To Philippe G. Ciarlet, with friendship and admiration

PERIODIC HOMOGENIZATION IN TERMS OF DIFFERENTIAL FORMS

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We embed the homogenization theory of second-order elliptic differential equations in a more general framework, where the unknown is a differential form of arbitrary degree. This allows us to unify some scattered facts as pieces of general statements. A central tool is the duality between the homogenization of forms of complementary degrees.

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1. INTRODUCTION

Most of the statements in this introduction are well-known. The reader may refer to classical books [1, 7, 11].

We limit ourselves to the context of uniformly elliptic second-order PDEs in space dimension n, when the underlying differential operator $\operatorname{div} A(x)\nabla$ is self-adjoint, that is $A(x)$ is positive definite (we write $A(x) \in \mathbf{SPD}_n$), uniformly in x. The theory of homogenization is concerned with sequences of such operators, when the tensor A^ϵ depends upon a small parameter ϵ. We assume that it remains elliptic, uniformly as $\epsilon \to 0$. In practice the coefficients do not converge almost everywhere, but only weakly-star in $L^\infty(\Omega)$. A typical example is that of periodic homogenization, where

$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right),$$

A being periodic according to a lattice Γ.

Given $f \in H^{-1}(\Omega)$, the solutions u^ϵ of a Dirichlet boundary-value problem

$$\operatorname{div}(A^\epsilon \nabla u^\epsilon) = f \quad \text{in } \Omega, \qquad u^\epsilon = 0 \quad \text{on } \partial\Omega$$

form a bounded sequence in $H_0^1(\Omega)$. L. Tartar [11] proved that, up to a subsequence, u^ϵ converges weakly in $H_0^1(\Omega)$ towards the solution u of a problem of the same family

$$\mathrm{div}(A_{\mathrm{eff}}\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where A_{eff} depends only upon the sequence A^ϵ. The tensor $A_{\mathrm{eff}}(x)$ is called the *effective* tensor. In periodic homogenization, it does not depend upon x. In general A_{eff} is significantly different from the weak limit of the sequence A^ϵ. This is obvious if $n = 1$ (then $A^\epsilon = a^\epsilon(x)$ is just a scalar function), where one finds

$$\frac{1}{a_{\mathrm{eff}}} = w * \lim_{\epsilon\to 0} \frac{1}{a^\epsilon}\,.$$

For higher dimensions, the situation is much more complex and we know a closed formula of effective tensors in only a very few cases.

For the sake of simplicity, we restrict to the periodic case. If Γ is a lattice of $\mathbb{R}^n$, functions h that are Γ-periodic admit an average, denoted

$$⨍_{\mathbb{R}^n/\Gamma} h(x)\,\mathrm{d}x.$$

We consider an equation

$$\mathrm{div}(A(x)\nabla u) = f, \tag{1}$$

where $x \mapsto A(x)$ is symmetric, measurable, uniformly elliptic in the sense that

$$\alpha I_n \le A(x) \le \beta I_n, \qquad \text{p.p. } x \in \mathbb{R}^n,$$

for some constants $0 < \alpha \le \beta < \infty$, and is Γ-periodic. We often call A a tensor. Our domain is the torus $\mathbb{R}^n/\Gamma$. The *effective tensor* $A_{\mathrm{eff}} \in \mathbf{SPD}_n$ associated with the sequence $A^\epsilon = A(\frac{\cdot}{\epsilon})$ can be calculated by the following procedure. Given a vector $p \in \mathbb{R}^n$, the elliptic equation

$$\mathrm{div}(A(x)(p + \nabla u)) = 0, \qquad x \in \mathbb{R}^n$$

admits a unique Γ-periodic solution u^p, up to an additive constant. Then

$$A_{\mathrm{eff}}p := ⨍_{\mathbb{R}^n/\Gamma} A(x)(p + \nabla u^p)\,\mathrm{d}x.$$

An integration by parts shows that

$$p^T A_{\mathrm{eff}}p = ⨍_{\mathbb{R}^n/\Gamma} (p + \nabla u^p)^T A(x)(p + \nabla u^p)\,\mathrm{d}x, \tag{2}$$

a formula that can be taken as an alternate definition of A_{eff}. Equivalently, we have

$$p^T A_{\mathrm{eff}}p = \inf_u ⨍_{\mathbb{R}^n/\Gamma} (p + \nabla u)^T A(x)(p + \nabla u)\,\mathrm{d}x, \tag{3}$$

where the infimum is taken on all Γ-periodic functions $u \in H^1_{loc}$. The latter formula implies immediately (take $u \equiv 0$ in (3)) an upper bound for the effective tensor, namely the average of A itself:

$$A_{\rm eff} \le A_+ := \fint_{\mathbb{R}^n/\Gamma} A(x)\,{\rm d}x. \tag{4}$$

Notice that A_+ is nothing but the weak limit of A^ϵ.

A slightly less obvious bound follows from the observation that the map $(S,w) \mapsto w^T S^{-1} w$ is convex[1] over $\mathbf{SPD}_n \times \mathbb{R}^n$. Writing this convexity at $S = A(x)$ and $w = p + \nabla u^p$, one derives the lower bound

$$A_{\rm eff} \ge A_- := \left(\fint_{\mathbb{R}^n/\Gamma} A(x)^{-1}\,{\rm d}x\right)^{-1}. \tag{5}$$

The bounds $A_\pm$ are respectively the arithmetic and harmonic means of the tensor A. They turn out to be sharp:

Proposition 1.1. *The equality case $A_{\rm eff} = A_+$ happens if and only if the rows of A are divergence-free:*

$$\sum_{j=1}^n \partial_j a_{ij} = 0, \qquad i = 1, \dots, n.$$

The equality case $A_{\rm eff} = A_-$ happens if and only if $A^{-1} = \nabla^2 \rho$ is the Hessian matrix of a convex function ρ.

We point out that in the latter case $\nabla^2\rho$ is Γ-periodic, but ρ itself is not.

Proof. The equality to A_+ corresponds obviously to the case where $\nabla u^p \equiv 0$ for every p. Taking for p the ith vector of the canonical basis, we obtain the divergence-free condition. And because $p \mapsto \nabla u^p$ is linear, the condition is also sufficient.

