3 \hookrightarrow X_1$.) Indeed, on the one hand we have $f_\varepsilon \to f$ in L^q as $\varepsilon \to 0$, and therefore, by dominated convergence,

$$\begin{aligned} \|f - g_\varepsilon\|_{L^q} &\leq \|(1 - \psi^\varepsilon) f\|_{L^q} + \|\psi^\varepsilon (f - f_\varepsilon)\|_{L^q} \\ &\lesssim \|(1 - \psi^\varepsilon) f\|_{L^q} + \|f - f_\varepsilon\|_{L^q} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

On the other, using the fact that $\nabla f_\varepsilon \to \nabla f$ in L^p as $\varepsilon \to 0$ and that

$$|\nabla \psi^\varepsilon(x)| \lesssim \begin{cases} \varepsilon, & \text{if } 1/\varepsilon < |x| < 2/\varepsilon, \\ 0, & \text{otherwise} \end{cases},$$

we obtain, via Hölder's inequality, that

$$\begin{aligned} \|\nabla f - \nabla g_\varepsilon\|_{L^p} &\leq \|(1 - \psi^\varepsilon) \nabla f\|_{L^p} + \|\psi^\varepsilon (\nabla f - \nabla f_\varepsilon)\|_{L^p} + \|f_\varepsilon \nabla \psi^\varepsilon\|_{L^p} \\ &\lesssim \|(1 - \psi^\varepsilon) \nabla f\|_{L^p} + \|\nabla f - \nabla f_\varepsilon\|_{L^p} + \varepsilon \|f_\varepsilon\|_{L^p(\{1/\varepsilon < |x| < 2/\varepsilon\})} \\ &\lesssim \|(1 - \psi^\varepsilon) \nabla f\|_{L^p} + \|\nabla f - \nabla f_\varepsilon\|_{L^p} + \|f_\varepsilon\|_{L^q(\{1/\varepsilon < |x| < 2/\varepsilon\})} \\ &\to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

This final estimate completes the proof of the lemma. □

By analogy with the case of $\dot{W}^{1,p}$, we define the homogeneous space $\dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{R}^n)$ as one of the spaces X_j, $j = 1, \ldots, 4$, below, with its natural norm.

Lemma 15. *Let* $0 < s < 1$ *and* $1 \le p < \infty$ *be such that* $sp < n$ *and let* $q := (np)/(n - sp)$. *Set*

$$X_1 := \text{the closure of } C_c^\infty(\mathbb{R}^n) \text{ equipped with the norm } f \mapsto |f|_{W^{s,p}}, \tag{3.30}$$

$$X_2 := \left\{ f : \mathbb{R}^n \to \mathbb{R};\ |f|_{W^{s,p}} < \infty \text{ and } \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{sp}}\, \mathrm{d}x < \infty \right\}, \tag{3.31}$$

$$X_3 := \{ f : \mathbb{R}^n \to \mathbb{R};\ |f|_{W^{s,p}} < \infty \text{ and } f \in L^q \}, \tag{3.32}$$

$$X_4 := \left\{ f : \mathbb{R}^n \to \mathbb{R};\ |f|_{W^{s,p}} < \infty \text{ and } \lim_{R\to\infty} \fint_{B_R(0)} f = 0 \right\}. \tag{3.33}$$

Then $X_1 = X_2 = X_3 = X_4$. *Moreover, if we endow* X_j, $j = 1, \dots, 4$, *with its "natural" norm*

$$\|f\|_{X_1}^p := |f|_{W^{s,p}}^p,\ \|f\|_{X_2}^p := |f|_{W^{s,p}}^p + \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\, \mathrm{d}x,$$
$$\|f\|_{X_3}^p := |f|_{W^{s,p}}^p + \|f\|_{L^q}^p,\ \|f\|_{X_4}^p := |f|_{W^{s,p}}^p,$$

then these norms are equivalent. In particular, each X_j *is complete.*

Proof. The embedding $X_1 \hookrightarrow X_2$ with equivalence of norms follows from Lemma 5. The embedding $X_2 \hookrightarrow X_3$ and the estimate

$$\|f\|_{X_3} \lesssim \|f\|_{X_1} \le \|f\|_{X_2},\ \forall\, f \in X_2,$$

are established via Lemma 8.

The embedding $X_3 \hookrightarrow X_4$ follows from Hölder's inequality.

In order to establish the embedding $X_4 \hookrightarrow X_3$, we rely on the following result, whose proof is postponed to the appendix.

Lemma 16. *Let* $0 < s < 1$, $1 \le p < \infty$ *and* $R > 0$. *Let*

$$Y_R := \left\{ f : B_R(0) \to \mathbb{R};\ |f|_{W^{s,p}(B_R(0))} < \infty \text{ and } \fint_{B_R(0)} f = 0 \right\}.$$

Then there exists an extension operator P_R *on* Y_R *such that:*

1. $P_R f \in W^{s,p}(\mathbb{R}^n)$, $\forall\, f \in Y_R$.
2. $P_R f = f$ *on* $B_R(0)$, $\forall\, f \in Y_R$.
3. $|P_R f|_{W^{s,p}} \le C_{s,p,n} |f|_{W^{s,p}(B_R(0))}$, $\forall\, f \in Y_R$.

(The main point in the above result is that the constant in item *3* does not depend on R.)

Granted Lemma 16, we proceed as follows. Let $f \in X_4$. Let us note that $f \in L^p_{loc}(\mathbb{R}^n)$ (since $|f|_{W^{s,p}} < \infty$). Set

$$f_R := \fint_{B_R(0)} f, \tag{3.34}$$

so that $f - f_R \in Y_R$. Applying Lemma 16 to $f - f_R$ and Corollary 2 to $P_R(f - f_R)$, we find that

$$\|f - f_R\|_{L^q(B_R(0))} \le \|P_R(f - f_R)\|_{L^q} \le C\,|f|_{W^{s,p}}. \tag{3.35}$$

Letting $R \to \infty$ in (3.35), we find that $f \in X_3$ and that $X_4 \hookrightarrow X_3$.

Finally, let $f \in X_3$. Let, as in the proof of Lemma 14, $g_\varepsilon := \psi^\varepsilon f_\varepsilon$. As there, in order to find that $X_3 \hookrightarrow X_1$ and to complete the proof of the lemma, it suffices to prove that $g_\varepsilon \to f$ in X_3 as $\varepsilon \to 0$. The fact that $g_\varepsilon \to f$ in L^q follows as in the proof of Lemma 14. It remains to prove that $|g_\varepsilon - f|_{W^{s,p}} \to 0$ as $\varepsilon \to 0$. This is the content of Lemma 17 below, whose proof is postponed to the appendix. □

LEMMA 17. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < n$. Set $q := (np)/(n - sp)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $|f|_{W^{s,p}} < \infty$ and $f \in L^q$. Define g_ε as in* (3.29)*. Then $|g_\varepsilon - f|_{W^{s,p}} \to 0$ as $\varepsilon \to 0$.*

Remark 5. Augusto Ponce [48] suggested to me another possible definition of $\dot{W}^{s,p}$ as follows. If f "vanishes at infinity", then for every $\delta > 0$ the set $\{x \in \mathbb{R}^n;\ |f(x)| > \delta\}$ has finite measure. It is thus natural to consider the space

$$X_5 := \{f : \mathbb{R}^n \to \mathbb{R};\ |f|_{W^{s,p}} < \infty \text{ and } |\{x \in \mathbb{R}^n;\ |f(x)| > \delta\}| < \infty,\ \forall\,\delta > 0\},$$

with the semi-norm $\|f\|^p_{X_5} := |f|^p_{W^{s,p}}$.

If $0 < s < 1$ and $1 \le p < \infty$ are such that $sp < n$, then $X_5 = X_1$. Indeed, if $f \in X_1$ then $f \in L^q$ and thus $f \in X_5$, by Markov's inequality. Conversely, let $f \in X_5$. We want to prove that $f \in X_1$. In view of Lemma 15, this amounts to $f_R \to 0$ as $R \to \infty$, where f_R is as in (3.34). We argue by contradiction an assume that $|f_{R_k}| \ge 2\delta > 0$ along a sequence $R_k \to \infty$. By (3.35), we have $\|f - f_R\|_{L^q(B_R(0))} \le C = C_f$, and thus for every $t > 0$ and $R > 0$ we have (using Markov's inequality)

$$|\{x \in B_R(0);\ |f(x) - f_R| \le t\}| \ge |B_R(0)| - \frac{C^q}{t^q}. \tag{3.36}$$

We apply (3.36) with $t = \delta$ and $R = R_k$. We find that

$$\begin{aligned} |\{x \in B_{R_k}(0);\ |f(x)| > \delta\}| &\ge |\{x \in B_{R_k}(0);\ |f(x) - f_{R_k}| \le \delta\}| \\ &\ge |B_{R_k}(0)| - \frac{C^q}{\delta^q} \to \infty \text{ as } k \to \infty, \end{aligned} \tag{3.37}$$

and therefore $|\{x \in \mathbb{R}^n;\ |f(x)| > \delta\}| = \infty$. This contradiction completes the proof of the equality $X_5 = X_1$.

An inspection of the above proof is the the analogous equality "$X_5 = X_1$" still holds for $s = 1$.

For more advanced considerations on homogeneous spaces and their realizations, see *e.g.* Bourdaud [1].

3.5. **Slicing (I)**

It will often be more convenient to work in $\mathbb{R}$ with functions of one variable instead of working in $\mathbb{R}^n$. This is possible thanks to a "Fubini type" property stated below. Such a property is reminiscent of the fact that, if $f \in L^p(\mathbb{R}^2)$, then for a.e. $y \in \mathbb{R}$ we have $f(\cdot, y) \in L^p(\mathbb{R})$. For simplicity, we state our next result in $\mathbb{R}^n$, but analogous ones hold in sufficiently smooth open sets $\Omega \subset \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$, we use the notation $\widehat{x}_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

LEMMA 18. *Let $0 < s < 1$ and $1 \le p < \infty$. Then for every $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$|f|^p_{W^{s,p}(\mathbb{R}^n)} \approx \sum_{j=1}^{n} \int_{\mathbb{R}^{n-1}} |f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)|^p_{W^{s,p}(\mathbb{R})} \, \mathrm{d}\widehat{x}_j. \tag{3.38}$$

The proof of the lemma is presented in the appendix.

Remark 6. Other forms of slicing are possible. Instead of fixing $(n-1)$ variables and considering functions of one variable, one may fix $(n-k)$ variables and consider functions of k variables. Then the analogue of (3.38) holds. This can be established by copying the proof of Lemma 18. See also Section 3.7.

3.6. **Higher order spaces**

There are several possible reasonable definitions of higher order fractional Sobolev spaces $W^{s,p}$. Consider for example some $s \in (1,2)$ and write $s = 1 + \sigma$ with $0 < \sigma < 1$. A first possible definition of $W^{s,p}(\mathbb{R})$ is

$$W^{s,p} = W^{s,p}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R};\ f \in W^{1,p} \text{ and } f' \in W^{\sigma,p}\}. \tag{3.39}$$

Another possibility consists of defining $W^{s,p}$ via adapted higher order average rates of change. Recalling that when $0 < s < 1$ spaces are defined via the first order rates $(f(x)-f(y))/(x-y)$, one may consider seconder order rates. It is actually more convenient to use, instead of rates of change, slightly different quantities. We consider the first order variation $\Delta_h^1 f(x) := f(x+h) - f(x)$, and then the second order variation given by

$$\Delta_h^2 f(x) := \Delta_h^1(\Delta_h^1 f)(x) = f(x+2h) - 2f(x+h) + f(x).$$

Higher order variations are defined by induction: we let

$$\Delta_h^M := \underbrace{\Delta_h^1 \circ \cdots \circ \Delta_h^1}_{M \text{ times}}.$$

For $1 < s < 2$, one may try the following alternative to (3.39).

$$W^{s,p} = W^{s,p}(\mathbb{R}) := \Big\{ f : \mathbb{R} \to \mathbb{R};\ f \in L^p \text{ and } \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\Delta_h^2 f(x)|^p}{|h|^{n+sp}} \,\mathrm{d}x\mathrm{d}h < \infty \Big\}. \tag{3.40}$$

It turns out that the definitions (3.39) and (3.40) lead to the same space and to equivalent "natural" norms. The situation is similar in higher dimensions and for higher order derivatives. For simplicity, we justify the equality of spaces and the equivalence of norms only when $n = 1$ and $1 < s < 2$, but with more work arguments can be adapted to the general case. We refer the interested reader to [57, Section 2.6.1] for a comprehensive list of equivalent definitions of $W^{s,p}$ with non-integer s. Since we want to keep this text of reasonable length, in the next sections we will take for granted the equivalence of some of these characterizations.

It will be useful later to have *at least one* definition of $W^{s,p}(\mathbb{R}^n)$. We adopt the following one. Let $s > 0$ be a non-integer and let $1 \leq p < \infty$. Let $M > s$ be an integer, and define

$$|f|^p_{W^{s,p}} = |f|^p_{W^{s,p}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\Delta_h^M f(x)|^p}{|h|^{n+sp}} \,\mathrm{d}x\mathrm{d}h. \tag{3.41}$$

Strictly speaking, this semi-norm depends not only on s, p and n, but also on M. However, in order to keep notation simple we omit the dependence on M. We let

$$W^{s,p} = W^{s,p}(\mathbb{R}^n) := \{ f : \mathbb{R}^n \to \mathbb{R};\ f \in L^p \text{ and } |f|_{W^{s,p}} < \infty \},$$

equipped with the "natural" norm

$$\|f\|^p_{W^{s,p}} := \|f\|^p_{L^p} + |f|^p_{W^{s,p}}. \tag{3.42}$$

Spaces on sufficiently smooth domains Ω are defined similarly. The double integral in x and h is performed over the set

$$\{(x, h) \in \Omega \times \mathbb{R}^n;\ [x, x + Mh] \subset \Omega\}.$$

Let us note that the standard space $W^{s,p}$ with $0 < s < 1$ corresponds to the choice $M = 1$. Incidentally, our above discussion reveals that we could have defined $W^{s,p}$ with $0 < s < 1$ via higher order variations. In order to illustrate this, we present in the appendix a proof of the equality of the spaces $W^{s,p}(\mathbb{R})$ with $0 < s < 1$, defined in one dimension via first, respectively second order variations; see Lemma 38.

We next justify the equivalence of the definitions (3.39) and (3.40). Our result in this direction is the following.

Lemma 19. *Let* $1 < s < 2$ *and* $1 \le p < \infty$. *Let* $\sigma := s - 1 \in (0,1)$. *Set*

$$Z_1 := \{f : \mathbb{R} \to \mathbb{R};\ f \in L^p \text{ and } |f|_{W^{s,p}} < \infty\},$$
$$Z_2 := \{f : \mathbb{R} \to \mathbb{R};\ f \in W^{1,p} \text{ and } \langle f\rangle_{W^{s,p}} := |f'|_{W^{\sigma,p}} < \infty\},$$

equipped respectively with the norms

$$\|f\|_{Z_1}^p := \|f\|_{L^p}^p + |f|_{W^{s,p}}^p,$$
$$\|f\|_{Z_2}^p := \|f\|_{L^p}^p + \langle f\rangle_{W^{s,p}}^p.$$

Then $Z_1 = Z_2$, *with equivalence of norms.*

In the above, $|\ |_{W^{s,p}}$ is the semi-norm given by (3.41) with $n = 1$ and $M = 2$. We define $W^{s,p} = W^{s,p}(\mathbb{R})$ as one of the spaces Z_1, Z_2 with its norm.

The main ingredient in the proof of Lemma 19 is the following.

Lemma 20. *Let* $1 < s < 2$ *and* $1 \le p < \infty$. *Let* $\sigma := s - 1 \in (0,1)$. *Let* $f \in C^1(\mathbb{R})$. *Then we have*

$$|f|_{W^{s,p}} \lesssim \langle f\rangle_{W^{s,p}}. \tag{3.43}$$

Assuming that $\langle f\rangle_{W^{s,p}} < \infty$, *we also have*

$$\langle f\rangle_{W^{s,p}} \lesssim |f|_{W^{s,p}}. \tag{3.44}$$

Proof. The proof relies only on a Hardy type inequality!

Step 1. Proof of (3.43). Let us note the identity

$$\Delta_h^2 f(x-h) = \int_0^h [f'(x+t) - f'(x-t)]\,\mathrm{d}t. \tag{3.45}$$

Using (3.45) and the Hardy inequality at 0 (2.3), we find that

$$\int_{\mathbb{R}} \frac{|\Delta_h^2 f(x-h)|^p}{|h|^{1+sp}}\,\mathrm{d}h \lesssim \int_{\mathbb{R}} \frac{|f'(x+t) - f'(x-t)|^p}{|t|^{1+\sigma p}}\,\mathrm{d}t,\ \forall\, x \in \mathbb{R}. \tag{3.46}$$

Integrating (3.46) over x, we obtain (3.43).

Step 2. Proof of (3.44). This time we start from the identity

$$\Delta_\varepsilon^2 f(x) = \int_{x+\varepsilon}^{x+2\varepsilon} f'(t)\,\mathrm{d}t - \int_x^{x+\varepsilon} f'(t)\,\mathrm{d}t. \tag{3.47}$$

Let k be a large integer to be chosen later. Using (3.47), we find that

$$\sum_{j=0}^{k-1} \Delta_\varepsilon^2 f(x+j\varepsilon) = \int_{x+k\varepsilon}^{x+(k+1)\varepsilon} f'(t)\,\mathrm{d}t - \int_x^{x+\varepsilon} f'(t)\,\mathrm{d}t. \tag{3.48}$$

Identity (3.48) is equivalent to

$$
\begin{aligned}
f'(x+k\varepsilon)-f'(x)=&\frac{1}{\varepsilon}\sum_{j=0}^{k-1}\Delta_\varepsilon^2 f(x+j\varepsilon)\\
&-\frac{1}{\varepsilon}\int_0^\varepsilon (f'(x+k\varepsilon+t)-f'(x+k\varepsilon))\,\mathrm{d}t\\
&+\frac{1}{\varepsilon}\int_0^\varepsilon (f'(x+t)-f'(x))\,\mathrm{d}t.
\end{aligned}
\tag{3.49}
$$

Taking absolute values in (3.49), we find that

$$
\begin{aligned}
|f'(x+k\varepsilon)-f'(x)|\le&\frac{1}{|\varepsilon|}\sum_{j=0}^{k-1}|\Delta_\varepsilon^2 f(x+j\varepsilon)|\\
&+\frac{1}{|\varepsilon|}\left|\int_0^\varepsilon (f'(x+k\varepsilon+t)-f'(x+k\varepsilon))\,\mathrm{d}t\right|\\
&+\frac{1}{|\varepsilon|}\left|\int_0^\varepsilon (f'(x+t)-f'(x))\,\mathrm{d}t\right|.
\end{aligned}
\tag{3.50}
$$

If we raise (3.50) to the pth power, divide by $|\varepsilon|^{1+\sigma p}$, integrate over x and ε and perform in the left-hand side integral the change of variable $h := k\varepsilon$, we find that

$$
\begin{aligned}
\langle f\rangle_{W^{s,p}}^p \le& C_{s,p,k}\,|f|_{W^{s,p}}^p\\
&+C_{s,p}\,k^{-\sigma p}\int_{\mathbb{R}}\int_{\mathbb{R}}|\varepsilon|^{-1-sp}\left|\int_0^\varepsilon (f'(x+t)-f'(x))\,\mathrm{d}t\right|^p\,\mathrm{d}x\mathrm{d}\varepsilon.
\end{aligned}
\tag{3.51}
$$

We now apply in (3.51) the Hardy inequality (2.3) to the integral in ε (with x fixed) and find that

$$
\langle f\rangle_{W^{s,p}}^p \le C_{s,p,k}\,|f|_{W^{s,p}}^p + C_{s,p}\,k^{-\sigma p}\,\langle f\rangle_{W^{s,p}}^p. \tag{3.52}
$$

Finally, if we choose k sufficiently large then $C_{s,p}\,k^{-\sigma p}<1/2$. For such k, (3.52) combined with the assumption $\langle f\rangle_{W^{s,p}}<\infty$ yields (3.44). □

Proof of Lemma 19. Let us note that we have $Z_j \hookrightarrow L^p$, $j=1,2$, and thus it suffices to prove the norm equivalence for $f:\mathbb{R}\to\mathbb{R}$, $f\in L^p$.

Step 1. Norm equivalence for $f*\rho_\varepsilon$. Set $f_\varepsilon := f*\rho_\varepsilon$, where ρ is a standard mollifier. We will prove that $\|f_\varepsilon\|_{Z_1}\approx\|f_\varepsilon\|_{Z_2}$ (with constants independent of ε). Indeed, on the one hand (3.43) implies that $\|f_\varepsilon\|_{Z_1}\lesssim\|f_\varepsilon\|_{Z_2}$.

For the opposite inequality, we claim that $f_\varepsilon\in Z_2$, and thus $\langle f_\varepsilon\rangle_{W^{s,p}}<\infty$. (This implies the validity of (3.44) for f_ε and completes Step 1.) We actually claim that

$$
(f_\varepsilon)^{(m)}\in L^p,\ \forall\, m\in\mathbb{N}, \tag{3.53}
$$

and

$$|(f_\varepsilon)^{(m)}|_{W^{t,p}} < \infty,\ \forall\, m \in \mathbb{N},\ \forall\, t \in (0,1). \tag{3.54}$$

Indeed, (3.53) follows from

$$\|(f_\varepsilon)^{(m)}\|_{L^p} = \|f * (\rho_\varepsilon)^{(m)}\|_{L^p} \le \|f\|_{L^p}\, \|(\rho_\varepsilon)^{(m)}\|_{L^1} \le C_{m,p,\varepsilon}\|f\|_{L^p}. \tag{3.55}$$

Estimate (3.55) yields

$$\begin{aligned} \left\|\Delta_h^1[(f_\varepsilon)^{(m)}]\right\|_{L^p} &= \left\|\left[(f_\varepsilon)^{(m)}(\cdot+\tau)\right]_{\tau=0}^{\tau=h}\right\|_{L^p} \\ &= \left\|\int_0^h (f_\varepsilon)^{(m+1)}(\cdot+\tau)\,\mathrm{d}\tau\right\|_{L^p} \\ &\le |h|\, \|(f_\varepsilon)^{(m+1)}\|_{L^p} \le C_{m,p,\varepsilon}|h|\, \|f\|_{L^p}. \end{aligned} \tag{3.56}$$

Using (3.55) and (3.56), we find that

$$\begin{aligned} |(f_\varepsilon)^{(m)}|^p_{W^{t,p}} =& \int_{|h|<1}\int_{\mathbb{R}} \frac{\left|\Delta_h^1[(f_\varepsilon)^{(m)}](x)\right|^p}{|h|^{1+tp}}\,\mathrm{d}x\mathrm{d}h \\ &+ \int_{|h|\ge 1}\int_{\mathbb{R}} \frac{\left|\Delta_h^1[(f_\varepsilon)^{(m)}](x)\right|^p}{|h|^{1+tp}}\,\mathrm{d}x\mathrm{d}h \\ \le& C_{m,p,\varepsilon}\|f\|^p_{L^p}\left(\int_{|h|\le 1}\frac{\mathrm{d}h}{|h|^{1-(1-t)p}} + \int_{|h|>1}\frac{\mathrm{d}h}{|h|^{1+tp}}\right) \\ \le& C_{m,p,\varepsilon}\|f\|^p_{L^p}, \end{aligned}$$

whence (3.54).

Step 2. A control for $\|f'\|_{L^p}$. Assuming $f \in C^1$, we will control $\|f'\|_{L^p}$ in terms of $\langle f\rangle_{W^{s,p}}$ and $\|f\|_{L^p}$. The starting point is the identity

$$f'(x) = f(x+1) - f(x) - \int_0^1 [f'(x+t) - f(x)]\,\mathrm{d}t,$$

which implies, in conjunction with Hölder's inequality, that

$$\|f'\|_{L^p} \le 2\,\|f\|_{L^p} + \int_0^1 \|\Delta_t^1 f'\|_{L^p}\,\mathrm{d}t \lesssim \|f\|_{L^p} + \langle f\rangle_{W^{s,p}}. \tag{3.57}$$

Step 3. $\varepsilon \to 0$. Assume first that $f \in Z_2$. Using the identity $\Delta_h^1(f' * \rho_\varepsilon) = (\Delta_h^1 f') * \rho_\varepsilon$, we find that $\|\Delta_h^1(f' * \rho_\varepsilon)\|_{L^p} \le \|\Delta_h^1 f'\|_{L^p}$, and therefore $\langle f_\varepsilon\rangle_{W^{s,p}} \le \langle f\rangle_{W^{s,p}}$. Using Step 1, we obtain

$$|f_\varepsilon|_{W^{s,p}} \lesssim |f|_{W^{s,p}},\ \forall\, \varepsilon > 0. \tag{3.58}$$

We next argue as follows. Since $f \in L^p$, we have $f_\varepsilon \to f$ in L^p as $\varepsilon \to 0$ and thus, for fixed h, we have $\|\Delta_h^2 f_\varepsilon\|_{L^p} \to \|\Delta_h^2 f\|_{L^p}$ as $\varepsilon \to 0$. Combining this with Fatou's lemma and letting $\varepsilon \to 0$ in the uniform estimate (3.58), we find that

$$|f|_{W^{s,p}} \le \liminf_{\varepsilon\to 0} |f_\varepsilon|_{W^{s,p}} \lesssim \langle f\rangle_{W^{s,p}},\ \forall\, f \in Z_2, \tag{3.59}$$

and in particular that $Z_2 \hookrightarrow Z_1$.

For the opposite inequality, let $f \in Z_1$. We will prove that

$$f' \in L^p \tag{3.60}$$

and

$$\langle f\rangle_{W^{s,p}} \le C_{s,p}|f|_{W^{s,p}}. \tag{3.61}$$

The key fact is the following variant of Lemma 17 (or, more precisely, of estimate (7.20) established during its proof), whose proof is postponed to the appendix.

Lemma 21. *Let $f \in Z_1$. Assume that ρ is an even mollifier. Then we have $|f_\varepsilon - f|_{W^{s,p}} \to 0$ as $\varepsilon \to 0$.*

Granted Lemma 21, we proceed as follows. Consider a sequence $\varepsilon_j \searrow 0$ such that

$$|f_{\varepsilon_0}|_{W^{s,p}} \le 2\,|f|_{W^{s,p}} \text{ and } |f_{\varepsilon_j} - f_{\varepsilon_{j-1}}|_{W^{s,p}} \le 2^{-j}\,|f|_{W^{s,p}},\ \forall\, j \ge 1, \tag{3.62}$$

$$\|f_{\varepsilon_0}\|_{L^p} \le 2\,\|f\|_{L^p} \text{ and } \|f_{\varepsilon_j} - f_{\varepsilon_{j-1}}\|_{L^p} \le 2^{-j}\,\|f\|_{L^p},\ \forall\, j \ge 1. \tag{3.63}$$

Combining Step 1 with (3.62), we find that

$$\langle f_{\varepsilon_0}\rangle_{W^{s,p}} + \sum_{j\ge 1}\langle f_{\varepsilon_j} - f_{\varepsilon_{j-1}}\rangle_{W^{s,p}} \lesssim |f|_{W^{s,p}}. \tag{3.64}$$

From (3.57), (3.62), (3.64) and Step 1, we obtain

$$\|(f_{\varepsilon_0})'\|_{L^p} + \sum_{j\ge 1}\|(f_{\varepsilon_j} - f_{\varepsilon_{j-1}})'\|_{L^p} \lesssim \|f\|_{W^{s,p}}. \tag{3.65}$$

Since, on the other hand, we have $f = f_{\varepsilon_0} + \sum_{j\ge 1}(f_{\varepsilon_j} - f_{\varepsilon_{j-1}})$ in L^p, we find from (3.65) that $f' = (f_{\varepsilon_0})' + \sum_{j\ge 1}(f_{\varepsilon_j} - f_{\varepsilon_{j-1}})' \in L^p$ and that $\|f'\|_{L^p} \lesssim \|f\|_{W^{s,p}}$. This is a quantitative form of (3.60).

Finally, arguing as for (3.58), we have

$$\langle f_\varepsilon\rangle_{W^{s,p}} \lesssim |f_\varepsilon|_{W^{s,p}} \le |f|_{W^{s,p}},\ \forall\, \varepsilon > 0. \tag{3.66}$$

Since now we know that $f' \in L^p$, we may rewrite (3.66) as

$$|f' * \rho_\varepsilon|_{W^{\sigma,p}} \lesssim |f|_{W^{s,p}},\ \forall\, \varepsilon > 0. \tag{3.67}$$

We now let $\varepsilon \to 0$ in (3.67) (using $f' * \rho_\varepsilon \to f'$ in L^p as $\varepsilon \to 0$ and Fatou's lemma, as in the proof of (3.59)), and obtain (3.61).

Granted Lemma 21, the proof of Lemma 19 is complete. □

3.7. **Slicing (II)**

We discuss here the extension of Lemma 18 to higher order spaces $W^{s,p}$, possibly with integer s. The first remark is that in general, Lemma 18 does not hold for large s. Indeed, a famous construction due to Ornstein [47] exhibits a compactly supported function $f = f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ such that $f, \partial_1\partial_1 f, \partial_2\partial_2 f \in L^1$ but $\partial_1\partial_2 f \notin L^1$. Thus for this f we have

$$\infty = \|f\|_{W^{2,1}(\mathbb{R}^2)} \not\lesssim \int_{\mathbb{R}} \|f(x,\cdot)\|_{W^{2,1}(\mathbb{R})}\,\mathrm{d}x + \int_{\mathbb{R}} \|f(\cdot,y)\|_{W^{2,1}(\mathbb{R})}\,\mathrm{d}y < \infty.$$

There exists however a form of slicing which holds for all regularity exponents $s > 0$, integer or not; see *e.g.* [4, formula (D.3)]. This is explained in our next result, whose proof is postponed to the appendix.

If $f : \mathbb{R}^n \to \mathbb{R}$ and $\omega \in \mathbb{S}^{n-1}$, let $\omega^\perp$ denote the hyperplane orthogonal to ω, *i.e.*, $\omega^\perp := \{x \in \mathbb{R}^n;\ \langle x, \omega\rangle = 0\}$, and consider the partial functions f_ω^x given by

$$f_\omega^x(t) := f(x + t\,\omega),\ \forall\,\omega \in \mathbb{S}^{n-1},\ \forall\,x \in \omega^\perp,\ \forall\,t \in \mathbb{R}. \tag{3.68}$$

LEMMA 22. *Let $s \geq 0$ and $1 \leq p < \infty$. Then*

$$\|f\|_{W^{s,p}(\mathbb{R}^n)}^p \approx \int_{\mathbb{S}^{n-1}} \int_{\omega^\perp} \|f_\omega^x\|_{W^{s,p}(\mathbb{R})}^p\,\mathrm{d}x\mathrm{d}\omega,\ \forall\,f : \mathbb{R}^n \to \mathbb{R}. \tag{3.69}$$

When s is not an integer, we also have

$$|f|_{W^{s,p}(\mathbb{R}^n)}^p \approx \int_{\mathbb{S}^{n-1}} \int_{\omega^\perp} |f_\omega^x|_{W^{s,p}(\mathbb{R})}^p\,\mathrm{d}x\mathrm{d}\omega,\ \forall\,f : \mathbb{R}^n \to \mathbb{R}. \tag{3.70}$$

(Strictly speaking, the integral in $x \in \omega^\perp$ is with respect to the $(n-1)$-dimensional Hausdorff measure on $\omega^\perp$.)

4. **SUPERPOSITION OPERATORS**

4.1. **Overview**

For $\Phi = \Phi(t) : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$, we set $T_\Phi f := \Phi \circ f$. T_Φ is a "superposition" or "Nemitzkii" operator. We discuss here the following question. Given some function space X, which is the regularity (common to all f) of $T_\Phi f$ with $f \in X$? A related question is the following: given this time two function spaces, X and Y, which are the functions Φ such that $T_\Phi f \in Y$, $\forall\,f \in X$? These are natural questions when dealing *e.g.*, with nonlinear partial differential equations or nonlinear nonlocal equations.

One could consider more general Φ's, depending not only on t, but also on the space variable x, but already the case of an "autonomous" Φ is difficult and not completely understood, even in the case where $Y = X$.

There exists an important literature on the subject. The interested reader may consult the monograph of Runst and Sickel [49, Chapter 5] for a detailed account of the results available in the mid 90's, and the vivid partial description by Bourdaud and Sickel [3] of the more recent developments. We focus in what follows on several results in whose proofs the Hardy type inequalities play a crucial role.

Before proceeding, and in order to warn the reader that life in Sobolev spaces is more complicated than the one in spaces of continuous functions, let us state without proof some relevant results in this context.

1. The first one is merely an exercise. If $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz with $\Phi(0) = 0$, and if $0 < s < 1$ and $1 \le p < \infty$, then T_Φ maps continuously $W^{s,p}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$.

2. It is slightly more difficult to see that, under the same assumptions on Φ, T_Φ also maps $W^{1,p}(\mathbb{R}^n)$ into $W^{1,p}(\mathbb{R}^n)$. It turns out (but this is a delicate result due to Marcus and Mizel [35]) that, for such Φ, T_Φ is continuous from $W^{1,p}(\mathbb{R}^n)$ into itself.

3. The above results suggest that if Φ is sufficiently smooth (smoothness depending on s), then T_Φ maps $W^{s,p}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$. However, the following result, due to Dahlberg [17], ruins such expectations. Let $n \ge 3$ and $1 < p < n/2$. If T_Φ maps $W^{2,p}$ into itself, then $\Phi(t) = ct$, $\forall t \in \mathbb{R}$, for some constant c. (The converse clearly holds, also.)

4. Assume that $n \ge 2$ and that $p > n/2$ (this assumption on p goes in the opposite direction with respect to Dahlberg's result). If $\Phi \in C^2(\mathbb{R})$ and $\Phi(0) = 0$, then T_Φ maps $W^{2,p}$ into itself. We will come back to this (and more) in Section 5.2.

The above suggest that, when $Y = X = W^{s,p}$, the interesting range is $s > 1$, and that for such s additional conditions may be necessary either on f, or on the triple (s, p, n), even if Φ is sufficiently smooth.

4.2. Mapping properties of $f \mapsto |f|$

The following beautiful result is due to Bourdaud and Meyer [2].

THEOREM 1. *Let* $1 \le p < \infty$ *and* $1 < s < 1 + 1/p$. *Then* $f \mapsto |f|$ *maps* $W^{s,p}(\mathbb{R}^n)$ *into itself.*

A preliminary result, before proceeding to the proof of the theorem.

LEMMA 23. *Let* $f \in W^{1,1}_{loc}(\mathbb{R}^n)$. *Then* $|f| \in W^{1,1}_{loc}(\mathbb{R}^n)$ *and*

$$\partial_j |f| = (\operatorname{sgn} f)\, \partial_j f, \ \forall\, j = 1, \ldots, n. \tag{4.1}$$

In one dimension, this result was essentially known to de la Vallée Poussin [18]. In a more general form, it is proved in Serrin and Varberg [51]. The n-dimensional version appears *e.g.* in Gilbarg and Trudinger [27, Lemma 7.6]. One can pass from one dimension to n dimensions via a standard slicing argument in $W^{1,p}$ (see *e.g.* Ziemer [59, Theorem 2.1.4]), and thus the heart of the matter is the validity of (4.1) in one dimension. We give in the appendix a very simple proof of this equality.

For more complicated Φ's, the chain rule for $\Phi \circ f$ is more delicate to establish. The chain rule and its higher-order analogue, the Faà di Bruno formula, play an essential role in the study of the superposition operators; see *e.g.* Dincă and Isaia [19–21].

The proof below of Theorem 1 is a variant of the one in [2].

Proof of Theorem 1. Write $s = 1 + \sigma$, with $0 < \sigma < 1/p$. It will be convenient to use on $W^{s,p}$ the following norm, suggested by Lemma 18 and equivalent to the usual ones:

$$\begin{aligned}\langle\langle f\rangle\rangle^p_{W^{s,p}} :=& \|f\|^p_{L^p} + \|\nabla f\|^p_{L^p}\\ &+ \sum_{j=1}^n \int_{\mathbb{R}^{n-1}} |\partial_j f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)|^p_{W^{\sigma,p}(\mathbb{R})}\, \mathrm{d}\widehat{x}_j.\end{aligned} \tag{4.2}$$

(The equivalence of norms is obtained by combining Lemma 18 with [56, Section 2.3.8, Theorem, pp. 58-59].)

In view of Lemma 23 and of (4.2), in order to obtain the conclusion of the theorem it suffices to obtain the estimate

$$||g|'|_{W^{\sigma,p}(\mathbb{R})} \lesssim |g'|_{W^{\sigma,p}(\mathbb{R})},\ \forall\, g \in W^{s,p}(\mathbb{R}). \tag{4.3}$$

To summarize, up to now we have reduced the proof of the theorem to the one of (4.3), which is equivalent to

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|(\operatorname{sgn} g(x))\, g'(x) - (\operatorname{sgn} g(y))\, g'(y)|^p}{|x-y|^{1+\sigma p}}\, \mathrm{d}x\mathrm{d}y\\ &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|g'(x) - g'(y)|^p}{|x-y|^{1+\sigma p}}\, \mathrm{d}x\mathrm{d}y,\ \forall\, g \in W^{s,p}(\mathbb{R}).\end{aligned} \tag{4.4}$$

Clearly, whenever $g(x)\, g(y) > 0$, the integrands on both sides of (4.4) coincide. On the other hand, if $g(x) = g(y) = 0$ then the integrand on the left-hand side vanishes. Therefore, it suffices to consider only couples (x, y) such that

$$g(x)\, g(y) \le 0 \text{ and } (g(x), g(y)) \neq (0, 0).$$

For such a couple (x, y), we use the estimate

$$\begin{aligned}&|(\operatorname{sgn} g(x))\, g'(x) - (\operatorname{sgn} g(y))\, g'(y)|\\ &\le |(\operatorname{sgn} g(x))\, g'(x)| + |\operatorname{sgn} g(y))\, g'(y)|.\end{aligned} \tag{4.5}$$

In view of (4.5) and by symmetry, in order to obtain (4.4) it thus suffices to establish the estimate

$$
\begin{aligned}
&\int_{g(x)>0}\int_{g(y)\le 0}\frac{|g'(x)|^p}{|x-y|^{1+\sigma p}}\,\mathrm{d}x\mathrm{d}y\\
&\lesssim\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|g'(x)-g'(y)|^p}{|x-y|^{1+\sigma p}}\,\mathrm{d}x\mathrm{d}y,\ \forall\, g\in W^{s,p}(\mathbb{R}).
\end{aligned}
\tag{4.6}
$$

Set $U:=\{x;\, g(x)>0\}$ and write U as a disjoint union of open intervals, $U=\cup_j I_j$. If $x\in I_j$ for some j, then

$$
\int_{g(y)\le 0}\frac{1}{|x-y|^{1+\sigma p}}\,\mathrm{d}y\le\int_{y\notin I_j}\frac{1}{|x-y|^{1+\sigma p}}\,\mathrm{d}y\le\frac{C_{s,p}}{[\mathrm{dist}(x,\partial I_j)]^{\sigma p}}.
\tag{4.7}
$$

In view of (4.7), in order to prove (4.6) it suffices to establish the estimate

$$
\int_{I_j}\frac{|g'(x)|^p}{[\mathrm{dist}(x,\partial I_j)]^{\sigma p}}\,\mathrm{d}x\le C_{s,p}\int_{I_j}\int_{I_j}\frac{|g'(x)-g'(y)|^p}{|x-y|^{1+\sigma p}}\,\mathrm{d}x\mathrm{d}y.
\tag{4.8}
$$

When I_j is unbounded, (4.8) follows from the fractional Hardy inequality (3.5) applied to g' in $W^{\sigma,p}(I_j)$ (recall that $\sigma\, p<1$).

When $I_j=(a_j,b_j)$ is bounded, we start by noting that $g(a_j)=g(b_j)=0$, and thus $\int_{I_j}g'(t)\,\mathrm{d}t=0$. We may now apply Lemma 16 and obtain the existence of some $h\in W^{\sigma,p}(\mathbb{R})$ such that $h=g'$ on I_j and

$$
|h|_{W^{\sigma,p}(\mathbb{R})}\lesssim|g'|_{W^{\sigma,p}(I_j)}.
\tag{4.9}
$$

Applying the fractional Hardy inequality (3.5) to h and using (4.9), we find that

$$
\begin{aligned}
\int_{I_j}\frac{|g'(x)|^p}{[\mathrm{dist}(x,\partial I_j)]^{\sigma p}}\,\mathrm{d}x &\lesssim\int_{I_j}\frac{|g'(x)|^p}{(x-a_j)^{\sigma p}}\,\mathrm{d}x+\int_{I_j}\frac{|g'(x)|^p}{(b_j-x)^{\sigma p}}\,\mathrm{d}x\\
&\le\int_{\mathbb{R}}\frac{|h(x)|^p}{|x-a_j|^{\sigma p}}\,\mathrm{d}x+\int_{\mathbb{R}}\frac{|h(x)|^p}{|x-b_j|^{\sigma p}}\,\mathrm{d}x\\
&\lesssim\int_{I_j}\int_{I_j}\frac{|g'(x)-g'(y)|^p}{|x-y|^{1+\sigma p}}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Therefore, (4.8) holds. This completes the proof of the theorem. □

4.3. Mapping properties of $f\mapsto|f|^a$, $0<a<1$

Let $\Phi(t):=|t|^a$, $\forall\, t\in\mathbb{R}$, where $0<a<1$. Since Φ is even, concave on $[0,\infty)$ and $\Phi(0)=0$, we have

$$
|\Phi(t)-\Phi(\tau)|=|\Phi(|t|)-\Phi(|\tau|)|\le\Phi(|t|-|\tau|),
$$

and thus

$$|\Phi(t)-\Phi(\tau)|^{1/a}\le[\Phi(|t|-|\tau|)]^{1/a}=||t|-|\tau||\le|t-\tau|,\ \forall\, t,\tau\in\mathbb{R}. \tag{4.10}$$

Let $0<s<1$ and let $f\in W^{s,p}(\mathbb{R}^n)$. In view of (4.10), we have

$$|\Phi(f(x))-\Phi(f(y))|^{p/a}\le|f(x)-f(y)|^p,\ \forall\, x,y\in\mathbb{R}^n, \tag{4.11}$$

and therefore

$$||f|^a|^{p/a}_{W^{as,p/a}}\le|f|^p_{W^{s,p}},\ \forall\, f\in W^{s,p}. \tag{4.12}$$

Using (4.12), we easily find that T_Φ maps $W^{s,p}$ into $W^{as,p/a}$, $\forall\, 0<s<1$, $\forall\, 1\le p<\infty$, $\forall\, 0<a<1$.

When $s=1$, the analogous conclusion does not follow from (4.11). This case is covered by the following result [41].

THEOREM 2. *Let* $0<a<1$ *and* $1<p<\infty$. *Then* $f\mapsto|f|^a$ *maps* $W^{1,p}$ *into* $W^{a,p/a}$.

Remark 7. The conclusion of the theorem is wrong when $p=1$ [41].

On the other hand, it is not known what happens in $W^{s,p}$ with $s>1$. The following conjecture seems plausible. Let $1\le p<\infty$, $1<s<1+1/p$ and let $0<a<1$. Then $f\mapsto|f|^a$ maps $W^{s,p}$ into $W^{as,p/a}$.