To treat the second equality case, we must go back to the proof of (5). The convex function $F(S,w) = w^T S^{-1} w$ satisfies

$$F(S,w) = F(S_0,w_0) + L_{S_0;w_0}(S - S_0, w - w_0) + R$$

where $L_{S_0;w_0}$ is a linear form (the differential of F at (S_0,w_0)) and

$$R = (S^{-1}w - S_0^{-1}w_0)^T S (S^{-1}w - S_0^{-1}w_0)$$

is non-negative. We infer a general formula

$$\fint_{\mathbb{R}^n/\Gamma} F(S,w)\,{\rm d}x = F(S_0,w_0) + \fint_{\mathbb{R}^n/\Gamma} R\,{\rm d}x \ge F(S_0,w_0).$$

[1]This convexity extends that of the map $(\rho, m) \mapsto \frac{|m|^2}{2\rho}$, which represents the kinetic energy of a fluid in terms of mass density and linear momentum.

Taking

$$S=A^{-1},\quad w=p+\nabla u^p,\quad S_0=(A_-)^{-1},\quad w_0=p,$$

we obtain the lower bound $A_{\rm eff}\ge A_-$. Then the equality case is characterized by $R=0$, namely $A(p+\nabla u^p)\equiv A_-p$. Let us denote w_i the function $p+\nabla u^p$ associated with the choice $p=\vec e_i$. Then the vector field $\vec z:=A_-^{-1}\vec w$ satisfies $A\nabla\vec z=I_n$ (we have used the symmetry of A_-). Therefore $\nabla\vec z=A(x)^{-1}$ is symmetric positive definite. This implies the existence of a convex function ρ such that $\vec z=\nabla\rho$. Then we have $A^{-1}=\nabla^2\rho$. □

We point out that the equality cases are of different nature. On the one hand, the equality $A_{\rm eff}=A_-$ is parametrized by only one function ρ, while the equality $A_{\rm eff}=A_+$ is parametrized by $\frac{n(n-1)}{2}$ independent functions. On another hand, we have

$$(A_{\rm eff}=A_-)\Longrightarrow\left(⨍_{\mathbb{R}^n/\Gamma}\frac{{\rm d}x}{\det A(x)}=\frac{1}{\det A_-}\right). \tag{6}$$

To see this, write

$$\rho(x)=\frac12\,x^TA_-^{-1}x+r\cdot x+g(x)$$

where g is Γ-periodic, then

$$⨍_{\mathbb{R}^n/\Gamma}\frac{{\rm d}x}{\det A(x)}=⨍_{\mathbb{R}^n/\Gamma}\det(A_-^{-1}+\nabla g)\,{\rm d}x,$$

and the determinant in the right is the sum of $\det A_-^{-1}$ and null-Lagrangians (for this notion, we refer to [3]), the average of the latter being zero. On the contrary, the equality $A_{\rm eff}=A_+$ does not imply an equality (except if $n=2$, for a special reason we shall explain below), but only an inequality: it was proved recently [10] that

$$(A_{\rm eff}=A_+)\Longrightarrow\left(⨍_{\mathbb{R}^n/\Gamma}(\det A(x))^{\frac{1}{n-1}}{\rm d}x\le(\det A_+)^{\frac{1}{n-1}}\right). \tag{7}$$

We point out that if moreover $A={\rm diag}(a_1,\dots,a_n)$ is diagonal and $\Gamma=\mathbb{Z}^n$, then $A_{\rm eff}=A_+$ means that $a_j=a_j(\widehat{x_j})$ does not depend upon the jth coordinate x_j. Then (7) reduces to the famous Gagliardo inequality [6]

$$\left\|\prod_{j=1}^n g_j(\widehat{x_j})\right\|_{L^1(K_n)}\le\prod_{j=1}^n\|g_j\|_{L^{n-1}(K_{n-1})},$$

where K_m is the unit cube in $\mathbb{R}^m$.

Remark 1.1. We warn the reader that (7) does not extend naively when $A_{\rm eff}$ differs from A_+. If

$$\fint_{\mathbb{R}^n/\Gamma}(\det A(x))^{\frac{1}{n-1}}\mathrm{d}x$$

was bounded by $(\det A_{\rm eff})^{\frac{1}{n-1}}$ for every tensor, then the bound (4), together with the monotony of the determinant, would again imply the same conclusion as in (7). This would imply the concavity of the map $S\mapsto(\det S)^{\frac{1}{n-1}}$ over $\mathbf{SPD}_n$; but this property is false, since this map is homogeneous of degree $\frac{n}{n-1}>1$.

Notice that the equality case $A_{\rm eff}=A_+$ is precisely that one for which the first corrector in the expansion of the solution u^ϵ of

$$\mathrm{div}\left(A(\frac{x}{\epsilon})\nabla u^\epsilon\right)=f(x),$$

with Dirichlet boundary condition for instance, is independent of the fast variable $\frac{x}{\epsilon}$. This yields the sharper estimate that the sequence u^ϵ remains in a bounded set of $H^2(\Omega)$ whenever $f\in L^2(\Omega)$ and $\epsilon\to 0+$; see theorem 4.7 of [4].

Let us end this short presentation with special situations where the effective matrix $A_{\rm eff}$ can be expressed in closed form. We have already mentioned the one-dimensional case, where $a_{\rm eff}=a_-$. Similar, though more involved, formulæ exist for laminated structures, that is when A depends upon only one spatial coordinate; see for instance Theorem 1.3.28 in [1]. Other formulæ can be established in two space dimensions, using a special trick that works only when $n=2$. The idea is that the rotation $\sigma(x):=(x_2,-x_1)$ of angle $\frac{\pi}{2}$ switches solenoidal and potential vector fields. The basic ingredient is the equivariance formula:

If $n=2$ and $B(y):=A(\sigma y)^{-1}$, then $B_{\rm eff}=(A_{\rm eff})^{-1}$.

This can be used to prove the following property:

If $n=2$ and $\det A\equiv\delta$ is a constant, then $\det A_{\rm eff}=\delta$.

This also allows one to calculate the effective matrix of a chessboard: suppose $A(x)=a_\pm I_2$, according to whether x belongs to a square $[k,k+1)\times[\ell,\ell+1)$ with $k+\ell$ even/odd. Then $A_{\rm eff}=\sqrt{a_-a_+}\,I_2$.

Goal of the paper. We intent to develop a framework in which many of the special situations described above become special cases of general statements. This will be done by replacing the unknown ∇u in (1) by a closed differential form of degree m (with $m=1$ in (1)) and $A\nabla u$ by another closed differential form of degree $n-m$. Let us summarize our results:

- There is a duality, which transforms a homogenization problem in degree m into another homogenization problem, in degree $n-m$. This duality is involutive.
- The map $A \longmapsto A_{\rm eff}$ is equivariant under this duality. An upper bound of $A_{\rm eff}$ in the primary problem yields a lower bound in the dual problem.
- Convex functions with periodic Hessian provide tensors $A(x)$ for which the theoretical upper bound actually equals the effective tensor.