We present below a variant of the proof of Theorem 2 in [41].

Proof of Theorem 2. More generally, we consider an increasing concave homeomphism $\Phi:[0,\infty)\to[0,M)$ and seek for an inequality of the form

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\Psi(|\Phi(|f(x)|)-\Phi(|f(y)|)|)}{|x-y|^{n+p}}\,\mathrm{d}x\mathrm{d}y\\ &\lesssim\int_{\mathbb{R}^n}|\nabla f(x)|^p\,\mathrm{d}x,\ \forall\, f\in W^{1,p}(\mathbb{R}^n).\end{aligned} \tag{4.13}$$

We will determine an appropriate increasing function $\Psi:[0,M)\to[0,\infty)$ (depending on the nonlinearity Φ) such that (4.13) holds and such that, in the special case where $\Phi(t)=t^a$, we have $\Psi(t)=C\,t^{p/a}$. Assuming that (4.13) holds for these particular Φ and Ψ, we find that

$$||f|^a|^{p/a}_{W^{as,p/a}}\lesssim\|\nabla f\|^p_{L^p},\forall\, f\in W^{1,p}(\mathbb{R}^n),$$

and this easily implies that

$$\||f|^a\|^{p/a}_{W^{as,p/a}}\lesssim\|f\|^p_{W^{1,p}},\forall\, f\in W^{1,p}(\mathbb{R}^n),$$

and leads to the conclusion of the theorem.

It will be more instructive not to give the formula defining Ψ from the beginning, but to derive it from a series of calculations. Let us note that a necessary condition for the validity of (4.13) is $\Psi(0) = 0$. Indeed, if $\Psi(0) \neq 0$, then (4.13) with $f \equiv 0$ is wrong.

Step 1. Slicing. Assume that we are able to prove (4.13) in dimension one. If we apply this estimate to f^x_ω (defined in (3.68)), integrate over $\omega \in \mathbb{S}^{n-1}$ and $x \in \omega^\perp$ and use the equivalences (7.42) and (7.44), we obtain that (4.13) holds in $\mathbb{R}^n$.

Therefore, from now on we work in one dimension.

Step 2. Replacing f by $|f|$. Clearly, the left-hand side of (4.13) does not change if we replace f by $|f|$. Nor does the right-hand side, by Lemma 4.1. We may thus assume, in what follows, that $f \in W^{1,p}(\mathbb{R}; [0,\infty))$.

Step 3. Use of a Hardy type inequality. Let $f \in W^{1,p}(\mathbb{R}; [0,\infty))$. Assuming Φ, Ψ sufficiently smooth in order to ensure the validity of the next calculations, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\Psi(|\Phi(f(x)) - \Phi(f(y))|)}{|x-y|^{1+p}}\,\mathrm{d}x\mathrm{d}y \\
&= 2\int_{\mathbb{R}}\int_{f(y)<f(x)} \frac{\Psi(\Phi(f(x)) - \Phi(f(y)))}{|x-y|^{1+p}}\,\mathrm{d}y\mathrm{d}x \\
&= 2\int_{\mathbb{R}}\int_{f(y)<f(x)} \frac{[-\Psi(\Phi(f(x)) - \Phi(t))]_{t=f(y)}^{t=f(x)}}{|x-y|^{1+p}}\,\mathrm{d}y\mathrm{d}x \\
&= 2\int_{\mathbb{R}}\int_0^{f(x)} \Phi'(t)\int_{f(y)<t} \frac{\Psi'(\Phi(f(x)) - \Phi(t))}{|x-y|^{1+p}}\,\mathrm{d}y\mathrm{d}t\mathrm{d}x.
\end{aligned}
\tag{4.14}
$$

Consider now, for $0 < t < M$, the open set $U_t := \{x \in \mathbb{R};\ f(x) > t\}$. We decompose, for each fixed t, $U_t = \cup I_{j,t}$, with $I_{j,t}$ mutually disjoint open intervals. Note that U_t has finite measure (by Markov's inequality) and thus each $I_{j,t}$ has finite length. By (4.14), we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\Psi(|\Phi(f(x)) - \Phi(f(y))|)}{|x-y|^{1+p}}\,\mathrm{d}x\mathrm{d}y \\
&= 2\int_0^M \Phi'(t)\int_{U_t}\int_{f(y)<t} \frac{\Psi'(\Phi(f(x)) - \Phi(t))}{|x-y|^{1+p}}\,\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&= 2\int_0^M \Phi'(t)\sum_j\int_{I_{j,t}}\int_{f(y)<t} \frac{\Psi'(\Phi(f(x)) - \Phi(t))}{|x-y|^{1+p}}\,\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&\le 2\int_0^M \Phi'(t)\sum_j\int_{I_{j,t}}\int_{\mathbb{R}\setminus I_{j,t}} \frac{\Psi'(\Phi(f(x)) - \Phi(t))}{|x-y|^{1+p}}\,\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&\lesssim \int_0^M \Phi'(t)\sum_j\int_{I_{j,t}} \frac{\Psi'(\Phi(f(x)) - \Phi(t))}{[\mathrm{dist}(x,\partial I_{j,t})]^p}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.15}
$$

We next intend to apply the Hardy inequality (2.56) to the inner integral $\int_{I_{j,t}} \ldots \mathrm{d}x$ in (4.15). For that purpose, we write

$$\Psi'(\Phi(f(x)) - \Phi(t)) = [\underbrace{(\Psi'(\Phi(f(x)) - \Phi(t)))^{1/p}}_{g_t(x)}]^p = [g_t(x)]^p. \tag{4.16}$$

We note that, at the endpoints of $I_{j,t}$, we have $f(x) = t$. Therefore, if we assume that $\Psi'(0) = 0$, then g_t vanishes at the endpoints of $I_{j,t}$. We are thus in position to apply (2.56) and find that

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\Psi(|\Phi(f(x)) - \Phi(f(y))|)}{|x-y|^{1+p}}\,\mathrm{d}x\mathrm{d}y \\ &\lesssim \int_0^M \Phi'(t) \sum_j \int_{I_{j,t}} |(g_t)'(x)|^p\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^M \Phi'(t) \int_{U_t} |(g_t)'(x)|^p\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}}\int_0^{f(x)} \Phi'(t)|(g_t)'(x)|^p\,\mathrm{d}t\mathrm{d}x. \end{aligned} \tag{4.17}$$

We next note that

$$\begin{aligned} (g_t)'(x) =& (\Psi')^{1/p-1}(\Phi(f(x)) - \Phi(t))\,\Psi''(\Phi(f(x)) - \Phi(t)) \\ &\times \Phi'(f(x))\,f'(x). \end{aligned} \tag{4.18}$$

Inserting (4.18) into (4.17), we obtain

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\Psi(|\Phi(f(x)) - \Phi(f(y))|)}{|x-y|^{1+p}}\,\mathrm{d}x\mathrm{d}y \\ &\lesssim \int_{\mathbb{R}} K(f(x))\,(\Phi')^p(f(x))\,|f'(x)|^p\,\mathrm{d}x, \end{aligned} \tag{4.19}$$

where we have set, for $A > 0$,

$$K(A) := \int_0^A \Phi'(t)(\Psi')^{1-p}(\Phi(A) - \Phi(t))\,|\Psi''|^p(\Phi(A) - \Phi(t))\,\mathrm{d}t. \tag{4.20}$$

Step 4. Choice of Ψ. In order to obtain (4.13) from (4.19)–(4.20), we seek for Ψ such that

$$K(A)\,(\Phi')^p(A) = C \in (0,\infty),\ \forall\, A > 0. \tag{4.21}$$

We next manipulate (4.21) in order to derive the expression of Ψ. Set $\xi := \Phi^{-1} : [0, M) \to [0,\infty)$, so that ξ is convex and increasing. If we perform, in the integral defining $K(A)$, the change of variable $\tau := \Phi(t)$ and we set $B := \Phi(A)$, then

$$\begin{aligned} K(A) &= \int_0^B (\Psi')^{1-p}(B-\tau)\,|\Psi''|^p(B-\tau)\,\mathrm{d}\tau \\ &= \int_0^B (\Psi')^{1-p}(\tau)\,|\Psi''|^p(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.22}$$

Since on the other hand we have $\Phi'(A) = 1/\xi'(B)$, we find, using (4.22), that (4.21) is equivalent to

$$\int_0^B (\Psi')^{1-p}(\tau)\,|\Psi''|^p(\tau)\,\mathrm{d}\tau = C\,(\xi')^p(B),\ \forall\, 0 < B < M. \tag{4.23}$$

We may now differentiate (4.23) with respect to B and find that

$$(\Psi')^{1-p}(B)\,|\Psi''|^p(B) = C\,(\xi')^{p-1}(B)\,\xi''(B). \tag{4.24}$$

Assuming that Ψ is convex, we obtain from (4.24) that

$$[(\Psi')^{1/p}]'(B) = C\,(\Psi')^{1/p-1}(B)\,\Psi''(B) = C\,(\xi')^{1-1/p}(B)\,(\xi'')^{1/p}(B). \tag{4.25}$$

Using (4.25) and the assumption $\Psi'(0) = 0$, we determine $(\Psi')^{1/p}$, and thus Ψ'. We next find Ψ from the formula Ψ' and the necessary condition $\Psi(0) = 0$. We end up with the fact that, up to a multiplicative constant, we have

$$\Psi(t) = \int_0^t \left(\int_0^r [\xi'(\tau)]^{1-1/p}[\xi''(\tau)]^{1/p}\,\mathrm{d}\tau \right)^p \mathrm{d}r,\ \forall\, 0 \le t < M. \tag{4.26}$$

In the special case where $\Phi(t) = t^a$, we have $\xi(t) = t^{1/a}$, and it is easy to see that $\Psi(t) = C\,t^{p/a}$.

Step 5. A generalization of Theorem 2. It remains to give sufficient conditions on Φ in order to justify *a posteriori* the above formal calculations. The bottom line is that the definition (4.26) has to make sense. In order to achieve this, we assume that Φ is continuous concave with $\Phi(0) = 0$, that Φ is increasing (and thus a homeomorphism onto its image $[0, M)$), and we require that its reciprocal $\xi : [0, M) \to [0, \infty)$ is twice differentiable and that $\xi'' \in L^1_{loc}([0, M))$. We thus guess the following extension of Theorem 2 (which slightly generalizes [41, Theorem 1.3]).

THEOREM 3. *Let $\Phi : [0, \infty) \to [0, M)$ be an increasing concave homeomorphism. Let $\xi := \Phi^{-1} : [0, M) \to [0, \infty)$. Assume that ξ is twice differentiable and that $\xi'' \in L^1_{loc}([0, M))$. Set*

$$\Psi(t) = \int_0^t \left(\int_0^r [\xi'(\tau)]^{1-1/p}[\xi''(\tau)]^{1/p}\,\mathrm{d}\tau \right)^p \mathrm{d}r,\ \forall\, 0 \le t < M. \tag{4.27}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Psi(|\Phi(|f(x)|) - \Phi(|f(y)|)|)}{|x-y|^{n+p}}\,\mathrm{d}x\mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^n} |\nabla f(x)|^p\,\mathrm{d}x,\ \forall\, f \in W^{1,p}(\mathbb{R}^n). \end{aligned} \tag{4.28}$$

Step 6. Proof of Theorem 3. As explained above, it suffices to prove the validity of (4.28) for $n = 1$ and $f \in W^{1,p}(\mathbb{R}; [0, \infty))$.

From the assumptions of Theorem 3, we have ξ increasing and concave, and thus

$$0 \le \xi'(\tau) \le \xi'(t), \ \forall\, 0 \le \tau \le t < M. \tag{4.29}$$

On the other hand, since ξ' is differentiable and its derivative is locally summable, we have

$$\xi'(t) = \xi'(\tau) + \int_\tau^t \xi''(r)\,\mathrm{d}r, \ \forall\, 0 \le \tau \le t < M; \tag{4.30}$$

see *e.g.*, Natanson [45, Chapter IX, § 7, Theorem 1]. In particular, we have

$$\xi \in C^1([0, M)). \tag{4.31}$$

Using (4.29), (4.30) and Hölder's inequality, we find that

$$\begin{aligned} &\int_{r_1}^{r_2} [\xi'(\tau)]^{1-1/p} [\xi''(\tau)]^{1/p}\,\mathrm{d}\tau \\ &\le \left(\int_{r_1}^{r_2} \xi'(\tau)\,\mathrm{d}\tau\right)^{1-1/p} \left(\int_{r_1}^{r_2} \xi''(\tau)\,\mathrm{d}\tau\right)^{1/p} \\ &\le (r_2 - r_1)^{1-1/p}\,\xi'(r_2), \ \forall\, 0 \le r_1 < r_2 < M. \end{aligned} \tag{4.32}$$

Estimate (4.32) implies that

$$[0, M) \ni r \mapsto F(r) := \int_0^r [\xi'(\tau)]^{1-1/p} [\xi''(\tau)]^{1/p}\,\mathrm{d}\tau \text{ is continuous.} \tag{4.33}$$

From (4.27) and (4.33), we obtain that

$$\Psi \in C^1([0, M)), \ \Psi(0) = 0, \ \Psi'(0) = 0 \tag{4.34}$$

and

$$\Psi'(t) = F^p(t) = \left(\int_0^t [\xi'(\tau)]^{1-1/p} [\xi''(\tau)]^{1/p}\,\mathrm{d}\tau\right)^p, \ \forall\, t \in [0, M). \tag{4.35}$$

On the other hand, since ξ is an increasing differentiable homeomorphism, we have $\xi'(t) > 0$ for a.e. $t \in [0, M)$. Combining this with (4.29), we find that $\xi'(t) > 0$, $\forall\, t \in (0, M)$, and thus (using also (4.31))

$$\Phi \in C^1((0, \infty)) \text{ and } \Phi'(t) > 0, \ \forall\, t > 0. \tag{4.36}$$

The validity of (4.34) and (4.36) implies the one of (4.14).

We next note that

$$\begin{aligned} &[\xi'(\tau)]^{1-1/p} [\xi''(\tau)]^{1/p} \\ &\le (1 - 1/p)\,\xi'(\tau) + (1/p)\,\xi''(\tau) \\ &\le (1 - 1/p)\,\xi'(t) + (1/p)\,\xi''(\tau), \ \forall\, 0 \le \tau < t < M, \end{aligned}$$

and thus the integrand defining F in (4.33) is locally summable. From Lebesgue's differentiation theorem (see *e.g.* [45, Chapter IX, § 4, Theorem 2]), we find that

(4.37) $$F' = [\xi']^{1-1/p}[\xi'']^{1/p} \text{ a.e. and in the distributions sense.}$$

On the other hand, (4.35) implies that the function g_t defined in (4.16) is given by

(4.38) $$g_t(x) = F(\Phi(f(x)) - \Phi(t)), \ \forall\, 0 < t < f(x).$$

Using (4.36), (4.37), (4.38) and the chain rule in $W^{1,1}_{loc}$ (see *e.g.* [51, Theorem 2]), we find that for every fixed $t > 0$ we have, a.e. and in the distributions sense,

(4.39) $$(g_t)'(x) = F'(\Phi(f(x)) - \Phi(t))\, \Phi'(f(x))\, f'(x).$$

From (4.17), (4.37) and (4.39), we obtain the validity of (4.19), with

(4.40) $$K(A) := \int_0^A \Phi'(t)\, (F')^p(\Phi(A) - \Phi(t))\, \mathrm{d}t.$$

In order to complete the proof, it remains to establish (4.21) for this K. The change of variable $\tau := \Phi(t)$ in (4.40) leads, as in (4.22), to

(4.41) $$K(A) = \int_0^{\Phi(A)} (\xi')^{p-1}(\tau)\, \xi''(\tau)\, \mathrm{d}\tau.$$

On the other hand, the chain rule in $W^{1,1}_{loc}$ yields

(4.42) $$[(\xi')^p]' = p\, (\xi')^{p-1}\, \xi'' \text{ a.e. and in the distributions sense.}$$

Since ξ' is locally bounded and ξ'' is locally summable, we find from (4.41) and (4.42) that

(4.43) $$K(A) = C\, (\xi')^p(\Phi(A)), \ \forall\, A > 0.$$

Identity (4.21) follows from (4.43) and the fact that Φ and ξ are reciprocal to each other.

The proof of Theorem 3 (and, in particular, of Theorem 2) is complete. □

Remark 8. Step 6 is significantly simpler if we weaken the assumptions on ξ in Theorem 3 to $\xi \in C^2$; see [41, proof of Theorem 1.3].

5. TRACE THEORY OF WEIGHTED SOBOLEV SPACES

5.1. Overview

In order to establish further properties of the superposition operators T_Φ, it will be convenient to rely on a new tool: the trace theory of (weighted)

Sobolev spaces. A striking fact is that this theory is essentially a consequence of the Hardy type inequalities, so that we have the following rough scheme

$$\text{Hardy inequalities} \implies \text{traces of weighted spaces} \implies \text{properties of } T_\Phi \\ \text{(and more)}.$$

The general philosophy of the trace theory is that a function in a half-space having some Sobolev regularity has a "trace" ("restriction") on the boundary of the half-space. Usually, this trace is defined by density, starting from smooth functions. We will work only with continuous (and even better) functions, and in this setting we will dispose of an equivalent but more tractable approach to the notion of trace.

First, some notation and the appropriate definition.

1. We set $\mathbb{R}^{n+1}_{+,*} := \mathbb{R}^n \times (0, \infty)$ and $\mathbb{R}^{n+1}_{+} := \mathbb{R}^n \times [0, \infty)$.

2. A generic point in these sets will be denoted (x, t) or (x, ε), with x in $\mathbb{R}^n$ and t, ε in $(0, \infty)$ or $[0, \infty)$.

3. Let $F : \mathbb{R}^{n+1}_{+,*} \to \mathbb{R}$ be a continuous function. We say that $f : \mathbb{R}^n \to \mathbb{R}$ is *the trace of* F (implicitly understood: on $\mathbb{R}^n \sim \mathbb{R}^n \times \{0\}$) if

$$\lim_{\varepsilon \to 0} F(x, \varepsilon) = f(x) \text{ for a.e. } x \in \mathbb{R}^n. \tag{5.1}$$

If (5.1) holds, then f is a.e. uniquely defined by (5.1), and we write $f = \operatorname{tr} F$.

4. Here is a fundamental example. Let $\rho \in C^\infty_c(\mathbb{R}^n)$ be a standard mollifier. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Set

$$F_f(x, \varepsilon) := f * \rho_\varepsilon(x), \ \forall\, x \in \mathbb{R}^n, \forall\, \varepsilon > 0. \tag{5.2}$$

(Strictly speaking, F_f depends not only on f, but also on ρ, but in practice ρ will be fixed independently of f and we omit this dependence.)

It is a standard exercise that F_f is smooth in $\mathbb{R}^{n+1}_{+,*}$. A more delicate result is that we have $f = \operatorname{tr} F_f$. Equivalently, if $f \in L^1_{loc}(\mathbb{R}^n)$ and ρ is a standard mollifier, then we have

$$\lim_{\varepsilon \to 0} f * \rho_\varepsilon(x) = f(x) \text{ for a.e. } x \in \mathbb{R}^n; \tag{5.3}$$

see *e.g.* Stein [52, formula (16), p. 23, and Chapter I, Section § 8.16] when $f \in L^p$ for some p, but the arguments there hold also for $f \in L^1_{loc}$.

We may now state (temporarily without proof) two basic results in the trace theory of Sobolev spaces, due to Gagliardo [25].

Theorem 4 (Direct trace theorem). *Let* $1 < p < \infty$. *Let* $F \in C^1(\mathbb{R}^{n+1}_{+,*})$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *be such that* $f = \operatorname{tr} F$. *Then*

$$|f|_{W^{1-1/p,p}(\mathbb{R}^n)} \le C_{p,n}\, \|\nabla F\|_{L^p(\mathbb{R}^{n+1}_{+,*})}. \tag{5.4}$$

THEOREM 5 (Inverse trace theorem). *Let* $f \in L^1_{loc}(\mathbb{R}^n)$. *Let* F_f *be as in* (5.2). *Then*

$$\|\nabla F_f\|_{L^p(\mathbb{R}^{n+1}_{+,*})} \le C_{p,n}\, |f|_{W^{1-1/p,p}(\mathbb{R}^n)}. \tag{5.5}$$

Let us give an application of the above results to the study of T_Φ. Although this trivial application could have been obtained directly and with little effort, its proof via Theorems 4 and 5 is instructive since it suggests a sound strategy that will be useful in more difficult problems. Assume that we want to estimate $|T_\Phi f|_{W^{1-1/p,p}}$ for some (at least C^1) Φ and some $f \in L^1_{loc}(\mathbb{R}^n)$. Consider F_f as in (5.2). Then $\operatorname{tr} T_\Phi(F_f) = T_\Phi f$. By Theorem 4, we have

$$|T_\Phi f|_{W^{1-1/p,p}} \lesssim \|\nabla T_\Phi(F_f)\|_{L^p}. \tag{5.6}$$

Assume now that Φ is Lipschitz. Then

$$|\nabla T_\Phi(F_f)| \lesssim |\nabla F_f| \text{ a.e.} \tag{5.7}$$

(this can be obtained *e.g.* from the chain rule).

From (5.5), (5.6) and (5.7), we obtain that $|T_\Phi f|_{W^{1-1/p,p}} \lesssim |f|_{W^{1-1/p,p}}$.

Let us pause and summarize the above strategy of proof. In order to estimate $T_\Phi f$ in $W^{1-1/p,p}$, we first estimate $\|\nabla T_\Phi(F_f)\|_{L^p}$. The direct theorem then yields an estimate of $T_\Phi f$ in $W^{1-1/p,p}$. Assume next that $\|\nabla T_\Phi(F_f)\|_{L^p}$ is controlled by $\|\nabla F_f\|_{L^p}$. Then the inverse theorem allows to estimate $\|\nabla T_\Phi(F_f)\|_{L^p}$ in terms of $|f|_{W^{1-1/p,p}}$. Combining the two, we estimate $|T_\Phi f|_{W^{1-1/p,p}}$ in terms $|f|_{W^{1-1/p,p}}$. The interesting point is that we estimate fractional semi-norms via calculations which involve L^p norms of derivatives – and in general it is easier to deal with integer derivatives instead of fractional ones. (The idea of increasing the space dimension in order to establish mapping properties of T_Φ appears already in [10].)

If we want to attack less academic problems, then we have to have at our disposal function spaces of integer Sobolev type having as traces $W^{s,p}$ maps for arbitrary non-integer s, and not only for $s = 1 - 1/p$. This can be achieved, but the price to pay is that we have to deal with weighted Sobolev spaces.

The theory of weighted Sobolev spaces has been established in the 60's. The results we present below are a light version of this theory, sufficient for our purposes. They are included in more general results due to Uspenskiĭ [58]. Before stating them, let us recall that when $s > 0$ is non-integer and $1 \le p < \infty$, we have defined in (3.41) a semi-norm $|\ |_{W^{s,p}}$ adapted to the space $W^{s,p}(\mathbb{R}^n)$. This semi-norm depends not only on s, p and n, but also on an integer $M > s$ that will explicitly be mentioned in the next statements.

Given M, set

$$\mathcal{M}_M := \{(\beta, 0);\ \beta \in \mathbb{N}^n \text{ and } |\beta| = M\} \cup \{(\underbrace{0, \dots, 0}_{n \text{ times}}, M)\} \subset \mathbb{N}^{n+1}.$$

When $M = 1$, we have $\mathcal{M}_1 = \{\alpha \in \mathbb{N}^{n+1};\ |\alpha| = 1\}$. On the other hand, when $M \geq 2$, $\mathcal{M}_M$ is clearly a strict subset of $\{\alpha \in \mathbb{N}^{n+1};\ |\alpha| = M\}$.

THEOREM 6 (Direct trace theorem (I)). *Let $s > 0$ be non-integer and let $1 \leq p < \infty$. Let M be an integer such that $M \geq s + 1/p$. Let $F \in C^M(\mathbb{R}^{n+1}_{+,*})$ and $f : \mathbb{R}^n \to \mathbb{R}$ be such that $f = \operatorname{tr} F$. Then*

$$|f|^p_{W^{s,p}(\mathbb{R}^n)} \leq C_{s,p,n} \sum_{\alpha \in \mathcal{M}_M} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F(x,\varepsilon)|^p \, \mathrm{d}x \mathrm{d}\varepsilon. \tag{5.8}$$

Note the technical assumption $M \geq s + 1/p$, which is stronger than the natural assumption $M > s$ required to in order to define $|\ |_{W^{s,p}}$. As explained in the next result, we may recover the condition $M > s$ if we adopt a more restrictive notion of trace.

THEOREM 7 (Direct trace theorem (II)). *Let $s > 0$ be non-integer and let $1 \leq p < \infty$. Let M be an integer such that $M > s$. Let $F \in C^M(\mathbb{R}^{n+1}_{+,*})$ and $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\lim_{\varepsilon \to 0} F(\cdot, \varepsilon) = f$ in $L^1_{loc}(\mathbb{R}^n)$. Then*

$$|f|^p_{W^{s,p}(\mathbb{R}^n)} \leq C_{s,p,n} \sum_{\alpha \in \mathcal{M}_M} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F(x,\varepsilon)|^p \, \mathrm{d}x \mathrm{d}\varepsilon. \tag{5.9}$$

In particular, (5.9) *holds for $F \in C^M(\mathbb{R}^{n+1}_{+,*}) \cap C(\mathbb{R}^{n+1}_{+})$.*

THEOREM 8 (Inverse trace theorem). *Let $s > 0$ be non-integer and let $1 \leq p < \infty$. Let M be an integer such that $M > s$. Let $f \in W^{s,p}(\mathbb{R}^n)$. Let F_f be as in* (5.2)*. Then*

$$\sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| = M}} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F_f(x,\varepsilon)|^p \, \mathrm{d}x \mathrm{d}\varepsilon \leq C_{s,p,n} \, \|f\|^p_{W^{s,p}(\mathbb{R}^n)}. \tag{5.10}$$

When $0 < s < 1$ and $f \in L^1_{loc}(\mathbb{R}^n)$, we have the stronger conclusion

$$\sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| = M}} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F_f(x,\varepsilon)|^p \, \mathrm{d}x \mathrm{d}\varepsilon \leq C_{s,p,n} \, |f|^p_{W^{s,p}(\mathbb{R}^n)}. \tag{5.11}$$

Remark 9. Theorems 4 and 5 are special cases of Theorems 6 and 8 (with $1 < p < \infty$, $s = 1 - 1/p$ and $M = 1$).

Remark 10. Estimate (5.10) still holds true – and this is a relatively difficult result – when $f \in L^p$ and we replace F_f by the harmonic extension of f, given by the Poisson formula. For this and similar results, see [58], Taibleson [54,55] and the more modern treatment in [43]; see also Leoni [34, Section 18.7].

We present below proofs of Theorems 6 and 7 which follow essentially [36, pp. 512–513] and [43, proof of Theorem 1.3].

Proof of Theorem 6. This result is a consequence of Theorem 7. In order to justify this assertion, assume that we have established (5.8) for every $M > s$ and every $F \in C^M(\mathbb{R}^n_+)$. Then we claim that, under the stronger assumption $M \geq s + 1/p$, we have (5.8) for every $F \in C^M(\mathbb{R}^{n+1}_{+,*})$. Indeed, we let $\delta > 0$ and we apply (5.8) to $(x, \varepsilon) \mapsto F(x, \varepsilon + \delta)$. We find that

$$(5.12)\quad |F(\cdot,\delta)|^p_{W^{s,p}} \leq C_{s,p,n} \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|=M}} \int_\delta^\infty \int_{\mathbb{R}^n} (\varepsilon-\delta)^{(M-s)p-1} |\partial^\alpha F(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon.$$

Letting $\delta \to 0$ in (5.12), we obtain (5.8) (using the definition of the trace and Fatou's lemma on the left-hand side, respectively the assumption $(M-s)p-1 \geq 0$ and the monotone convergence theorem on the right-hand side). □

Proof of Theorem 7. Step 1. Proof of (5.9) for $F \in C^M(\mathbb{R}^{n+1}_+)$. This is the main step of the proof, and it consists (again!) of an application of a Hardy type inequality.

For such F, have $f(x) = F(x, 0)$. The proof of (5.9) relies on the following elementary lemma, whose proof is postponed to the appendix.

Lemma 24. *Let $M > 0$ be an integer. We set*

$$|D_M F(x,\varepsilon)| := \sum_{\alpha\in\mathcal{M}_M} |\partial^\alpha F(x,\varepsilon)|, \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon \geq 0.$$

Let $h \in \mathbb{R}^n$ and set $r := |h|$. Then for every $x \in \mathbb{R}^n$ we have

$$(5.13)\quad \begin{aligned} |\Delta_h^M f(x)| \lesssim & r^M \sum_{j=1}^M \int_0^M t^{M-1} |D_M F(x+th, jr)| \,\mathrm{d}t \\ & + r^M \sum_{j=0}^M \int_0^M t^{M-1} |D_M F(x+jh, tr)| \,\mathrm{d}t. \end{aligned}$$

Granted Lemma 24, we proceed to the proof of the theorem. Set $g(\varepsilon) := \|D_M F(\cdot,\varepsilon)\|_{L^p(\mathbb{R}^n)}$. Integrating (5.13) in x, we obtain (with $r := |h|$)

$$(5.14)\quad \begin{aligned} \|\Delta_h^M f\|_{L^p(\mathbb{R}^n)} &\lesssim r^M \sum_{j=1}^M g(jr) + r^M \int_0^M t^{M-1} g(tr) \,\mathrm{d}t \\ &\approx r^M \sum_{j=1}^M g(jr) + \int_0^{Mr} t^{M-1} g(t) \,\mathrm{d}t. \end{aligned}$$

In view of (3.41) and (5.14), in order to establish (5.9) it suffices to prove that

$$\int_{\mathbb{R}^n} |h|^{(M-s)p-n} [g(j|h|)]^p \, \mathrm{d}h \le C_{s,p,n,M,j} \int_0^\infty \varepsilon^{(M-s)p-1} [g(\varepsilon)]^p \, \mathrm{d}\varepsilon \tag{5.15}$$

and

$$\int_{\mathbb{R}^n} \left(\int_0^{M|h|} t^{M-1} g(t) \, \mathrm{d}t \right)^p \frac{\mathrm{d}h}{|h|^{n+sp}} \lesssim \int_0^\infty \varepsilon^{(M-s)p-1} [g(\varepsilon)]^p \, \mathrm{d}\varepsilon. \tag{5.16}$$

Passing to spherical coordinates and performing on the left-hand side of (5.15) the change of variable $\varepsilon := j\,|h|$, we see that the two integrals in (5.15) are proportional, and thus (5.15) holds.

Also in spherical coordinates, (5.16) amounts to

$$\int_0^\infty \frac{1}{\varepsilon^{sp+1}} \left(\int_0^{M\varepsilon} t^{M-1} g(t) \, \mathrm{d}t \right)^p \mathrm{d}\varepsilon \lesssim \int_0^\infty \varepsilon^{(M-s)p-1} [g(\varepsilon)]^p \, \mathrm{d}\varepsilon. \tag{5.17}$$

In turn, after the change of variable $\delta := M\,\varepsilon$ on the left-hand side, (5.17) follows from Hardy's inequality at 0 (2.3) applied with $r \rightsquigarrow sp$, $q \rightsquigarrow p$ and $g(u) \rightsquigarrow \varepsilon^{M-1}\, g(\varepsilon)$.

Granted Lemma 24, the proof of Step 1 is complete.

Step 2. Proof of (5.9) in the general case. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier. Let, for $\delta > 0$,

$$F_\delta(x, \varepsilon) := \int_{\mathbb{R}^n} F(x - y, \varepsilon) \eta_\delta(y) \, \mathrm{d}y, \ \forall\, x \in \mathbb{R}^n, \forall\, \varepsilon > 0.$$

Then clearly

$$\partial^\alpha F_\delta(x, \varepsilon) = \int_{\mathbb{R}^n} (\partial^\alpha F(x - y, \varepsilon)) \eta_\delta(y) \, \mathrm{d}y, \ \forall\, \alpha \in \mathbb{N}^{n+1}, \forall\, x \in \mathbb{R}^n, \forall\, \varepsilon > 0,$$

and thus

$$\|\partial^\alpha F_\delta(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} \le \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}, \ \forall\, \alpha \in \mathbb{N}^{n+1}, \forall\, \varepsilon, \delta > 0. \tag{5.18}$$

We find that

$$\begin{aligned} &\sum_{\alpha \in \mathcal{M}_M} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F_\delta(x, \varepsilon)|^p \, \mathrm{d}x \mathrm{d}\varepsilon \\ &\le \sum_{\alpha \in \mathcal{M}_M} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F(x, \varepsilon)|^p \, \mathrm{d}x \mathrm{d}\varepsilon, \ \forall\, \delta > 0. \end{aligned} \tag{5.19}$$

On the other hand, we have

$$\lim_{\varepsilon \searrow 0} F_\delta(\cdot, \varepsilon) = f * \eta_\delta \text{ in } L^\infty_{loc}(\mathbb{R}^n),$$

and thus F_δ extends by continuity to $\mathbb{R}^{n+1}_+$ by setting $F_\delta(x, 0) := f * \eta_\delta(x)$.

We next note that the proof of (5.9), and in particular, the proof of Lemma 24, still work if we weaken the assumption $F \in C^M(\mathbb{R}^{n+1}_+)$ to $F \in C^M(\mathbb{R}^{n+1}_{+,*}) \cap C(\mathbb{R}^{n+1}_+)$. (Indeed, for such F we estimate $\Delta_h^M F(x,\tau)$, $\tau > 0$, as in (5.13), then we let $\tau \to 0$, and we recover the conclusion of Lemma 24.)

This observation implies that (5.9) holds for F_δ. Using this remark, (5.9) and (5.18), we find that

$$
\begin{aligned}
|f * \eta_\delta|^p_{W^{s,p}(\mathbb{R}^n)} &\lesssim \sum_{\alpha\in\mathcal{M}_M} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F_\delta(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon \\
&\le \sum_{\alpha\in\mathcal{M}_M} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon.
\end{aligned}
\tag{5.20}
$$

We obtain (5.9) from (5.3), which implies that $f * \eta_\delta \to f$ a.e. as $\delta \to 0$, Fatou's lemma, and (5.20). □

Remark 11. In the proof of Theorem 7, we did not use the assumption $M > s$! However, when $M \le s$ the theorem is of limited interest. Indeed, if $F \in C^M(\mathbb{R}^{n+1}_{+,*} \cap C^M(\mathbb{R}^{n+1})$ with $M \le s$ and if the right-hand side of (5.9) is finite, then f is a polynomial of degree $\le M-1$, and thus $|f|_{W^{s,p}} = 0$. This follows by combining the proof of (5.9) (which holds, as we have noticed, also for $M \le s$) with [43, Proposition 5.1]. Thus, when $M \le s$, the information conveyed by (5.8) is merely $|f|_{W^{s,p}} = 0$.

We now turn to the proof of Theorem 8. Its main ingredients are three simple results, Lemmas 25, 26 and 27 below.

LEMMA 25. *Let $\xi \in L^\infty(\mathbb{R}^n)$ be such that* $\operatorname{supp}\xi \subset B_1(0)$ *and* $\int_{\mathbb{R}^n}\xi = 0$. *Let $0 < s < 1$ and $1 \le p < \infty$. Given $f \in L^1_{loc}(\mathbb{R}^n)$, set $G_f(x,\varepsilon) := f * \xi_\varepsilon(x)$, $\forall\, x \in \mathbb{R}^n$, $\forall\, \varepsilon > 0$. Then we have*

$$
\int_0^\infty \varepsilon^{-sp-1} \int_{\mathbb{R}^n} |G_f(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon \le C_{s,p,n,\xi} |f|^p_{W^{s,p}(\mathbb{R}^n)}.
\tag{5.21}
$$

Proof. We have

$$
\begin{aligned}
|G_f(x,\varepsilon)| &= \varepsilon^{-n} \left| \int_{|y|<\varepsilon} f(x-y)\,\xi(y/\varepsilon)\,\mathrm{d}y \right| \\
&= \varepsilon^{-n} \left| \int_{|y|<\varepsilon} [f(x-y) - f(x)]\,\xi(y/\varepsilon)\,\mathrm{d}y \right| \\
&\lesssim \varepsilon^{-n} \int_{|y|<\varepsilon} |\Delta^1_y f(x)|\,\mathrm{d}y.
\end{aligned}
\tag{5.22}
$$

Using (5.22) and Hölder's inequality, we find that

$$\begin{aligned}
&\int_0^\infty \varepsilon^{-sp-1}\int_{\mathbb{R}^n}|G_f(x,\varepsilon)|^p\,\mathrm{d}x\mathrm{d}\varepsilon\\
&\lesssim \int_0^\infty \varepsilon^{-sp-np-1}\int_{\mathbb{R}^n}\left(\int_{|y|<\varepsilon}|\Delta_y^1 f(x)|\,\mathrm{d}y\right)^p\mathrm{d}x\mathrm{d}\varepsilon\\
&\lesssim \int_0^\infty \varepsilon^{-sp-n-1}\int_{\mathbb{R}^n}\int_{|y|<\varepsilon}|\Delta_y^1 f(x)|^p\,\mathrm{d}y\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{|y|}^\infty \varepsilon^{-sp-n-1}\,|\Delta_y^1 f(x)|^p\,\mathrm{d}\varepsilon\mathrm{d}y\mathrm{d}x\\
&=C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|y|^{-sp-n}\,|\Delta_y^1 f(x)|^p\,\mathrm{d}y\mathrm{d}x = C\,|f|^p_{W^{s,p}},
\end{aligned}$$

whence (5.21). □

LEMMA 26. *Let $\rho\in C_c^\infty(\mathbb{R}^n)$, $f\in L^1_{loc}(\mathbb{R}^n)$ and F_f be given by (5.2). For every $\alpha\in\mathbb{N}^{n+1}\setminus\{0\}$, there exists some $\xi=\xi^\alpha\in C_c^\infty(\mathbb{R}^n)$ (depending on n, α and ρ) such that:*

1. $\operatorname{supp}\xi\subset\operatorname{supp}\rho$,
2. $\int_{\mathbb{R}^n}\xi=0$,
3. $\partial^\alpha F_f(x,\varepsilon)=\varepsilon^{-|\alpha|}\,f*\xi_\varepsilon(x)$, $\forall\,x\in\mathbb{R}^n$, $\forall\,\varepsilon>0$.

Proof. The proof is by induction, based on the following calculations.

If $a\in\mathbb{R}$ and $\eta\in C_c^\infty(\mathbb{R}^n)$, set $H(x,\varepsilon):=\varepsilon^{-a}\,f*\eta_\varepsilon(x)$.

When $j=1,\ldots,n$, we have

$$\begin{aligned}
\partial_j H(x,\varepsilon)&=\partial_j\left(\varepsilon^{-a-n}\int_{\mathbb{R}^n}f(y)\,\eta((x-y)/\varepsilon)\,\mathrm{d}y\right)\\
&=\varepsilon^{-a-n-1}\int_{\mathbb{R}^n}f(y)\,(\partial_j\eta)((x-y)/\varepsilon)\,\mathrm{d}y\\
&=\varepsilon^{-a-1}\,f*(\partial_j\eta)_\varepsilon.
\end{aligned}\tag{5.23}$$

When $j=n+1$ and thus $\partial_{n+1}=\dfrac{\partial}{\partial\varepsilon}$, we have

$$\begin{aligned}
\partial_{n+1}H(x,\varepsilon)=&-\varepsilon^{-a-n-1}\int_{\mathbb{R}^n}f(y)\sum_{k=1}^n\frac{x_k-y_k}{\varepsilon}\,(\partial_k\eta)((x-y)/\varepsilon)\,\mathrm{d}y\\
&-(a+n)\,\varepsilon^{-a-n-1}\int_{\mathbb{R}^n}f(y)\,\eta((x-y)/\varepsilon)\,\mathrm{d}y\\
=&-\varepsilon^{-a-1}\,f*(a\,\eta+\operatorname{div}(x\,\eta))_\varepsilon.
\end{aligned}\tag{5.24}$$

On the other hand, we clearly have

$$\int_{\mathbb{R}^n}\partial_j\zeta=0 \text{ and } \int_{\mathbb{R}^n}\operatorname{div}\zeta=0,\ \forall\,j=1,\ldots,n,\ \forall\,\zeta\in C_c^\infty(\mathbb{R}^n).\tag{5.25}$$

The existence of ξ satisfying *1–3* follows easily by induction on $|\alpha|$, using (5.23)–(5.25). □

When f has additional differentiability properties, we may improve the conclusion of Lemma 26 as follows.

LEMMA 27. *Let $m \in \mathbb{N}$, $m \geq 1$. Assume that $f \in W^{m,1}_{loc}(\mathbb{R}^n)$. Let $\rho \in C^\infty_c(\mathbb{R}^n)$ and let F_f be given by (5.2). Let $\alpha \in \mathbb{N}^{n+1}$ be such that $|\alpha| > m$. Then there exist $\zeta^{\alpha,\beta} \in C^\infty_c(\mathbb{R}^n)$, $\forall\, \beta \in \mathbb{N}^n$ with $|\beta| = m$ (depending on n, α, β, ρ) such that:*

1. $\operatorname{supp} \zeta^{\alpha,\beta} \subset \operatorname{supp} \rho$,
2. $\int_{\mathbb{R}^n} \zeta^{\alpha,\beta} = 0$

and

$$\partial^\alpha F_f(x,\varepsilon) = \varepsilon^{-|\alpha|+m} \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=m}} (\partial^\beta f) * (\zeta^{\alpha,\beta})_\varepsilon(x),\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \tag{5.26}$$

Proof. By (5.23) and (5.24), for $j = 1, \dots, n+1$, we have

$$\partial_j F_f(x,\varepsilon) = \frac{1}{\varepsilon} f * \sum_{k=1}^{n} (\partial_k \psi_{j,k})_\varepsilon(x), \tag{5.27}$$

for appropriate $\psi_{j,k} \in C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp} \psi_{j,k} \subset \operatorname{supp} \rho$. Using the fact that $(1/\varepsilon)\,(\partial_k \psi_{j,k})_\varepsilon = \partial_k[(\psi_{j,k})_\varepsilon]$, we find from (5.27) that

$$\partial_j F_f(x,\varepsilon) = \sum_{k=1}^{n} (\partial_k f) * (\psi_{j,k})_\varepsilon(x). \tag{5.28}$$

Starting from (5.28) and repeating the above argument, we find (by induction on the length $|\gamma| \leq m$) that for every $\gamma \in \mathbb{N}^{n+1}$ with $|\gamma| \leq m$ we have

$$\partial^\gamma F_f(x,\varepsilon) = \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=|\gamma|}} (\partial^\beta f) * (\psi_{\gamma,\beta})_\varepsilon(x) \tag{5.29}$$

for some appropriate $\psi_{\gamma,\beta} \in C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp} \psi_{\gamma,\beta} \subset \operatorname{supp} \rho$.

We obtain properties *1*, *2* and (5.27) from (5.29) and Lemma 26. □

Proof of Theorem 8. Step 1. Proof of (5.11). Without loss of generality, we may assume that the mollifier ρ defining F_f in (5.2) satisfies $\operatorname{supp} \rho \subset B_1(0)$. If $\alpha \in \mathbb{N}^{n+1}$ is such that $|\alpha| = M$, we write $\partial^\alpha F_f(x,\varepsilon) = \varepsilon^{-M} f * \xi_\varepsilon(x)$, as in Lemma 26. Using Lemma 25, we find that

$$\int_0^\infty \varepsilon^{(M-s)p-1} \int_{\mathbb{R}^n} |\partial^\alpha F_f(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon = \int_0^\infty \varepsilon^{-sp-1} \int_{\mathbb{R}^n} |f * \xi_\varepsilon(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon$$
$$\lesssim |f|^p_{W^{s,p}},$$

i.e., (5.11) holds.