When $n=2m$, the duality is a little more powerful in that it relates two distinct homogenization problems on forms of same degree. For instance, if $n=2$ and $m=1$, we obtain the formulæ mentioned above.

2. HOMOGENIZING DIFFERENTIAL FORMS

2.1. Differential forms over $\mathbb{R}^n$

Again, we recall basics of differential forms, without proofs.

Let U be an open subset of $\mathbb{R}^n$. A differential form over U, of degree m (we call it an m-form), is a map $x \mapsto \alpha(x)$, where $\alpha(x)$ is an alternate m-linear form over $\mathbb{R}^n$, that is $\alpha(x) \in \Lambda^{m*}(\mathbb{R}^n)$. The space of m-forms is denoted $\Omega^m(U)$. In practice, we need some control of the coefficients of $x \mapsto \alpha(x)$ over some (arbitrary) basis of $\Lambda^{m*}(\mathbb{R}^n)$; at least, they must be measurable. When they belong to some Lebesgue, Sobolev or Hölder space, we write $\alpha \in L^p\Omega^m(U), H^s\Omega^m(U)$ or $C^s\Omega^m(U)$. If the coefficients are bounded measures, we speak of $\mathcal{M}\Omega^m(U)$. We recall that 0-forms are just scalar functions over U and 1-forms

$$a_1(x)\mathrm{d}x_1 + \cdots + a_n(x)\mathrm{d}x_n, \qquad \mathrm{d}x_j := \vec{e}_j^{\,*}$$

are meant to be integrated along curves. More generally, m-forms can be integrated over m-dimensional oriented submanifolds. Finally, $\Omega^m(U)=\{0\}$ when $m>n$.

A canonical basis of $\Lambda^{m*}(\mathbb{R}^n)$ is given by the elements

$$\mathrm{d}x_I := \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m}, \qquad I=\{i_1<\cdots<i_m\}.$$

Its dimension is therefore the binomial $\binom{n}{m}$. An m-form decomposes in a unique way as

$$\sum_{|I|=m} a_I(x)\mathrm{d}x_I.$$

For instance, every n-form is written in a unique way $g(x)\mathbf{vol}$, where $\mathbf{vol} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is the canonical volume form.

The direct sum

$$\Omega(U) = \oplus_{m=0}^n \Omega^m(U)$$

is an associative, graded algebra under the exterior product $\alpha\wedge\beta$. The product of an m-form α and an ℓ-form β is an $(m+\ell)$-form; in particular $\alpha\wedge\beta=0$ if $m+\ell>n$. There holds

$$\beta\wedge\alpha=(-1)^{m\ell}\alpha\wedge\beta.$$

For instance, 1-forms satisfy $\alpha\wedge\alpha=0$, and the subspace spanned by the forms of even degree is a commutative[2] subalgebra.

An exterior derivative d is defined from $\Omega^m(U)$ into[3] $\Omega^{m+1}(U)$, which displays the following properties:

$$\mathrm{d}\circ\mathrm{d}=0,\qquad \mathrm{d}(\alpha\wedge\beta)=(\mathrm{d}\alpha)\wedge\beta+(-1)^m\alpha\wedge(\mathrm{d}\beta),\qquad \mathrm{d}f=\sum_{j=1}^n\frac{\partial f}{\partial x_j}\,\mathrm{d}x_j,$$

when f is a function and α,β are an m-form and an ℓ-form. One therefore has

$$\mathrm{d}\left(\sum_{|I|=m}a_I(x)\mathrm{d}x_I\right)=\sum_{|I|=m}\mathrm{d}a_I(x)\wedge\mathrm{d}x_I=\sum_{|I|=m}\sum_{j\notin I}\frac{\partial a_I}{\partial x_j}\,\mathrm{d}x_j\wedge\mathrm{d}x_I.$$

An m-form α is *closed* if $\mathrm{d}\alpha=0$; it is *exact* if there exists an $(m-1)$-form β such that $\alpha=\mathrm{d}\beta$. An exact form is always closed, but the converse is not necessarily true (its validity depends upon m and the topology of U).

The formula above shows that the gradient of a function f can be identified with $\mathrm{d}f$. Likewise, the differential of a vector field identifies with its rotational (curl operator). For an $(n-1)$-form

$$\alpha=q_1\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_n-q_2\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\cdots\wedge\mathrm{d}x_n+\cdots+(-1)^{n-1}q_n\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{n-1},$$

the exterior derivative $\mathrm{d}\alpha$ equals $(\operatorname{div}\vec{q})\mathbf{vol}$. We warn the reader that these identifications depend upon the choice of a euclidian structure over $\mathbb{R}^n$ (here we choose the standard one).

2.2. **Elliptic second-order equations for differential forms**

Adopting the formalism of differential forms, we see that an equation (1) can be rewritten as

$$\mathrm{d}\alpha=0,\qquad \mathrm{d}(A\alpha)=0,$$

where $\alpha=\nabla u$ is a 1-form and $A\alpha$ is viewed as an $(n-1)$-form. The tensor $A(x)$ is now identified with a linear map $A(x):\Lambda^{1*}(\mathbb{R}^n)\to\Lambda^{n-1,*}(\mathbb{R}^n)$. Defining a bilinear form b_x by

$$b_x(\alpha,\beta)\mathbf{vol}=(A(x)\alpha)\wedge\beta,$$

[2]This fact is used in the proof of the Amitsur–Levitski Theorem; see [9], chapter 4.

[3]Here, we do not mention the regularities needed for the definition. The regularity must be high enough that products make sense.

the symmetry and positivity of $A(x)$ amount to saying that b_x is symmetric and positive definite.

If A is Γ-periodic, we observe that the effective tensor can be described as follows. We begin by setting a positive definite quadratic form over $\Lambda^{1*}(\mathbb{R}^n)$,

$$b_{\text{eff}}(p,p) := \inf \left\{ \fint_{\mathbb{R}^n/\Gamma} (A\alpha)\wedge\alpha \quad | \quad \alpha = p + du, \quad u \in H^1(\mathbb{R}^n/\Gamma) \right\}.$$

Then $A_{\text{eff}} : \Lambda^{1*}(\mathbb{R}^n) \to \Lambda^{n-1,*}(\mathbb{R}^n)$ is the unique symmetric linear map such that

$$(A_{\text{eff}}p)\wedge p = b_{\text{eff}}(p,p)\,\mathbf{vol}, \qquad \forall p \in \Lambda^{1*}(\mathbb{R}^n).$$

Remark that in the procedure above, one minimizes over those 1-forms α that are square integrable, closed, and such that $\fint_{\mathbb{R}^n/\Gamma} \alpha\,\mathrm{d}x = p$. This formulation has the advantage that the minimizer α^p is unique, contrary to the classical formulation, where u^p is unique only up to an additive constant.