Step 2. Proof of (5.10). In view of Step 1, we may assume that $s > 1$. We write $s = m + \sigma$, with $m \in \mathbb{N}$, $m \geq 1$, and $0 < \sigma < 1$. We choose on $W^{s,p}(\mathbb{R}^n)$ the norm

$$\langle\langle f\rangle\rangle_{W^{s,p}}^p := \|f\|_{L^p}^p + \langle f\rangle_{W^{s,p}}^p, \text{ with } \langle f\rangle_{W^{s,p}}^p := \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=m}} |\partial^\beta f|_{W^{\sigma,p}}^p.$$

When $1 < s < 2$ and $n = 1$, we have proved that this norm is equivalent to the standard one, given by (3.41)–(3.42); see Lemma 19. The same holds for any s, p and $M > s$. However, we will not need the full strength of this assertion, but only the weaker property

$$\langle f\rangle_{W^{s,p}} \lesssim \|f\|_{W^{s,p}}, \tag{5.30}$$

for which we refer the reader to [56, Section 2.3.8, Theorem, pp. 58–59]. In view of (5.30), in order to complete Step 2 it suffices to establish the estimate

$$\sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|=M}} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F_f(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon \leq C_{s,p,n} \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=m}} |\partial^\beta f|_{W^{\sigma,p}}^p. \tag{5.31}$$

By Lemmas 27 and 25, we have

$$\begin{aligned}
&\sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|=M}} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(M-s)p-1} |\partial^\alpha F_f(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon\\
&\lesssim \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|=M}} \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=m}} \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{-\sigma p-1} |(\partial^\beta f) * (\zeta^{\alpha,\beta})_\varepsilon(x)|^p \,\mathrm{d}x\mathrm{d}\varepsilon\\
&\lesssim \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=m}} |\partial^\beta f|_{W^{\sigma,p}}^p = \langle f\rangle_{W^{s,p}}^p,
\end{aligned}$$

and thus (5.31) holds. □

5.2. **Two applications to superposition operators**

We continue here the discussion initiated at the end of Section 4.1. We let $s > 1$ and seek for conditions ensuring that if $f \in W^{s,p}(\mathbb{R}^n)$, then $T_\Phi f \in W^{s,p}$. We have noticed there that, even for smooth Φ, the conclusion $T_\Phi f \in W^{s,p}$ may require additional conditions either on f, or on the triple (s, p, n).

We present here two main results in this direction, together with a consequence. (The interested reader may find in [43, Section 6] more applications of the trace theory of weighted Sobolev spaces to the study of the mapping properties of T_Φ.) It turns out that these results hold also (but they are less interesting) for $s \le 1$. They are equally true for integer s.

First, a notation. If s is real number, $\lceil s \rceil$ denotes the smallest integer $k \ge s$.

THEOREM 9. *Let $s > 0$ and $1 \le p < \infty$. Let $M := \lceil s \rceil$. Let $\Phi \in C^M(\mathbb{R})$ be such that $\Phi(0) = 0$. Then T_Φ maps $W^{s,p} \cap L^\infty(\mathbb{R}^n)$ into itself.*

COROLLARY 6. *Let $s > 0$ and $1 \le p < \infty$ be such that $sp > n$. Let $M := \lceil s \rceil$. Let $\Phi \in C^M(\mathbb{R})$ be such that $\Phi(0) = 0$. Then T_Φ maps $W^{s,p}(\mathbb{R}^n)$ into itself.*

THEOREM 10. *Let $s > 0$ and $1 \le p < \infty$ be such that $sp = n$. Let $M := \lceil s \rceil$. Let $\Phi \in C^M(\mathbb{R})$ be such that $\Phi(0) = 0$ and $\Phi^{(j)} \in L^\infty$, $\forall j = 1, \dots, M$. Then T_Φ maps $W^{s,p}(\mathbb{R}^n)$ into itself.*

When s is an integer, Theorem 9 is due to Moser [44]. Its proof relies on the Gagliardo-Nirenberg inequalities that we will recall below. The need of such inequalities in this context is obvious from the proof. When s is not an integer, there are two standard proofs of Theorem 9. The first one uses the para-differential calculus and an ingenious identity due to Meyer [38]. The second one is elementary, but relies on a tedious identity which is quite difficult both to check and guess; see Escobedo [23]. We will see below that when we prove this result using the theory of weighted Sobolev spaces, we only need an obvious Gagliardo-Nirenberg type inequality!

We start with some important results that we will use in the proof. First, let us recall the following fundamental interpolation inequality, due to Gagliardo [26] and Nirenberg [46].

LEMMA 28 (Gagliardo-Nirenberg inequalities). *Let $0 \le m_1 < m < m_2$ be integers, and $1 \le p_1, p_2 \le \infty$. Define the number $\theta \in (0,1)$ by $m = (1-\theta)\, m_1 + \theta\, m_2$ and let $1 \le p \le \infty$ be given by $\dfrac{1}{p} = \dfrac{1-\theta}{p_1} + \dfrac{\theta}{p_2}$. Then, for some $C = C_{m_1,m_2,m,p_1,p_2,n}$, we have*

$$\|D^m u\|_{L^p} \le C\, \|D^{m_1} u\|_{L^{p_1}}^{1-\theta}\, \|D^{m_2} u\|_{L^{p_2}}^{\theta}, \ \forall\, u \in C^\infty(\mathbb{R}^n). \tag{5.32}$$

In the above, we use the compact notation

$$|D^m u| := \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} |\partial^\alpha u|. \tag{5.33}$$

When $m = 0$ (respectively $m = 1$), we write $|u|$ instead of $|D^0 u|$ (respectively $|\nabla u|$ instead of $|D^1 u|$).

We now present an interpolation inequality, of Gagliardo-Nirenberg type, involving fractional Sobolev spaces; see [12] for the comprehensive list of the Gagliardo-Nirenberg type inequalities valid in the full scale of Sobolev spaces.

LEMMA 29. *Let* $0 < t < s < \infty$ *be non-integers, and let* $1 \le p < \infty$. *Let* $q := sp/t \in (p, \infty)$ *and* $\theta := t/s \in (0,1)$. *Then we have* $W^{s,p} \cap L^\infty(\mathbb{R}^n) \subset W^{t,q}(\mathbb{R}^n)$. *More specfically, with* $C = C_{s,t,p,n}$, *we have*

$$\|f\|_{W^{t,q}} \le C\, \|f\|_{W^{s,p}}^{\theta}\, \|f\|_{L^\infty}^{1-\theta}, \forall\, f \in W^{s,p} \cap L^\infty(\mathbb{R}^n). \tag{5.34}$$

Proof. Let $M > s$ be an integer, and let $|\ |_{W^{s,p}}$, $|\ |_{W^{t,q}}$ be the semi-norms defined via M as in (3.41). We consider on $W^{s,p}$ and $W^{t,q}$ the norms given by (3.42). Using the inequalities

$$|f|^q \le |f|^p\, \|f\|_{L^\infty}^{q-p} \text{ and } |\Delta_h^M f|^q \lesssim |\Delta_h^M f|^p\, \|f\|_{L^\infty}^{q-p},$$

we immediately obtain (5.34). □

We next state and establish two special cases of the Sobolev embeddings.

LEMMA 30. *Let* $s > 0$ *and* $1 \le p < \infty$ *be such that* $sp > n$. *Then* $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^\infty$.

Proof. When s is an integer, see *e.g.* Brezis [9, Corollary 9.13].

Assume next that $0 < s < 1$. By Corollary 5, we have

$$|f(y) - f(x)| \lesssim |f|_{W^{s,p}},\ \forall\, f \in W^{s,p}(\mathbb{R}^n), \forall\, x \in \mathbb{R}^n, \forall\, y \in B_1(x). \tag{5.35}$$

On the other hand, for every $x \in \mathbb{R}^n$ there exists some $y \in B_1(x)$ such that

$$|f(y) \lesssim \|f\|_{L^p(B_1(x))} \le \|f\|_{L^p}. \tag{5.36}$$

From (5.35) and (5.36), we obtain that $|f(x)| \lesssim \|f\|_{W^{s,p}}$, and thus $W^{s,p} \hookrightarrow L^\infty$.

Finally, assume that $s > 1$ is non-integer. Write $s = m + \sigma$, with $m \in \mathbb{N}$, $m \ge 1$ and $0 < \sigma < 1$. We consider on $W^{s,p}$ the norm $\langle\langle f \rangle\rangle_{W^{s,p}}^p := \|D^m f\|_{L^p}^p + |D^m f|_{W^{\sigma,p}}^p$. Let $0 < \sigma' < \sigma$. By Lemma 13, we have $W^{m+\sigma,p} \hookrightarrow W^{m+\sigma',p}$. Therefore, by lowering σ if necessary, we may assume that $sp > n$ and $\sigma p \ne n$.

Applying repeatedly Lemma 13, we find that $W^{s,p} \hookrightarrow W^{\sigma,p}$. Thus, if $\sigma p > n$ then $W^{s,p} \hookrightarrow W^{\sigma,p} \hookrightarrow L^\infty$.

On the other hand, if $\sigma p < n$, then, by Corollary 4 applied to $D^m f$, we find that $W^{s,p} \hookrightarrow W^{m,q}$, with $q := (np)/(n - \sigma p)$. It is easy to see that $mq > n$, and thus $W^{s,p} \hookrightarrow W^{m,q} \hookrightarrow L^\infty$.

The proof of Lemma 30 is complete. □

LEMMA 31. *Let* $s > 1$ *and* $1 \le p < \infty$ *be such that* $sp = n$. *Let* $1 \le k < s$ *be an integer. Then* $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{k,n/k}$.

Proof. When s is an integer, see [9, Corollary 9.13].

Assume that s is non-integer and write as above $s = m + \sigma$. In view of [9, Corollary 9.13], it suffices to obtain the conclusion when $k = m$. In that case, the conclusion follows from Corollary 4 applied to $D^m f$. $\square$

In the proofs of the main results announced in this section, we will consider only relatively small values of s (we take $s \leq 2$). Although this limitation is not important for the validity of the arguments, the reason is the following. We will have to estimate $D^m(\Phi \circ f)$, with $m := \lceil s \rceil$. In order to calculate $D^m(\Phi \circ f)$, we rely on the Faà di Bruno's formula for the higher order derivatives of composite functions. This formula becomes cumbersome when $m \geq 3$. Since, apart from this complexification of the calculations, the arguments are similar for all $s > 1$, we took the party of limiting the arguments to $1 < s \leq 2$. We refer the interested reader to [43, Section 6] for full proofs of the above results (using slightly different arguments). In what follows, the case where $s \leq 1$ is much easier; it was briefly discusses at the beginning of the Section 4.1, and is left to the reader.

Proof of Theorem 9 when $1 < s \leq 2$. *Step 1.* Proof when $s = 2$. Let $p > n/2$. Let $f \in C^\infty(\mathbb{R}^n)$. Consider a number $r \geq \|f\|_{L^\infty}$. On the one hand we clearly have $|T_\Phi f| \leq \sup\{|\Phi'(t)|;\ |t| \leq r\}\, |f|$, and thus

$$\|T_\Phi f\|_{L^p} \leq \sup\{|\Phi'(t)|;\ |t| \leq r\}\, \|f\|_{L^p}. \tag{5.37}$$

On the other hand, we have the pointwise inequality

$$|D^2 T_\Phi f| \lesssim \sup\{|\Phi'(t)|;\ |t| \leq r\}\, |D^2 f| + \sup\{|\Phi''(t)|;\ |t| \leq r\}\, |\nabla f|^2. \tag{5.38}$$

Using the Gagliardo-Nirenberg inequality (5.32) with $m_1 := 0$, $m_2 := 2$, $m := 1$, $p_1 := \infty$ and $p_2 := p$, as well as (5.38), we find that

$$\begin{aligned} \|D^2 T_\Phi f\|_{L^p} \lesssim & \sup\{|\Phi'(t)|;\ |t| \leq r\}\, \|D^2 f\|_{L^p} \\ & + \sup\{|\Phi''(t)|;\ |t| \leq r\}\, r\, \|D^2 f\|_{L^p}. \end{aligned} \tag{5.39}$$

Using again the Gagliardo-Nirenberg inequalities, this time in conjunction with (5.37) and (5.39), we find that

$$\begin{aligned} \|\nabla T_\Phi f\|_{L^p} \lesssim & \sup\{|\Phi'(t)|;\ |t| \leq r\}\, \|f\|_{L^p}^{1/2}\, \|D^2 f\|_{L^p}^{1/2} \\ & + \sup\{|\Phi'(t)|;\ |t| \leq r\}^{1/2} \sup\{|\Phi''(t)|;\ |t| \leq r\}^{1/2} \\ & \times r^{1/2}\, \|f\|_{L^p}^{1/2}\, \|D^2 f\|_{L^p}^{1/2}. \end{aligned} \tag{5.40}$$

Consider now some $f \in W^{2,p} \cap L^\infty$ and set $r := \|f\|_{L^\infty}$. Set $f_\varepsilon := f * \rho_\varepsilon$, where ρ is a standard mollifier. Note that $\|f_\varepsilon\|_{L^\infty} \leq \|f\|_{L^\infty}$, $\forall\, \varepsilon > 0$. We may thus apply (5.37), (5.39) and (5.40) to f_ε and obtain uniform L^p bounds for

$D^j T_\Phi f_\varepsilon$, $j = 0, 1, 2$. By Fatou's lemma, we find that $D^j T_\Phi f \in L^p$, $j = 0, 1, 2$, and thus $T_\Phi f \in W^{2,p}$ (and clearly $T_\Phi f \in L^\infty$.)

Step 2. Proof when $1 < s < 2$. Let $f \in W^{s,p} \cap L^\infty$. Let F_f be as in (5.2). Since $f \in L^p$, we have $F_f(\cdot, \varepsilon) \to f$ in L^p as $\varepsilon \to 0$. On the other hand, we have $|F_f| \le \|f\|_{L^\infty}$, and thus

$$\|T_\Phi F_f(\cdot,\varepsilon) - T_\Phi f\|_{L^p} \le \sup\{|\Phi'(t)|;\ |t| \le \|f\|_{L^\infty}\}\, \|F_f(\cdot,\varepsilon) - f\|_{L^p} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, the function $T_\Phi F_f$ has trace $T_\Phi f$ in the stronger sense of Theorem 7. In view of Theorem 7 it follows that, in order to prove that $T_\Phi f \in W^{s,p}$, it suffices to prove that

$$T_\Phi f \in L^p, \tag{5.41}$$

$$I := \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} |D^2 T_\Phi F_f(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon < \infty. \tag{5.42}$$

(5.41) being clear, we proceed to the proof of (5.42). As in Step 1, using the fact that $\|F_f\|_{L^\infty} \le \|f\|_{L^\infty}$, we obtain

$$\begin{aligned} I \le & C_f \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} |D^2 F_f(x,\varepsilon)|^p \,\mathrm{d}x\mathrm{d}\varepsilon \\ & + C_f \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} |\nabla F_f(x,\varepsilon)|^{2p} \,\mathrm{d}x\mathrm{d}\varepsilon. \end{aligned} \tag{5.43}$$

In view of (5.43), of Theorem 8 (applied twice) and of Lemma 29, we have

$$\begin{aligned} I \le & C_f \,\|f\|^p_{W^{s,p}} + C_f \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} |\nabla F_f(x,\varepsilon)|^{2p} \,\mathrm{d}x\mathrm{d}\varepsilon \\ = & C_f \,\|f\|^p_{W^{s,p}} + C_f \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(1-s/2)\,(2p)-1} |\nabla F_f(x,\varepsilon)|^{2p} \,\mathrm{d}x\mathrm{d}\varepsilon \\ \le & C_f \,\|f\|^p_{W^{s,p}} + C_f \,\|f\|^{2p}_{W^{s/2,2p}} \le C_f \,\|f\|^p_{W^{s,p}}. \end{aligned} \tag{5.44}$$

This completes the proof of Theorem 9 when $1 < s \le 2$. ☐

Proof of Corollary 6. We combine Theorem 9 with Lemma 30. ☐

Proof of Theorem 10. By Lemma 31, we have $W^{s,p} \hookrightarrow W^{s,p} \cap W^{1,sp}$. Therefore, Theorem 10 is a special case of Theorem 11 stated and proved in the next section. ☐

5.3. Superposition operators in $W^{s,p} \cap W^{1,sp}$

Let us take a closer look at the proof of Theorem 9 when $s = 2$. It relies on the following ingredients.

1. $\Phi(f) \in L^p$.
2. $\Phi'(f)\, D^2 f \in L^p$.
3. $\Phi''(f)\, |\nabla f|^2 \in L^p$.

Let us now make the following assumptions on Φ: $\Phi \in C^2$, $\Phi(0) = 0$, $\Phi^{(j)}$ is bounded, $j = 1, 2$. Then item 1 above holds if $f \in L^p$. Item 2 holds if $f \in W^{2,p}$. Finally, item 3 holds if $\nabla f \in L^{2p}$. By the Gagliardo-Nirenberg inequalities (5.32), the third requirement is satisfied if $f \in W^{2,p} \cap L^\infty$. However, if we replace the assumption $f \in W^{2,p} \cap L^\infty$ by the weaker assumptions $f \in W^{2,p}$ and $\nabla f \in L^{2p}$, we still obtain the conclusion of Theorem 10 (with $s = 2$). These considerations and Lemma 31 suggest the following improvement of Theorems 9 and 10.

THEOREM 11. *Let $s > 1$ and $1 \le p < \infty$. Let $M := \lceil s \rceil$. Let $\Phi \in C^M(\mathbb{R})$ be such that $\Phi(0) = 0$ and $\Phi^{(j)} \in L^\infty$, $\forall\, j = 1, \dots, M$. Set*

$$X := \{f \in W^{s,p}(\mathbb{R}^n);\ \nabla f \in L^{sp}\}.$$

Then T_Φ maps X into itself.

This result was initially obtained in [12], with a proof using Fefferman-Stein type vector-valued maximal inequalities [24] and Littlewood-Paley theory. A more elementary proof, using fractional maximal inequalities, was found by Maz'ya and Shaposhnikova [37]. We present below a very natural proof, using trace theory. It relies only on the maximal function theorem in L^p, $p > 1$, and on the following simple observation.

LEMMA 32. *Let $g \in L^1_{loc}(\mathbb{R}^n)$ and $\eta \in C_c^\infty(\mathbb{R}^n)$. Then*

$$|g * \eta_\varepsilon(x)| \le C_\eta\, \mathcal{M}g(x),\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \tag{5.45}$$

Proof. Let $R > 0$ be such that $\operatorname{supp} \eta \subset B_R(0)$. Then

$$\begin{aligned} |g * \eta_\varepsilon(x)| &\le \sup |\eta|\, \frac{1}{\varepsilon^n} \int_{B_{R\varepsilon}(x)} |g(y)|\, \mathrm{d}y \\ &= C_n \sup |\eta| \fint_{B_{R\varepsilon}(x)} |g(y)|\, \mathrm{d}y \lesssim \mathcal{M}g(x), \end{aligned}$$

whence (5.45). □

The interested reader may find a useful generalization of (5.45) in [52, Chapter II, Section 2.1, formula (16), p. 54].

Proof of Theorem 11 when $1 < s \le 2$. The case where $s = 2$ has been discussed at the beginning of this section. We may thus assume that $1 < s < 2$. Let $f \in W^{s,p}$ be such that $\nabla f \in L^{sp}$. We have

$$|T_\Phi f| \le \|\Phi'\|_{L^\infty}\, |f| \text{ and } |\nabla T_\Phi f| \le \|\Phi'\|_{L^\infty}\, |\nabla f|,$$

so that $f \in L^p$ and $\nabla f \in L^p \cap L^{sp}$.