2.2.1. GENERALIZING TO FORMS OF HIGHER DEGREE

This suggests to adopt the same presentation when the degree $1 \le m \le n-1$ of the unknown form is arbitrary. We start from a field of bilinear symmetric forms b_x over $\Lambda^{m*}(\mathbb{R}^n)$, measurable in $x \in \mathbb{R}^n/\Gamma$. We suppose that b_x is positive definite, uniformly in $x \in \mathbb{R}^n/\Gamma$: there exists a finite constant C such that

$$\frac{1}{C}\|p\|^2 \le b_x(p,p) \le C\|p\|^2, \qquad \forall\, p \in \Lambda^{m*}(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n/\Gamma,$$

where a norm has been chosen once for all in $\Lambda^{m*}(\mathbb{R}^n)$. This bilinear form defines a self-adjoint operator

$$T(x) \in \mathcal{L}(\Lambda^{m*}(\mathbb{R}^n);\Lambda^{n-m,*}(\mathbb{R}^n))$$

by the formula

$$(T(x)p)\wedge p' = b_x(p,p')\mathbf{vol} = (T(x)p')\wedge p.$$

The functional

$$\begin{aligned} L^2\Omega^m(\mathbb{R}^n/\Gamma) &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \fint_{\mathbb{R}^n/\Gamma} b_x(\alpha(x),\alpha(x))\,\mathrm{d}x \end{aligned}$$

is convex, continuous and coercive. When restricted to the affine subspace F_p of closed m-form whose average equal a given $p \in \Lambda^{m*}(\mathbb{R}^n)$, it achieves a minimum at a unique element α^p. This one is the unique solution of the elliptic problem

$$(8)\qquad \mathrm{d}\alpha^p = 0, \qquad \mathrm{d}(T\alpha^p) = 0, \qquad \fint_{\mathbb{R}^n/\Gamma} \alpha^p \,\mathrm{d}x = p.$$

The map $p \mapsto \alpha^p$ is obviously linear. Then the effective operator $T_{\text{eff}} \in \mathcal{L}(\Lambda^{m*}(\mathbb{R}^n); \Lambda^{n-m,*}(\mathbb{R}^n))$, which is self-adjoint too, is defined by

$$T_{\text{eff}}p := \fint_{\mathbb{R}^n/\Gamma} T\alpha^p \mathrm{d}x$$

or equivalently

$$(9)\quad b_{\text{eff}}(p,p) = (T_{\text{eff}}p) \wedge p := \fint_{\mathbb{R}^n/\Gamma} b_x(\alpha^p(x), \alpha^p(x))\,\mathrm{d}x$$
$$= \inf \left\{ \fint_{\mathbb{R}^n/\Gamma} b_x(\alpha,\alpha)\,\mathrm{d}x \quad \middle| \quad \mathrm{d}\alpha = 0, \right.$$
$$\left. \alpha \in L^2\Omega^m(\mathbb{R}^n/\Gamma), \quad \fint_{\mathbb{R}^n/\Gamma} \alpha\,\mathrm{d}x = p \right\}.$$

It is clear that $b_{\text{eff}}(p,p)$ satisfies the same inequalities

$$\frac{1}{C}\|p\|^2 \le b_{\text{eff}}(p,p) \le C\|p\|^2, \qquad \forall\, p \in \Lambda^{m*}(\mathbb{R}^n).$$

We justify the terminology T_{eff} or b_{eff} in the next paragraphs. We begin by defining an appropriate boundary value problem in bounded domains. Then we prove a homogenization result.

2.2.2. A DIRICHLET BOUNDARY VALUE PROBLEM

Let U be a bounded open domain in $\mathbb{R}^n$, and $T(x) \in \mathcal{L}(\Lambda^{m*}(U); \Lambda^{n-m,*}(U))$ be self-adjoint and positive definite, uniformly in x. We define the symmetric, positive definite bilinear form

$$B(\alpha,\alpha') = \int_U (T_x\alpha) \wedge \alpha', \qquad \alpha, \alpha' \in L^2\Omega^m(U),$$

where we use the fact that $(T_x\alpha) \wedge \alpha'$ is a volume form over U and therefore can be integrated.

If $\beta \in L^2\Omega^{n-m}(U)$, the functional

$$\mathcal{N}[\alpha] := \frac{1}{2}\,B(\alpha,\alpha) + \int_U \alpha \wedge \beta$$

is continuous, coercive and strictly convex over $L^2\Omega^m(U)$. Given a closed vector space (or even an affine one), it achieves a unique minimum. We apply this principle to the subspace $\mathrm{d}(H^1_0\Omega^m(U))$ of exact forms $\alpha = \mathrm{d}\xi$ whose potential is of class

$$\xi \in H^1_0\Omega^m(U) = \overline{\mathcal{D}\Omega(U)}^{H^1}.$$

Here above, H^1 is the usual Sobolev space and $\mathcal{D}$ denotes the space of $\mathcal{C}^\infty$-functions with compact support. Let α be this minimum. The Euler–Lagrange equation for this minimization problem is

$$\alpha \in \mathrm{d}(H^1_0\Omega^m(U)), \qquad B(\alpha,\alpha') = \int_U \beta \wedge \alpha', \quad \forall\, \alpha' \in \mathrm{d}(H^1_0\Omega^m(U)).$$

The second equality is

$$\int_U (T_x\alpha - \beta)\wedge \mathrm{d}\beta' = 0, \qquad \forall\, \beta' \in H^1_0\Omega^m(U).$$

Since the left-hand side is

$$\int_U (\mathrm{d}(T_x\alpha - \beta))\wedge \beta',$$

this amounts to saying that $\mathrm{d}(T_x\alpha - \beta) = 0$. We summarize this analysis in the following statement.

PROPOSITION 2.1. *Let U be a bounded open domain in $\mathbb{R}^n$, and*

$$T(x) \in \mathcal{L}(\Lambda^{m*}(U);\Lambda^{n-m,*}(U))$$

be self-adjoint and positive definite, uniformly in x.

Given an exact form $\gamma \in H^{-1}\Omega^{n-m+1}(U)$, there exists one, and only one exact form $\alpha \in \mathrm{d}(H^1_0\Omega^m(U))$, satisfying

$$\mathrm{d}(T\alpha) = \gamma.$$

We emphasize that the boundary condition is encoded in the requirement that $\alpha \in \mathrm{d}(H^1_0\Omega^m(U))$. When $\mu \in L^2\Omega^m(U)$ is closed, $\mathrm{d}\mu = 0$, it admits a tangential trace $\mu_\tau \in H^{-1/2}\Omega^m(\partial U)$, by a duality argument similar to that used for the definition of the normal trace of elements of the space $H_{\mathrm{div}}(U)$. If U_1 is a contractile subdomain and $\xi \in H^1\Omega^m(U_1)$ is a potential in U_1, $\mu = \mathrm{d}\xi$, then μ_τ coincides with the exterior derivative of the tangential part of the standard trace of ξ over $\partial U_1 \cap \partial U$. In the case of α, the potential vanishes over the boundary and we have therefore

$$\alpha_\tau \equiv 0 \qquad \text{over} \quad \partial U.$$

2.2.3. HOMOGENIZATION OF THE DIRICHLET BVP

THEOREM 2.1. *Let U be a bounded open domain in $\mathbb{R}^n$, and*

$$T(x) \in \mathcal{L}(\Lambda^{m*}(\mathbb{R}^n); \Lambda^{n-m,*}(\mathbb{R}^n))$$

be self-adjoint, Γ-periodic and positive definite, uniformly in x.

Let $\gamma \in L^2\Omega^{n-m+1}(U)$ be an exact form. Denote α^ϵ the unique solution of

$$\alpha^\epsilon \in \mathrm{d}(H_0^1\Omega^m(U)), \qquad \mathrm{d}(T\left(\frac{x}{\epsilon}\right)\alpha^\epsilon) = \gamma \qquad \text{in } \Omega.$$

When $\epsilon \to 0+$, α^ϵ converges weakly-star in $L^2\Omega^m(U)$ towards the unique solution of

$$\alpha \in \mathrm{d}(H_0^1\Omega^m(U)), \qquad \mathrm{d}(T_{\mathrm{eff}}\alpha) = \gamma \qquad \text{in } \Omega.$$

Proof. We follow Tartar's strategy (see Section 1.3 of [7]), which employs the compensated compactness in its version for differential forms (Theorem 1.1 in [8] or Lemma 9.1 in [11]).

Let us denote $p^\epsilon = T\left(\frac{x}{\epsilon}\right)\alpha^\epsilon = T^\epsilon\alpha^\epsilon$, which is a closed $(n-m)$-form. Because of the variational formulation and the uniform ellipticity, α^ϵ is a bounded sequence in $L^2\Omega^m(U)$. Since T^ϵ is uniformly bounded, p^ϵ is also a bounded sequence, in $L^2\Omega^{n-m}(U)$. Up to an extraction, we may assume that both sequences have weak-star limits:

$$\alpha^\epsilon \rightharpoonup \alpha, \qquad p^\epsilon \rightharpoonup p \qquad \text{weakly-star in} \quad L^2.$$

By continuity and closeness, we still have $\alpha \in \mathrm{d}(H_0^1\Omega^m(U))$ and $\mathrm{d}p = \gamma$.

Let $\xi \in \Lambda^{m*}(\mathbb{R}^n)$ be given. We denote ω the unique solution of the cell problem

$$\mathrm{d}\omega = 0, \qquad \mathrm{d}(T\omega) = 0 \quad \text{in} \quad \mathbb{R}^n, \qquad \fint_{\mathbb{R}^n/\Gamma} \omega(x)\,\mathrm{d}x = \xi.$$

Then we define

$$\omega^\epsilon(x) := \omega\left(\frac{x}{\epsilon}\right).$$

We still have $\mathrm{d}\omega^\epsilon = 0$ and $\mathrm{d}(T^\epsilon\omega^\epsilon) = 0$. In addition, the following weak convergence holds

$$\omega^\epsilon \rightharpoonup \fint_{\mathbb{R}^n/\Gamma} \omega(x)\,\mathrm{d}x = \xi, \qquad T^\epsilon\omega^\epsilon \rightharpoonup \fint_{\mathbb{R}^n/\Gamma} T(x)\omega(x)\,\mathrm{d}x = T_{\mathrm{eff}}\xi,$$

by the definition of T_{eff}.

Finally, the self-adjointness of T gives us

$$p^\epsilon \wedge \omega^\epsilon = (T^\epsilon\alpha^\epsilon) \wedge \omega^\epsilon = \alpha^\epsilon \wedge (T^\epsilon\omega^\epsilon). \tag{10}$$

We are now in position to apply compensated compacted to both sides of equality (10). All the factors in action are sequences of closed forms, but one, namely p^ϵ. The sequence $\mathrm{d}p^\epsilon$ is independent of ϵ, therefore remains in a compact subset of $H^{-1}\Omega^{n-m+1}(U)$. All of them are weakly converging in $L^2\Omega$. The version of the *div-curl* Lemma for differential forms tells us that the exterior product weakly-star converges in $\mathcal{M}\Omega^n$, towards the product of weak-star limits. We deduce the equality $p \wedge \xi = \alpha \wedge (T_{\rm eff}\xi)$, that is

$$p \wedge \xi = (T_{\rm eff}\alpha) \wedge \xi.$$

Since this equality holds true for every $\xi \in \Lambda^{m*}(\mathbb{R}^n)$, we infer $p = T_{\rm eff}\alpha$. Therefore α is the (unique) solution of the Dirichlet problem in U associated with the operator $T_{\rm eff} \in \mathcal{L}(\Lambda^{m*}(\mathbb{R}^n); \Lambda^{n-m,*}(\mathbb{R}^n))$.

The uniqueness of the limit α ensures that the whole sequence α^ϵ converges. □

2.3. Duality in homogenization

When the operator T_x is as above, it is invertible, because of positive definiteness. Its inverse

$$T(x)^{-1} \in \mathcal{L}(\Lambda^{n-m,*}(\mathbb{R}^n); \Lambda^{m*}(\mathbb{R}^n))$$

is still self-adjoint and positive definite. We associate with T^{-1} an elliptic PDE similar to (8), but whose unknown is an $(n-m)$-form. Of course, homogenization yields an effective tensor $(T^{-1})_{\rm eff}$, for which we establish an equivariance property.