Write $s = 1 + \sigma$, with $0 < \sigma < 1$. In view of the above, in order to complete the proof of the theorem we have to prove that $\nabla T_\Phi f \in W^{\sigma,p}$. We fix some $1 \le j \le n$, and prove that $g := \partial_j T_\Phi f = \Phi'(f)\, \partial_j f \in W^{\sigma,p}$. Let $f \mapsto F_f$ be the operator defined in (5.2). Set

$$\begin{aligned} G(x,\varepsilon) :=&F_{\Phi'(f)}(x,\varepsilon)\, \partial_j F_f(x,\varepsilon) \\ =&F_{\Phi'(f)}(x,\varepsilon)\, F_{\partial_j f}(x,\varepsilon),\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \end{aligned} \tag{5.46}$$

We let to the reader the proof of the fact that $\lim_{\varepsilon\to 0} G(\cdot,\varepsilon) = g$ in L^p, and thus that g is the trace of G in the strong sense of Theorem 7. Note also that $1-\sigma = 2-s$. From these observations and Theorem 7, we find that, when $1 < s < 2$, the conclusion of Theorem 11 amounts to

$$\int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1}\, |\nabla G(x,\varepsilon)|^p\, \mathrm{d}x \mathrm{d}\varepsilon < \infty. \tag{5.47}$$

By (5.46) and the assumption that $\Phi' \in L^\infty$, we have

$$|\nabla G| \lesssim |D^2 F_f| + |\nabla F_{\Phi'(f)}|\, |\nabla F_f|. \tag{5.48}$$

The heart of the proof consists of estimating $|\nabla F_{\Phi'(f)}|$ in two different ways. On the one hand, since Φ' is bounded, we have $\Phi'(f) \in L^\infty$ and therefore, by Lemma 26 item *3*, we have

$$|\nabla F_{\Phi'(f)}(x,\varepsilon)| \lesssim \frac{1}{\varepsilon},\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \tag{5.49}$$

On the other hand, using successively (5.26) with $|\alpha| = 1$, the fact that Φ'' is bounded and Lemma 32, we obtain

$$\begin{aligned} |\nabla F_{\Phi'(f)}(x,\varepsilon)| \lesssim& \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|=1}} \sum_{\substack{\beta\in\mathbb{N}^{n}\\ |\beta|=1}} |(\partial^\beta f) * (\zeta^{\alpha,\beta})_\varepsilon(x)| \\ \lesssim&\mathcal{M}|\nabla f|(x),\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \end{aligned} \tag{5.50}$$

Similarly, we have

$$|\nabla F_f(x,\varepsilon)| \lesssim \mathcal{M}|\nabla f|(x),\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \tag{5.51}$$

Combining (5.48)–(5.51) , we find that

$$\begin{aligned} |\nabla G(x,\varepsilon)| \lesssim |D^2 F_f(x,\varepsilon)| +& \left(\frac{1}{\varepsilon} \wedge \mathcal{M}|\nabla f|(x)\right), \\ &\times \mathcal{M}|\nabla f|(x),\ \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0. \end{aligned} \tag{5.52}$$

Using (5.52), Theorem 8 and the maximal function theorem, we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} \, |\nabla G(x,\varepsilon)|^p \, \mathrm{d}x\mathrm{d}\varepsilon \\
&\lesssim \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} \, |D^2 F_f(x,\varepsilon)|^p \, \mathrm{d}x\mathrm{d}\varepsilon \\
&\quad + \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} \left(\frac{1}{\varepsilon} \wedge \mathcal{M}|\nabla f|(x)\right)^p (\mathcal{M}|\nabla f|(x))^p \, \mathrm{d}x\mathrm{d}\varepsilon \\
&= \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} \, |D^2 F_f(x,\varepsilon)|^p \, \mathrm{d}x\mathrm{d}\varepsilon \\
&\quad + \int_{\mathbb{R}^n} \int_0^{1/\mathcal{M}|\nabla f|(x)} \varepsilon^{(2-s)p-1} \, (\mathcal{M}|\nabla f|(x))^{2p} \, \mathrm{d}\varepsilon\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} \int_{1/\mathcal{M}|\nabla f|(x)}^\infty \varepsilon^{(2-s)p-1} \, \varepsilon^{-p} \, (\mathcal{M}|\nabla f|(x))^p \, \mathrm{d}\varepsilon\mathrm{d}x \\
&\approx \int_0^\infty \int_{\mathbb{R}^n} \varepsilon^{(2-s)p-1} \, |D^2 F_f(x,\varepsilon)|^p \, \mathrm{d}x\mathrm{d}\varepsilon + \int_{\mathbb{R}^n} (\mathcal{M}|\nabla f|(x))^{sp} \, \mathrm{d}x \\
&\lesssim \|f\|_{W^{s,p}}^p + \|\nabla f\|_{L^{sp}}^{sp}.
\end{aligned}$$

This yields (5.47) and completes the proof of Theorem 11 when $1 < s < 2$. □

Remark 12. Theorem 11 is, in some sense, optimal. Indeed, assume that $f \in W^{s,p}$ and that, for *every* Φ as in Theorem 11, we have $T_\Phi f \in W^{s,p}$. In particular, by taking $\Phi =$id, we find that $f \in W^{s,p}$. Similarly, we have $\sin f, (\cos f - 1) \in W^{s,p}$. Since $\sin f, (\cos f - 1) \in L^\infty$, we find from the general form of the Gagliardo-Nirenberg inequalities (see *e.g.* [12]) that $\sin f, (\cos f - 1) \in W^{1,sp}$. Using the chain rule for composite functions, we obtain $\cos f \ \nabla f, \sin f \ \nabla f \in L^{sp}$, and thus $|\nabla f| = |(\cos f, \sin f) \ \nabla f| \in L^{sp}$. We have thus obtained that the assumptions on f in Theorem 11 are essentially necessary.

6. MAPS WITH VALUES INTO MANIFOLDS

6.1. Overview

Let Σ be a smooth r-dimensional manifold and let ω be a smooth k-form on Σ. If $f : \mathbb{R}^n \to \Sigma$ is sufficiently smooth (say, $f \in C^\ell$ for some $\ell \geq 1$), then we may define the pullback $f^\sharp\omega$ of ω by f, which is a k-form of class $C^{\ell-1}$ on $\mathbb{R}^n$. More specifically, if $(y^1, \ldots, y^r)$ is a system of local coordinates on Σ and if we write, near $f(a)$ with $a \in \mathbb{R}^n$,

$$\omega = \sum_{1\le i_1<\ldots<i_k\le r} \alpha_{i_1,\ldots,i_k}(y)\,\mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k},$$

then near a we have

$$\omega = \sum_{1\le i_1<\ldots<i_k\le r} \alpha_{i_1,\ldots,i_k}(f(x))\, df_{i_1} \wedge \cdots \wedge df_{i_r}$$

(with $(f_1(x), \ldots, f_r(x))$ the coordinates of $f(x)$ in the coordinate system $(y^1, \ldots, y^r)$).

The question we address here is the possibility of defining $f^\sharp\omega$ when f has less regularity, say f is not even C^1. This is already an issue when we assume that $f \in W^{1,p}_{loc}(\mathbb{R}^n)$. In that case, $f^\sharp\omega$ is well-defined a.e. as a k-form with measurable coefficients. This form is a useful analytical object (a form distribution, or *current*) only when its coefficients are in L^1_{loc}. Since clearly the coefficients are in $L^{p/k}_{loc}(\mathbb{R}^n)$, we find that $f^\sharp\omega$ is a current when $p \ge k$. However, we will see below that in some situations it is possible to define $f^\sharp\omega$ when f has a regularity below $W^{1,k}_{loc}$.

A thorough discussion about these topics would require a considerable amount of auxiliary results. Therefore, we will focus on some results in this direction that require little additional technology, and refer the interested reader to a series of articles dealing with the case where $\Sigma = \mathbb{S}^r$ and ω is the canonical volume form on $\mathbb{S}^r$ (or the Jacobian in $\mathbb{R}^{r+1}$): Jerrard and Soner [31, 32], Hang and Lin [28], Brezis and Nguyen [15], and also [6, 7, 40].

The arguments we present below rely on two types of ingredients: "null Lagrangians" (or "cancellation phenomena") and the trace theory. In order to make clear the role of each ingredient, we start with continuous (or, more generally, *VMO*) maps, for which the null Lagrangians play a key role. We next turn to the $W^{s,p}$ setting, which requires combining both tools. While the questions discussed in Section 6.2 are rather simple and could have been tackled by other methods, the approach we use to answer them will prove to be useful in the more complicated situations investigated in Sections 6.3 and 6.4, and even beyond.

6.2. **Winding number (I)**

We discuss here the possibility of defining through a convenient integral formula the winding number of maps $f : \mathbb{S}^1 \to \mathbb{S}^1$. This turns out to be possible when f is continuous (and even slightly less than continuous). We mention that it is possible to extend this approach to higher dimensions, and define the degree of continuous maps $f : \mathbb{S}^n \to \mathbb{S}^n$ via an integral formula similar to (6.18) below (see [39]).

Since at some point we would like to address this question in the context of Sobolev maps and we want to avoid working with Sobolev spaces of maps defined on manifolds, we rather consider maps $f : \mathbb{R} \to \mathbb{S}^1$. In order to further simplify the discussion, we make the following assumption:

$$f \equiv 1 \text{ for } |x| \geq R = R_f. \tag{6.1}$$

Assume temporarily that f is continuous. Identifying f with a complex-valued function, we may write $f = e^{\imath\varphi}$, with φ continuous and φ constant on $(-\infty, -R]$ and on $[R, \infty)$. In addition, $\varphi(-R), \varphi(R) \in 2\pi\mathbb{Z}$ (since $f(\pm R) = 1$). Therefore, the "winding number" (or "index", or "degree")

$$\deg f := \frac{\varphi(R) - \varphi(-R)}{2\pi} \tag{6.2}$$

is an integer, and one can prove that this integer does not depend on the choice of R as in (6.1) or on the specific continuous lifting φ.

Let us recall the following standard property of the degree:

$$\text{if } f, g \text{ satisfy (6.1) and if } |f - g| < 2, \text{ then } \deg f = \deg g. \tag{6.3}$$

Assume next that f is smoother, say $f \in C^1$. Then $\varphi \in C^1$, and thus we have $f' = \imath\, \varphi'\, e^{\imath\varphi}$. We claim that

$$\varphi' = \frac{1}{\imath\, f}\, f' = f \wedge f'. \tag{6.4}$$

In the second equality in (6.4), we have identified f with an $\mathbb{R}^2$-valued map, and we let $(a_1, a_2) \wedge (b_1, b_2) := a_2 b_1 - a_1 b_2$. In order to justify (6.4), we note that

$$\frac{1}{\imath\, f}\, f' = -\,\imath\, \overline{f}\, f' = f \wedge f' - \imath\, (f_1\, f_1' + f_2\, f_2') = f \wedge f';$$

the latter equality follows from the fact that

$$f_1\, f_1' + f_2\, f_2' = \frac{1}{2}(|f|^2)' = 0.$$

Let f satisfy

$$f \in C^1(\mathbb{R}; \mathbb{S}^1) \text{ and } f(x) \equiv 1, \forall\, |x| \geq R = R_f. \tag{6.5}$$

Combining (6.2) and (6.4), we recover the Cauchy formula

$$\deg f = \frac{1}{2\pi} \int_{\mathbb{R}} f \wedge f', \ \forall\, f \text{ as in (6.5)}. \tag{6.6}$$

The connection between this formula and the pullback of forms is the following. Let

$$\omega := \frac{1}{2\pi}\, (x^1\, \mathrm{d}x^2 - x^2\, \mathrm{d}x^1) = \frac{1}{2\pi}\, d\theta$$

denote the canonical volume form on $\mathbb{S}^1$. Then

$$f^\sharp \omega = \frac{1}{2\pi}\,(f_1\, f_2' - f_2\, f_1') = \frac{1}{2\pi}\, f \wedge f'.$$

Therefore, (6.6) reads

$$\deg f = \int_{\mathbb{R}} f^\sharp \omega = \langle f^\sharp \omega, 1\rangle, \tag{6.7}$$

the latter quantity being the duality bracket between the compactly supported distribution $f^\sharp\omega$ and the smooth test function 1.

Starting from (6.7), one may address the question of the existence of the distribution $f^\sharp\omega$ when f is less than C^1. We do not follow this route, for which we refer the reader to [15]. We consider instead the more modest task of finding an analogue of (6.7) valid when f is merely continuous. For this purpose we let $u = (u_1, u_2) : \mathbb{R}^2_{+,*} \to \mathbb{R}^2$, $u := F_f$, with F_f as in (5.2). Although f is $\mathbb{S}^1$-valued, u is merely $\mathbb{R}^2$-valued, and not $\mathbb{S}^1$-valued (unless $f \equiv 1$). We let Ju denote the Jacobian of u,

$$Ju = \partial_1 u \wedge \partial_2 u = \nabla u_1 \wedge \nabla u_2.$$

The following formula goes back to Poincaré.

Lemma 33. *Let $f \in C^2(\mathbb{R}; \mathbb{S}^1)$ satisfy* (6.5). *Then*

$$\deg f = \frac{1}{\pi} \int_{\mathbb{R}^2_{+,*}} Ju. \tag{6.8}$$

Proof. By Lemma 27 (with $m := 2$), u extends to a map in $C^2(\mathbb{R}^2_+)$. On the other hand, the assumption (6.5), Lemma 26 and Lemma 27 (with $m := 1$) lead to

$$|\nabla u(x_1, x_2)| \lesssim \begin{cases} 0, & \text{if } |x_1| \ge R + x_2 \\ 1 \wedge (1/(x_2)^2), & \text{if } |x_1| < R + x_2 \end{cases}. \tag{6.9}$$

In view of (6.9), we have

$$\int_{\mathbb{R}^2_{+,*}} |Ju| < \infty, \tag{6.10}$$

$$\lim_{r\to\infty} \int_{C_r^+(0)} |\nabla u|\, \mathrm{d}\ell = 0. \tag{6.11}$$

Here,

$$C_r^+(0) := \{x = (x_1, x_2) \in \mathbb{R}^2;\ x_2 > 0 \text{ and } |x| = r\}.$$

Since $u \in C^2$, the following two identities hold in $\mathbb{R}^2_+$:

$$Ju = \partial_1(u_1\, \partial_2 u_2) - \partial_2(u_1\, \partial_1 u_2), \tag{6.12}$$

$$Ju = \partial_2(u_2\,\partial_1 u_1) - \partial_1(u_2\,\partial_2 u_1). \tag{6.13}$$

Combining (6.12) and (6.13), we find that

$$Ju = \frac{1}{2}[\partial_1(u \wedge \partial_2 u) - \partial_2(u \wedge \partial_1 u)]. \tag{6.14}$$

Let $\Omega_r := \mathbb{R}^2_{+,*} \cap B_r(0)$, $r > 0$, and let ν denote the unit outward normal to $\partial\Omega_r$. Note that, on $(-r, r)$, we have $\nu = (0, -1)$. Using successively (6.10), (6.14), (6.11) and (6.6), we find that

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{R}^2_{+,*}} Ju =& \frac{1}{\pi}\lim_{r\to\infty}\int_{\Omega_r} Ju = \frac{1}{2\pi}\lim_{r\to\infty}\int_{\Omega_r}[\partial_1(u\wedge\partial_2 u) - \partial_2(u\wedge\partial_1 u)]\\
=& \frac{1}{2\pi}\lim_{r\to\infty}\int_{\partial\Omega_r}[\nu_1(u\wedge\partial_2 u) - \nu_2(u\wedge\partial_1 u)]\\
=& \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r}[\nu_1(u\wedge\partial_2 u) - \nu_2(u\wedge\partial_1 u)]\\
=& \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} u\wedge\partial_1 u = \frac{1}{2\pi}\int_{-R}^{R} u\wedge\partial_1 u\\
=& \frac{1}{2\pi}\int_{-R}^{R} f\wedge f' = \deg f.
\end{aligned}$$

This completes the proof of Lemma 33. □

It will be convenient later to have a variant of Lemma 33, Lemma 34 below, whose proof, very similar to the one of Lemma 33, is left to the reader.

LEMMA 34. *Let $f \in C^2(\mathbb{R}; \mathbb{S}^1)$ satisfy $f(x) \equiv 1$ for $|x| \geq R = R_f$. Let $w \in C^2(\mathbb{R}^2_+; \mathbb{R}^2)$ be any extension of f to $\mathbb{R}^2_+$ such that*

$$\int_{\mathbb{R}^2_{+,*}} |Jw| < \infty \tag{6.15}$$

and

$$\lim_{r\to\infty}\int_{C_r^+(0)} |\nabla w|\,\mathrm{d}\ell = 0. \tag{6.16}$$

Then

$$\deg f = \frac{1}{\pi}\int_{\mathbb{R}^2_{+,*}} Jw. \tag{6.17}$$

Our next task, consisting of extending (6.17) to maps f which are merely continuous, is more subtle. Indeed, Lemma 34 asserts that, when f is smooth, $\deg f$ can be calculated via any smooth extension w of f that has sufficient decay at infinity. In the case of a continuous f, one has to take care not only of the decay at infinity, but also of the behavior of w near $\mathbb{R}$.

We let $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$ be an "approximate projection" onto $\mathbb{S}^1$, *i.e.*, a map satisfying

1. $\Pi \in C^\infty$.
2. $\Pi(x) = x/|x| \in \mathbb{S}^1$ when $|x| \geq 1/2$.

Thus Π is the radial projection onto $\mathbb{S}^1$ except near the origin, where it is modified in order to obtain a smooth function.

LEMMA 35. *Let $f \in C(\mathbb{R}; \mathbb{S}^1)$ satisfy* (6.1). *Let $u := F_f$ (as in* (5.2)*) and set $w := \Pi(u)$. Then*

$$\deg f = \frac{1}{\pi} \int_{\mathbb{R}^2_{+,*}} Jw. \tag{6.18}$$

The proof of the lemma relies on a cancellation phenomenon described below.

Proof. In view of (6.1), f is uniformly continuous. Therefore, there exists some $\varepsilon_0 > 0$ such that

$$|u(x,\varepsilon) - f(x)| \leq 1/2, \ \forall\, x \in \mathbb{R}, \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{6.19}$$

In view of (6.19) and of the fact that $|f| = 1$, we find that

$$|u(x,\varepsilon)| \geq 1/2, \ \forall\, x \in \mathbb{R}, \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{6.20}$$

In turn, (6.20) implies that

$$|w(x,\varepsilon)| = 1, \ \forall\, x \in \mathbb{R}, \forall\, 0 < \varepsilon \leq \varepsilon_0, \tag{6.21}$$

$$|w(x,\varepsilon) - f(x)| \leq 1/2, \ \forall\, x \in \mathbb{R}, \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{6.22}$$

Now comes the crucial observation. We claim that the Jacobian of a smooth map $g : \Omega \to \mathbb{S}^1$, with $\Omega \subset \mathbb{R}^2$, vanishes. Indeed, differentiating the identity $|g|^2 \equiv 1$, we find that $g \cdot \partial_1 g = 0$ and $g \cdot \partial_2 g = 0$. This implies that the vectors $\partial_1 g$ and $\partial_2 g$ are both orthogonal to g, thus parallel. In conclusion, $Jg = \partial_1 g \wedge \partial_2 g = 0$, as claimed.

Using this observation and (6.21), we obtain the fundamental cancellation property

$$Jw(x,\varepsilon) = 0, \ \forall\, x \in \mathbb{R}, \forall\, 0 < \varepsilon < \varepsilon_0. \tag{6.23}$$

On the other hand, the assumption (6.1) and Lemma 26 yield

$$|\nabla u(x_1,x_2)| \lesssim \begin{cases} 0, & \text{if } |x_1| \geq R + x_2 \\ (1/(x_2)^2), & \text{if } |x_1| < R + x_2 \end{cases}. \tag{6.24}$$

Combining (6.23) and (6.24), we find that

$$\int_{\mathbb{R}^2_{+,*}} |Jw| < \infty, \tag{6.25}$$

$$\lim_{r\to\infty} \int_{C_r^+(0)+\{(0,\varepsilon_0)\}} |\nabla w| \, \mathrm{d}\ell = 0. \tag{6.26}$$

Using the cancellation property (6.23) and applying Lemma 34 in $\mathbb{R} \times (\varepsilon_0, \infty)$, we find that

$$\frac{1}{\pi}\int_{\mathbb{R}^2_{+,*}} Jw = \frac{1}{\pi}\int_{\mathbb{R}\times(\varepsilon,\infty)} Jw = \deg(w(\cdot,\varepsilon_0)). \tag{6.27}$$

Combining (6.3), (6.22) and (6.27), we obtain (6.18). □

Remark 13. We briefly explain here the possibility of defining $\deg f$ when f is slightly less than continuous. In this context, a natural class of maps is the class *VMO* of functions of "vanishing mean oscillations" (introduced by Sarason [50]), and defined on $\mathbb{R}$ as follows.

$$VMO(\mathbb{R}) := \left\{ f \in L^1_{loc}(\mathbb{R});\ \lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}} \fint_{x-\varepsilon}^{x+\varepsilon}\fint_{x-\varepsilon}^{x+\varepsilon} |f(y)-f(z)| \, \mathrm{d}y\mathrm{d}z = 0 \right\}.$$

We adopt the same notation as in Lemma 35. One may prove (see *e.g.* Brezis and Nirenberg [16, formula (7)]) that if $f : \mathbb{R} \to \mathbb{S}^1$ satisfies $f \in VMO$, then $u := F_f$ satisfies

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}} ||u(x,\varepsilon)| - 1| = 0, \tag{6.28}$$

or, equivalently, that $|u(\cdot,\varepsilon)| \to 1$ uniformly as $\varepsilon \to 0$. Assume in addition that $f(x) \equiv 1$ for $|x| \geq R = R_f$. Repeating the proof of Lemma 35, we obtain that there exists some ε_0 such that

$$\frac{1}{\pi}\int_{\mathbb{R}^2_{+,*}} Jw = \deg w(\cdot,\varepsilon),\ \forall\, 0 < \varepsilon \leq \varepsilon_0. \tag{6.29}$$

At this stage, we have the non-trivial information that the left-hand side of (6.29) is an integer. But we cannot continue and claim that $\frac{1}{\pi}\int_{\mathbb{R}^2_{+,*}} Jw = \deg f$. Indeed, we have not defined $\deg f$! However, we may take this equality as the *definition* of $\deg f$. It is not difficult to see that this definition coincides with the one in [16]. To summarize, maps in $VMO(\mathbb{R};\mathbb{S}^1)$ satisfying $f(x) \equiv 1$ for $|x| \geq R = R_f$ have a well-defined degree. This degree can be calculated via the integral formula

$$\deg f = \frac{1}{\pi}\int_{\mathbb{R}^2_{+,*}} Jw. \tag{6.30}$$

6.3. **Winding number (II)**

We return here to the Sobolev context and investigate the existence of $\deg f$ when $f : \mathbb{R} \to \mathbb{S}^1$ has some Sobolev regularity. More specifically, we assume that

$$f(x) \equiv 1 \text{ for } |x| \geq R = R_f \text{ and } f - 1 \in W^{s,p}(\mathbb{R}). \tag{6.31}$$

When $sp > 1$ or $s = p = 1$, f is continuous and thus it has a degree. When $sp < 1$, there is no reasonable definition of degree [39]. The "critical" case is the one of the spaces $W^{1/p,p}$, with $1 < p < \infty$. It turns out that these spaces are embedded into *VMO* (see *e.g.* [16, § 1.2, Example 2]), and thus, as explained in Remark 13, we may define $\deg f$, which is given by formula (6.30). We address here the question of an *estimate* for $\deg f$. The answer is provided by the following result, originally established in [6] with a slightly different argument.