Proposition 2.2 (Equivariance). *Let the tensor $T(x) \in \mathcal{L}(\Lambda^{m*}(\mathbb{R}^n); \Lambda^{n-m,*}(\mathbb{R}^n))$ be self-adjoint and positive definite. Then we have the formula*

$$(T^{-1})_{\rm eff} = (T_{\rm eff})^{-1}. \tag{11}$$

Proof. It is enough to remark that (8) and the definition of $T_{\rm eff}$ are equivalent to

$$\mathrm{d}(T\alpha^p) = 0, \qquad \mathrm{d}(T^{-1}(T\alpha^p)) = 0, \qquad ⨍_{\mathbb{R}^n/\Gamma} T\alpha^p \,\mathrm{d}x = T_{\rm eff}p,$$

together with

$$⨍_{\mathbb{R}^n/\Gamma} T^{-1}(T\alpha^p)\,\mathrm{d}x = p.$$

This shows that $(T^{-1})_{\rm eff}(T_{\rm eff}p) = p$. Since $T_{\rm eff}$ is positive definite and therefore one-to-one, this proves (11). □

Warning. When $m = 1$ and we start from a PDE in classical formulation (1), then the dual PDE governs $(n-1)$-forms, instead of 1-forms. Both PDEs are of a completely different nature, except if $n = 2$.

The case $n = 2m$. When m is the half of the space dimension, both $T(x)$ and $T(x)^{-1}$ belong to $\mathcal{L}(\Lambda^m(\mathbb{R}^n))$. Then T and T^{-1} define two elliptic PDEs over the same space of m-forms. If it happens that these PDEs are identical up to an isometry of the domain $\mathbb{R}^n/\Gamma$, then we have a linear relation between T_{eff} and $(T^{-1})_{\text{eff}}$. For instance, say that $T_{\text{eff}} = (T^{-1})_{\text{eff}}$; thanks to (11) and because T_{eff} is self-adjoint and positive definite, we conclude that T_{eff} is nothing but the identity of $\Lambda^m(\mathbb{R}^n)$. This is exactly the way that the effective tensor of a chessboard was identified (when $m = 1$ and $n = 2$). We thus have a similar result for generalized chessboards in even-dimensional spaces. For this, we say that a cube $z + [0,1)^n$ with $z \in \mathbb{Z}^n$ is even/odd if $\sum_j z_j$ is even/odd. If $|I| = m$ and $n = 2m$, we denote $\epsilon_I = \pm 1$ the sign in $\mathrm{d}x_I \wedge \mathrm{d}x_{I^c} = \pm\mathbf{vol}$.

Proposition 2.3 ($n = 2m$.). *Let $T(x)$ be defined by*

$$T(x)\mathrm{d}x_I = a_\pm \epsilon_I \mathrm{d}x_I^c, \qquad \forall\, |I| = m,$$

equivalently

$$b_x(\mathrm{d}x_I, \mathrm{d}x_J) = a_\pm \delta_I^J, \qquad \forall\, |I| = |J| = m,$$

according to whether x belongs to an even/odd cube. Then

$$T_{\text{eff}} = \sqrt{a_- a_+}\, T_1,$$

where $T_1 \mathrm{d}x_I = \epsilon_I \mathrm{d}x_I^c$, that is $b_1(\mathrm{d}x_I, \mathrm{d}x_J) = \delta_I^J$.

2.3.1. BOUNDS FOR THE EFFECTIVE TENSOR

An obvious bound is obtained by choosing $\alpha \equiv p$ in (9):

$$b_{\text{eff}}(p,p) \le \fint_{\mathbb{R}^n/\Gamma} b_x(p,p)\,\mathrm{d}x,$$

which we write simply and by analogy with the case of 1-forms

$$b_{\text{eff}} \le \fint_{\mathbb{R}^n/\Gamma} b_x \mathrm{d}x =: b_+, \quad \text{or} \quad T_{\text{eff}} \le \fint_{\mathbb{R}^n/\Gamma} T_x \mathrm{d}x =: T_+. \tag{12}$$

By duality, and because the map $T \mapsto T^{-1}$ is monotonous decreasing over the self-adjoint positive definite operators, we infer the lower bound

$$T_{\text{eff}} \ge T_- := \left(\fint_{\mathbb{R}^n/\Gamma} T_x^{-1} \mathrm{d}x \right)^{-1}. \tag{13}$$

2.4. **The equality** $T_{\rm eff} = T_+$

Because the minimum in (9) is achieved at a unique point α^p, the equality $T_{\rm eff} = T_+$ happens if, and only if $\alpha^p \equiv p$ for every p and almost every $x \in \mathbb{R}^n/\Gamma$. Looking at (8), this amounts to saying that for every $|I| = m$ one has

$$\mathrm{d}(T\mathrm{d}x_I) \equiv 0.$$

This condition is denoted $\mathrm{d}T = 0$, by analogy with the case $m = 1$, where it was written $\mathrm{Div}\, A = 0$, the divergence acting over the rows (or columns) of the symmetric matrix $A(x)$.

By duality, we infer that the equality of $T_{\rm eff}$ to T_- is equivalent to $\mathrm{d}(T^{-1}) = 0$.

We now give two important examples where $T_{\rm eff}$ coincides with T_+.

2.4.1. THE DIAGONAL CASE

We say that T is diagonal if $T\mathrm{d}x_I = \epsilon_I f_I(x)\mathrm{d}x_{I^c}$ for some scalar function f_I, for every index set I of cardinality m. In other words, $b_x(\mathrm{d}x_I, \mathrm{d}x_J)$ equals f_I if $J = I$ and 0 otherwise. Then one has

$$\mathrm{d}(T\mathrm{d}x_I) = \mathrm{d}(f_I \mathrm{d}x_{I^c}) = \mathrm{d}f_I \wedge \mathrm{d}x_{I^c} = \sum_{j \in I} \frac{\partial f}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{I^c}.$$

We deduce that $\mathrm{d}T = 0$ if and only if f_I depends only upon the coordinates x_i for $i \in I^c$; we write $f_I = f_I(x_{I^c})$. Remark that for this to be compatible with the periodicity, one needs that $\Gamma = D\mathbb{Z}^n$, where D is a diagonal matrix.

The discriminant of b_x in the canonical basis of the $\mathrm{d}x_I$'s, is the product of the functions f_I. It turns out that a generalization of Gagliardo's inequality provides an inequality for this discriminant. The following result is due to Finner [5] and was rediscovered later on in [2].

PROPOSITION 2.4 (Generalized Gagliardo inequality). *For each index subset I of cardinality ℓ, let $F_I : \mathbb{R}^\ell \to \mathbb{R}$ be a non-negative, measurable function of the variables $x_I = (x_i)_{i \in I}$. Let us form the product*

$$F(x) = \prod_{|I|=\ell} F_I(x_I).$$

Then there holds

$$\int_{\mathbb{R}^n} F(x)\,\mathrm{d}x \le \prod_{|I|=\ell} \|F_I\|_{L^q(\mathbb{R}^\ell)}, \tag{14}$$

where $q = \binom{n-1}{\ell-1}$ is a binomial coefficient.

The same inequality, where integrals are replaced by averages, is valid in the periodic case.

Applying (14) to the functions $F_J = f_{J^c}$, with $\ell = n - m$, we obtain:

Corollary 2.1. *Suppose that $T_{\text{eff}} = T_+$ and that $T(x)$ is diagonal. Then one has*

$$\fint_{\mathbb{R}^n/\Gamma} (\operatorname{disc} b_x)^{1/q} \mathrm{d}x \le (\operatorname{disc} b_+)^{1/q}, \qquad q = \binom{n-1}{m}. \tag{15}$$

Remark that the function $b \mapsto (\operatorname{disc} b)^{\frac{1}{q}}$ is homogeneous of degree

$$\frac{1}{q}\binom{n}{m} = \frac{n}{n-m} > 1,$$

and therefore is not concave. Thus (15) is not a consequence of Jensen's inequality.

2.4.2. ANOTHER CASE OF EQUALITY

Proposition 2.5. *Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a strongly convex function such that $\nabla^2\theta$ is Γ-periodic. Let us define $T(x) : \Lambda^{m*}(\mathbb{R}^n) \to \Lambda^{n-m,*}(\mathbb{R}^n)$ by*

$$T\mathrm{d}x_I = \epsilon_I \bigwedge_{j \in I^c} \mathrm{d}\partial_j\theta \qquad \textit{whenever} \quad |I| = m.$$

Then T is symmetric positive definite and satisfies the constraint $\mathrm{d}T = 0$. One thus has $T_{\text{eff}} = T_+$.

In addition, T satisfies

$$\fint_{\mathbb{R}^n/\Gamma} (\operatorname{disc} b_x)^{1/q} \mathrm{d}x = (\operatorname{disc} b_+)^{1/q}, \tag{16}$$

where $\operatorname{disc} b_x$ is the discriminant of the quadratic form b_x, and $q = \binom{n-1}{m}$ is the same binomial coefficient as above.

This generalizes the observation made in [10] that for $m = 1$, a tensor $A(x) = \widehat{\nabla^2\theta}$ defined as the cofactor matrix of the Hessian of a strongly convex function is symmetric positive definite and row-wise divergence-free.

An equivalent definition of T can be given in terms of the m-forms

$$\alpha_K := \bigwedge_{k \in K} \mathrm{d}\partial_k\theta.$$

Using the symmetry of T, we have

$$(T\alpha_K) \wedge \mathrm{d}x_I = (T\mathrm{d}x_I) \wedge \alpha_K = \left(\epsilon_I \bigwedge_{j \in I^c} \mathrm{d}\partial_j\theta\right) \wedge \alpha_K.$$

This yields $(T\alpha_K)\wedge \mathrm{d}x_I = 0$ if $K \neq I$, and

$$(T\alpha_I)\wedge \mathrm{d}x_I = \bigwedge_{i=1}^{n} \mathrm{d}\partial_i\theta = \det \nabla^2\theta \cdot \mathbf{vol}$$

otherwise. Equivalently,

$$b_x(\alpha_K, \mathrm{d}x_I) = \begin{cases} 0 & \text{if} \quad K \neq I, \\ \det \nabla^2\theta & \text{if} \quad K = I. \end{cases}$$

This means also

$$T\alpha_I = \epsilon_I(\det \nabla^2\theta)\,\mathrm{d}x_{I^c}. \tag{17}$$

Proof. Because of $d \circ d = 0$, and the Leibniz rule, one obviously has $\mathrm{d}(T\mathrm{d}x_I) = 0$ for every I.

Decomposing m-forms over the canonical basis, $\omega = \sum_I \omega_I \mathrm{d}x_I$, the bilinear form associated with T_x writes

$$b_x(\omega, \omega') = \sum_{|I|=|J|=p} a_{IJ}\omega_I\omega'_J,$$

where the coefficients are Jacobian determinants

$$a_{IJ} = \mathrm{Jac}(\partial_k\theta|_{k\in I^c}\,;\, x_\ell|_{\ell\in J^c}) = \det\left(\partial_k\partial_\ell\theta\right)_{k\in I^c, \ell\in J^c}.$$

Because $a_{IJ} = a_{JI}$, the form b_x is symmetric, as required.

The rest of the proof involves the following result, whose proof is postponed for a minute.

Lemma 2.1. *Let $H \in \mathbf{Sym}_n(\mathbb{R})$ be given. Let R be the quadratic form over $\Lambda^{r*}(\mathbb{R}^n)$, defined by*

$$\eta = \sum_{|K|=r} \eta_K \mathrm{d}x_K \longmapsto R(\eta) = \sum_{|K|=|L|=r} H\binom{K}{L}\eta_K\eta_L,$$

where the coefficient of $\eta_K\eta_L$ is the minor of H whose rows (resp. columns) have indices in the set K (resp. L).

The discriminant of R in the canonical basis equals $(\det H)^N$ where $N = \binom{n-1}{r-1}$.

If H is positive definite, then R is positive definite.

Actually, the quadratic form

$$Q_x(\xi) = \sum_{|I|=|J|=m} a_{IJ}(x)\omega_I\omega_J$$

can be written $R(\eta)$ with $r = n - m$ and $H = \nabla^2\theta(x)$, after the change of variable $\eta_I := \omega_{I^c}$. According to the Lemma, it is positive definite, and its discriminant equals

$$(\det \nabla^2\theta)^N, \qquad N := \binom{n-1}{n-m-1} = \binom{n-1}{m} = q.$$

One deduces

$$\fint_{\mathbb{R}^n/\Gamma} (\operatorname{disc} T(x))^{1/q} \mathrm{d}x = \fint_{\mathbb{R}^n/\Gamma} \det \nabla^2\theta \,\mathrm{d}x = \det \fint_{\mathbb{R}^n/\Gamma} \nabla^2\theta \,\mathrm{d}x,$$

where the last equality follows from the fact that $\det \nabla^2\theta$ is a null-Lagrangian. Remarking now that, for the same reason,

$$\left(\fint_{\mathbb{R}^n/\Gamma} \nabla^2\theta \,\mathrm{d}x\right)\binom{K}{L} = \fint_{\mathbb{R}^n/\Gamma} \nabla^2\theta \binom{K}{L} \mathrm{d}x, \qquad \forall\, |K| = |L| = n-m,$$

one sees that the quadratic form associated with A_+ is nothing but the R of the lemma when choosing

$$H = \fint_{\mathbb{R}^n/\Gamma} \nabla^2\theta \,\mathrm{d}x.$$

One has therefore

$$\operatorname{disc} A_+ = \left(\det \fint_{\mathbb{R}^n/\Gamma} \nabla^2\theta \,\mathrm{d}x\right)^q,$$

from which we deduce the equality in (16). □

There remains to prove Lemma 2.1.