LEMMA 36. *Let $1 < p < \infty$. Let $f : \mathbb{R} \to \mathbb{S}^1$ satisfy $f(x) \equiv 1$ for $|x| \geq R = R_f$ and $f - 1 \in W^{1/p,p}$. Then*

$$|\deg f| \leq C_p \, |f|^p_{W^{1/p,p}}. \tag{6.32}$$

Proof. Let, for $x \in \mathbb{R}$,

$$d(x) := \inf\{\varepsilon > 0; \, |u(x,\varepsilon)| \leq 1/2\}. \tag{6.33}$$

By (6.28), we know that $d(x) > 0$. Consider the open set

$$U := \{(x,\varepsilon); \, x \in \mathbb{R}, \, 0 < \varepsilon < d(x)\}. \tag{6.34}$$

By the proof of (6.23), we have

$$Jw = 0 \text{ in } U. \tag{6.35}$$

On the other hand, Lemma 26 implies that

$$|Jw(x,\varepsilon)| \leq \frac{C}{\varepsilon^2}, \; \forall f : \mathbb{R} \to \mathbb{S}^1, \forall x \in \mathbb{R}, \forall \varepsilon > 0. \tag{6.36}$$

Combining (6.30), (6.35) and (6.36), we obtain

$$|\deg f| \leq \frac{1}{\pi} \int_{\mathbb{R}^2_{+,*}} |Jw| \lesssim \int_{\mathbb{R}} \int_{d(x)}^{\infty} \frac{1}{\varepsilon^2} \, \mathrm{d}\varepsilon \mathrm{d}x \approx \int_{\mathbb{R}} \frac{1}{d(x)} \, \mathrm{d}x. \tag{6.37}$$

We complete the proof of Lemma 36 combining (6.37) with Lemma 37 below (with $s = 1/p$). □

LEMMA 37. *Let $0 < s < 1$ and let $f : \mathbb{R} \to \mathbb{S}^1$ be such that $f - 1 \in W^{s,p}$. Let $d(x)$ be as in (6.33). Then*

$$\int_{\mathbb{R}} \frac{1}{[d(x)]^{sp}} \, \mathrm{d}x \lesssim |f|^p_{W^{s,p}}. \tag{6.38}$$

Proof. In view of (5.3) and of Theorem 8, for a.e. $x \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = f(x) \tag{6.39}$$

and

$$\int_0^\infty \varepsilon^{(1-s)p-1} \, |\nabla u(x, \varepsilon)|^p \, \mathrm{d}\varepsilon < \infty. \tag{6.40}$$

Let x satisfy (6.39)–(6.40). Since (6.39) holds, we have either $d(x) = \infty$, or $d(x) < \infty$ and then $|u(x, d(x))| = 1/2$. Assume that $d(x) < \infty$. Using Hölder's inequality when $p > 1$ (and a trivial argument when $p = 1$) we find that

$$\begin{aligned}(1/2)^p &= ||u(x, d(x))| - |f(x)||^p \leq |u(x, d(x)) - f(x)|^p \\ &\leq \left(\int_0^{d(x)} \left| \frac{\partial}{\partial \varepsilon} u(x, \varepsilon) \right| \, \mathrm{d}\varepsilon \right)^p \\ &\leq C_{s,p} \, [d(x)]^{sp} \int_0^{d(x)} \varepsilon^{(1-s)p-1} \, |\nabla u(x, \varepsilon)|^p \, \mathrm{d}\varepsilon \\ &\leq C_{s,p} \, [d(x)]^{sp} \int_0^\infty \varepsilon^{(1-s)p-1} \, |\nabla u(x, \varepsilon)|^p \, \mathrm{d}\varepsilon.\end{aligned} \tag{6.41}$$

Consequently, we have

$$\frac{1}{[d(x)]^{sp}} \leq C_{s,p} \int_0^\infty \varepsilon^{(1-s)p-1} \, |\nabla u(x, \varepsilon)|^p \, \mathrm{d}\varepsilon, \text{ for a.e. } x \in \mathbb{R}. \tag{6.42}$$

We obtain (6.38) by combining (6.42) with Theorem 8. □

6.4. **Factorization**

We first summarize what we have achieved in Sections 6.2 and 6.3. We have an integral formula for $\deg f$ when f is continuous, or merely VMO. If, in addition, f has some Sobolev regularity, then we also have an estimate of $\deg f$. In terms of pullback of forms, we gave a meaning to $\langle f^\sharp \omega, 1 \rangle$ for $f \in VMO$ and we also have an estimate of this quantity when $f \in W^{1/p,p}$.

It is much more difficult to give a robust meaning to $f^\sharp \omega$ when $f : \mathbb{R}^n \to \mathbb{S}^1$. It is beyond the scope of this presentation to explain in detail how can this be achieved (and we refer to [13, Chapter 8] for the complete proofs). However, we will explain the definition of $f^\sharp \omega$ and the main ingredient used in the definition. Assume first that $f = e^{\imath \varphi}$, with smooth φ. Then (see the proof of (6.4)) we have

$$f^\sharp \omega = \frac{1}{2\pi} d\varphi. \tag{6.43}$$

Similarly, if $f, g : \mathbb{R}^n \to \mathbb{S}^1$ are sufficiently smooth, then

$$(f\, g)^\sharp \omega = f^\sharp \omega + g^\sharp \omega. \tag{6.44}$$

Another easy observation is that

$$f^\sharp \omega = \frac{1}{2\pi} f \wedge df \text{ is well-defined when } \nabla f \in L^1. \tag{6.45}$$

Put together, the three above observations lead to the following reasonable definition. If

$$f = g\, h, \text{ where } g = \mathrm{e}^{\mathrm{i}\varphi} \text{ and } \nabla h \in L^1, \tag{6.46}$$

then we define

$$f^\sharp \omega := \frac{1}{2\pi} \mathrm{d}\varphi + \frac{1}{2\pi} h \wedge \mathrm{d}h. \tag{6.47}$$

It is possible to follow this route and give a robust meaning to $f^\sharp \omega$ when $f - 1 \in W^{s,p}$ with $sp \geq 1$. The main ingredient is the "factorization" theorem, asserting the possibility of decomposing f as in (6.46). More specifically, we have the following result, valid in any dimension [13, Chapter 8].

Theorem 12. *Assume that* $n \geq 1$, $s > 0$ *and* $1 \leq p < \infty$. *Let* $f : \mathbb{R}^n \to \mathbb{S}^1$ *be such that* $f - 1 \in W^{s,p}$ *and*

$$f(x) \equiv 1 \text{ for } |x| \geq R = R_f. \tag{6.48}$$

Then we may write $f = \mathrm{e}^{\mathrm{i}\varphi} h$, *where* $\varphi \in W^{s,p}(\mathbb{R}^n; \mathbb{R})$ *and* $h - 1 \in W^{sp,1}(\mathbb{R}^n; \mathbb{R}^2)$.

In particular, when $sp \geq 1$, Theorem 12 allows to define

$$f \wedge df = f^\sharp \omega := \frac{1}{2\pi} \mathrm{d}\varphi + \frac{1}{2\pi} h \wedge \mathrm{d}h, \tag{6.49}$$

the result being a 1-form with coefficients in $W^{s-1,p} + W^{sp-1,1}$.

The proof of Theorem 12 is too long to be given here. Let us simply mention that it relies heavily on the trace theory of weighed Sobolev spaces and on cancellation phenomena. It is simpler when $sp < 1$, and in this specific case we refer the reader to [42].

APPENDIX. SOME DETAILED CALCULATIONS

Proof of Lemma 16. By scaling, it suffices to prove the lemma when $R = 1$.

We start with a useful preliminary observation. By the mean value theorem, there exists some $y \in B_1(0)$ such that

$$\int_{B_1(0)} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \,\mathrm{d}x \lesssim |f|^p_{W^{s,p}(B_1(0))}.$$

For any such y, we have

$$\|f - f(y)\|^p_{L^p(B_1(0))} \lesssim \int_{B_1(0)} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \lesssim |f|^p_{W^{s,p}(B_1(0))}. \tag{7.1}$$

We now recall the following elementary inequality. Let μ be a measure on the set A such that $0 < \mu(A) < \infty$. Then

$$\left\| f - \fint_A f \, \mathrm{d}\mu \right\|_{L^p(A)} \le 2\|f - c\|_{L^p(A)}, \ \forall f \in L^p(A), \forall c \in \mathbb{R}. \tag{7.2}$$

Using (7.1), (7.2) (with $A := B_1(0)$ and μ the Lebesgue measure), and the assumption $\int_{B_1(0)} f = 0$, $\forall f \in Y_1$, we obtain

$$\|f\|_{L^p(B_1(0))} \lesssim |f|_{W^{s,p}(B_1(0))}, \ \forall f \in Y_1. \tag{7.3}$$

For $x \in \mathbb{R}^n$ such that $|x| > 1$, let $x^* := x/|x|^2 \in B_1(0)$. Fix some $\psi \in C_c^\infty(B_2(0))$ such that $\psi \equiv 1$ in $\overline{B}_1(0)$. Let $f \in Y_1$. We set

$$f^*(x) := \begin{cases} f(x), & \text{if } |x| < 1 \\ f(x^*), & \text{if } |x| > 1 \end{cases} \text{ and } P_1 f := \psi f^*. \tag{7.4}$$

Noting that

$$P_1 f(x) - P_1 f(y) = 0 \text{ if } |x| > 2 \text{ and } |y| > 2$$

and that

$$|P_1 f(x) - P_1 f(y)| \le |f^*(x)| \, |\psi(x) - \psi(y)| + |\psi(y)| \, |f^*(x) - f^*(y)|, \tag{7.5}$$

we find that

$$|P_1 f|^p_{W^{s,p}} \lesssim |f^*|^p_{W^{s,p}(B_2(0))} + \int_{B_2(0)} \int_{\mathbb{R}^n} |f^*(x)|^p \frac{|\psi(x) - \psi(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y\mathrm{d}x. \tag{7.6}$$

On the other hand, since $\psi \in C_c^\infty(\mathbb{R}^n)$, it is easy to see that

$$\int_{\mathbb{R}^n} \frac{|\psi(x) - \psi(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \le C = C_\psi, \ \forall x \in \mathbb{R}^n. \tag{7.7}$$

Combining (7.6) and (7.7), we find that

$$|P_1 f|^p_{W^{s,p}} \lesssim |f^*|^p_{W^{s,p}(B_2(0))} + \|f^*\|^p_{L^p(B_2(0))}. \tag{7.8}$$

Next, using the definition of f^* and performing in $B_2(0) \backslash B_1(0)$ the change of variable $x \mapsto x^*$, we find that

$$\|f^*\|_{L^p(B_2(0))} \lesssim \|f\|_{L^p(B_1(0))}. \tag{7.9}$$

By (7.8), (7.9) and (7.3), we obtain

$$|P_1 f|^p_{W^{s,p}} \lesssim |f^*|^p_{W^{s,p}(B_2(0))} + |f|^p_{W^{s,p}(B_1(0))}, \tag{7.10}$$

and thus the conclusion of the lemma amounts to

$$|f^*|_{W^{s,p}(B_2(0))} \lesssim |f|_{W^{s,p}(B_1(0))}. \tag{7.11}$$

In turn, (7.11) is obtained as follows. We have

$$\begin{aligned} |f^*|^p_{W^{s,p}(B_2(0))} \lesssim & |f|^p_{W^{s,p}(B_1(0))} \\ & + \int_{B_2(0)\setminus B_1(0)} \int_{B_2(0)\setminus B_1(0)} \frac{|f(x^*)-f(y^*)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y\mathrm{d}x \\ & + \int_{B_1(0)} \int_{B_2(0)\setminus B_1(0)} \frac{|f(x)-f(y^*)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y\mathrm{d}x. \end{aligned} \tag{7.12}$$

Using the change of variable $x^* \mapsto x$ and $y^* \mapsto y$ and noting that

$$|x^*-y^*| \approx |x-y|, \ \forall\, x, y \in B_2(0) \setminus B_1(0),$$

and

$$|x-y^*| \approx |x-y|, \ \forall\, x \in B_1(0), \forall\, y \in B_2(0) \setminus B_1(0),$$

we obtain (7.11) from (7.12).

The proof of Lemma 16 is complete. $\square$

Proof of Lemma 17. We will use repeatedly the following straightforward consequences of Hölder's inequality. If $0 < a < \infty$, then for some $C = C_{a,p,n} < \infty$ we have

$$\int_{\mathbb{R}^n\setminus B_{aR}(0)} \frac{|f(x)|^p}{|x|^{n+sp}}\, \mathrm{d}x \le C\, R^{-n}\, \|f\|^p_{L^q(\mathbb{R}^n\setminus B_{aR}(0))}, \ \forall\, f : \mathbb{R}^n \to \mathbb{R}. \tag{7.13}$$

Similarly, we have, with $C = C_{a,p,n} < \infty$,

$$\int_{B_{aR}(0)} |f|^p \le C\, R^{sp}\, \|f\|^p_{L^q(B_{aR}(0))}. \tag{7.14}$$

Using the fact that $\psi \in C_c^\infty(\mathbb{R}^n)$, we find that the function $\xi : \mathbb{R}^n \to \mathbb{R}$,

$$\xi(x) := \int_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y,$$

satisfies

$$\xi(x) \lesssim \begin{cases} 1, & \text{if } |x| < 1 \\ |x|^{-(n+sp)}, & \text{if } |x| \ge 1 \end{cases}. \tag{7.15}$$

Set $\eta^\varepsilon := 1 - \psi^\varepsilon$. Combining (7.15) with the fact that

$$\int_{\mathbb{R}^n} \frac{|\psi^\varepsilon(x)-\psi^\varepsilon(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y = \int_{\mathbb{R}^n} \frac{|\eta^\varepsilon(x)-\eta^\varepsilon(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y = \varepsilon^{sp}\, \xi(\varepsilon x),$$

we obtain

$$
\int_{\mathbb{R}^n} \frac{|\psi^\varepsilon(x)-\psi^\varepsilon(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y = \int_{\mathbb{R}^n} \frac{|\eta^\varepsilon(x)-\eta^\varepsilon(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y \lesssim \begin{cases} \varepsilon^{sp}, & \text{if } |x|<1/\varepsilon \\ \varepsilon^{-n}|x|^{-(n+sp)}, & \text{if } |x|\geq 1/\varepsilon \end{cases}. \tag{7.16}
$$

For the convenience of the reader, we split the remaining part of the proof into three steps. Clearly, these steps lead to the conclusion of the lemma.

Step 1. We have

$$
|\psi^\varepsilon f - f|_{W^{s,p}} \to 0 \text{ as } \varepsilon \to 0. \tag{7.17}
$$

Indeed, noting that

$$
\psi^\varepsilon(x)\, f(x) - f(x) = \eta^\varepsilon(x)\, f(x) = 0 \text{ if } |x| \leq 1/\varepsilon,
$$

we find that

$$
\begin{aligned} |\psi^\varepsilon f - f|^p_{W^{s,p}} \lesssim & \int_{|x|>1/\varepsilon}\int_{\mathbb{R}^n} |f(x)|^p \frac{|\eta^\varepsilon(x)-\eta^\varepsilon(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\mathrm{d}x \\ & + \int_{|x|>1/\varepsilon}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\mathrm{d}x. \end{aligned} \tag{7.18}
$$

Using successively (7.16) and (7.13), we find that

$$
\begin{aligned} |\psi^\varepsilon f - f|^p_{W^{s,p}} \lesssim & \|f\|^p_{L^q(\mathbb{R}^n\setminus B_{1/\varepsilon}(0))} \\ & + \int_{|x|>1/\varepsilon}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\mathrm{d}x \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{7.19}
$$

The first step is complete.

Step 2. We have

$$
|f_\varepsilon - f|_{W^{s,p}} \to 0 \text{ as } \varepsilon \to 0. \tag{7.20}
$$

With no loss of generality, we assume that $\operatorname{supp}\rho \subset B_1(0)$. Set $H_\varepsilon := f_\varepsilon - f$. Then (7.20) amounts to

$$
\begin{aligned} I_\varepsilon := & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|H_\varepsilon(x)-H_\varepsilon(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y \\ = & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|H_\varepsilon(x+h)-H_\varepsilon(x)|^p}{|h|^{n+sp}}\,\mathrm{d}x\mathrm{d}h \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{7.21}
$$

In order to estimate I_ε, we start by noting that

$$
|H_\varepsilon(x+h) - H_\varepsilon(x)| \lesssim \int_{|y|<\varepsilon} \frac{|f(x+h-y)-f(x+h)-f(x-y)+f(x)|}{\varepsilon^n}\,\mathrm{d}y.
$$

Using this observation, we estimate the integrand in (7.21) as follows.

1. If $|h| < \varepsilon$, we use

$$|H_\varepsilon(x+h) - H_\varepsilon(x)| \lesssim \int_{|y|<\varepsilon} \frac{|f(x+h-y) - f(x-y)|}{\varepsilon^n}\,\mathrm{d}y + \int_{|y|<\varepsilon} \frac{|f(x) - f(x+h)|}{\varepsilon^n}\,\mathrm{d}y.$$

2. If $|h| \geq \varepsilon$, we write

$$|H_\varepsilon(x+h) - H_\varepsilon(x)| \lesssim \int_{|y|<\varepsilon} \frac{|f(x+h-y) - f(x+h)|}{\varepsilon^n}\,\mathrm{d}y + \int_{|y|<\varepsilon} \frac{|f(x) - f(x-y)|}{\varepsilon^n}\,\mathrm{d}y.$$

Thus

$$I_\varepsilon \lesssim \varepsilon^{-np}\,(K_1 + K_2 + K_3 + K_4) = \varepsilon^{-np}\,(K_{1,\varepsilon} + K_{2,\varepsilon} + K_{3,ve} + K_{4,\varepsilon}),$$

where

$$K_1 := \int_{\mathbb{R}^n}\int_{|h|<\varepsilon}\left(\int_{|y|<\varepsilon} |f(x+h-y) - f(x-y)|\,\mathrm{d}y\right)^p |h|^{-(n+sp)}\,\mathrm{d}h\mathrm{d}x,$$

$$K_2 := \int_{\mathbb{R}^n}\int_{|h|<\varepsilon}\left(\int_{|y|<\varepsilon} |f(x+h) - f(x)|\,\mathrm{d}y\right)^p |h|^{-(n+sp)}\,\mathrm{d}h\mathrm{d}x,$$

$$K_3 := \int_{\mathbb{R}^n}\int_{|h|\geq\varepsilon}\left(\int_{|y|<\varepsilon} |f(x+h-y) - f(x+h)|\,\mathrm{d}y\right)^p |h|^{-(n+sp)}\,\mathrm{d}h\mathrm{d}x,$$

$$K_4 := \int_{\mathbb{R}^n}\int_{|h|\geq\varepsilon}\left(\int_{|y|<\varepsilon} |f(x-y) - f(x)|\,\mathrm{d}y\right)^p |h|^{-(n+sp)}\,\mathrm{d}h\mathrm{d}x.$$

We will prove that $\varepsilon^{-np}\,K_j \to 0$ as $\varepsilon \to 0$, $j = 1, \ldots, 4$. The only ingredient we use in the proof is the straightforward fact that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{|y|<\varepsilon} \frac{|f(x+y) - f(x)|^p}{|y|^{n+sp}}\,\mathrm{d}y\mathrm{d}x = 0. \tag{7.22}$$

We start with K_2. Noting that

$$\left(\int_{|y|<\varepsilon} |f(x+h) - f(x)|\,\mathrm{d}y\right)^p = C\,\varepsilon^{np}\,|f(x+h) - f(x)|^p,$$

we find that

$$\varepsilon^{-np}\,K_2 = C\int_{\mathbb{R}^n}\int_{|h|<\varepsilon} \frac{|f(x+h) - f(x)|^p}{|h|^{n+sp}}\,\mathrm{d}h \to 0 \text{ as } \varepsilon \to 0.$$

For K_1, Hölder's inequality implies that

$$\left(\int_{|y|<\varepsilon} |f(x+h-y)-f(x-y)|\,\mathrm{d}y\right)^p \lesssim \varepsilon^{n(p-1)} \int_{|y|<\varepsilon} |f(x+h-y)-f(x-y)|^p\,\mathrm{d}y, \tag{7.23}$$

and thus

$$\varepsilon^{-np} K_1 \lesssim \int_{\mathbb{R}^n} \int_{|h|<\varepsilon} \int_{|y|<\varepsilon} \frac{|f(x+h-y)-f(x-y)|^p}{\varepsilon^n\,|h|^{n+sp}}\,\mathrm{d}y\mathrm{d}h\mathrm{d}x.$$

For fixed y and h, the change of variable $x-y=z$ leads to

$$\varepsilon^{-np} K_1 \lesssim \int_{\mathbb{R}^n} \int_{|h|<\varepsilon} \frac{|f(z+h)-f(z)|^p}{|h|^{n+sp}}\,\mathrm{d}h\mathrm{d}z \to 0 \text{ as } \varepsilon \to 0.$$

We next estimate K_3; the calculation for K_4 is similar and will be omitted. Inequality (7.23) implies that

$$\varepsilon^{-np} K_3 \lesssim \varepsilon^{-n} \int_{\mathbb{R}^n} \int_{|h|\geq\varepsilon} \int_{|y|<\varepsilon} \frac{|f(x+h-y)-f(x+h)|^p}{|h|^{n+sp}}\,\mathrm{d}y\mathrm{d}h\mathrm{d}x.$$

In the above integral, we fix y and h and make the change of variable $x+h=z$. Next we integrate in h and find that

$$\begin{aligned}\varepsilon^{-np} K_3 &\lesssim \int_{\mathbb{R}^n} \int_{|y|<\varepsilon} \frac{|f(z-y)-f(z)|^p}{\varepsilon^{n+sp}}\,\mathrm{d}y\mathrm{d}z \\ &\lesssim \int_{\mathbb{R}^n} \int_{|y|<\varepsilon} \frac{|f(z-y)-f(z)|^p}{|y|^{n+sp}}\,\mathrm{d}y\mathrm{d}z \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

The second step is complete.