Proof. Let us diagonalize H in an orthogonal basis, $H = U^T D U$ where D is diagonal and U is orthogonal. The Cauchy–Binet formula (see in [9], the chapter about square matrices) yields

$$H\binom{K}{L} = \sum_{|M|=|N|=r} U^T\binom{K}{M} D\binom{M}{N} U\binom{N}{L}$$
$$= \sum_{|M|=|N|=r} U\binom{M}{K} D\binom{M}{N} U\binom{N}{L}.$$

We infer

$$R(\eta) = S(\rho) := \sum_{|M|=|N|=r} D\binom{M}{N}\rho_M\rho_N, \qquad \rho_M := \sum_{|K|=r} U\binom{M}{K}\eta_K.$$

The change of variable $\eta \mapsto \rho$ is an isomorphism, because of

$$\eta_K = \sum_{|K|=r} U\binom{M}{K}\rho_M.$$

Since D is diagonal, one has

$$S(\rho)=\sum_{|M|=r}d_M\rho_M^2,\qquad d_M:=\prod_{i\in M}d_i.$$

When H is positive definite, the d_i's are >0, the d_M's are positive too and the quadratic form S is positive definite.

Eventually, we compute the discriminant of R. The reciprocal of $\eta\mapsto\rho$ being equal to its transpose, it is an orthogonal transformation. The discriminant of R is thus equal to that of S. And the latter is just

$$\operatorname{disc}S=\prod_{|M|=r}d_M=\left(\prod_{i=1}^n d_i\right)^s=(\det H)^s$$

where, by homogeneity,

$$s=\frac{r}{n}\binom{n}{r}=\binom{n-1}{r}.\quad\square$$

3. FUNCTIONAL INEQUALITIES

Let us assemble the results that we know so far, concerning the equality case $T_{\rm eff}=T_+$:

- If $m=1$, that is $T(x)$ is represented by a symmetric matrix $A(x)>0_n$, which satisfies $\operatorname{Div}A\equiv 0$, then the following inequality is shown in [10]:

$$\fint_{\mathbb{R}^n/\Gamma}(\det A(x))^{\frac{1}{n-1}}\mathrm{d}x\le\left(\det\fint_{\mathbb{R}^n/\Gamma}A(x)\,\mathrm{d}x\right)^{\frac{1}{n-1}}.$$

- By duality, the case $m=1$ and $A_{\rm eff}=A_-$ yields a situation where $m=n-1$ and $T_{\rm eff}=T_+$. Then (6) transforms into

$$\fint_{\mathbb{R}^n/\Gamma}\operatorname{disc}b_x\mathrm{d}x=\operatorname{disc}\fint_{\mathbb{R}^n/\Gamma}b_x\mathrm{d}x.$$

- In the diagonal case, we have inequality (15):

$$\fint_{\mathbb{R}^n/\Gamma}(\operatorname{disc}b_x)^{1/q}\mathrm{d}x\le(\operatorname{disc}b_+)^{1/q},\qquad q=\binom{n-1}{m}.$$

- In the special case built from the Hessian of a convex function, we have the equality (16):

$$\fint_{\mathbb{R}^n/\Gamma}(\operatorname{disc}b_x)^{1/q}\mathrm{d}x=(\operatorname{disc}b_+)^{1/q},\qquad q=\binom{n-1}{m}.$$

We point out that the second case is a particular one of the latter. These convergent clues lead us to a more general statement, which contains all of them. We leave it as an *open question*:

Let $T(x) : \Lambda^{m*}(\mathbb{R}^n) \to \Lambda^{n-m,*}(\mathbb{R}^n)$ *be* Γ*-periodic, and* b_x *defined by*

$$b_x(\omega, \omega')\,\mathbf{vol} = (T(x)\omega) \wedge \omega'$$

be symmetric, positive semi-definite and uniformly bounded.

If $\mathrm{d}T \equiv 0$, *is it true that*

$$\fint_{\mathbb{R}^n/\Gamma} (\operatorname{disc} b_x)^{1/q} \mathrm{d}x \le (\operatorname{disc} b_+)^{1/q}, \qquad q = \binom{n-1}{m} \quad ? \tag{18}$$

Remark. Let

$$q_1 = \binom{n}{m}$$

be the dimension of $\Lambda^{m*}(\mathbb{R}^n)$. If the exponent $\frac{1}{q}$ is replaced by $\frac{1}{q_1}$, then the map $b \mapsto (\operatorname{disc} b)^{\frac{1}{q_1}}$ is concave over the cone of positive definite quadratic forms, and the inequality

$$\fint_{\mathbb{R}^n/\Gamma} (\operatorname{disc} b_x)^{1/q_1} \mathrm{d}x \le (\operatorname{disc} b_+)^{1/q_1}$$

is valid, even without the constraint $\mathrm{d}T = 0$, as a consequence of Jensen's inequality. Instead, the inequality (18), if it was true, would be non-trivial because the map $b \mapsto (\operatorname{disc} b)^{\frac{1}{q}}$ is not concave; it is actually superlinear along the rays passing through the origin.

By duality, every statement about the case $T_{\text{eff}} = T_+$ is equivalent to a statement about the case $T_{\text{eff}} = T_-$. Here, an equivalent *question* is

Let $T(x) : \Lambda^{m*}(\mathbb{R}^n) \to \Lambda^{n-m,*}(\mathbb{R}^n)$ *be* Γ*-periodic, and* b_x *defined by* $b_x(\omega, \omega')\,\mathbf{vol} = (T(x)\omega) \wedge \omega'$ *be symmetric, positive semi-definite and uniformly bounded below.*

If $T_{\text{eff}} = T_-$, *is it true that*

$$\fint_{\mathbb{R}^n/\Gamma} \frac{\mathrm{d}x}{(\operatorname{disc} b_x)^{1/r}} \le \frac{1}{(\operatorname{disc} b_-)^{1/r}}, \qquad r = \binom{n-1}{m-1} \quad ? \tag{19}$$

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