Step 3. We have

$$|\psi^\varepsilon (f_\varepsilon - f)|_{W^{s,p}} \to 0 \text{ as } \varepsilon \to 0. \tag{7.24}$$

Set $L_\varepsilon := f_\varepsilon - f$, so that

$$|L_\varepsilon|_{W^{s,p}} \to 0 \text{ and } \|L_\varepsilon\|_{L^q} \to 0 \text{ as } \varepsilon \to 0. \tag{7.25}$$

In order to prove (7.24), we start from the straightforward estimate

$$|\psi^\varepsilon L_\varepsilon|^p_{W^{s,p}} \lesssim |L_\varepsilon|^p_{W^{s,p}} + \int_{\mathbb{R}^n} |L_\varepsilon(x)|^p \int_{\mathbb{R}^n} \frac{|\psi^\varepsilon(x)-\psi^\varepsilon(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\mathrm{d}x. \tag{7.26}$$

Combining (7.26) with (7.13), (7.14) and (7.16), we obtain (7.24).

The third step is complete. □

Proof of Lemma 18. Let $f:\mathbb{R}^n \to \mathbb{R}$. Let $x,y \in \mathbb{R}^n$. Set

$$z^0 := x,\ z^1 = (y_1, x_2, \ldots, x_n),\ z^2 := (y_1, y_2, x_3 \ldots, x_n), \ldots, z^n := y.$$

Then

$$|f(x)-f(y)|^p \lesssim \sum_{j=1}^{n} |f(z^j)-f(z^{j-1})|^p. \tag{7.27}$$

Dividing (7.27) by $|x-y|^{n+sp}$ and integrating over x and y, we find that

$$|f|_{W^{s,p}}^p \lesssim \sum_{j=1}^{n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(z^j)-f(z^{j-1})|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y. \tag{7.28}$$

Next, we note that

$$\int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^{n+sp}}\,\mathrm{d}x_1\ldots\mathrm{d}x_{j-1}\mathrm{d}y_{j+1}\ldots\mathrm{d}y_n = \frac{C}{|x_j-y_j|^{1+sp}}, \tag{7.29}$$

for some finite constant C. Inserting (7.29) into (7.28), we find that "$\lesssim$" holds in (3.38).

For the reverse inequality, we fix some j, say $j=1$. For $x_1 \neq y_1 \in \mathbb{R}$, set $t := (x_1+y_1)/2$, $X := (t, x_2, \ldots, x_n)$ and $r := |x_1-y_1|/4$. We start from the inequality

$$\begin{aligned}|f(x_1,x_2,\ldots,x_n)-f(y_1,x_2,\ldots,x_n)|^p \lesssim & |f(x_1,x_2,\ldots,x_n)-f(z)|^p \\ & + |f(y_1,x_2,\ldots,x_n)-f(z)|^p.\end{aligned} \tag{7.30}$$

We divide (7.30) by $|x_1-y_1|^{n+1+sp}$ and integrate over $x \in \mathbb{R}^n$, $y_1 \in \mathbb{R}$ and $z \in B_r(X)$. We find that

$$\int_{\mathbb{R}^{n-1}} |f(\cdot, x_2,\ldots,x_n)|_{W^{s,p}(\mathbb{R})}^p\,\mathrm{d}\widehat{x}_1 \lesssim \int_{\mathbb{R}^n} |f(x)-f(z)|^p\,F(x,z)\,\mathrm{d}x\mathrm{d}z, \tag{7.31}$$

with

$$F(x,z) := \int_{z\in B_r(X)} \frac{1}{|x_1-y_1|^{n+1+sp}}\,\mathrm{d}y_1.$$

Using the fact that, whenever $z \in B_r(X)$, we have $|x-z| \leq (3/4)\,|x_1-y_1|$, we find that $F(x,z) \lesssim |x-z|^{-(n+sp)}$. Inserting this into (7.31), we find that "$\gtrsim$" holds in (3.38) for $j=1$. The calculation for other values of j is similar and will be omitted. ☐

LEMMA 38. *Let* $0 < s < 1$ *and* $1 \leq p < \infty$. *Let*

$$Z_1 := \left\{ f:\mathbb{R}\to\mathbb{R};\ f\in L^p,\ |f|_{W^{s,p}}^p := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\Delta_h^1 f(x)|^p}{|h|^{1+sp}}\,\mathrm{d}x\mathrm{d}h < \infty \right\}, \tag{7.32}$$

$$Z_2 := \left\{ f:\mathbb{R}\to\mathbb{R};\ f\in L^p,\ \langle f\rangle_{W^{s,p}}^p := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\Delta_h^2 f(x)|^p}{|h|^{1+sp}}\,\mathrm{d}x\mathrm{d}h < \infty \right\}, \tag{7.33}$$

equipped respectively with the norms

$$\|f\|_{Z_1}^p := \|f\|_{L^p}^p + |f|_{W^{s,p}}^p,$$
$$\|f\|_{Z_2}^p := \|f\|_{L^p}^p + \langle f\rangle_{W^{s,p}}^p.$$

Then $Z_1 = Z_2$, *with equivalence of norms.*

Proof. As explained in the proof of Lemma 19, it suffices to establish the semi-norm equivalence $|f_\varepsilon|_{W^{s,p}} \approx \langle f_\varepsilon\rangle_{W^{s,p}}$, which in turn amounts to establishing the semi-norm equivalence

$$|g|_{W^{s,p}} \approx \langle g\rangle_{W^{s,p}},\ \forall\, g \in Z_1 \cap Z_2. \tag{7.34}$$

The identity $\Delta_h^2 g(x) = \Delta_h^1 g(x+h) - \Delta_h^1 g(x)$ leads to $\|\Delta_h^2 g\|_{L^p} \le 2\|\Delta_h^1 g\|_{L^p}$, which in turn implies the inequality $\|g\|_{Z_2} \le 2\|g\|_{Z_1}$ and the embedding $Z_1 \hookrightarrow Z_2$.

In order to obtain the opposite inequality

$$|g|_{W^{s,p}} \lesssim \langle g\rangle_{W^{s,p}},\ \forall\, g \in Z_1 \cap Z_2, \tag{7.35}$$

we let $k \ge 2$ be a large integer to be fixed later. We have the identity

$$\sum_{j=1}^{k-1} j\, \Delta_\varepsilon^2 g(x + (j-1)\varepsilon) = k\, \Delta_\varepsilon^1 g(x + (k-1)\varepsilon) - \Delta_{(k-1)\varepsilon}^1 g(x),$$

and thus

$$|\Delta_\varepsilon^1 g(x + (k-1)\varepsilon)| \le \frac{1}{k}|\Delta_{(k-1)\varepsilon}^1 g(x)| + \frac{1}{k}\sum_{j=1}^{k-1} j\, |\Delta_\varepsilon^2 g(x + (j-1)\varepsilon)|. \tag{7.36}$$

If we raise (7.36) to the pth power, divide by $|\varepsilon|^{1+sp}$, integrate over x and ε and perform in the first right-hand side integral the change of variable $h := (k-1)\varepsilon$, we find that

$$|g|_{W^{s,p}}^p \le C_{s,p}\, k^{-(1-s)p}\, |g|_{W^{s,p}}^p + C_{s,p,k}\langle g\rangle_{W^{s,p}}^p. \tag{7.37}$$

Let k satisfy $C_{s,p}\, k^{-(1-s)p} < 1/2$. If we apply (7.37) with such k and use the fact that $|g|_{W^{s,p}} < \infty$, we obtain (7.35). □

Proof of Lemma 21. The argument is similar to the one in Step 2 in the proof of Lemma 17. We take advantage of the compact notation for variations and present a shorter argument. With no loss of generality, we may assume that $\operatorname{supp} \rho \subset [-1, 1]$.

Let $h, \tau, x \in \mathbb{R}$. Then we have the identity

$$\begin{aligned} &\Delta_h^2 f(x+\tau) + \Delta_h^2 f(x-\tau) - 2\Delta_h^2 f(x) \\ &= \Delta_\tau^2 f(x - \tau + 2h) + \Delta_\tau^2 f(x-\tau) - 2\Delta_\tau^2 f(x - \tau + h). \end{aligned} \tag{7.38}$$

Multiplying (7.38) by $\rho_\varepsilon(\tau)$, integrating over $\sup \rho_\varepsilon \subset [-\varepsilon, \varepsilon]$ and taking into account the fact that ρ_ε is even, we find that

$$\begin{aligned} &2\Delta_h^2(f_\varepsilon - f) \\ &= \int [\Delta_\tau^2 f(\cdot - \tau + 2h) + \Delta_\tau^2 f(\cdot - \tau) - 2\Delta_\tau^2 f(\cdot - \tau + h)]\, \rho_\varepsilon(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{7.39}$$

Using the fact that $|\rho_\varepsilon| \lesssim 1/\varepsilon$, we obtain from (7.39) that

$$\|\Delta_h^2(f_\varepsilon - f)\|_{L^p} \lesssim \frac{1}{\varepsilon}\int_{|\tau|\le\varepsilon} \|\Delta_\tau^2 f\|_{L^p}. \tag{7.40}$$

On the other hand, we clearly have

$$\|\Delta_h^2 f_\varepsilon\| = \left\|(\Delta_h^2 f) * \rho_\varepsilon\right\|_{L^p} \le \left\|\Delta_h^2 f\right\|_{L^p} \|\rho_\varepsilon\|_{L^p} = \left\|\Delta_h^2 f\right\|_{L^p},$$

and thus

$$\|\Delta_h^2(f_\varepsilon - f)\| \le 2\|\Delta_h^2 f\|_{L^p}. \tag{7.41}$$

Using (7.40) when $|h| > \varepsilon$ and (7.41) when $|h| \le \varepsilon$ and Hölder's inequality, we find that

$$\begin{aligned}
|f_\varepsilon - f|_{W^{s,p}}^p \lesssim & \frac{1}{\varepsilon^p}\int_{|h|>\varepsilon}\left(\int_{|\tau|\le\varepsilon} \|\Delta_\tau^2 f\|_{L^p}\,\mathrm{d}\tau\right)^p |h|^{-1-sp}\,\mathrm{d}h \\
& + \int_{|h|\le\varepsilon} \frac{\|\Delta_h^2 f\|_{L^p}^p}{|h|^{1+sp}}\,\mathrm{d}h \\
\approx & \varepsilon^{-p-sp}\left(\int_{|\tau|\le\varepsilon} \|\Delta_\tau^2 f\|_{L^p}\,\mathrm{d}\tau\right)^p + \int_{|h|\le\varepsilon} \frac{\|\Delta_h^2 f\|_{L^p}^p}{|h|^{1+sp}}\,\mathrm{d}h \\
\lesssim & \varepsilon^{-1-sp}\int_{|\tau|\le\varepsilon} \|\Delta_\tau^2 f\|_{L^p}^p\,\mathrm{d}\tau + \int_{|h|\le\varepsilon} \frac{\|\Delta_h^2 f\|_{L^p}^p}{|h|^{1+sp}}\,\mathrm{d}h \\
\lesssim & \int_{|h|\le\varepsilon} \frac{\|\Delta_h^2 f\|_{L^p}^p}{|h|^{1+sp}}\,\mathrm{d}h \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

The proof of Lemma 21 is complete. □

Proof of Lemma 22. As explained in the proof of Lemma 19, it suffices to establish, for smooth $f : \mathbb{R}^n \to \mathbb{R}$ the following semi-norm equivalences:

$$\|D^m f\|_{L^p}^p \approx \int_{\mathbb{S}^{n-1}}\int_{\omega^\perp} \left\|(f_\omega^x)^{(m)}\right\|_{L^p}^p \mathrm{d}x\mathrm{d}\omega,\ \forall\, m \in \mathbb{N}, \tag{7.42}$$

$$|f|_{W^{s,p}}^p \approx \int_{\mathbb{S}^{n-1}}\int_{\omega^\perp} |f_\omega^x|_{W^{s,p}}^p\,\mathrm{d}x\mathrm{d}\omega,\ \forall\, s > 0 \text{ non-integer}, \forall\, M > s. \tag{7.43}$$

Estimate (7.43) is actually an identity, up to a multiplicative constant.

Indeed, we have

$$
\begin{aligned}
|f|^p_{W^{s,p}} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Delta_h^M f(y)|^p}{|h|^{n+sp}}\,\mathrm{d}y\mathrm{d}h\\
&=\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \frac{|\Delta_{r\,\omega}^M f(y)|^p}{|r|^{1+sp}}\,\mathrm{d}y\mathrm{d}r\mathrm{d}s_\omega\\
&=\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\Delta_{r\,\omega}^M f(x+t\,\omega)|^p}{|r|^{1+sp}}\,\mathrm{d}t\mathrm{d}r\mathrm{d}x\mathrm{d}s_\omega\\
&=\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\Delta_{r}^M f_\omega^x(t)|^p}{|r|^{1+sp}}\,\mathrm{d}t\mathrm{d}r\mathrm{d}x\mathrm{d}s_\omega\\
&=\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp} |f_\omega^x|^p_{W^{s,p}}\,\mathrm{d}x\mathrm{d}s_\omega,
\end{aligned}
$$

whence (7.43). In the above, we first expressed h in spherical coordinates (with $r\in\mathbb{R}$), next we performed the change of variables $y=x+t\,\omega$, $x\in\omega^\perp$, $t\in\mathbb{R}$, whose Jacobian is 1.

We now turn to the proof of (7.42). We let to the reader the easy case where $m=0$ and we assume that $m\geq 1$. The starting point of the proof is the following observation. If A is a symmetric m-linear form on $\mathbb{R}^n$, and if

$$
A(\underbrace{\omega,\ldots,\omega}_{m\text{ times}})=0,\ \forall\,\omega\in\mathbb{S}^{n-1},
$$

then $A=0$. This is a consequence of the polarization formula for symmetric forms. It follows that

$$
A\mapsto\langle A\rangle_p := \left(\int_{\mathbb{S}^{n-1}}|A(\omega,\ldots,\omega)|^p\,\mathrm{d}s_\omega\right)^{1/p},
$$

is a norm on the space of symmetric m-linear form on $\mathbb{R}^n$.

Applying the above to $A:=D^m f(x)$, $x\in\mathbb{R}^n$, we find that

$$
\begin{aligned}
\|D^m f\|^p_{L^p} &\approx \int_{\mathbb{R}^n}\langle D^m f(y)\rangle_p^p\,\mathrm{d}y\\
&=\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}^n}|D^m f(y)(\omega,\ldots,\omega)|^p\,\mathrm{d}y\mathrm{d}s_\omega\\
&=\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}|D^m f(x+t\,\omega)(\omega,\ldots,\omega)|^p\,\mathrm{d}t\mathrm{d}x\mathrm{d}s_\omega\\
&=\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\left|(f_\omega^x)^{(m)}(t)\right|^p\,\mathrm{d}t\mathrm{d}x\mathrm{d}s_\omega\\
&=\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\left\|(f_\omega^x)^{(m)}\right\|^p_{L^p}\,\mathrm{d}x\mathrm{d}s_\omega,
\end{aligned}
$$

whence (7.42). $\square$

Remark 14. The proof of (7.43) yields the following more general identity. If $g : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}(7.44)\qquad &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{g(x,y)}{|x-y|^{n+\alpha}}\,\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{g(x+t\,\omega, x+\tau\,\omega)}{|t-\tau|^{1+\alpha}}\,\mathrm{d}t\mathrm{d}\tau\mathrm{d}x\mathrm{d}s_\omega.\end{aligned}$$

Proof of Lemma 23 when $n=1$. Let $f \in W^{1,1}_{loc}(\mathbb{R})$. We will prove that

$$(7.45)\qquad |f|' = (\operatorname{sgn} f)\, f' \in L^1_{loc}.$$

Since f is continuous, the conclusion is clear if f does not vanish. We may thus assume that f vanishes at some point, say $f(0) = 0$. Then (7.45) amounts to

$$(7.46)\qquad |f(x)| = \int_0^x (\operatorname{sgn} f(t))\, f'(t)\,\mathrm{d}t,\ \forall\, x \in \mathbb{R}.$$

We prove *e.g.* (7.46) when $x > 0$. Assume first that $f(x) = 0$. Let $U := \{y \in (0,x);\ f(y) \neq 0\}$. We write the open set U as a disjoint union $U = \cup_j I_j$, with each $I_j = (a_j, b_j)$ an open interval. Since f has constant sign on I_j and we have $f(a_j) = f(b_j) = 0$, we find that

$$\int_{I_j} (\operatorname{sgn} f(t))\, f'(t)\,\mathrm{d}t = \pm\int_{I_j} f'(t)\,\mathrm{d}t = \pm(f(b_j) - f(a_j)) = 0,\ \forall\, j.$$

Therefore,

$$\begin{aligned}\int_0^x (\operatorname{sgn} f(t))\, f'(t)\,\mathrm{d}t &= \int_U (\operatorname{sgn} f(t))\, f'(t)\,\mathrm{d}t\\ &= \sum_j \int_{I_j} (\operatorname{sgn} f(t))\, f'(t)\,\mathrm{d}t = 0 = f(x),\end{aligned}$$

as desired.

When $f(x) \neq 0$, set $z := \sup\{y \in [0,x);\ f(y) = 0\}$. Then $\operatorname{sgn} f = \operatorname{sgn} f(x)$ on (z,x) and $f(z) = 0$. By the previous calculation, we have $\int_0^z (\operatorname{sgn} f(t))\, f'(t$ 0. On the other hand, we clearly have

$$\int_z^x (\operatorname{sgn} f(t))\, f'(t)\,\mathrm{d}t = \operatorname{sgn} f(x) \int_z^x f'(t)\,\mathrm{d}t = (\operatorname{sgn} f(x))\, f(x) = |f(x)|,$$

so that (7.46) holds. $\square$

Proof of Lemma 24. Step 1. An identity. We claim that, for $x, h \in \mathbb{R}^n$

and $M>0$ we have, with $r:=|h|$:

$$\begin{aligned}\Delta_h^M f(x) =& \sum_{j=0}^{M}\binom{M}{j}(-1)^j \Delta^M_{re_{n+1}} F(x+jh,0)\\ &+\sum_{j=1}^{M}\binom{M}{j}(-1)^{j+1}\Delta_h^M F(x,jr).\end{aligned} \tag{7.47}$$

In order to prove (7.47), we start from the identity

$$f(x)=\sum_{j=0}^{M}\binom{M}{j}(-1)^j F(x,jr)+\sum_{j=1}^{M}\binom{M}{j}(-1)^{j+1}F(x,jr),\ \forall\, x\in\mathbb{R}^n.$$

As a consequence,

$$\begin{aligned}\Delta_h^M f(x) =& \sum_{k=0}^{M}\binom{M}{k}(-1)^{M-k} f(x+kh)\\ =& \sum_{k=0}^{M}\binom{M}{k}(-1)^{M-k}\sum_{j=0}^{M}\binom{M}{j}(-1)^j F(x+kh,jr)\\ &+\sum_{k=0}^{M}\binom{M}{k}(-1)^{M-k}\sum_{j=1}^{M}\binom{M}{j}(-1)^{j+1} F(x+kh,jr)\\ =& \sum_{k=0}^{M}\binom{M}{k}(-1)^{k}\Delta^M_{re_{n+1}}F(x+kh,0)+\sum_{j=1}^{M}\binom{M}{j}(-1)^{j+1}\Delta_h^M F(x,jr).\end{aligned}$$

(In the second term of the last equality, we have inverted the sums over M and j.) Therefore, (7.47) holds, as claimed.

Step 2. From the identity (7.47) to the estimate (5.13)

In view of (7.47) and of the desired estimate (5.13), it suffices to establish the estimate (7.48) below. ☐

LEMMA 39. *Let $M>0$ be an integer. Let $X:=(x,\varepsilon)\in\mathbb{R}^{n+1}_+$ and let $H=(h,t)\in\mathbb{R}^{n+1}$ be such that $[X,X+MH]\subset\mathbb{R}^{n+1}_+$. Assume that either $h=0$ or $t=0$. Then we have*

$$|\Delta_H^M F(X)|\lesssim |H|^M\int_0^M t^{M-1}|D_M F(X+tH)|\,\mathrm{d}t. \tag{7.48}$$

Proof. Set $G(t):=F(X+tH)$, $t\in[0,M]$. Then clearly

$$\Delta_H^M F(X)=\Delta_1^M G(0) \text{ and } |G^{(M)}(t)|\lesssim r^M|D_M F(X+tH)|.$$

Therefore, it suffices to prove that

$$|\Delta_1^M G(0)|\lesssim\int_0^M t^{M-1}|G^{(M)}(t)|\,\mathrm{d}t. \tag{7.49}$$

In turn, estimate (7.49) is obtained as follows. Let $\chi_1 := \mathbb{1}_{[-1,0]}$ and, for $j \geq 2$, set

$$\chi_j := \underbrace{\chi_1 * \chi_1 * \cdots * \chi_1}_{j \text{ times}}.$$

By a straightforward induction on j, the distributional derivative $\chi_j^{(j-1)}$ is bounded, and $\chi_j(t) = 0$ when $t \geq 0$ or when $t \leq -j$. This leads to the inequality

$$|\chi_j(-t)| \leq C_j t^{j-1}, \ \forall\, j \geq 1, \ \forall\, t \geq 0. \tag{7.50}$$

On the other hand, by a straightforward induction on M, we have

$$\Delta_1^M G(0) = G^{(M)} * \chi_M(0) = \int_0^M G^{(M)}(t)\chi_M(-t)\,\mathrm{d}t. \tag{7.51}$$

We obtain (7.49) from (7.50) and (7.51). □

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